# Supplement to

# "Rank Based Tests for High Dimensional White Noise"

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## 1. Chatterjee's rank Correlation

For a pair of continuous random variables $(X, Y)$, Chatterjee (2021) proposed a new rank correlation, i.e.

$$\xi_n(X, Y) = 1 - \frac{3\sum_{i=1}^{n-1} |r_{i+1} - r_i|}{n^2 - 1}$$

where $r_i$ is the rank of $Y_{(i)}$. Here we rearrange the data as $(X_{(1)}, Y_{(1)}), \cdots, (X_{(n)}, Y_{(n)})$ such that $X_{(1)} \leq \cdots \leq X_{(n)}$. So, we can also consider the test based on Chatterjee's rank correlation of the form

$$\Xi_{ij}(k) = 1 - \frac{3\sum_{t=1}^{n-k} |R^{ij}_{n-k,t+k+1}(k) - R^{ij}_{n-k,t+k}(k)|}{(n-k+1)^2 - 1} \quad (1.1)$$

By Theorem 2.2 in Chatterjee (2021), $\sqrt{n-k+1}\Xi_{ij}(k) \xrightarrow{d} N(0, 2/5)$ as $n \to \infty$ under the null hypothesis. Thus, we propose the following statistics for testing $H_0$:

$$\Xi_n = \max_{1\le k\le K} \max_{1\le i,j\le p} \sqrt{\frac{5(n-k+1)}{2}} |\Xi_{ij}(k)| \tag{1.2}$$

Next, we state the theoretical result about the limiting null distribution of $\Xi_n$.

**THEOREM S1.** *If $(\varepsilon_{t1}, \cdots, \varepsilon_{tp})$ are mutually independent continuous random variables, under $H_0$, for any $y \in \mathbb{R}$, we have*

$$P\left(\Xi_n^2 - 2\log(N) + \log\log(N) \le y\right) = \exp\left\{-\pi^{-1/2}\exp(-y/2)\right\} + o(1)$$

*as $n, p \to \infty$.*

Based on Theorem S1, we proposed the following high dimensional white noise test based on Chatterjee's Correlation

$$T_\alpha^\xi \doteq I\left(\Xi_n^2 - 2\log(Kp^2) + \log\log(Kp^2) \ge q_\alpha\right), \tag{1.3}$$

where $q_\alpha = -\log(\pi) - 2\log\log(1-\alpha)^{-1}$.

Table 1 show the empirical sizes of the proposed test $T_\alpha^\xi$ under the same settings as subsection 3.1 in the main text. We observe that the empirical sizes of $T_\alpha^\xi$ is a little conservative in most cases. Additionally, we also show the power of the proposed test $T_\alpha^\xi$ and the seven tests in the main text in Table 2 with $n = 100, p = 30, \rho = 0.5, k_0 = 2$. The other settings are all the same as subsection 3.2 in the main text. We found that $T_\alpha^\xi$ does not perform very well in most cases, which is consistent with many recent studies (Cao and Bickel, 2020; Shi et al., 2021; Lin and Han, 2023). They showed that independence tests based on Chatterjee's rank correlation are unfortunately rate-inefficient against various local alternatives.

## 2. L-statistic

As shown in the main text, the max-type test statistics performs very well under sparse alternative. Motivated by Chang et al. (2023), we consider an L-statistic for high dimensional white noise test, which combines the first several largest signals together. That is,

$$L_V = \sum_{l=1}^{L} V_{(l)}, \quad L_U = \sum_{l=1}^{L} U_{(l)} \tag{2.4}$$

where $V_{(l)}$ and $U_{(l)}$ are the $l$-th largest maximum of $\{|V_{ij}(k)|\}_{1\le i,j\le p, 1\le k\le K}$ and $\{|U_{ij}(k)|\}_{1\le i,j\le p, 1\le k\le K}$, respectively.

It is difficult to establish the limit null distribution of $L_V$ and $L_U$. So we adopt the permutation test to calculate the critical value of each test. We randomly rearrange $\{\varepsilon_1, \cdots, \varepsilon_T\}$ as $\{\varepsilon_{\pi(1)}, \cdots, \varepsilon_{\pi(T)}\}$ where $\pi$ is a permutation of $\{1, \cdots, T\}$. The permutation test statistic $\tilde{L}_V$ and $\tilde{L}_U$ are accordingly built from the permutation sample $\{\varepsilon_{\pi(1)}, \cdots, \varepsilon_{\pi(T)}\}$. When this procedure is repeated many times, the permutation critical value $z_\alpha^L$

Table 1: Empirical sizes of $L_\xi$ under Models (i)-(viii). ($L_\xi$: the max-type test defined in (1.3).)

| | Models | i | ii | iii | iv | v | vi | vii | viii |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | $p$ | $K = 2$ | | | | | | | |
| 100 | 30 | 0.022 | 0.03 | 0.026 | 0.029 | 0.026 | 0.032 | 0.022 | 0.019 |
| 100 | 60 | 0.026 | 0.019 | 0.028 | 0.026 | 0.027 | 0.02 | 0.019 | 0.024 |
| 100 | 120 | 0.032 | 0.019 | 0.023 | 0.023 | 0.027 | 0.025 | 0.031 | 0.022 |
| 100 | 240 | 0.031 | 0.018 | 0.02 | 0.027 | 0.03 | 0.022 | 0.023 | 0.019 |
| 200 | 30 | 0.033 | 0.025 | 0.039 | 0.03 | 0.039 | 0.03 | 0.038 | 0.035 |
| 200 | 60 | 0.033 | 0.029 | 0.031 | 0.03 | 0.029 | 0.032 | 0.035 | 0.028 |
| 200 | 120 | 0.031 | 0.030 | 0.029 | 0.029 | 0.035 | 0.037 | 0.029 | 0.038 |
| 200 | 240 | 0.030 | 0.035 | 0.039 | 0.027 | 0.025 | 0.033 | 0.038 | 0.027 |
| | | $K = 4$ | | | | | | | |
| 100 | 30 | 0.029 | 0.022 | 0.023 | 0.022 | 0.025 | 0.031 | 0.032 | 0.016 |
| 100 | 60 | 0.016 | 0.022 | 0.021 | 0.025 | 0.02 | 0.031 | 0.028 | 0.022 |
| 100 | 120 | 0.018 | 0.023 | 0.032 | 0.015 | 0.029 | 0.019 | 0.029 | 0.028 |
| 100 | 240 | 0.03 | 0.023 | 0.03 | 0.029 | 0.018 | 0.02 | 0.031 | 0.031 |
| 200 | 30 | 0.036 | 0.033 | 0.038 | 0.033 | 0.04 | 0.032 | 0.032 | 0.033 |
| 200 | 60 | 0.04 | 0.03 | 0.023 | 0.032 | 0.023 | 0.037 | 0.028 | 0.023 |
| 200 | 120 | 0.022 | 0.033 | 0.023 | 0.032 | 0.02 | 0.028 | 0.034 | 0.023 |
| 200 | 240 | 0.02 | 0.038 | 0.035 | 0.036 | 0.028 | 0.024 | 0.022 | 0.02 |
| | | $K = 6$ | | | | | | | |
| 100 | 30 | 0.03 | 0.027 | 0.022 | 0.023 | 0.027 | 0.018 | 0.016 | 0.028 |
| 100 | 60 | 0.015 | 0.028 | 0.028 | 0.032 | 0.03 | 0.03 | 0.032 | 0.018 |
| 100 | 120 | 0.021 | 0.022 | 0.021 | 0.022 | 0.029 | 0.021 | 0.018 | 0.021 |
| 100 | 240 | 0.021 | 0.027 | 0.025 | 0.024 | 0.016 | 0.016 | 0.022 | 0.029 |
| 200 | 30 | 0.023 | 0.028 | 0.02 | 0.013 | 0.026 | 0.027 | 0.024 | 0.032 |
| 200 | 60 | 0.012 | 0.022 | 0.022 | 0.012 | 0.02 | 0.019 | 0.017 | 0.018 |
| 200 | 120 | 0.021 | 0.018 | 0.02 | 0.012 | 0.02 | 0.011 | 0.019 | 0.018 |
| 200 | 240 | 0.023 | 0.021 | 0.012 | 0.010 | 0.022 | 0.012 | 0.025 | 0.022 |

Table 2: Power of tests with $n = 100, p = 30, \rho = 0.5, k_0 = 2$ under Models (I)-(VIII). ($L_\xi$: the max-type test defined in (1.3).)

| | Methods | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Models | $L_r$ | $L_\tau$ | $L_\rho$ | $L_{\tau^*}$ | $L_D$ | $L_R$ | $S_r$ | $L_\xi$ |
| | $K = 2$ | | | | | | | |
| I | 0.22 | 0.22 | 0.22 | 0.38 | 0.42 | 0.37 | 0.22 | 0.03 |
| II | 0.41 | 0.43 | 0.44 | 0.81 | 0.83 | 0.79 | 0.43 | 0.02 |
| III | 0.45 | 0.48 | 0.48 | 0.79 | 0.8 | 0.8 | 0.41 | 0.06 |
| IV | 0.64 | 0.63 | 0.64 | 0.88 | 0.9 | 0.88 | 0.49 | 0.21 |
| V | 0.15 | 0.1 | 0.12 | 0.26 | 0.27 | 0.25 | 0.05 | 0.02 |
| VI | 0.29 | 0.26 | 0.23 | 0.45 | 0.46 | 0.44 | 0.1 | 0.01 |
| VII | 0.38 | 0.33 | 0.37 | 0.54 | 0.55 | 0.55 | 0.16 | 0.06 |
| VIII | 0.26 | 0.24 | 0.23 | 0.42 | 0.49 | 0.43 | 0.17 | 0.02 |
| | $K = 4$ | | | | | | | |
| I | 0.21 | 0.21 | 0.2 | 0.37 | 0.42 | 0.34 | 0.03 | 0.02 |
| II | 0.42 | 0.46 | 0.44 | 0.7 | 0.75 | 0.71 | 0.12 | 0.08 |
| III | 0.44 | 0.42 | 0.42 | 0.76 | 0.81 | 0.75 | 0.16 | 0.03 |
| IV | 0.58 | 0.56 | 0.56 | 0.83 | 0.86 | 0.83 | 0.28 | 0.1 |
| V | 0.11 | 0.06 | 0.06 | 0.21 | 0.26 | 0.19 | 0 | 0.04 |
| VI | 0.17 | 0.17 | 0.17 | 0.37 | 0.38 | 0.36 | 0.03 | 0 |
| VII | 0.18 | 0.18 | 0.18 | 0.47 | 0.51 | 0.45 | 0.06 | 0.01 |
| VIII | 0.13 | 0.16 | 0.16 | 0.39 | 0.42 | 0.38 | 0.04 | 0.03 |
| | $K = 6$ | | | | | | | |
| I | 0.19 | 0.17 | 0.17 | 0.36 | 0.42 | 0.35 | 0.04 | 0.05 |
| II | 0.31 | 0.39 | 0.42 | 0.71 | 0.72 | 0.68 | 0.08 | 0.07 |
| III | 0.42 | 0.45 | 0.45 | 0.75 | 0.78 | 0.75 | 0.02 | 0.01 |
| IV | 0.49 | 0.51 | 0.53 | 0.87 | 0.88 | 0.87 | 0.12 | 0.12 |
| V | 0.1 | 0.07 | 0.06 | 0.13 | 0.19 | 0.12 | 0 | 0.04 |
| VI | 0.07 | 0.11 | 0.09 | 0.26 | 0.3 | 0.24 | 0 | 0.03 |
| VII | 0.2 | 0.19 | 0.2 | 0.38 | 0.38 | 0.37 | 0.02 | 0.04 |
| VIII | 0.14 | 0.18 | 0.16 | 0.31 | 0.34 | 0.31 | 0.01 | 0.02 |

and $z_\alpha^U$ are the empirical $1-\alpha$ quantile of the permutation test statistic, respectively. The tests with rejection region $L_V \geq z_\alpha^L$ and $L_U \geq z_\alpha^U$ are our proposal.

Here we give a simulation study of the $L$-statistics. Let $\tilde{L}_\tau$, $\tilde{L}_\rho$, $\tilde{L}_{\tau^*}$, $\tilde{L}_D$, $\tilde{L}_R$ denote the corresponding $L$-statistics based on Kendall's tau, Spearman's rho, Bergsma-Dassios-Yanagimoto's $\tau^*$, Hoeffding's $D$, Blum-Kiefer-Rosenblatt's $R$, respectively. Table 3 reports the empirical sizes of the above test statistics with $K = 2$ and different $L$ under Model (i). We found that the permutation procedure can control all the empirical sizes of these tests very well. To show the performance of $L$-statistics, we consider a power comparison of $\tilde{L}_{\tau^*}$ with different $L$ because the Bergsma-Dassios-Yanagimoto's $\tau^*$ statistic performs very well in most cases in the simulation studies in the main text. We consider the same settings as subsection 3.2 in the main text except that $k_0 = 1, \cdots, 10$ and $\rho = 0.98k_0^{2/3}$. From Figure 1, we observe that when the number of non-zero autocorrelations is small, $\tilde{L}_{\tau^*}$ with small $L$s have better performance than large $L$s and vice versa. So the optimal $L$ depends on the sparsity of the autocorrelations. Generally speaking, $\tilde{L}_{\tau^*}$ with $L > 1$ outperforms $\tilde{L}_{\tau^*}$ with $L = 1$, i.e. $L_{\tau^*}$ in most cases. So how to derive the limit null distribution of rank based $L$-statistics and choose the optimal $L$ for high dimensional white noise test deserves further studies.

## 3. Additional Simulation Results of Section 3 in the main document

The empirical sizes of the seven test statistics listed in the beginning of Section 3 in the main document are reported in Table 4 and 5, respectively. Figures 2 and 3 report the power curves of the seven test statistics with different $\rho$ for $K = 4$ and 6, respectively.

