Simultaneous Inference for Mean Curves of Functional and Longitudinal Data: A Unified Theory

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Supplementary Material

In this material, we provide technical proofs for the results presented in the previous sections.

S1 Simulation results

All the simulation results are summarized in Tables 3–5.

S2 Some useful Lemmas

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^p$ be independent random vectors. Throughout this section, we use C to denote generic constant whose value may vary from place to place. For convenience, write $\mathbb{E}_0(X) = X - \mathbb{E}(X)$ and write $a_n \lesssim b_n$ if $a_n/b_n \leq C$. For a vector $\mathbf{x} = (x_1, \ldots, x_p)^\top \in \mathbb{R}^p$, denote the L_∞ norm $|\mathbf{x}|_\infty = \max_{j \leq p} |x_j|$. Lemma 1 below follows from Lemma 0.1 in Zhang and Wu (2018).

		Gmb		Bonf		Point		Gumbel	
c_0	$1-\alpha$	90%	95%	90%	95%	90%	95%	90%	95%
0.9	case 1	89.15	94.95	98.40	99.05	17.75	40.30	93.10	97.20
	case 2	89.95	94.60	97.90	98.90	14.85	37.55	94.25	97.40
	case 3	88.80	94.05	98.45	99.05	21.25	44.70	94.75	97.90
1.0	case 1	87.45	93.50	98.05	98.75	19.15	43.45	91.20	96.35
	case 2	88.80	94.20	98.70	99.25	19.90	42.10	93.40	97.55
	case 3	89.40	94.80	98.55	99.20	27.45	50.00	95.55	98.55
1.1	case 1	88.30	94.25	98.45	99.00	22.50	45.80	90.70	96.40
	case 2	88.95	93.95	98.65	99.30	23.10	45.45	93.70	97.35
	case 3	89.20	94.15	98.35	99.10	28.85	50.60	95.15	97.55

Table 3: Empirical coverage probabilities for the four different confidence bands of the mean curve μ with $\zeta \sim N(0,1)$

		Gmb		Bonf		Point		Gumbel	
c_0	$1-\alpha$	90%	95%	90%	95%	90%	95%	90%	95%
0.9	case 1	90.10	95.00	98.85	99.40	19.45	42.65	93.30	97.65
	case 2	90.50	94.55	98.10	98.80	17.60	40.55	94.20	97.35
	case 3	89.20	94.60	98.45	99.30	22.25	45.10	95.45	98.10
1.0	case 1	88.55	93.80	98.40	99.20	21.90	45.95	91.15	96.75
	case 2	89.50	94.65	98.35	99.15	19.45	43.05	93.65	97.70
	case 3	89.25	94.60	98.75	99.15	27.65	50.15	94.70	98.20
1.1	case 1	88.25	93.30	98.15	98.90	22.50	46.25	89.35	96.10
	case 2	87.95	94.65	98.60	99.45	22.40	44.30	93.45	97.10
	case 3	87.95	93.50	98.45	98.90	29.20	50.20	94.65	98.05

Table 4: Empirical coverage probabilities for the four different confidence bands of the mean curve μ with $\zeta \sim t_5$.

		Gmb		Bonf		Point		Gumbel	
c_0	$1-\alpha$	90%	95%	90%	95%	90%	95%	90%	95%
0.9	case 1	87.90	93.80	98.10	98.65	19.05	42.15	91.75	97.00
	case 2	89.00	94.70	98.75	99.35	19.15	40.20	94.30	98.10
	case 3	88.65	94.00	98.05	98.75	22.65	46.05	95.15	97.65
1.0	case 1	87.40	93.35	98.45	99.00	21.40	43.10	91.35	97.05
	case 2	87.75	93.50	98.50	99.20	21.65	43.10	93.50	97.60
	case 3	90.05	95.15	98.60	99.25	26.05	51.60	95.95	98.35
1.1	case 1	87.55	93.35	98.00	98.80	25.95	48.95	90.95	97.20
	case 2	86.60	93.35	97.95	98.70	24.10	45.70	92.55	97.25
	case 3	89.55	94.10	98.60	99.15	30.85	53.25	95.45	98.30

Table 5: Empirical coverage probabilities for the four different confidence bands of the mean curve μ with $\zeta \sim (\chi_5^2 - 5)$