## 4. Proof of Theorems

### 4.1 Proof of Theorems of Simple Linear Rank Statistics

**Lemma 1.** *Suppose that $X, Y$ are two independent continuous random variables. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be independent observations of $X$ and $Y$. Let $\left\{Q_i^X, i = 1, \ldots, n\right\}$ and $\left\{Q_i^Y, i = 1, \ldots, n\right\}$ be the rank of $X_i$ and $Y_i$ in the samples $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$. Let $\{R_{ni}\}_{i=1}^n$ represent the relative ranks:*

$$R_{ni} = Q_{i'}^Y \text{ subject to } Q_{i'}^X = i$$

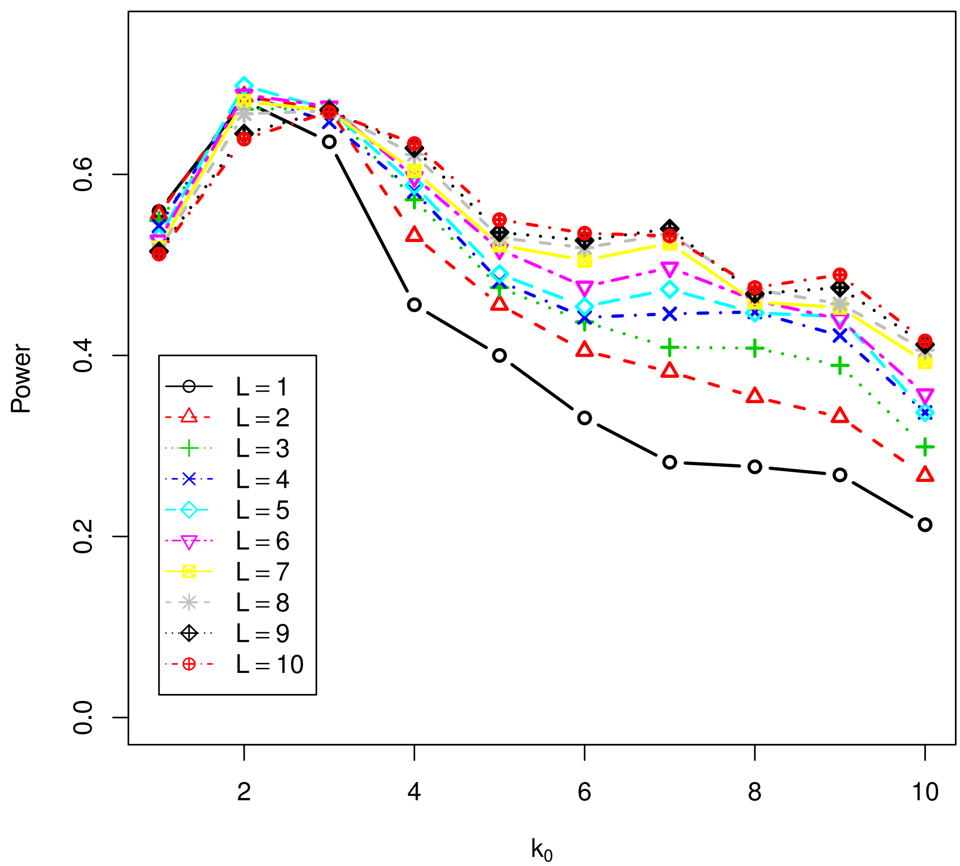


Figure 1: Power curves of different $L$-statistics with different $L$ and $n = 100, p = 30, K = 2$ under Model I.

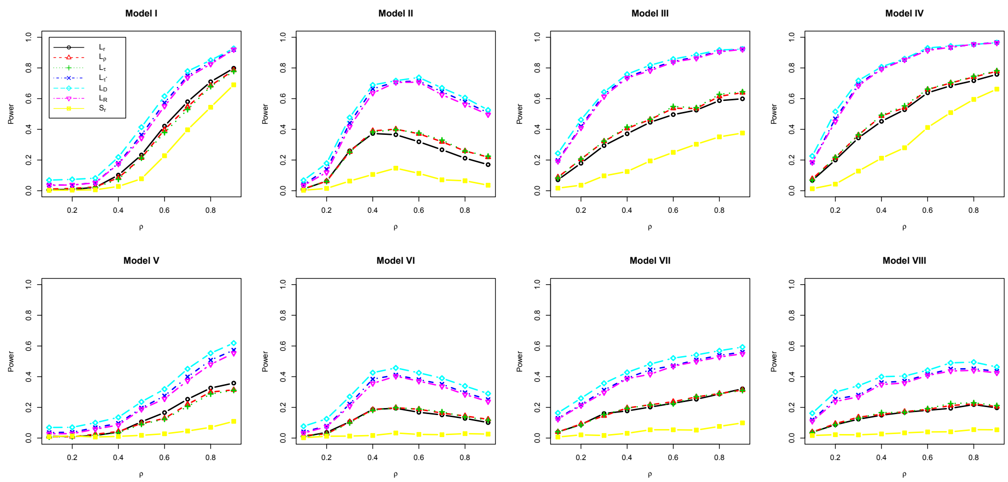


Figure 2: Power curves of different methods with different $\rho$ and $k_0 = 2, n = 100, p = 30, K = 4$.

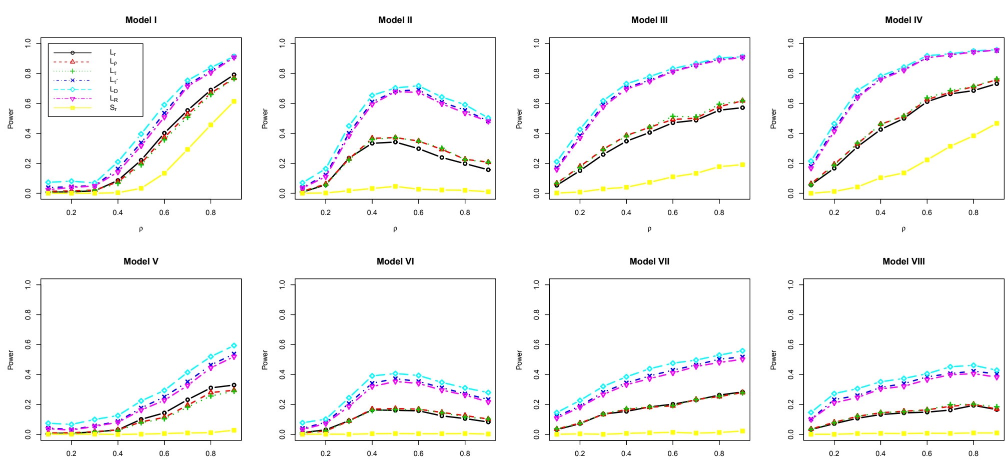


Figure 3: Power curves of different methods with different $\rho$ and $k_0 = 2, n = 100, p = 30, K = 6$.

*We then have* $\{R_{n1}, \ldots, R_{nn}\}$ *are uniformly distributed in all permutations of* $\{1, \ldots, n\}$ *with*

$$\mathrm{pr}\left(R_{n1} = i_1, \ldots, R_{nn} = i_n\right) = \frac{1}{n!}$$

*for any permeation* $\{i_1, \ldots, i_n\}$ *of* $\{1, \ldots, n\}$. *Here n! represents the factorial of n.*

**Lemma 2.** *(Concentration Inequality For Simple Linear Rank Statistics) . Assume the setting and notation in Lemma 1 . Consider the simple linear rank statistic*

$$V \equiv \sum_{i=1}^{n} c_{ni} g\left(\frac{R_{ni}}{n+1}\right) = \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{Q_i^X}{n+1}\right) g\left(\frac{Q_i^Y}{n+1}\right)$$

*where* $f(\cdot)$ *and* $g(\cdot)$ *are Lipschitz functions with Lipschitz constant* $\Delta < \infty$ *and* $\max\{|f(0)|, |g(0)|\} \leq A_2$. *We have, for any* $t > 0$

$$\mathrm{pr}(|V - EV| > t) \leq 2 \exp\left(-Cnt^2\right)$$

*for some scalar* $C$ *only depending on* $\Delta$ *and* $A_2$.

**Lemma 3.** *Suppose that the regularity conditions in Theorem 1 hold. Under the null hypothesis* $H_0$ *holds, we have in the region* $x \in \left(0, O\left(n^{1/6-\epsilon}\right)\right)$ *for some* $\epsilon > 0$

$$\mathrm{pr}\left[\frac{V_{ij}(k) - E_{H_0}\left(V_{ij}(k)\right)}{\left\{\mathrm{var}_{H_0}\left(V_{ij}(k)\right)\right\}^{1/2}} > x\right] = \{1 - \Phi(x)\}\left\{1 + O\left(\frac{1+x^3}{n^{1/2}} + \frac{x}{n^{1/6}}\right)\right\}$$

### 4.1.1 Proof of Theorem 1

And without loss of generality, we assume that $\sum_{i=1}^{n} c_{ni} = 0$. Thus, $E_{H_0}(V_{ij}(k)) = 0$. Let $\{\nu_i\}_{i=1}^{N} = \{V_{ij}(k)/\sigma_V\}_{1\le k\le K, 1\le i,j\le p}$. Define $z = (2\log(N) - \log\log(N) + y)^{1/2}$. By Lemma 3, we have

$$P(|\nu_i| \ge z) = 2\{1-\Phi(z)\}\{1+o(1)\} \sim \frac{1}{\sqrt{\pi}}\frac{e^{-y/2}}{N}$$

Thus,

$$P\left(\max_{i\in C_N}|\nu_i| > z\right) \le |C_N|\cdot P(|\nu_i|\ge z) \to 0$$

as $p\to\infty$. Set $D_N := \{1\le i\le N; |B_{N,i}| < N^{\varsigma}\}$. By assumption, $|D_N|/N \to 1$ as $N\to\infty$ Easily,

$$\begin{aligned} P\left(\max_{i\in D_N}|\nu_i| > z\right) &\le P\left(\max_{1\le i\le N}|\nu_i| > z\right) \\ &\le P\left(\max_{i\in D_N}|\nu_i| > z\right) + P\left(\max_{i\in C_N}|\nu_i| > z\right) \end{aligned}$$

Therefore, to prove Theorem 1, it is enough to show

$$\lim_{N\to\infty} P\left(\max_{i\in D_N}|\nu_i| > z\right) = 1 - \exp\left(-\frac{1}{\sqrt{\pi}}e^{-x/2}\right)$$

as $N\to\infty$. Define

$$\alpha_t = \sum{}^{*} P\left(|\nu_{i_1}| > z, \cdots, |\nu_{i_t}| > z\right)$$

for $1 \le t \le N$, where the sum runs over all $i_1 < \cdots < i_t$ and $i_1 \in D_N, \cdots, i_t \in D_N$. First, we will prove next that

$$\lim_{N\to\infty}\alpha_t = \frac{1}{t!}\pi^{-t/2}e^{-ty/2}$$

for each $t \geq 1$. Because $g(F_j(\varepsilon_{t+k,j}))$ is bounded by a constant $C_g$, thus all the assumptions in Theorem 1.1 in Zaitsev (1987) are satisfied. Thus, we have

$$\begin{aligned}
&\sum^{*} P\left(|Z_{i_1}| > z + \epsilon_n(\log(N))^{-1}, \cdots, |Z_{i_t}| > z + \epsilon_n(\log(N))^{-1}\right) \\
&\quad - \begin{pmatrix} |D_N| \\ t \end{pmatrix} c_1 t^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 t^3 (\log N)^{1/2}}\right) \\
&\leq \sum^{*} P\left(|\nu_{i_1}| > z, \cdots, |\nu_{i_t}| > z\right) \\
&\leq \sum^{*} P\left(|Z_{i_1}| > z - \epsilon_n(\log(N))^{-1}, \cdots, |Z_{i_t}| > z - \epsilon_n(\log(N))^{-1}\right) \\
&\quad + \begin{pmatrix} |D_N| \\ t \end{pmatrix} c_1 t^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 t^3 (\log N)^{1/2}}\right)
\end{aligned}$$

where $(Z_{i_1}, \cdots, Z_{i_t})$ follows a multivariate normal distribution with mean zero and the same covariance matrix with $(\nu_{i_1}, \cdots, \nu_{i_t})$. By the proof of Theorem 2 in Feng et al. (2022a), we have

$$\begin{aligned}
&\sum^{*} P\left(|Z_{i_1}| > z + \epsilon_n(\log(N))^{-1}, \cdots, |Z_{i_t}| > z + \epsilon_n(\log(N))^{-1}\right) \to \frac{1}{t!}\pi^{-t/2} e^{-ty/2} \\
&\sum^{*} P\left(|Z_{i_1}| > z - \epsilon_n(\log(N))^{-1}, \cdots, |Z_{i_t}| > z - \epsilon_n(\log(N))^{-1}\right) \to \frac{1}{t!}\pi^{-t/2} e^{-ty/2}
\end{aligned}$$

with $\epsilon_n \to 0$ and $N \to \infty$. Additionally,

$$\begin{pmatrix} |D_N| \\ t \end{pmatrix} c_1 t^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 t^3 (\log N)^{1/2}}\right) \leq C \begin{pmatrix} N \\ t \end{pmatrix} t^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 t^3 (\log N)^{1/2}}\right) \to 0$$

for $\epsilon_n \to 0$ sufficiently slow. Thus, we have

$$\sum^{*} P\left(|\nu_{i_1}| > z, \cdots, |\nu_{i_t}| > z\right) \to \frac{1}{t!}\pi^{-t/2}e^{-ty/2}.$$

Then, by Bonferroni inequality,

$$\sum_{t=1}^{2k}(-1)^{t-1}\alpha_t \leq P\left(\max_{i\in D_N}|\nu_i| > z\right) \leq \sum_{t=1}^{2k+1}(-1)^{t-1}\alpha_t$$

for any $k \geq 1$. let $N \to \infty$, we have

$$\sum_{t=1}^{2k}(-1)^{t-1}\frac{1}{t!}\left(\frac{1}{\sqrt{\pi}}e^{-x/2}\right)^t \leq \liminf_{N\to\infty} P\left(\max_{i\in D_N}|\nu_i| > z\right)$$
$$\leq \limsup_{N\to\infty} P\left(\max_{i\in D_N}|\nu_i| > z\right) \leq \sum_{t=1}^{2k+1}(-1)^{t-1}\frac{1}{t!}\left(\frac{1}{\sqrt{\pi}}e^{-x/2}\right)^t$$

for each $k \geq 1$. By letting $k \to \infty$ and using the Taylor expansion of the function $1 - e^{-x}$, so we obtain the result.

### 4.1.2 Proof of Theorem 2

By Lemma 2, there exist a constant $c$ such that, for any $t > 0$,

$$P\left(|\hat{V}_{ij}(k) - E(\hat{V}_{ij}(k))| > t\right) \leq 2e^{-t^2/c}.$$

Then,

$$P\left(\max_{1\leq i,j\leq p, 1\leq k\leq K}|\hat{V}_{ij}(k) - E(\hat{V}_{ij}(k))| > t\right) \leq N2e^{-t^2/c}.$$

which implies that, with probability at least $1 - N^{-1}$,

$$\max_{1\leq i,j\leq p, 1\leq k\leq K}|\hat{V}_{ij}(k) - E(\hat{V}_{ij}(k))| \leq \sqrt{3c\log N}.$$

So, for large enough $n$, we have

$$V_n^2/\sigma_V^2 = \max_{1\le i,j\le p,1\le k\le K} \widehat{V}_{ij}(k)^2 \ge \left(\max_{1\le i,j\le p,1\le k\le K}\left|E(\hat{V}_{ij}(k))\right| - \max_{1\le i,j\le p,1\le k\le K}\left|\widehat{V}_{ij}(k) - E(\hat{V}_{ij}(k))\right|\right)^2$$
$$\ge (2+\varrho)\log N$$

for some small positive constant $\varrho$. Accordingly, for any given $q_\alpha$, with probability tending to one,

$$V_n^2/\sigma_V^2 > 2\log N - \log\log N - q_\alpha.$$

Then we complete the proof. □

### 4.1.3 Proof of Theorem 3

According to Theorem 3 in Feng et al. (2022b) and Assumption (A2), we can easily obtain the result. □

## 4.2 Proof of Theorems of Non-Degenerate U-Statistics

**Lemma 4.** *Suppose that $U$ is a $U$-statistic with degree $m$ and bounded kernel $|h(\cdot)| \le M$. We then have, for any $t > 0$,*

$$P(|U - EU| > t) \le 2\exp\left\{-nt^2/\left(2mM^2\right)\right\}.$$

**Lemma 5.** *Suppose that the boundedness assumption in Theorem 4 hold.*

*We then have, in a region* $x \in \left(0, o\left(n^{1/6}\right)\right)$

$$P\left[\frac{U_{ij}(k) - E\left(U_{ij}(k)\right)}{\left\{\operatorname{var}\left(U_{ij}(k)\right)\right\}^{1/2}} > x\right] = \{1 - \Phi(x)\}\left\{1 + O\left(\frac{1 + x^3}{n^{1/2}}\right)\right\}.$$

### 4.2.1 Proof of Theorem 4

First, we consider the following U-statistics with bounded and symmetric kernels, i.e.

$$U_{ij}(k) = \frac{1}{C_{n-k}^m} \sum_{1 \le t_1 < t_2, \cdots, < t_m \le n-k} h((\varepsilon_{t_1,i}, \varepsilon_{t_1+k,j})^\top, \cdots, (\varepsilon_{t_m,i}, \varepsilon_{t_m+k,j})^\top) \tag{4.5}$$

Here we define $\{u_s\}_{s=1}^N = \{U_{ij}(k)\}_{1 \le i,j \le p, 1 \le k \le K}$ and $\{\boldsymbol{X}_{tijk}\} = \{(\varepsilon_{t,i}, \varepsilon_{t+k,j})^\top\}_{1 \le t \le n-k}$.

So we rewrite $U_{ij}(k)$ in the following forms

$$u_s = \frac{1}{C_{n-k_s}^m} \sum_{1 \le t_1 < t_2, \cdots, < t_m \le n-k_s} h(\boldsymbol{X}_{t_1, i_s j_s k_s}, \cdots, \boldsymbol{X}_{t_m, i_s j_s k_s}) \tag{4.6}$$

Without loss of generality, we assume that $E(u_s) = 0$. By the condition, we have

$$\begin{aligned} \mu_q \doteq & E|h(\boldsymbol{X}_{t_1}, \cdots, \boldsymbol{X}_{t_m})|^q < \infty, \\ \psi_s(x) = & E(h(\boldsymbol{X}_{t_1}, \cdots, \boldsymbol{X}_{t_m})|\boldsymbol{X}_{t_1} = x), \quad \sigma_\psi^2 = \operatorname{var}(\psi_s(\boldsymbol{X}_{t_1})) > 0. \end{aligned}$$

for any $q \ge 2$. By Lemma 1 in Malevich and Abdalimov (1979), we can rewrite $u_s$ as follow

$$u_s = S_s + \eta_s, \quad S_s = \frac{m}{n - k_s} \sum_{i=1}^{n-k_s} \psi_s(\boldsymbol{X}_i), \quad \eta_s = \sum_{l=2}^{m} C_m^l u_{s,l}$$

where $u_{s,l}$ is a $U$-statistics of the form (4.6) with kernel $\psi^{(l)}(x_1, \cdots, x_l)$ such that

$$\mathbf{P}\left\{\mathbf{E}\left[\psi^{(l)}\left(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_l\right) \mid \boldsymbol{X}_1, \cdots, \boldsymbol{X}_{l-1}\right] = 0\right\} = 1,$$

$$\mathbf{E}\left|\psi^{(l)}\left(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_l\right)\right|^q \leqq 2^{mq}\mu_q.$$

By Lemma 2 in Malevich and Abdalimov (1979), we have

$$E|C_{n-k_s}^l u_{s,l}|^q \le C(n-k_s)^{lq/2}.$$

Thus, by the Markov inequality,

$$\begin{aligned}
&P(\max_{1\le s\le N}(n-k_s)^{1/2}|\eta_s| > (\log N)^{-1}) \\
&\le NP(|\eta_s| > (\log N)^{-1}(n-k_s)^{-1/2}) \\
&\le N(\log N)^{2q}(n-k_s)^q E(|\eta_s|^{2q}) \\
&\le N(\log N)^{2q}(n-k_s)^q E\left(\left|\sum_{l=2}^m C_m^l u_{s,l}\right|^{2q}\right) \\
&= N(\log N)^{2q}(n-k_s)^q m^{2q} E\left(\left|\frac{1}{m}\sum_{l=2}^m C_m^l u_{s,l}\right|^{2q}\right) \\
&\le N(\log N)^{2q}(n-k_s)^q m^{2q-1}\left\{\sum_{l=2}^m (C_m^l)^{2q} E(|u_{s,l}|^{2q})\right\} \\
&\le CN(\log N)^{2q}(n-k_s)^q m^{2q-1}\left\{\sum_{l=2}^m (C_m^l)^{2q}(n-k_s)^{-lq}\right\} \\
&\le Cm^{2q-1}\left\{\sum_{l=2}^m (C_m^l)^{2q}\right\} N(\log N)^{2q}(n-k_s)^q(n-k_s)^{-2q}
\end{aligned}$$

$$= C_0(m,q)N(\log N)^{2q}(n-k_s)^{-q} \to 0,$$

for some positive integer $q$ by $N = o(n^{\epsilon})$. Thus, by

$$\left|\max_{1\le s\le N}(n-k_s)u_s^2 - \max_{1\le s\le N}(n-k_s)S_s^2\right| \le 2\max_{1\le s\le N}(n-k_s)^{1/2}|S_s|\max_{1\le s\le N}(n-k_s)^{1/2}|\eta_s| + \max_{1\le s\le N}(n-k_s)\eta_s^2$$

we only need to show that

$$\mathrm{P}\left(\max_{1\le s\le N}(n-k_s)S_s^2/\sigma_U^2 - 2\log(Kp^2) + \log\log(Kp^2) \leqslant y\right) \to \exp\left\{-\pi^{-1/2}\exp(-y/2)\right\}$$

Here we define $\upsilon_s = (n-k_s)^{1/2}S_s/\sigma_U$ and $z = (2\log(N) - \log\log(N) + y)^{1/2}$. Since $(n-k_s)^{1/2}\eta_s$ is negligible, the tail behavior of $\upsilon_s$ is the same as that of $(n-k_s)^{1/2}u_s/\sigma_U$. Therefore, by Lemma 5, we have

$$P(|\upsilon_i| \ge z) = 2\{1-\Phi(z)\}\{1+o(1)\} \sim \frac{1}{\sqrt{\pi}}\frac{e^{-y/2}}{N}$$

Thus,

$$P\left(\max_{i\in C_N}|\upsilon_i| > z\right) \le |C_N| \cdot P(|\upsilon_i| \ge z) \to 0$$

as $p \to \infty$. Set $D_N := \{1 \le i \le N; |B_{N,i}| < N^{\varsigma}\}$. By assumption, $|D_N|/N \to 1$ as $N \to \infty$ Easily,

$$\begin{aligned} P\left(\max_{i\in D_N}|\upsilon_i| > z\right) &\le P\left(\max_{1\le i\le N}|\upsilon_i| > z\right) \\ &\le P\left(\max_{i\in D_N}|\upsilon_i| > z\right) + P\left(\max_{i\in C_N}|\upsilon_i| > z\right) \end{aligned}$$

Therefore, to prove Theorem 4, it is enough to show

$$\lim_{N\to\infty} P\left(\max_{i\in D_N}|\upsilon_i| > z\right) = 1 - \exp\left(-\frac{1}{\sqrt{\pi}}e^{-x/2}\right)$$

as $N \to \infty$. Define

$$\alpha_t = \sum^{*} P\left(|\upsilon_{i_1}| > z, \cdots, |\upsilon_{i_t}| > z\right)$$

for $1 \leq t \leq N$, where the sum runs over all $i_1 < \cdots < i_t$ and $i_1 \in D_N, \cdots, i_t \in D_N$. First, we will prove next that

$$\lim_{N\to\infty} \alpha_t = \frac{1}{t!}\pi^{-t/2}e^{-ty/2}$$

for each $t \geq 1$. Because $\psi_s(\boldsymbol{X}_i)$ is bounded, thus all the assumptions in Theorem 1.1 in Zaitsev (1987) are satisfied. Thus, we have

$$\begin{aligned}
&\sum^{*} P\left(|Z_{i_1}| > z + \epsilon_n(\log(N))^{-1/2}, \cdots, |Z_{i_t}| > z + \epsilon_n(\log(N))^{-1/2}\right) \\
&- \begin{pmatrix} |D_N| \\ t \end{pmatrix} c_1 t^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 t^3(\log N)^{1/2}}\right) \\
\leq &\sum^{*} P\left(|\upsilon_{i_1}| > z, \cdots, |\upsilon_{i_t}| > z\right) \\
\leq &\sum^{*} P\left(|Z_{i_1}| > z - \epsilon_n(\log(N))^{-1/2}, \cdots, |Z_{i_t}| > z - \epsilon_n(\log(N))^{-1/2}\right) \\
&+ \begin{pmatrix} |D_N| \\ t \end{pmatrix} c_1 t^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 t^3(\log N)^{1/2}}\right)
\end{aligned}$$

where $(Z_{i_1}, \cdots, Z_{i_t})$ follows a multivariate normal distribution with mean zero and the same covariance matrix with $(\upsilon_{i_1}, \cdots, \upsilon_{i_t})$. By the proof of Theorem 2 in Feng et al. (2022a), we have

$$\sum^{*} P\left(|Z_{i_1}| > z + \epsilon_n(\log(N))^{-1/2}, \cdots, |Z_{i_t}| > z + \epsilon_n(\log(N))^{-1/2}\right) \to \frac{1}{t!}\pi^{-t/2}e^{-ty/2}$$

$$\sum^{*} P\left(|Z_{i_1}| > z - \epsilon_n(\log(N))^{-1/2}, \cdots, |Z_{i_t}| > z - \epsilon_n(\log(N))^{-1/2}\right) \to \frac{1}{t!}\pi^{-t/2}e^{-ty/2}$$

with $\epsilon_n \to 0$ and $N \to \infty$. Additionally,

$$\begin{pmatrix} |D_N| \\ t \end{pmatrix} c_1 t^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 t^3 (\log N)^{1/2}}\right) \le C \begin{pmatrix} N \\ t \end{pmatrix} t^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 t^3 (\log N)^{1/2}}\right) \to 0$$

for $\epsilon_n \to 0$ sufficiently slow. Thus, we have

$$\sum^{*} P\left(|\upsilon_{i_1}| > z, \cdots, |\upsilon_{i_t}| > z\right) \to \frac{1}{t!}\pi^{-t/2}e^{-ty/2}.$$

Then, by Bonferroni inequality,

$$\sum_{t=1}^{2k}(-1)^{t-1}\alpha_t \le P\left(\max_{i\in D_N}|\upsilon_i| > z\right) \le \sum_{t=1}^{2k+1}(-1)^{t-1}\alpha_t$$

for any $k \ge 1$. let $N \to \infty$, we have

$$\begin{aligned}\sum_{t=1}^{2k}(-1)^{t-1}\frac{1}{t!}\left(\frac{1}{\sqrt{\pi}}e^{-x/2}\right)^t &\le \liminf_{N\to\infty} P\left(\max_{i\in D_N}|\upsilon_i| > z\right) \\ &\le \limsup_{N\to\infty} P\left(\max_{i\in D_N}|\upsilon_i| > z\right) \le \sum_{t=1}^{2k+1}(-1)^{t-1}\frac{1}{t!}\left(\frac{1}{\sqrt{\pi}}e^{-x/2}\right)^t\end{aligned}$$

for each $k \ge 1$. By letting $k \to \infty$ and using the Taylor expansion of the function $1 - e^{-x}$, so we obtain the result. □

### 4.2.2 Proof of Theorem 5

By Lemma 4, we have

$$P\left(|\hat{U}_{ij}(k) - E(\hat{U}_{ij}(k))| > t\right) \le 2e^{t^2/c}$$

for some positive constant $c$. Taking the same procedure as Theorem 2, we can also obtain the result. $\square$

### 4.2.3 Proof of Theorem 6

According to Theorem 3 in Feng et al. (2022b) and Assumption (A3), we can easily obtain the result. $\square$

## 4.3 Proof of Theorems of Degenerate U-Statistics

### 4.3.1 Proof of Theorem 7

**Lemma 6.** *For $N \geq 1$, let $\iota_N$ be positive integers with $\lim_{N\to\infty} \iota_N/N = 1$. Let $Y_{iv}, i = 1, \cdots, \iota_N, v = 1, \cdots, M$ be $N(0,1)$-distributed random variables and $\mathrm{cov}(Y_{iv}, Y_{is}) = 0$ for $v \neq s$. Let $\boldsymbol{Y}_i = (Y_{i1}, \cdots, Y_{iM})^\top$ and $\Xi_{ij} = \mathrm{cov}(\boldsymbol{Y}_i, \boldsymbol{Y}_j)$. Assume $|\lambda_{max}(\Xi_{ij}\Xi_{ij}^\top)| \leq \delta_N^{2+2c}$ for $c > 0$ and all $1 \leq i < j \leq \iota_N$ , where $\{\delta_N; N \geq 1\}$ are constants satisfying $0 < \delta_N = o(1/\log N)$. Define $W_{i_k} = \sum_{v=1}^M \lambda_v Y_{i_k v}^2$. Given $x \in \mathbb{R}$, set $z = 2\lambda_1 \log(N) + \lambda_1 (\mu_1 - 2) \log\log(N) + \lambda_1 y + o(1/\log(N))$. Then, for any fixed $m \geq 1$, we have*

$$\left(\frac{\Gamma(\mu_1/2)}{\kappa}\frac{N}{e^{-y/2}}\right)^m \cdot P(W_{i_1} > z, \cdots, W_{i_m} > z) \to 1 \tag{4.7}$$

*as $N \to \infty$ uniformly for all $1 \leq i_1 < \cdots < i_m \leq \iota_N$.*

**Proof of Lemma 6** For $m = 1$, (4.7) is followed by Equation (6) in Zolotarev (1962). Assume Equation (4.7) holds with $m = k - 1$. We will prove it also holds with $m = k$.