Lemma 1. *Denote* $\mathbf{S}_n = \sum_{i=1}^n \mathbf{x}_i$, $\sigma_{\mathbf{x}}^2 = \max_{j\le p}\sum_{i=1}^n \mathbb{E}(\mathbf{x}_{ij}^2)$ *and* $M_{\mathbf{x}} = \max_{i\le n}|\mathbf{x}_i|_\infty$. *Assume that* $\mathbb{E}(M_{\mathbf{x}}^q) < \infty$ *for some* $q \ge 2$. *Then we have*

$$\||\mathbf{S}_n - \mathbb{E}(\mathbf{S}_n)|_\infty\|_q \lesssim \sigma_{\mathbf{x}}(\log p)^{1/2} + \|M_{\mathbf{x}}\|_q \log p, \tag{S2.1}$$

where the constant in $\lesssim$ *only depends on* q. *If* $\mathbf{x}_{ij} \ge 0$, *then*

$$\||\mathbf{S}_n|_\infty\|_q \lesssim \max_{j\le p}\sum_{i=1}^n \mathbb{E}(\mathbf{x}_{ij}) + \|M_{\mathbf{x}}\|_q \log p.$$

Proof sketch of Lemma 1. Let $\epsilon_1, \ldots, \epsilon_n \in \mathbb{R}$ be i.i.d. Rademacher random variables independent of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Then

$$\||\mathbb{E}_0(\mathbf{S}_n)|_\infty\|_q \lesssim \||\sum_{i=1}^n \epsilon_i \mathbf{x}_i|_\infty\|_q \lesssim \sqrt{(\log p)\|\max_{j\le p}\sum_{i=1}^n \mathbf{x}_{ij}^2\|_{q/2}}.$$

It is straightforward to derive that

$$\|\max_{j\le p}\sum_{i=1}^n \mathbf{x}_{ij}^2\|_{q/2} \le \sigma_{\mathbf{x}}^2 + \|M_{\mathbf{x}}\|_q^2 \log p.$$

Combining these two pieces together yields (S2.1). □

Assume that now $\mathbb{E}(\mathbf{x}_i) = \mathbf{0}$. Let $\mathbf{y}_1, \ldots, \mathbf{y}_n \in \mathbb{R}^p$ be independent centered Gaussian random vectors such that $\mathrm{cov}(\mathbf{y}_i) = \mathrm{cov}(\mathbf{x}_i)$ for all $i = 1, \ldots, n$. Define $L_n = \max_{j\le p} n^{-1}\sum_{i=1}^n \mathbb{E}|\mathbf{x}_{ij}|^3$ and

$$M_{n,\mathbf{x}}(\phi) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\max_{j\le p}|\mathbf{x}_{ij}|^3 \mathbf{1}\left\{\max_{j\le p}|\mathbf{x}_{ij}| > \left(\frac{n}{4\phi\log p}\right)^{1/2}\right\}\right] \quad \text{for} \ \ \phi > 1. \tag{S2.2}$$

Similarly, define $M_{n,\mathbf{y}}(\phi)$ with $\mathbf{x}_{ij}$'s replaced by $\mathbf{y}_{ij}$'s in (S2.2). Let $M_n(\phi) = M_{n,\mathbf{x}}(\phi) + M_{n,\mathbf{y}}(\phi)$. Now we are ready to state Lemma 2, which presents a nonasymptotic bound of the Kolmogorov distance $\rho_{\mathbf{x},\mathbf{y}}$ between the distribution functions of $|n^{-1/2}\sum_{i=1}^n \mathbf{x}_i|_\infty$ and $|n^{-1/2}\sum_{i=1}^n \mathbf{y}_i|_\infty$, that is,

$$\rho_{\mathbf{x},\mathbf{y}} = \sup_{y\geq 0}\left|\mathbb{P}\left\{\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbf{x}_i\right|_\infty \leq y\right\} - \mathbb{P}\left\{\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbf{y}_i\right|_\infty \leq y\right\}\right|.$$

Lemma 2 (Theorem 2.1 in Chernozhukov et al. (2017))**.** *Suppose that there exists a constant $\kappa_0 > 0$ such that $n^{-1}\sum_{i=1}^n \mathbb{E}(\mathbf{x}_{ij}^2) \geq \kappa_0$ for all $1 \leq j \leq p$. Then there exist constants $\kappa_1, \kappa_2 > 0$ depending only on κ_0 such that for every $\bar{L}_n \geq L_n$, we have*

$$\rho_{\mathbf{x},\mathbf{y}} \leq \kappa_1\left\{\left(\frac{\bar{L}_n^2\log^7 p}{n}\right)^{1/6} + \frac{M_n(\phi_n)}{\bar{L}_n}\right\} \text{ with } \phi_n = \kappa_2\left(\frac{\bar{L}_n^2\log^4 p}{n}\right)^{-1/6}. \tag{S2.3}$$