Define $\boldsymbol{Y}_S = (\boldsymbol{Y}_{i_1}^\top, \cdots, \boldsymbol{Y}_{i_{m-1}}^\top)^\top$, $\boldsymbol{\Sigma}_{i_m S} = \text{cov}(\boldsymbol{Y}_{i_m}, \boldsymbol{Y}_S)$ and $\boldsymbol{\Sigma}_{S i_m} = \boldsymbol{\Sigma}_{i_m S}^\top$. So $\boldsymbol{Y}_{i_m} = (\boldsymbol{Y}_{i_m} - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{Y}_S) + \boldsymbol{\Sigma}_{i_m S}\boldsymbol{Y}_S \doteq U_Y + V_Y$ . Thus, by the conditional distribution of multivariate normal distributions, we have $\boldsymbol{Y}_{i_m} - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{Y}_S \sim N(\mathbf{0}, \mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})$ is independent of $\boldsymbol{Y}_S$. Define $\mathbf{A} = \text{diag}\{\lambda_1, \cdots, \lambda_M\}$. Thus, we have

$$
\begin{aligned}
& P(W_{i_1} > z, \cdots, W_{i_m} > z) \\
=& P\left(U_Y^\top \mathbf{A} U_Y + 2U_Y^\top \mathbf{A} V_Y + V_Y^\top \mathbf{A} V_Y \geq z, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A} \boldsymbol{Y}_{i_l} \geq z\right) \\
=& P\left(U_Y^\top \mathbf{A} U_Y + 2U_Y^\top \mathbf{A} V_Y + V_Y^\top \mathbf{A} V_Y \geq z, 2U_Y^\top \mathbf{A} V_Y + V_Y^\top \mathbf{A} V_Y \leq C\delta_N, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A} \boldsymbol{Y}_{i_l} \geq z\right) \\
& + P\left(U_Y^\top \mathbf{A} U_Y + 2U_Y^\top \mathbf{A} V_Y + V_Y^\top \mathbf{A} V_Y \geq z, 2U_Y^\top \mathbf{A} V_Y + V_Y^\top \mathbf{A} V_Y > C\delta_N, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A} \boldsymbol{Y}_{i_l} \geq z\right) \\
\leq& P\left(U_Y^\top \mathbf{A} U_Y \geq z - C\delta_N, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A} \boldsymbol{Y}_{i_l} \geq z\right) + P\left(2U_Y^\top \mathbf{A} V_Y + V_Y^\top \mathbf{A} V_Y > C\delta_N\right) \\
\leq& P\left(U_Y^\top \mathbf{A} U_Y \geq z - C\delta_N\right) P\left(\min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A} \boldsymbol{Y}_{i_l} \geq z\right) + P\left(U_Y^\top \mathbf{A} V_Y > C\delta_N/4\right) + P\left(V_Y^\top \mathbf{A} V_Y > C\delta_N/2\right)
\end{aligned}
$$

By $U_Y \sim N(\mathbf{0}, \mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})$, we have

$$
U_Y^\top \mathbf{A} U_Y \sim \xi_U^\top (\mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})^{1/2} \mathbf{A} (\mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})^{1/2} \xi_U
$$

where $\xi_U \sim N(\mathbf{0}, \mathbf{I}_M)$. Define the eigenvalues of $(\mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})^{1/2}\mathbf{A}(\mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})^{1/2}$ are $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_M$ and $\tilde{\Lambda}, \tilde{\kappa}, \tilde{\mu}_1$ are the corresponding parameters as in Proposition 3.2 in Drton et al. (2020). Because $\lambda_{max}(\mathbf{\Sigma}_{i_m S}\mathbf{\Sigma}_{S i_m}) \leq \delta_N^{2+2c}$, $\tilde{\lambda}_1 = \lambda_1(1 + o(\delta_N))$. So does $\tilde{\Lambda}, \tilde{\kappa}, \tilde{\mu}_1$. So by Equation (6) in Zolotarev (1962), we have

$$P\left(U_Y^\top \mathbf{A} U_Y \geq z - C\delta_N\right) \to \frac{\tilde{\kappa}}{\Gamma(\tilde{\mu}_1/2)}\left(\frac{z - C\delta_N + \tilde{\Lambda}}{2\tilde{\lambda}_1}\right)^{\tilde{\mu}_1/2-1} \exp\left(-\frac{z - C\delta_N + \tilde{\Lambda}}{2\tilde{\lambda}_1}\right)$$
$$\to \frac{\kappa}{\Gamma\left(\mu_1/2\right)}\frac{e^{-y/2}}{N}$$

by the assumption $\delta_N = o(1/\log N)$. So by the

$$P\left(U_Y^\top \mathbf{A} U_Y \geq z - C\delta_N\right) P\left(\min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A} \boldsymbol{Y}_{i_l} \geq z\right) \to \left(\frac{\kappa}{\Gamma\left(\mu_1/2\right)}\frac{e^{-y/2}}{N}\right)^m$$

Next, we will show that $P\left(U_Y^\top \mathbf{A} V_Y > C\delta_N/4\right) + P\left(V_Y^\top \mathbf{A} V_Y > C\delta_N/2\right) = o(N^{-m})$. Similarly,

$$V_Y^\top \mathbf{A} V_Y \sim \xi_V^\top \mathbf{\Sigma}_{S i_m} \mathbf{A} \mathbf{\Sigma}_{i_m S} \xi_V$$

where $\xi_V \sim N(\mathbf{0}, \mathbf{I}_{(m-1)M})$. Define the eigenvalues of $\mathbf{\Sigma}_{S i_m}\mathbf{A}\mathbf{\Sigma}_{i_m S}$ are $\zeta_1, \cdots, \zeta_M$. So

$$V_Y^\top \mathbf{A} V_Y \sim \sum_{k=1}^{(m-1)M} \zeta_k \xi_k^2$$

where $\xi_k$ are all independently distributed as $N(0,1)$. Thus, for small enough constant $\varpi$,

$$E(\exp(\varpi\delta_N^{-1-c} V_Y^\top \mathbf{A} V_Y)) = \prod_{k=1}^{M} E e^{\varpi\delta_N^{-1-c}\zeta_k\xi_k^2} = \exp\left\{-\frac{1}{2}\sum_{k=1}^{(m-1)M} \log[1 - 2\varpi\delta_N^{-1-c}\zeta_k]\right\}$$

$$\leq \exp\left(2\varpi\delta_N^{-1-c}\sum_{k=1}^{(m-1)M}\zeta_k\right)$$

In addition,

$$\sum_{k=1}^{(m-1)M}\zeta_k = \mathrm{tr}(\mathbf{\Sigma}_{Si_m}\mathbf{A}\mathbf{\Sigma}_{i_mS}) \leq \lambda_{max}(\mathbf{\Sigma}_{Si_m}\mathbf{\Sigma}_{i_mS})\mathrm{tr}(\mathbf{A}) \leq \delta_N^{2+2c}\Lambda.$$

So $E(\exp(\varpi\delta_N^{-1-c}V_Y^\top\mathbf{A}V_Y)) \leq \exp\left(2\varpi\delta_N^{1+c}\Lambda\right) \leq C_2$ for some constant $C_2 > 0$. By the Markov inequality, we have

$$\begin{aligned}P\left(V_Y^\top\mathbf{A}V_Y > C\delta_N/2\right) &= P\left(\varpi\delta_N^{-1-c}V_Y^\top\mathbf{A}V_Y > C\varpi\delta_N^{-c}/2\right)\\ &\leq \exp\left(-C\varpi\delta_N^{-c}/2\right)E(\exp(\varpi\delta_N^{-1-c}V_Y^\top\mathbf{A}V_Y))\\ &\leq C_2\exp\left(-C\varpi\delta_N^{-c}/2\right) = o(N^{-m})\end{aligned}$$

for large enough constant $C$. Similarly,

$$U_Y^\top\mathbf{A}V_Y \sim \sum_{k=1}^{M}\rho_k\xi_k\eta_k$$

where $\eta_k$ are all independently distributed as $N(0,1)$, $\rho_1,\cdots,\rho_M$ are the singular values of $(\mathbf{I}_M - \mathbf{\Sigma}_{i_mS}\mathbf{\Sigma}_{Si_m})^{1/2}\mathbf{A}\mathbf{\Sigma}_{i_mS}$. And then

$$\begin{aligned}&P\left(U_Y^\top\mathbf{A}V_Y > C\delta_N/4\right)\\ =&P\left(\varpi\delta_N^{-1-c}U_Y^\top\mathbf{A}V_Y > C\varpi\delta_N^{-c}/4\right)\\ \leq&\exp\left(-C\varpi\delta_N^{-c}/4\right)E(\exp(\varpi\delta_N^{-1-c}U_Y^\top\mathbf{A}V_Y))\\ \leq&\exp\left(-C\varpi\delta_N^{-c}/4\right)\exp\left(-\frac{1}{2}\sum_{k=1}^{M}\log[1-\varpi^2\delta_N^{-2-2c}\rho_k^2]\right)\end{aligned}$$

$$\leq \exp\left(-C\varpi\delta_N^{-c}/4\right)\exp\left(\varpi^2\delta_N^{-2-2c}\sum_{k=1}^{M}\rho_k^2\right)$$

$$\leq \exp\left(-C\varpi\delta_N^{-c}/4\right)\exp\left(\varpi^2\delta_N^{-2-2c}\mathrm{tr}(((\mathbf{I}_M-\mathbf{\Sigma}_{i_mS}\mathbf{\Sigma}_{Si_m})^{1/2}\mathbf{A}\mathbf{\Sigma}_{i_mS})^2)\right)$$

$$\leq \exp\left(-C\varpi\delta_N^{-c}/4\right)\exp\left(\varpi^2\delta_N^{-2-2c}\lambda_{max}(\mathbf{\Sigma}_{Si_m}\mathbf{\Sigma}_{i_mS})\mathrm{tr}(\mathbf{A}^2)\right)$$

$$\leq \exp\left(-C\varpi\delta_N^{-c}/4\right)\exp\left(\varpi^2\mathrm{tr}(\mathbf{A}^2)\right)=o(N^{-m})$$

for large enough constant $C>0$. Thus, we have

$$P(W_{i_1}>z,\cdots,W_{i_m}>z)\leq\left(\frac{\kappa}{\Gamma\left(\mu_1/2\right)}\frac{e^{-y/2}}{N}\right)^m+o(N^{-m}). \qquad (4.8)$$

Further more,

$$P(W_{i_1}>z,\cdots,W_{i_m}>z)$$
$$=P\left(U_Y^\top\mathbf{A}U_Y+2U_Y^\top\mathbf{A}V_Y+V_Y^\top\mathbf{A}V_Y\geq z,\min_{1\leq l\leq m-1}\boldsymbol{Y}_{i_l}^\top\mathbf{A}\boldsymbol{Y}_{i_l}\geq z\right)$$
$$\geq P\left(U_Y^\top\mathbf{A}U_Y\geq z+C\delta_N,\min_{1\leq l\leq m-1}\boldsymbol{Y}_{i_l}^\top\mathbf{A}\boldsymbol{Y}_{i_l}\geq z\right)-P\left(2U_Y^\top\mathbf{A}V_Y+V_Y^\top\mathbf{A}V_Y\leq -C\delta_N\right)$$
$$\geq P\left(U_Y^\top\mathbf{A}U_Y\geq z+C\delta_N\right)P\left(\min_{1\leq l\leq m-1}\boldsymbol{Y}_{i_l}^\top\mathbf{A}\boldsymbol{Y}_{i_l}\geq z\right)-P\left(2U_Y^\top\mathbf{A}V_Y+V_Y^\top\mathbf{A}V_Y\leq -C\delta_N\right),$$

where the first inequality is based on the fact that:

$$P\left(\min_{1\leq l\leq m-1}\boldsymbol{Y}_{i_l}^\top\mathbf{A}\boldsymbol{Y}_{i_l}\geq z\right)-P\left(U_Y^\top\mathbf{A}U_Y+2U_Y^\top\mathbf{A}V_Y+V_Y^\top\mathbf{A}V_Y\geq z,\min_{1\leq l\leq m-1}\boldsymbol{Y}_{i_l}^\top\mathbf{A}\boldsymbol{Y}_{i_l}\geq z\right)$$
$$=P\left(U_Y^\top\mathbf{A}U_Y+2U_Y^\top\mathbf{A}V_Y+V_Y^\top\mathbf{A}V_Y<z,\min_{1\leq l\leq m-1}\boldsymbol{Y}_{i_l}^\top\mathbf{A}\boldsymbol{Y}_{i_l}\geq z\right)$$
$$=P\left(U_Y^\top\mathbf{A}U_Y+2U_Y^\top\mathbf{A}V_Y+V_Y^\top\mathbf{A}V_Y<z,2U_Y^\top\mathbf{A}V_Y+V_Y^\top\mathbf{A}V_Y<-C\delta_N,\min_{1\leq l\leq m-1}\boldsymbol{Y}_{i_l}^\top\mathbf{A}\boldsymbol{Y}_{i_l}\geq z\right)$$
$$+P\left(U_Y^\top\mathbf{A}U_Y+2U_Y^\top\mathbf{A}V_Y+V_Y^\top\mathbf{A}V_Y<z,2U_Y^\top\mathbf{A}V_Y+V_Y^\top\mathbf{A}V_Y\geq -C\delta_N,\min_{1\leq l\leq m-1}\boldsymbol{Y}_{i_l}^\top\mathbf{A}\boldsymbol{Y}_{i_l}\geq z\right)$$

$$\leq P\left(2U_Y^\top \mathbf{A} V_Y + V_Y^\top \mathbf{A} V_Y < -C\delta_N\right) + P\left(U_Y^\top \mathbf{A} U_Y < z + C\delta_N, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A} \boldsymbol{Y}_{i_l} \geq z\right).$$

Obviously,

$$P\left(U_Y^\top \mathbf{A} U_Y \geq z + C\delta_N\right) P\left(\min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A} \boldsymbol{Y}_{i_l} \geq z\right) \to \left(\frac{\kappa}{\Gamma\left(\mu_1/2\right)} \frac{e^{-y/2}}{N}\right)^m$$

by $\delta_N = o(1/\log N)$. Next,

$$\begin{aligned} &P\left(2U_Y^\top \mathbf{A} V_Y + V_Y^\top \mathbf{A} V_Y \leq -C\delta_N\right) \\ \leq &P\left(U_Y^\top \mathbf{A} V_Y \leq -C\delta_N/2\right) = P\left(U_Y^\top \mathbf{A} V_Y \geq C\delta_N/2\right) = o(N^{-m}). \end{aligned}$$

So

$$P(W_{i_1} > z, \cdots, W_{i_m} > z) \geq \left(\frac{\kappa}{\Gamma\left(\mu_1/2\right)} \frac{e^{-y/2}}{N}\right)^m + o(N^{-m}). \tag{4.9}$$

Then, we obtain the result by (4.8) and (4.9). □

**Lemma 7.** *Let* $W_i = \sum_{v=1}^M \lambda_v Y_{iv}^2$ *where* $Y_{iv} \overset{i.i.d}{\sim} N(0,1)$ *for* $v = 1, \cdots, M$. *Define* $\boldsymbol{Y}_i = (Y_{i1}, \cdots, Y_{iM})^\top$ *and* $\boldsymbol{Y}_S = (\boldsymbol{Y}_{i_1}^\top, \cdots, \boldsymbol{Y}_{i_{m-1}}^\top)^\top$ *where* $S = \{i_1, \cdots, i_{m-1}\}$ *which for any* $i, j \in S$, $\lambda_{max}(\Xi_{ij}\Xi_{ij}^\top) \leq \delta_N^{2+2c_0}$ *for some constant* $c_0 > 0$. *Let* $\Xi_{ij} = \mathrm{cov}(\boldsymbol{Y}_i, \boldsymbol{Y}_j)$. *Let* $\boldsymbol{Y}_{i_m}$ *satisfy* $\max_{1\leq j\leq m-1} |\lambda_{max}(\Xi_{i_m j}\Xi_{i_m j}^\top)| > \delta_N^{2+2c_0}$ *and* $\lambda_{max}(\boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m}) \leq \delta \in (0,1)$ *where* $\boldsymbol{\Sigma}_{i_m S} = \mathrm{cov}(\boldsymbol{Y}_{i_m}, \boldsymbol{Y}_S)$ *and* $\boldsymbol{\Sigma}_{S i_m} = \boldsymbol{\Sigma}_{i_m S}^\top$. *Then, we have*

$$P\left(W_{i_m} \geq z, \min_{1\leq l\leq m-1} W_{i_l} \geq z\right) \leq C(\log N)^c N^{\varrho - m} \tag{4.10}$$

*where* $\varrho \in (0,1)$, $c, C$ *are some positive constants.*

**Proof.** Define $\boldsymbol{Y}_{i_m} = (\boldsymbol{Y}_{i_m} - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{Y}_S) + \boldsymbol{\Sigma}_{i_m S}\boldsymbol{Y}_S \doteq U_Y + V_Y$ . Thus, by the conditional distribution of multivariate normal distributions, we have $\boldsymbol{Y}_{i_m} - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{Y}_S \sim N(\mathbf{0}, \mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})$ is independent of $\boldsymbol{Y}_S$. Define $\mathbf{A} = \mathrm{diag}\{\lambda_1, \cdots, \lambda_M\}$. Thus, we have

$$
\begin{aligned}
& P\left(W_{i_m} \geq z, \min_{1\leq l\leq m-1} W_{i_l} \geq z\right) \\
=& P\left(\boldsymbol{Y}_{i_m}^\top \mathbf{A}\boldsymbol{Y}_{i_m} \geq z, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A}\boldsymbol{Y}_{i_l} \geq z\right) \\
=& P\left(U_Y^\top \mathbf{A} U_Y + 2U_Y^\top \mathbf{A} V_Y + V_Y^\top \mathbf{A} V_Y \geq z, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A}\boldsymbol{Y}_{i_l} \geq z\right) \\
\leq& P\left(2U_Y^\top \mathbf{A} U_Y + 2V_Y^\top \mathbf{A} V_Y \geq z, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A}\boldsymbol{Y}_{i_l} \geq z\right) \\
\leq& P\left(U_Y^\top \mathbf{A} U_Y \geq \frac{1}{4}z, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A}\boldsymbol{Y}_{i_l} \geq z\right) + P\left(V_Y^\top \mathbf{A} V_Y \geq \frac{1}{4}z, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A}\boldsymbol{Y}_{i_l} \geq z\right) \\
=& P\left(U_Y^\top \mathbf{A} U_Y \geq \frac{1}{4}z\right) P\left(\min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A}\boldsymbol{Y}_{i_l} \geq z\right) + P\left(V_Y^\top \mathbf{A} V_Y \geq \frac{1}{4}z, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A}\boldsymbol{Y}_{i_l} \geq z\right)
\end{aligned}
$$

By Lemma 6, we have

$$
P\left(\min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A}\boldsymbol{Y}_{i_l} \geq z\right) \leq (1+2\epsilon)^{m-1}\left(\frac{\kappa}{\Gamma(\mu_1/2)}\frac{e^{-y/2}}{N}\right)^{m-1} \leq CN^{1-m}
$$

By $U_Y \sim N(\mathbf{0}, \mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})$, we have

$$
U_Y^\top \mathbf{A} U_Y \sim \xi_U^\top (\mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})^{1/2}\mathbf{A}(\mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})^{1/2}\xi_U
$$

where $\xi_U \sim N(\mathbf{0}, \mathbf{I}_M)$. So

$$
\tilde{\lambda}_1 \doteq \lambda_{max}(((\mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})^{1/2}\mathbf{A}(\mathbf{I}_M - \boldsymbol{\Sigma}_{i_m S}\boldsymbol{\Sigma}_{S i_m})^{1/2})
$$

$$\geq \lambda_{max}(\mathbf{A})\lambda_{\min}(\mathbf{I}_M - \mathbf{\Sigma}_{i_m S}\mathbf{\Sigma}_{S i_m}) \geq \lambda_1(1-\delta)$$

by the assumption. By Equation (6) in Zolotarev (1962), we have

$$\begin{aligned} P\left(U_Y^\top \mathbf{A} U_Y \geq \frac{1}{4}z\right) \leq& (1+\epsilon)\frac{\tilde{\kappa}}{\Gamma(\tilde{\mu}_1/2)}\left(\frac{z/4+\tilde{\Lambda}}{2\tilde{\lambda}_1}\right)^{\tilde{\mu}_1/2-1}\exp\left(-\frac{z/4+\tilde{\Lambda}}{2\tilde{\lambda}_1}\right) \\ \leq& C\left(\frac{z/4+\tilde{\Lambda}}{2\lambda_1(1-\delta)}\right)^{\tilde{\mu}_1/2-1}\exp\left(-\frac{z/4+\tilde{\Lambda}}{2\lambda_1(1-\delta)}\right) \\ \leq& C(\log N)^c N^{-(1-\delta)/4} \end{aligned}$$

where $c = (\tilde{\mu}_1 - \mu_1)/2$. Thus, we have

$$P\left(U_Y^\top \mathbf{A} U_Y \geq \frac{1}{4}z\right) P\left(\min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A} \boldsymbol{Y}_{i_l} \geq z\right) \leq C(\log N)^c N^{-m+(3+\delta)/4} \tag{4.11}$$

Define $\tilde{\mathbf{A}} = \mathrm{diag}\{\mathbf{A}, \cdots, \mathbf{A}\} \in \mathbb{R}^{(m-1)M\times(m-1)M}$. Next, we consider

$$\begin{aligned} & P\left(V_Y^\top \mathbf{A} V_Y \geq \frac{1}{4}z, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A}\boldsymbol{Y}_{i_l} \geq z\right) \\ =& P\left(\boldsymbol{Y}_S^\top \mathbf{\Sigma}_{S i_m}\mathbf{A}\mathbf{\Sigma}_{i_m S}\boldsymbol{Y}_S \geq \frac{1}{4}z, \min_{1\leq l\leq m-1} \boldsymbol{Y}_{i_l}^\top \mathbf{A}\boldsymbol{Y}_{i_l} \geq z\right) \\ \leq& P\left(\boldsymbol{Y}_S^\top \mathbf{\Sigma}_{S i_m}\mathbf{A}\mathbf{\Sigma}_{i_m S}\boldsymbol{Y}_S \geq \frac{1}{4}z, \boldsymbol{Y}_S^\top \tilde{\mathbf{A}}\boldsymbol{Y}_S \geq (m-1)z\right) \\ \leq& P\left(\boldsymbol{Y}_S^\top((1-\epsilon)\tilde{\mathbf{A}} + \epsilon\mathbf{\Sigma}_{S i_m}\mathbf{A}\mathbf{\Sigma}_{i_m S})\boldsymbol{Y}_S \geq ((m-1)(1-\epsilon) + \frac{1}{4}\epsilon)z\right) \\ =& P\left(\xi_S^\top \mathbf{\Sigma}_S^{1/2}((1-\epsilon)\tilde{\mathbf{A}} + \epsilon\mathbf{\Sigma}_{S i_m}\mathbf{A}\mathbf{\Sigma}_{i_m S})\mathbf{\Sigma}_S^{1/2}\xi_S \geq ((1-\epsilon)m - 1 + \frac{5}{4}\epsilon)z\right) \end{aligned}$$

where $\xi_S \sim N(\mathbf{0}, \mathbf{I}_{(m-1)M})$. We have

$$\breve{\lambda}_1 \doteq \lambda_{max}\left(\mathbf{\Sigma}_S^{1/2}((1-\epsilon)\tilde{\mathbf{A}} + \epsilon\mathbf{\Sigma}_{S i_m}\mathbf{A}\mathbf{\Sigma}_{i_m S})\mathbf{\Sigma}_S^{1/2}\right)$$

$$\geq (1-(m-1)\delta_N)\lambda_{max}\left((1-\epsilon)\tilde{\mathbf{A}} + \epsilon \boldsymbol{\Sigma}_{Si_m}\mathbf{A}\boldsymbol{\Sigma}_{i_mS}\right)$$

$$\geq (1-(m-1)\delta_N)\left(\lambda_{max}\left((1-\epsilon)\tilde{\mathbf{A}}\right) - \lambda_{max}\left(\epsilon \boldsymbol{\Sigma}_{Si_m}\mathbf{A}\boldsymbol{\Sigma}_{i_mS}\right)\right)$$

$$\geq \lambda_1(1-(m-1)\delta_N)(1-(1+\delta)\epsilon) \geq (1-\delta/2)\lambda_1$$

for a small enough positive real number $\epsilon$.

By Equation (6) in Zolotarev (1962), we have

$$P\left(\xi_S^\top \boldsymbol{\Sigma}_S^{1/2}((1-\epsilon)\tilde{\mathbf{A}} + \epsilon \boldsymbol{\Sigma}_{Si_m}\mathbf{A}\boldsymbol{\Sigma}_{i_mS})\boldsymbol{\Sigma}_S^{1/2}\xi_S \geq ((1-\epsilon)m - 1 + \frac{5}{4}\epsilon)z\right)$$
$$\leq (1+\epsilon)\frac{\breve{\kappa}}{\Gamma(\breve{\mu}_1/2)}\left(\frac{((1-\epsilon)m-1+\frac{5}{4}\epsilon)z + \breve{\Lambda}}{2\breve{\lambda}_1}\right)^{\breve{\mu}_1/2-1}\exp\left(-\frac{((1-\epsilon)m-1+\frac{5}{4}\epsilon)z + \breve{\Lambda}}{2\breve{\lambda}_1}\right)$$
$$\leq C\left(\frac{((1-\epsilon)m-1+\frac{5}{4}\epsilon)z + \breve{\Lambda}}{2\lambda_1(1-\delta/2)}\right)^{\breve{\mu}_1/2-1}\exp\left(-\frac{((1-\epsilon)m-1+\frac{5}{4}\epsilon)z + \breve{\Lambda}}{2\lambda_1(1-\delta/2)}\right)$$
$$\leq C(\log N)^c N^{-\frac{(1-\epsilon)m-1+\frac{5}{4}\epsilon}{1-\delta/2}} \leq C(\log N)^c N^{-m+(3+\delta)/4}$$

by setting $\frac{(1-\epsilon)m-1+\frac{5}{4}\epsilon}{1-\delta/2} \geq m-(3+\delta)/4$. □

**Lemma 8.** *Define* $\upsilon_t = \sum_{v=1}^M \lambda_v \left((n-k_t)^{-1/2}\sum_{i=1}^{n-k_t}\phi_v(Z_{i,t})\right)^2$ *where* $Z_{i,t}$ *is the corresponding random variable in Condition (C6) with respect to* $t \in \{1, \cdots, N\}$. *Let*

$$\beta_s = \sum^* P\left(\upsilon_{i_1} > z, \cdots, \upsilon_{i_s} > z\right)$$

*for* $1 \leq s \leq N$, *where the sum runs over all* $i_1 < \cdots < i_s$ *and* $i_1 \in D_N, \cdots, i_s \in D_N$. *Then,*

$$\lim_{N\to\infty}\beta_s = \frac{1}{s!}\left(\frac{\kappa}{\Gamma(\mu_1/2)}e^{-\frac{y}{2}}\right)^{-s}$$

*for each* $s \geq 1$.

**Proof.** According to Theorem 1.1 in Zaitsev (1987), we have

$$\sum^{*} P\left(\sum_{v=1}^{M} \lambda_v Y_{i_1 v}^2 > z + \epsilon_n (\log(N))^{-1}, \cdots, \sum_{v=1}^{M} \lambda_v Y_{i_s v}^2 > z + \epsilon_n (\log(N))^{-1}\right)$$
$$- \begin{pmatrix} |D_N| \\ s \end{pmatrix} c_1 s^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 s^3 (\log N)}\right)$$
$$\leq \sum^{*} P\left(\upsilon_{i_1} > z, \cdots, \upsilon_{i_s} > z\right)$$
$$\leq \sum^{*} P\left(\sum_{v=1}^{M} \lambda_v Y_{i_1 v}^2 > z - \epsilon_n (\log(N))^{-1}, \cdots, \sum_{v=1}^{M} \lambda_v Y_{i_s v}^2 > z - \epsilon_n (\log(N))^{-1}\right)$$
$$+ \begin{pmatrix} |D_N| \\ s \end{pmatrix} c_1 s^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 s^3 (\log N)}\right)$$

where $(Y_{i_1 v}, \cdots, Y_{i_s v})$ follows a multivariate normal distribution with mean zero and the same covariance matrix with $((n - k_{i_1})^{-1/2} \sum_{j=1}^{n-k_{i_1}} \phi_v(Z_{j,i_1}), \cdots, (n - k_{i_s})^{-1/2} \sum_{j=1}^{n-k_{i_s}} \phi_v(Z_{j,i_s}$

By the condition $N = o(n^{\epsilon})$, there exist small enough $\epsilon_n \to 0$ satisfy

$$\begin{pmatrix} |D_N| \\ s \end{pmatrix} c_1 s^{5/2} \exp\left(-\frac{n^{1/2}\epsilon_n}{c_2 s^3 (\log N)}\right) \to 0.$$

Define $W_i = \sum_{v=1}^{M} \lambda_v Y_{iv}^2$ and $\boldsymbol{Y}_i = (Y_{i1}, \cdots, Y_{iM})$. So we only need to show that

$$\sum^{*} P\left(W_{i_1} > z, \cdots, W_{i_s} > z\right) \to \frac{1}{s!}\left(\frac{\kappa}{\Gamma(\mu_1/2)} e^{-\frac{y}{2}}\right)^{-s}.$$

Recalling $D_N := \{1 \leq i \leq N;\, |B_{N,i}| < N^{\kappa}\}$, we write

$$\big\{(i_1, \cdots, i_t) \in (D_N)^t;\, i_1 < \cdots < i_t\big\} = F_t \cup G_t$$

where $\sigma_{i_r i_s} = \lambda_{max}(\Xi_{i_r i_s}\Xi_{i_r i_s}^{\top})$ and $\Xi_{i_r i_s} = \mathrm{cov}(\boldsymbol{Y}_{i_r}, \boldsymbol{Y}_{i_s})$.

$$\begin{aligned} F_t &:= \big\{(i_1, \cdots, i_t) \in (D_N)^t;\, i_1 < \cdots < i_t \text{ and } |\sigma_{i_r i_s}| \leq \delta_N^{2+2c} \text{ for all } 1 \leq r < s \leq t\big\}; \\ G_t &:= \big\{(i_1, \cdots, i_t) \in (D_N)^t;\, i_1 < \cdots < i_t \text{ and } |\sigma_{i_r i_s}| > \delta_N^{2+2c} \text{ for a pair } (i_r, i_s) \text{ with} \\ &\qquad 1 \leq r < s \leq t\big\}. \end{aligned} \tag{4.12}$$

Now, think $D_N$ as graph with $|D_N|$ vertices. Keep in mind that $|D_N| \leq N$ and $|D_N|/N \to 1$. Any two different vertices from them, say, $i$ and $j$ are connected if $|\sigma_{ij}| > \delta_N^{2+2c}$. In this case we also say there is an edge between them. By the definition $D_N$, each vertex in the graph has at most $N^{\varsigma}$ neighbors. Replacing "$n$", "$q$" and "$t$" in Lemma 7.1 in Feng et al. (2022a) with "$|D_N|$", "$N^{\varsigma}$" and "$t$", respectively, we have that $|G_t| \leq N^{t+\varsigma-1}$ for each $2 \leq t \leq N$. Therefore $\binom{|D_N|}{t} \geq |F_t| \geq \binom{|D_N|}{t} - N^{t+\varsigma-1}$. Since $D_N/N \to 1$ and $\varsigma = \varsigma_N \to 0$ as $N \to \infty$, we know

$$\lim_{N\to\infty} \frac{|F_t|}{N^t} = \frac{1}{t!}. \tag{4.13}$$

Here

$$\beta_t = \sum_{(i_1, \cdots, i_t) \in F_t} P(W_{i_1} > z, \cdots, W_{i_t} > z) +$$

$$\sum_{(i_1,\cdots,i_t)\in G_t} P(W_{i_1} > z, \cdots, W_{i_t} > z).$$

From Lemma 6 and (4.13) we have

$$\sum_{(i_1,\cdots,i_t)\in F_t} P(W_{i_1} > z, \cdots, W_{i_t} > z) \to \frac{1}{t!}\Big(\frac{\kappa}{\Gamma(\mu_1/2)} e^{-\frac{y}{2}}\Big)^t$$

as $N \to \infty$. As a consequence, it remains to show

$$\sum_{(i_1,\cdots,i_t)\in G_t} P(W_{i_1} > z, \cdots, W_{i_t} > z) \to 0 \tag{4.14}$$

as $N \to \infty$ for each $t \geq 2$.

Next, we will prove (4.14). If $t = 2$, the sum of probabilities in (4.14) is bounded by $|G_2| \cdot \max_{1\leq i<j\leq N} P(W_i > z, W_j > z)$. By Lemma 7.1 in Feng et al. (2022a), $|G_2| \leq N^{\varsigma+1}$. Since $|\sigma_{ij}| \leq \varrho$, by Lemma 7,

$$P(W_i > z, W_j > z) \leq \frac{(\log N)^C}{N^{(5-\varrho)/4}} \tag{4.15}$$

uniformly for all $1 \leq i < j \leq N$ as $N$ is sufficiently large, where $C > 0$ is a constant not depending on $N$. We then know (4.14) holds. So the remaining job is to show (4.14) for $t \geq 3$.

Let $N \geq 2$ and $(\sigma_{ij})_{N\times N}$ be a non-negative definite matrix. For $\delta_N > 0$ and a set $A \subset \{1, 2, \cdots, m\}$ with $2 \leq m \leq N$, define

$$\wp(A) = \max\left\{|S|; S \subset A \text{ and } \max_{i\in S, j\in S, i\neq j} |\sigma_{ij}| \leq \delta_N^{2+2c}\right\}$$

Easily, $\wp(A)$ takes possible values $0, 2\cdots, |A|$, where we regard $|\varnothing| = 0$. If $\wp(A) = 0$, then $|\sigma_{ij}| > \delta_N^{2+2c}$ for all $i \in A$ and $j \in A$.

Now we will look at $G_t$ closely. To do so, we classify $G_t$ into the following subsets

$$G_{t,j} = \left\{(i_1, \cdots, i_t) \in G_t;\ \wp(\{i_1, \cdots, i_t\}) = j\right\}$$

for $j = 0, 2, \cdots, t-1$. By the definition of $G_t$, we see $G_t = \cup G_{t,j}$ for $j = 0, 2, \cdots, t-1$. Since $t \geq 3$ is fixed, to show (4.14), it suffices to prove

$$\sum_{(i_1,\cdots,i_t)\in G_{t,j}} P(W_{i_1} > z, \cdots, W_{i_t} > z) \to 0 \tag{4.16}$$

for any $j \in \{0, 2, \cdots, t-1\}$.

Assume $(i_1, \cdots, i_t) \in G_{t,0}$. This implies that $|\sigma_{i_r i_s}| > \delta_N^{2+2c}$ for all $1 \leq r < s \leq t$. Therefore, the subgraph $\{i_1, \cdots, i_t\} \in G_t$ is a clique. Taking $n = |D_N| \leq N$, $t = t$ and $q = N^{\varsigma}$. Then by Lemma 7.1 in Feng et al. (2022a), $|G_{t,0}| \leq N^{1+\varsigma(t-1)} \leq N^{1+t\varsigma}$. Thus, the sum from (4.16) is bounded by

$$N^{1+t\varsigma} \cdot \max_{1\leq i<j\leq N} P(W_i > z, W_j > z) \leq N^{1+t\varsigma} \cdot \frac{(\log N)^C}{N^{(5-\varrho)/4}} \to 0 \tag{4.17}$$

as $N \to \infty$ by using (4.15). So (4.16) holds with $j = 0$.

Now we assume $(i_1, \cdots, i_t) \in G_{t,j}$ with $j \in \{2, \cdots, t-1\}$. By definition, there exits $S \subset \{i_1, \cdots, i_t\}$ such that $\max_{i\in S, j\in S, i\neq j} |\sigma_{ij}| \leq \delta_N^{2+2c}$ and for each $k \in \{i_1, \cdots, i_t\} \backslash S$, there exists $i \in S$ satisfying $|\sigma_{ik}| > \delta_N^{2+2c}$. Looking at the last statement we see two possibilities: (i) for each $k \in \{i_1, \cdots, i_t\} \backslash S$, there exist at least two indices, say, $i \in S$, $j \in S$ with $i \neq j$ satisfying $|\sigma_{ik}| > \delta_N^{2+2c}$ and $|\sigma_{jk}| > \delta_N^{2+2c}$; (ii) there exists $k \in \{i_1, \cdots, i_t\} \backslash S$ such that $|\sigma_{ik}| > \delta_N^{2+2c}$ for an unique $i \in S$. However, for $(i_1, \cdots, i_t) \in G_{t,j}$, (i) and (ii) could happen at the same time for different $S$, say, (i) holds for $S_1$ and (ii) holds for $S_2$ simultaneously. Thus, to differentiate the two cases, we introduce following two definitions. Set

$$\begin{aligned} H_{t,j} \;\; = \;\; & \big\{(i_1, \cdots, i_t) \in G_{t,j}; \;\text{ there exist } S \subset \{i_1, \cdots, i_t\} \text{ with } |S| = j \text{ and} \\ & \max_{i \in S, j \in S, i \neq j} |\sigma_{ij}| \leq \delta_N^{2+2c} \text{ such that for any } k \in \{i_1, \cdots, i_t\} \backslash S \text{ there exist } r \in S, s \in S, \\ & r \neq s \text{ satisfying } \min\{|\sigma_{kr}|, |\sigma_{ks}|\} > \delta_N^{2+2c}\big\}. \end{aligned} \tag{4.18}$$

Replacing "$n$", "$q$" and "$t$" in Lemma 7.1 in Feng et al. (2022a) with "$|D_N|$", "$N^{\varsigma}$" and "$t$", respectively, we have that $|H_{t,j}| \leq t^t \cdot N^{j-1+(t-j+1)\varsigma}$ for each $t \geq 3$. Again, set

$$\begin{aligned} H'_{t,j} \;\; = \;\; & \big\{(i_1, \cdots, i_t) \in G_{t,j}; \;\text{ for any } S \subset \{i_1, \cdots, i_t\} \text{ with } |S| = j \text{ and} \\ & \max_{i \in S, j \in S, i \neq j} |\sigma_{ij}| \leq \delta_N^{2+2c} \text{ there exists } k \in \{i_1, \cdots, i_t\} \backslash S \text{ such that } |\sigma_{kr}| > \delta_N^{2+2c} \\ & \text{for a unique } r \in S\big\}. \end{aligned} \tag{4.19}$$

From Lemma 7.1 in Feng et al. (2022a), we see $|H'_{t,j}| \leq t^t \cdot N^{j+(t-j)\varsigma}$. It is easy to see $G_{t,j} = H_{t,j} \cup H'_{t,j}$. Therefore, to show (4.16), we only need to prove

$$\sum_{(i_1,\cdots,i_t)\in H_{t,j}} P(W_{i_1} > z, \cdots, W_{i_t} > z) \to 0 \tag{4.20}$$

and

$$\sum_{(i_1,\cdots,i_t)\in H'_{t,j}} P(W_{i_1} > z, \cdots, W_{i_t} > z) \to 0 \tag{4.21}$$

as $N \to \infty$ for $j = 2, \cdots, t-1$. In fact, let $S$ be as in (4.18), then by using Lemma 6, the probability in (4.20) is bounded by $P(\cap_{l\in S}\{W_l > z\}) \le C \cdot N^{-j}$ uniformly for all $S$ as $N$ is sufficiently large, where $C$ is a constant not depending on $N$. Thus,

$$\begin{aligned}\sum_{(i_1,\cdots,i_t)\in H_{t,j}} P(W_{i_1} > z, \cdots, W_{i_t} > z) &\le t^t \cdot N^{j-1+(t-j+1)\varsigma} \cdot \big(C \cdot N^{-j}\big) \\ &\le (Ct^t) \cdot N^{-1+t\varsigma}\end{aligned}$$

as $N$ is sufficiently large. By assumption $\varsigma = \varsigma_N \to 0$, we then get (4.20).

Now we show (4.21). Recall the definition of $H'_{t,j}$. For $(i_1, \cdots, i_t) \in H'_{t,j}$, pick $S \subset \{i_1, \cdots, i_t\}$ with $|S| = j$, $\max_{i\in S, j\in S, i\ne j} |\sigma_{ij}| \le \delta_N^{2+2c}$ and $k \in \{i_1, \cdots, i_t\}\backslash S$ such that $\delta_N^{2+2c} < |\sigma_{kr}| \le \varrho$ for a unique $r \in S$. Then the probability from (4.21) is bounded by

$$P\Big(W_k > z, \bigcap_{l\in S}\{W_l > z\}\Big)$$

for $2 \le j \le t-1$. Taking $m = j+1$ in Lemma 7, then the probability above is dominated by

$$\frac{2^{j+1}}{z} \cdot \exp\Big\{ -\frac{z^2}{2}\Big(j + \frac{1-\varrho}{4}\Big)\Big\} = O\Big(\frac{(\log N)^{c_1}}{N^{j+(1-\varrho)/4}}\Big)$$

for some constant $c_1$ not depending on $N$. As stated earlier, $|H'_{t,j}| \le t^t \cdot N^{j+(t-j)\varsigma}$. Multiplying the two quantities, since $\varsigma = \varsigma_N \to 0$, we see the sum from (4.21) is of order $O(N^{-(1-\varrho)/8})$. Therefore (4.21) holds. We then have proved (4.16) for any $j \in \{0, 2, \cdots, t-1\}$. The proof is completed. □

**Proof of Theorem 7** We proceed in two steps, proving first the case $m = 2$ and then generalizing to $m \ge 2$. For notational convenience we introduce the constants $b_1 := \|h\|_\infty < \infty$ and $b_2 := \sup_v \|\phi_v\|_\infty < \infty$. Similar to the proof of Theorem 4, we define $\{u_s\}_{s=1}^N = \{U_{ij}(k)\}_{1\le i,j\le p, 1\le k\le K}$ and $\{\boldsymbol{X}_{tijk}\} = \{(\varepsilon_{t,i}, \varepsilon_{t+k,j})^\top\}_{1\le t\le n-k}$. So we rewrite $U_{ij}(k)$ in the following forms

$$u_s = \frac{1}{C_{n-k_s}^m} \sum_{1\le t_1 < t_2, \cdots, < t_m \le n-k_s} h(\boldsymbol{X}_{t_1, i_s j_s k_s}, \cdots, \boldsymbol{X}_{t_m, i_s j_s k_s}). \tag{4.22}$$

**Step I.** Suppose $m = 2$. We start with the scenario that there are infinitely many nonzero eigenvalues. For a large enough integer $M$ to be specified later, we define the "truncated" kernel of $h_2(z_1, z_2; \mathbb{P}_Z)$ as $h_{2,M}(z_1, z_2; \mathbb{P}_Z) =$

$\sum_{v=1}^{M} \lambda_v \phi_v(z_1) \phi_v(z_2)$, with corresponding U-statistic

$$u_{M,s} := \begin{pmatrix} n-k_s \\ 2 \end{pmatrix}^{-1} \sum_{1 \le i < j \le n-k_s} h_{2,M}(Z_i, Z_j; \mathbb{P}_Z)$$

For simpler presentation, define $Y_{v,i} = \phi_v(Z_i)$ for all $v = 1, 2, \ldots$ and $i \in [n-k_s]$. In view of the expansions of $h_{2,M}(\cdot)$ and $h_2(\cdot)$, $u_{M,s}$ and $u_s$ can be written as

$$u_{M,s} = \frac{1}{n-k_s-1} \left\{ \sum_{v=1}^{M} \lambda_v \left( (n-k_s)^{-1/2} \sum_{i=1}^{n-k_s} Y_{v,i} \right)^2 - \sum_{v=1}^{M} \lambda_v \left( \frac{\sum_{i=1}^{n-k_s} Y_{v,i}^2}{n-k_s} \right) \right\}$$

$$u_s = \frac{1}{n-k_s-1} \left\{ \sum_{v=1}^{\infty} \lambda_v \left( (n-k_s)^{-1/2} \sum_{i=1}^{n-k_s} Y_{v,i} \right)^2 - \sum_{v=1}^{\infty} \lambda_v \left( \frac{\sum_{i=1}^{n-k_s} Y_{v,i}^2}{n-k_s} \right) \right\}$$

Define $M = [n^{(1-3\theta)/5}]$. By the definition of $\theta$, there exist a positive absolute constant $C_\theta$ such that $\sum_{v=M+1}^{\infty} \lambda_v \le C_\theta n^{-\theta}$ for all sufficiently large $n$. Thus, for any $\epsilon > 0$, we have

$$\begin{aligned} P\left( \max_{1 \le s \le N} (n-k_s-1)|u_{M,s} - u_s| \ge \epsilon \right) \le & NP\left( (n-k_s-1)|u_{M,s} - u_s| \ge \epsilon \right) \\ \le & 2Ne^{1/12} \exp\left( -\frac{\epsilon}{12b_2^2 \sum_{v=M+1}^{\infty} \lambda_v} \right) \\ \le & 2Ne^{1/12} \exp\left( -\frac{\epsilon n^\theta}{12b_2^2 C_\theta} \right) \to 0 \end{aligned}$$

by $\log N = o(n^\theta)$. Here the second inequality are followed by (A.9) in Drton et al. (2020). Thus, by

$$\left| \max_{1 \le s \le N} (n-k_s-1)u_s - \max_{1 \le s \le N} (n-k_s-1)u_{M,s} \right| \le \max_{1 \le s \le N} (n-k_s-1)|u_s - u_{M,s}| \to 0$$

we only need to show that

$$\mathbb{P}\left\{\max_{1\le s\le N}(n-k_s-1)u_{M,s}-2\lambda_1\log(N)-\lambda_1(\mu_1-2)\log\log(N)+\Lambda\le\lambda_1 y\right\}$$
$$\to\exp\left\{-\frac{\kappa}{\Gamma(\mu_1/2)}\exp\left(-\frac{y}{2}\right)\right\} \tag{4.23}$$

Define

$$\tilde{u}_{M,s}=\frac{1}{n-k_s-1}\left\{\sum_{v=1}^{M}\lambda_v\left((n-k_s)^{-1/2}\sum_{i=1}^{n-k_s}Y_{v,i}\right)^2-\sum_{v=1}^{M}\lambda_v\right\}.$$

Thus, for any $\epsilon>0$, we have

$$\begin{aligned}P\left(\max_{1\le s\le N}(n-k_s-1)|u_{M,s}-\tilde{u}_{M,s}|\ge\epsilon\right)\le&NP\left((n-k_s-1)|u_{M,s}-\tilde{u}_{M,s}|\ge\epsilon\right)\\ \le&NP\left(\left|\sum_{v=1}^{M}\lambda_v\frac{\sum_{i=1}^{n-k_s}(Y_{v,i}^2-1)}{n-k_s}\right|\ge\epsilon\right)\\ \le&2N\exp\left(-\frac{(n-k_s)\epsilon^2}{48\Lambda^2(b_2^2+1)^2}\right)\to 0\end{aligned}$$

by $\log N=o(n^\theta)$. Here the second inequality are followed by (A.10) in Drton et al. (2020). Thus, by

$$\left|\max_{1\le s\le N}(n-k_s-1)\tilde{u}_{M,s}-\max_{1\le s\le N}(n-k_s-1)u_{M,s}\right|\le\max_{1\le s\le N}(n-k_s-1)|\tilde{u}_{M,s}-u_{M,s}|\to 0$$

we only need to show that

$$\mathbb{P}\left\{\max_{1\le s\le N}(n-k_s-1)\tilde{u}_{M,s}-2\lambda_1\log(N)-\lambda_1(\mu_1-2)\log\log(N)+\Lambda\le\lambda_1 y\right\}$$
$$\to\exp\left\{-\frac{\kappa}{\Gamma(\mu_1/2)}\exp\left(-\frac{y}{2}\right)\right\} \tag{4.24}$$

Define $\upsilon_s = \sum_{v=1}^{M} \lambda_v \left( (n-k_s)^{-1/2} \sum_{i=1}^{n-k_s} Y_{v,i} \right)^2 = (n-k_s-1)\tilde{u}_{M,s} + \sum_{v=1}^{M} \lambda_v$. By the definition of $\Lambda$, we have $\Lambda - \sum_{v=1}^{M} \lambda_v = \sum_{v=M+1}^{\infty} \lambda_v = O(n^{-\theta})$, thus, we only need to show that

$$\mathbb{P}\left\{ \max_{1\leq t\leq N} \upsilon_t - 2\lambda_1 \log(N) - \lambda_1 (\mu_1 - 2) \log\log(N) \leq \lambda_1 y \right\} \\ \to \exp\left\{ -\frac{\kappa}{\Gamma(\mu_1/2)} \exp\left( -\frac{y}{2} \right) \right\} \tag{4.25}$$

Define $z = 2\lambda_1 \log(N) + \lambda_1 (\mu_1 - 2) \log\log(N) + \lambda_1 y$.

By Theorem 4.1 in Drton et al. (2020) and $\log N = o(n^{\theta})$, we have

$$P(\upsilon_i \geq z) = \frac{\kappa}{\Gamma(\mu_1/2)} \left( \frac{z}{2\lambda_1} \right)^{\mu_1/2-1} \exp\left( -\frac{z}{2\lambda_1} \right) \{1 + o(1)\} \sim \frac{\kappa}{\Gamma(\mu_1/2)} \frac{e^{-y/2}}{N}$$

Thus,

$$P\left( \max_{i\in C_N} \upsilon_i > z \right) \leq |C_N| \cdot P(\upsilon_i \geq z) \to 0$$

as$p \to \infty$. Set $D_N := \{1 \leq i \leq N; |B_{N,i}| < N^{\varsigma}\}$. By assumption, $|D_N|/N \to 1$ as $N \to \infty$ Easily,

$$P\left( \max_{i\in D_N} \upsilon_i > z \right) \leq P\left( \max_{1\leq i\leq N} \upsilon_i > z \right) \\ \leq P\left( \max_{i\in D_N} \upsilon_i > z \right) + P\left( \max_{i\in C_N} \upsilon_i > z \right)$$

Therefore, to prove Theorem 7, it is enough to show

$$\lim_{N\to\infty} P\left( \max_{i\in D_N} \upsilon_i > z \right) = 1 - \exp\left\{ -\frac{\kappa}{\Gamma(\mu_1/2)} \exp\left( -\frac{y}{2} \right) \right\}$$

as $N \to \infty$. Define

$$\beta_t = \sum^{*} P\left( \upsilon_{i_1} > z, \cdots, \upsilon_{i_t} > z \right)$$

for $1 \le t \le N$, where the sum runs over all $i_1 < \cdots < i_t$ and $i_1 \in D_N, \cdots, i_t \in D_N$. By Lemma 8,

$$\lim_{N\to\infty} \beta_t = \frac{1}{t!}\left(\frac{\kappa}{\Gamma(\mu_1/2)} e^{-\frac{y}{2}}\right)^{-t}$$

for each $t \ge 1$. Then, by Bonferroni inequality,

$$\sum_{t=1}^{2k} (-1)^{t-1}\beta_t \le P\left(\max_{i\in D_N} \upsilon_i > z\right) \le \sum_{t=1}^{2k+1} (-1)^{t-1}\beta_t$$

for any $k \ge 1$. let $N \to \infty$, we have

$$\sum_{t=1}^{2k}(-1)^{t-1}\frac{1}{t!}\left(\frac{\kappa}{\Gamma(\mu_1/2)}e^{-\frac{y}{2}}\right)^t \le \liminf_{N\to\infty} P\left(\max_{i\in D_N}\upsilon_i > z\right)$$
$$\le \limsup_{N\to\infty} P\left(\max_{i\in D_N}\upsilon_i > z\right) \le \sum_{t=1}^{2k+1}(-1)^{t-1}\frac{1}{t!}\left(\frac{\kappa}{\Gamma(\mu_1/2)}e^{-\frac{y}{2}}\right)^t$$

for each $k \ge 1$. By letting $k \to \infty$ and using the Taylor expansion of the function $1 - e^{-x}$, so we obtain the result.

**Step II.** For $m \ge 2$, by the Hoeffding decomposition, we have

$$u_s = C_m^2 H_{n-k_s}^{(2)}(\cdot, \mathbb{P}_{Z_s}) + \sum_{\ell=3}^{m} C_m^\ell H_{n-k_s}^{(\ell)}(\cdot, \mathbb{P}_{Z_s})$$

where for any measure $\mathbb{P}_{Z_s}$ and kernel $h$, $H_{n-k_s}^{(\ell)}(\cdot;\mathbb{P}_{Z_s})$ is the U-statistic based on the completely degenerate kernel $h^{(\ell)}(\cdot;\mathbb{P}_{Z_s})$ from (2.8) :

$$H_{n-k_s}^{(\ell)}(\cdot;\mathbb{P}_{Z_s}) := \binom{n-k_s}{\ell}^{-1} \sum_{1\le i_1<i_2<\cdots<i_\ell\le n-k_s} h^{(\ell)}\left(Z_{i_1},\ldots,Z_{i_\ell};\mathbb{P}_{Z_s}\right).$$

To prove the result, we only need to show that $\max_{1\le s\le N}(n-k_s-1)H^{(\ell)}_{n-k_s}(\cdot,\mathbb{P}_{Z_s}) = o(1)$ for $\ell \ge 3$.

By Proposition 2.3(c) in Arcones and Giné (1993), there exist positive constant $C_1, C_2$ such that for all $\epsilon_n > 0$,

$$P\left((n-k_s)^{\ell/2}|H^{(\ell)}_{n-k_s}(\cdot,\mathbb{P}_{Z_s})| \ge \epsilon_n\right) \le C_1 \exp\left(-C_2\left(\frac{\epsilon_n}{2^\ell b_1}\right)^{2/\ell}\right) \quad (4.26)$$

So, for any $\epsilon_1 > 0$,

$$\begin{aligned} P\left(\max_{1\le s\le N}(n-k_s-1)H^{(\ell)}_{n-k_s}(\cdot,\mathbb{P}_{Z_s}) \ge \epsilon_1\right) \le& NP\left((n-k_s-1)H^{(\ell)}_{n-k_s}(\cdot,\mathbb{P}_{Z_s}) \ge \epsilon_1\right) \\ \le& C_1 N \exp\left(-C_2\left(\frac{(n-k_s)^{\ell/2-1}\epsilon_1}{2^\ell b_1}\right)^{2/\ell}\right) \to 0 \end{aligned}$$

by the condition $\log N = o(n^\theta)$. Here we complete the proof. □

### 4.3.2 Proof of Theorem 8

The proof is similar to the proof of Theorem 4.3 in Drton et al. (2020). So we omit it here.

### 4.3.3 Proof of Theorem 9

According to Theorem 3 in Feng et al. (2022b) and Assumption (A4), we can easily obtain the result. □

## 4.4 Proof of Theorem S.1

First, we restate the following lemma in Arratia et al. (1989).

**Lemma 9.** *Let $I$ be an index set and $\{B_\alpha, \alpha \in I\}$ be a set of subsets of $I$; that is, $B_\alpha \subset I$ for each $\alpha \in I$. Let also $\{\eta_\alpha, \alpha \in I\}$ be random variables. For a given $t \in \mathcal{R}$, set $\lambda = \sum_{\alpha \in I} \mathrm{pr}\left(\eta_\alpha > t\right)$. Then*

$$\left|\mathrm{pr}\left(\max_{\alpha \in I} \eta_\alpha \le t\right) - e^{-\lambda}\right| \le \min\left(1, \lambda^{-1}\right)\left(b_1 + b_2 + b_3\right),$$

where

$$b_1 \equiv \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} \mathrm{pr}\left(\eta_\alpha > t\right) \mathrm{pr}\left(\eta_\beta > t\right), b_2 \equiv \sum_{\alpha \in I} \sum_{\beta \neq \alpha, \beta \in B_\alpha} \mathrm{pr}\left(\eta_\alpha > t, \eta_\beta > t\right),$$

$$b_3 \equiv \sum_{\alpha \in I} E\left|\mathrm{pr}\left\{\eta_\alpha > t \mid \sigma\left(\eta_\beta, \beta \notin B_\alpha\right)\right\} - \mathrm{pr}\left(\eta_\alpha > t\right)\right|$$

where $\sigma\left(\eta_\beta, \beta \notin B_\alpha\right)$ is the $\sigma$-algebra generated by $\{\eta_\beta, \beta \notin B_\alpha\}$. In particular, if $\eta_\alpha$ is independent of $\{\eta_\beta, \beta \notin B_\alpha\}$ for each $\alpha$, then $b_3 = 0$.

Next, we adopt Lemma 9 to prove Theorem S1.

**Proof.** In Lemma 9, let $I = \{(i,j,k) : 1 \le i,j \le p, 1 \le k \le K\}$. For $u = \{(i,j,k) \in I\}$, set $B_u = \{(l,m,q) \in I : \{i,j\} \cap \{l,m\} \neq \emptyset\}$, $\eta_u = \{|\psi_{ij}(k)|\}$, $\psi_{ij}(k) = \sqrt{\frac{5(n-k+1)}{2}}|\Xi_{ij}(k)|$ and $A_u = \{\psi_{ij}(k) > t_y\}$, $t_y = 2\log N - \log\log N + y$. By Lemma 9, we have $b_3 = 0$ by the independence assumption. Thus,

$$\left|P\left(\Xi_n^2 \le t_y\right) - e^{-\lambda_n}\right| \le b_{1,n} + b_{2,n}$$

where $\lambda_n = NP(\psi_{ij}(k) > t_y)$ and

$$b_{1,n} \leq 2K^2p^3P^2(\psi_{ij}(k) > t_y) = O(p^{-1}) \to 0$$

since

$$\begin{aligned} P(\psi_{ij}(k) > t_y) &= P(|N(0,1)| > t_y)(1 + o(1)) \\ &= 2(1 - \Phi(t))(1 + o(1)) = \sqrt{\frac{2}{\pi}}\frac{e^{-t^2/2}}{t}(1 + o(1)) \end{aligned}$$

by Theorem 2.1 in Chatterjee (2021). Additionally, by Lemma C4 in Han et al. (2017), we have $\Xi_{ij}(k)$ is independent of $\Xi_{is}(l)$ if $i \neq j \neq s$ or $k \neq l$. So

$$\begin{aligned} b_{2,n} &\leq 2K^2p^3P(\psi_{ij}(k) > t_y, \psi_{is}(l) > t_y) + K^2pP(\psi_{ii}(k) > t_y, \psi_{ii}(l) > t_y) \\ &\leq 2K^2p^3P^2(\psi_{ij}(k) > t_y) + K^2pP(\psi_{ii}(k) > t_y) = O(p^{-1}) \to 0. \end{aligned}$$

Obviously, we have $\lambda_n \to \pi^{-1/2}\exp(-y/2)$ as $p \to \infty$. So we obtain the result. □

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Table 3: Sizes of $L$-statistic tests with $K = 2$ under Model (i).

| | $L$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | $(n, p) = (100, 30)$ | | | | | | | | | |
| $\tilde{L}_\tau$ | 0.041 | 0.046 | 0.059 | 0.052 | 0.046 | 0.058 | 0.05 | 0.052 | 0.059 | 0.048 |
| $\tilde{L}_\rho$ | 0.046 | 0.059 | 0.056 | 0.048 | 0.052 | 0.059 | 0.042 | 0.05 | 0.05 | 0.054 |
| $\tilde{L}_{\tau^*}$ | 0.059 | 0.046 | 0.05 | 0.043 | 0.047 | 0.045 | 0.047 | 0.047 | 0.056 | 0.05 |
| $\tilde{L}_D$ | 0.046 | 0.045 | 0.05 | 0.05 | 0.059 | 0.043 | 0.05 | 0.058 | 0.045 | 0.043 |
| $\tilde{L}_R$ | 0.054 | 0.051 | 0.059 | 0.052 | 0.056 | 0.048 | 0.045 | 0.056 | 0.054 | 0.045 |
| | $(n, p) = (100, 60)$ | | | | | | | | | |
| $\tilde{L}_\tau$ | 0.054 | 0.051 | 0.043 | 0.05 | 0.04 | 0.054 | 0.054 | 0.04 | 0.046 | 0.057 |
| $\tilde{L}_\rho$ | 0.044 | 0.04 | 0.057 | 0.047 | 0.048 | 0.051 | 0.04 | 0.054 | 0.055 | 0.048 |
| $\tilde{L}_{\tau^*}$ | 0.04 | 0.06 | 0.048 | 0.041 | 0.052 | 0.049 | 0.056 | 0.045 | 0.056 | 0.058 |
| $\tilde{L}_D$ | 0.04 | 0.059 | 0.056 | 0.049 | 0.044 | 0.049 | 0.058 | 0.042 | 0.041 | 0.058 |
| $\tilde{L}_R$ | 0.046 | 0.044 | 0.056 | 0.058 | 0.047 | 0.04 | 0.048 | 0.05 | 0.043 | 0.05 |
| | $(n, p) = (100, 90)$ | | | | | | | | | |
| $\tilde{L}_\tau$ | 0.047 | 0.041 | 0.053 | 0.048 | 0.058 | 0.049 | 0.048 | 0.043 | 0.041 | 0.049 |
| $\tilde{L}_\rho$ | 0.047 | 0.057 | 0.042 | 0.057 | 0.046 | 0.058 | 0.055 | 0.043 | 0.059 | 0.052 |
| $\tilde{L}_{\tau^*}$ | 0.058 | 0.053 | 0.041 | 0.048 | 0.053 | 0.049 | 0.045 | 0.053 | 0.051 | 0.041 |
| $\tilde{L}_D$ | 0.054 | 0.054 | 0.052 | 0.05 | 0.044 | 0.057 | 0.047 | 0.05 | 0.06 | 0.06 |
| $\tilde{L}_R$ | 0.051 | 0.057 | 0.06 | 0.056 | 0.057 | 0.054 | 0.049 | 0.057 | 0.058 | 0.049 |
| | $(n, p) = (100, 120)$ | | | | | | | | | |
| $\tilde{L}_\tau$ | 0.052 | 0.049 | 0.049 | 0.057 | 0.049 | 0.043 | 0.041 | 0.048 | 0.045 | 0.04 |
| $\tilde{L}_\rho$ | 0.058 | 0.054 | 0.044 | 0.049 | 0.056 | 0.058 | 0.06 | 0.049 | 0.049 | 0.045 |
| $\tilde{L}_{\tau^*}$ | 0.044 | 0.05 | 0.059 | 0.059 | 0.049 | 0.06 | 0.055 | 0.05 | 0.052 | 0.049 |
| $\tilde{L}_D$ | 0.058 | 0.059 | 0.044 | 0.044 | 0.054 | 0.041 | 0.05 | 0.045 | 0.041 | 0.044 |
| $\tilde{L}_R$ | 0.06 | 0.05 | 0.053 | 0.055 | 0.046 | 0.049 | 0.042 | 0.045 | 0.047 | 0.042 |

Table 4: Sizes of tests with $K = 4$ under Model (i)-(viii).

| $n$ | $p$ | i | | | | | | | ii | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $L_r$ | $L_\tau$ | $L_\rho$ | $L_{\tau^*}$ | $L_D$ | $L_R$ | $S_r$ | $L_r$ | $L_\tau$ | $L_\rho$ | $L_{\tau^*}$ | $L_D$ | $L_R$ | $S_r$ |
| 100 | 30 | 0.042 | 0.016 | 0.016 | 0.055 | 0.087 | 0.047 | 0.013 | 0.038 | 0.007 | 0.007 | 0.04 | 0.074 | 0.037 | 0.014 |
| 100 | 60 | 0.037 | 0.012 | 0.008 | 0.04 | 0.07 | 0.036 | 0.001 | 0.031 | 0.006 | 0.005 | 0.031 | 0.073 | 0.03 | 0.002 |
| 100 | 120 | 0.036 | 0.011 | 0.008 | 0.036 | 0.09 | 0.026 | 0 | 0.027 | 0.009 | 0.004 | 0.04 | 0.075 | 0.03 | 0 |
| 100 | 240 | 0.029 | 0.009 | 0.001 | 0.037 | 0.091 | 0.018 | 0 | 0.024 | 0.006 | 0.004 | 0.028 | 0.08 | 0.021 | 0 |
| 200 | 30 | 0.047 | 0.017 | 0.014 | 0.047 | 0.055 | 0.041 | 0.022 | 0.051 | 0.016 | 0.015 | 0.036 | 0.047 | 0.034 | 0.034 |
| 200 | 60 | 0.031 | 0.016 | 0.013 | 0.036 | 0.048 | 0.028 | 0.012 | 0.037 | 0.014 | 0.01 | 0.038 | 0.058 | 0.035 | 0.012 |
| 200 | 120 | 0.029 | 0.013 | 0.013 | 0.042 | 0.057 | 0.038 | 0.002 | 0.033 | 0.015 | 0.016 | 0.047 | 0.066 | 0.04 | 0.001 |
| 200 | 240 | 0.031 | 0.018 | 0.014 | 0.038 | 0.059 | 0.036 | 0 | 0.028 | 0.008 | 0.005 | 0.045 | 0.066 | 0.033 | 0 |
| | | iii | | | | | | | iv | | | | | | |
| 100 | 30 | 0.047 | 0.011 | 0.011 | 0.039 | 0.06 | 0.035 | 0.06 | 0.052 | 0.008 | 0.006 | 0.04 | 0.071 | 0.036 | 0.036 |
| 100 | 60 | 0.036 | 0.013 | 0.009 | 0.039 | 0.077 | 0.034 | 0.044 | 0.045 | 0.009 | 0.009 | 0.056 | 0.092 | 0.041 | 0.033 |
| 100 | 120 | 0.038 | 0.01 | 0.008 | 0.031 | 0.081 | 0.025 | 0.028 | 0.047 | 0.011 | 0.009 | 0.039 | 0.094 | 0.03 | 0.023 |
| 100 | 240 | 0.028 | 0.004 | 0.003 | 0.027 | 0.082 | 0.02 | 0.008 | 0.051 | 0.006 | 0.003 | 0.037 | 0.104 | 0.021 | 0.015 |
| 200 | 30 | 0.031 | 0.017 | 0.018 | 0.04 | 0.049 | 0.036 | 0.06 | 0.057 | 0.017 | 0.016 | 0.045 | 0.057 | 0.037 | 0.059 |
| 200 | 60 | 0.042 | 0.014 | 0.012 | 0.038 | 0.047 | 0.034 | 0.056 | 0.053 | 0.015 | 0.012 | 0.043 | 0.057 | 0.032 | 0.045 |
| 200 | 120 | 0.034 | 0.014 | 0.008 | 0.037 | 0.057 | 0.036 | 0.046 | 0.038 | 0.014 | 0.018 | 0.041 | 0.056 | 0.037 | 0.035 |
| 200 | 240 | 0.030 | 0.01 | 0.01 | 0.043 | 0.074 | 0.033 | 0.012 | 0.048 | 0.012 | 0.012 | 0.045 | 0.072 | 0.041 | 0.029 |
| | | v | | | | | | | vi | | | | | | |
| 100 | 30 | 0.039 | 0.011 | 0.01 | 0.038 | 0.071 | 0.041 | 0.013 | 0.039 | 0.013 | 0.011 | 0.039 | 0.072 | 0.029 | 0.026 |
| 100 | 60 | 0.041 | 0.013 | 0.014 | 0.055 | 0.09 | 0.047 | 0.003 | 0.033 | 0.007 | 0.005 | 0.034 | 0.067 | 0.03 | 0.001 |
| 100 | 120 | 0.031 | 0.016 | 0.006 | 0.044 | 0.086 | 0.036 | 0.001 | 0.032 | 0.013 | 0.007 | 0.037 | 0.083 | 0.027 | 0 |
| 100 | 240 | 0.027 | 0.006 | 0.004 | 0.025 | 0.08 | 0.019 | 0 | 0.034 | 0.008 | 0.005 | 0.035 | 0.106 | 0.029 | 0 |
| 200 | 30 | 0.053 | 0.024 | 0.023 | 0.049 | 0.064 | 0.043 | 0.027 | 0.052 | 0.017 | 0.016 | 0.047 | 0.051 | 0.042 | 0.031 |
| 200 | 60 | 0.044 | 0.013 | 0.015 | 0.029 | 0.053 | 0.03 | 0.016 | 0.039 | 0.014 | 0.014 | 0.043 | 0.06 | 0.038 | 0.017 |
| 200 | 120 | 0.032 | 0.023 | 0.021 | 0.053 | 0.077 | 0.049 | 0.011 | 0.031 | 0.015 | 0.013 | 0.047 | 0.066 | 0.038 | 0.003 |
| 200 | 240 | 0.029 | 0.008 | 0.006 | 0.029 | 0.067 | 0.024 | 0 | 0.029 | 0.011 | 0.009 | 0.032 | 0.053 | 0.029 | 0 |
| | | vii | | | | | | | viii | | | | | | |
| 100 | 30 | 0.053 | 0.008 | 0.007 | 0.038 | 0.059 | 0.034 | 0.052 | 0.053 | 0.01 | 0.013 | 0.043 | 0.07 | 0.04 | 0.037 |
| 100 | 60 | 0.042 | 0.01 | 0.007 | 0.042 | 0.071 | 0.038 | 0.041 | 0.044 | 0.012 | 0.01 | 0.039 | 0.067 | 0.026 | 0.031 |
| 100 | 120 | 0.037 | 0.012 | 0.006 | 0.035 | 0.083 | 0.02 | 0.023 | 0.031 | 0.009 | 0.004 | 0.028 | 0.073 | 0.019 | 0.016 |
| 100 | 240 | 0.026 | 0.009 | 0.003 | 0.036 | 0.101 | 0.025 | 0.006 | 0.043 | 0.005 | 0.005 | 0.028 | 0.108 | 0.017 | 0.017 |
| 200 | 30 | 0.043 | 0.025 | 0.025 | 0.052 | 0.062 | 0.049 | 0.073 | 0.046 | 0.016 | 0.014 | 0.045 | 0.057 | 0.044 | 0.047 |
| 200 | 60 | 0.036 | 0.016 | 0.015 | 0.046 | 0.057 | 0.041 | 0.046 | 0.033 | 0.011 | 0.01 | 0.054 | 0.058 | 0.047 | 0.024 |
| 200 | 120 | 0.031 | 0.01 | 0.008 | 0.034 | 0.05 | 0.032 | 0.022 | 0.043 | 0.015 | 0.012 | 0.043 | 0.066 | 0.033 | 0.036 |
| 200 | 240 | 0.028 | 0.012 | 0.01 | 0.04 | 0.059 | 0.042 | 0.005 | 0.034 | 0.015 | 0.024 | 0.039 | 0.053 | 0.026 | 0.034 |

Table 5: Sizes of tests with $K=6$ under Model (i)-(viii).

| $n$ | $p$ | i | | | | | | | ii | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $L_r$ | $L_\tau$ | $L_\rho$ | $L_{\tau^*}$ | $L_D$ | $L_R$ | $S_r$ | $L_r$ | $L_\tau$ | $L_\rho$ | $L_{\tau^*}$ | $L_D$ | $L_R$ | $S_r$ |
| 100 | 30 | 0.049 | 0.016 | 0.012 | 0.051 | 0.082 | 0.046 | 0.002 | 0.055 | 0.013 | 0.014 | 0.041 | 0.066 | 0.038 | 0.004 |
| 100 | 60 | 0.047 | 0.009 | 0.006 | 0.038 | 0.078 | 0.032 | 0.001 | 0.053 | 0.012 | 0.01 | 0.046 | 0.099 | 0.035 | 0 |
| 100 | 120 | 0.036 | 0.016 | 0.011 | 0.053 | 0.105 | 0.038 | 0 | 0.043 | 0.008 | 0.003 | 0.028 | 0.083 | 0.027 | 0 |
| 100 | 240 | 0.034 | 0.005 | 0.002 | 0.023 | 0.077 | 0.014 | 0 | 0.041 | 0.007 | 0.002 | 0.026 | 0.095 | 0.018 | 0 |
| 200 | 30 | 0.055 | 0.011 | 0.013 | 0.043 | 0.047 | 0.04 | 0.015 | 0.045 | 0.015 | 0.015 | 0.036 | 0.046 | 0.032 | 0.013 |
| 200 | 60 | 0.056 | 0.016 | 0.014 | 0.037 | 0.051 | 0.036 | 0.002 | 0.042 | 0.012 | 0.01 | 0.046 | 0.054 | 0.046 | 0.008 |
| 200 | 120 | 0.037 | 0.009 | 0.007 | 0.045 | 0.062 | 0.036 | 0 | 0.041 | 0.014 | 0.012 | 0.032 | 0.053 | 0.032 | 0 |
| 200 | 240 | 0.039 | 0.019 | 0.013 | 0.043 | 0.071 | 0.04 | 0 | 0.036 | 0.01 | 0.007 | 0.046 | 0.078 | 0.041 | 0 |
| | | iii | | | | | | | iv | | | | | | |
| 100 | 30 | 0.058 | 0.01 | 0.01 | 0.041 | 0.071 | 0.034 | 0.043 | 0.046 | 0.011 | 0.007 | 0.037 | 0.074 | 0.024 | 0.021 |
| 100 | 60 | 0.046 | 0.008 | 0.007 | 0.036 | 0.071 | 0.027 | 0.02 | 0.048 | 0.01 | 0.007 | 0.053 | 0.091 | 0.038 | 0.015 |
| 100 | 120 | 0.048 | 0.009 | 0.005 | 0.038 | 0.089 | 0.029 | 0.009 | 0.039 | 0.011 | 0.008 | 0.03 | 0.081 | 0.023 | 0.009 |
| 100 | 240 | 0.039 | 0.007 | 0.006 | 0.023 | 0.087 | 0.018 | 0.003 | 0.041 | 0.01 | 0.007 | 0.034 | 0.096 | 0.021 | 0.001 |
| 200 | 30 | 0.042 | 0.009 | 0.009 | 0.034 | 0.05 | 0.035 | 0.053 | 0.038 | 0.017 | 0.019 | 0.044 | 0.058 | 0.039 | 0.04 |
| 200 | 60 | 0.048 | 0.015 | 0.012 | 0.039 | 0.055 | 0.031 | 0.043 | 0.065 | 0.014 | 0.012 | 0.038 | 0.055 | 0.035 | 0.036 |
| 200 | 120 | 0.051 | 0.01 | 0.006 | 0.037 | 0.058 | 0.032 | 0.012 | 0.057 | 0.008 | 0.007 | 0.033 | 0.056 | 0.03 | 0.028 |
| 200 | 240 | 0.054 | 0.021 | 0.018 | 0.045 | 0.076 | 0.042 | 0.003 | 0.049 | 0.013 | 0.01 | 0.031 | 0.06 | 0.03 | 0.014 |
| | | v | | | | | | | vi | | | | | | |
| 100 | 30 | 0.049 | 0.014 | 0.01 | 0.036 | 0.06 | 0.032 | 0.007 | 0.049 | 0.018 | 0.01 | 0.05 | 0.082 | 0.046 | 0.007 |
| 100 | 60 | 0.038 | 0.01 | 0.007 | 0.043 | 0.072 | 0.033 | 0.002 | 0.036 | 0.01 | 0.01 | 0.037 | 0.076 | 0.029 | 0 |
| 100 | 120 | 0.032 | 0.012 | 0.007 | 0.033 | 0.086 | 0.026 | 0 | 0.032 | 0.009 | 0.008 | 0.026 | 0.069 | 0.022 | 0 |
| 100 | 240 | 0.031 | 0.011 | 0.006 | 0.028 | 0.093 | 0.021 | 0 | 0.033 | 0.008 | 0.003 | 0.035 | 0.096 | 0.019 | 0 |
| 200 | 30 | 0.042 | 0.019 | 0.014 | 0.044 | 0.065 | 0.04 | 0.02 | 0.047 | 0.017 | 0.014 | 0.038 | 0.047 | 0.036 | 0.021 |
| 200 | 60 | 0.041 | 0.016 | 0.014 | 0.051 | 0.071 | 0.043 | 0.006 | 0.035 | 0.014 | 0.011 | 0.044 | 0.06 | 0.037 | 0.001 |
| 200 | 120 | 0.032 | 0.014 | 0.011 | 0.034 | 0.053 | 0.03 | 0 | 0.041 | 0.006 | 0.005 | 0.033 | 0.049 | 0.031 | 0 |
| 200 | 240 | 0.037 | 0.012 | 0.007 | 0.035 | 0.065 | 0.029 | 0 | 0.037 | 0.013 | 0.013 | 0.04 | 0.073 | 0.032 | 0 |
| | | vii | | | | | | | viii | | | | | | |
| 100 | 30 | 0.052 | 0.016 | 0.014 | 0.045 | 0.079 | 0.04 | 0.025 | 0.051 | 0.01 | 0.011 | 0.042 | 0.081 | 0.035 | 0.025 |
| 100 | 60 | 0.038 | 0.007 | 0.008 | 0.041 | 0.084 | 0.035 | 0.013 | 0.042 | 0.01 | 0.008 | 0.03 | 0.069 | 0.026 | 0.015 |
| 100 | 120 | 0.036 | 0.008 | 0.005 | 0.03 | 0.078 | 0.02 | 0.005 | 0.038 | 0.005 | 0.001 | 0.031 | 0.084 | 0.025 | 0.005 |
| 100 | 240 | 0.033 | 0.008 | 0.006 | 0.032 | 0.103 | 0.022 | 0 | 0.033 | 0.003 | 0.002 | 0.028 | 0.101 | 0.018 | 0.005 |
| 200 | 30 | 0.046 | 0.022 | 0.021 | 0.044 | 0.059 | 0.044 | 0.052 | 0.047 | 0.016 | 0.015 | 0.048 | 0.058 | 0.04 | 0.027 |
| 200 | 60 | 0.043 | 0.017 | 0.013 | 0.041 | 0.063 | 0.038 | 0.033 | 0.052 | 0.016 | 0.015 | 0.054 | 0.068 | 0.046 | 0.041 |
| 200 | 120 | 0.038 | 0.015 | 0.014 | 0.032 | 0.061 | 0.027 | 0.016 | 0.033 | 0.014 | 0.012 | 0.042 | 0.066 | 0.034 | 0.016 |
| 200 | 240 | 0.037 | 0.013 | 0.008 | 0.047 | 0.07 | 0.041 | 0.001 | 0.037 | 0.01 | 0.011 | 0.039 | 0.066 | 0.031 | 0.013 |