Lemma 3. *Let Assumptions 1–5 be satisfied with $q \geq 2$. Then we have*

$$|\sigma(t) - \sigma(s)| \leq \frac{C|t-s|}{b^2}, \ \ t, s \in \mathcal{T}.$$

Proof of Lemma 3. By Assumptions 2 and 3, for every $1 \leq i \leq n$,

$$\begin{aligned}\|\eta_i(t) - \eta_i(s)\|_2 &\leq \frac{1}{m_i}\sum_{j=1}^{m_i}\|\{K_b(t_{ij}-t) - K_b(t_{ij}-s)\}e_{ij}\|_2 \\ &\leq \frac{1}{m_i}\sum_{j=1}^{m_i}\frac{L_K|t-s|\|e_{ij}\|_2}{b^2} \leq \frac{C|t-s|}{b^2}.\end{aligned} \tag{S2.4}$$

Recall that $\sigma^2(t) = n^{-1}\sum_{i=1}^n \mathrm{var}\{\eta_i(t)\} \asymp \vartheta^2$ uniformly for $t \in \mathcal{T}$. Hence it follows that

$$|\sigma(t)-\sigma(s)| = \frac{|\sum_{i=1}^n \mathbb{E}\{\eta_i(t)+\eta_i(s)\}\{\eta_i(t)-\eta_i(s)\}|}{n\{\sigma(t)+\sigma(s)\}} \leq \frac{\sum_{i=1}^n \|\eta_i(t)+\eta_i(s)\|_2\|\eta_i(t)-\eta_i(s)\|_2}{n\{\sigma(t)+\sigma(s)\}} \leq \frac{C|t-s|}{b^2}.$$

□

S3 Proof of Theorem 1

Proof of Theorem 1. Let $t_\ell = \ell/\mathcal{L}$, $\ell = 1, \ldots, \mathcal{L} = n^4$, be an equidistant partition of the interval $\mathcal{T} = [0,1]$. By the triangle inequality, it follows that $\rho_{\mathcal{T}} \leq \widetilde{\rho}_1 + \widetilde{\rho}_2 + \widetilde{\rho}_3$, where

$$\widetilde{\rho}_1 = \sup_{y\geq 0}\left|\mathbb{P}\Big\{\sup_{t\in\mathcal{T}}\sqrt{n}|\bar{\pi}_n(t)| \leq y\Big\} - \mathbb{P}\Big\{\max_{\ell\leq\mathcal{L}}\sqrt{n}|\bar{\pi}_n(t_\ell)| \leq y\Big\}\right|,$$

$$\widetilde{\rho}_2 = \sup_{y\geq 0}\left|\mathbb{P}\Big\{\max_{\ell\leq\mathcal{L}}\sqrt{n}|\bar{\pi}_n(t_\ell)| \leq y\Big\} - \mathbb{P}\Big\{\max_{\ell\leq\mathcal{L}}|\mathcal{G}(t_\ell)| \leq y\Big\}\right|,$$

$$\widetilde{\rho}_3 = \sup_{y\geq 0}\left|\mathbb{P}\Big\{\max_{\ell\leq\mathcal{L}}|\mathcal{G}(t_\ell)| \leq y\Big\} - \mathbb{P}\Big\{\sup_{t\in\mathcal{T}}|\mathcal{G}(t)| \leq y\Big\}\right|.$$

Hence it suffices to show that $\widetilde{\rho}_k \to 0$ for all $k = 1, 2, 3$.

Step 1. First we show that $\widetilde{\rho}_2 \to 0$. For simplicity of notation, we define the random vector $\pi_i = (\pi_{i1}, \ldots, \pi_{i\mathcal{L}})^\top \in \mathbb{R}^{\mathcal{L}}$, where

$$\pi_{i\ell} = \pi_i(t_\ell) = \frac{\sum_{j=1}^{m_i} K_b(t_{ij}-t_\ell)\nu_i(t_{ij})}{m_i\sigma(t_\ell)} + \frac{\sum_{j=1}^{m_i} K_b(t_{ij}-t_\ell)\varepsilon_{ij}}{m_i\sigma(t_\ell)} =: \pi_{i\ell}^\star + \pi_{i\ell}^\diamond.$$

Let $g_i = (g_{i1}, \ldots, g_{i\mathcal{L}})^\top \in \mathbb{R}^{\mathcal{L}}$, $i = 1, \ldots, n$, be independent centered Gaussian random vectors such that $\mathrm{cov}(g_i) = \mathrm{cov}(\pi_i)$ for all $1 \leq i \leq n$. Since

$\max_{\ell\le\mathcal{L}}|n^{-1/2}\sum_{i=1}^n g_{i\ell}|$ has the same distribution as $\max_{\ell\le\mathcal{L}}|\mathcal{G}(t_\ell)|$, it is equivalent to show that

$$\sup_{y\ge 0}\Big|\mathbb{P}\Big\{\max_{\ell\le\mathcal{L}}\sqrt{n}|\bar{\pi}_{n\ell}|\le y\Big\}-\mathbb{P}\Big\{\max_{\ell\le\mathcal{L}}\sqrt{n}|\bar{g}_{n\ell}|\le y\Big\}\Big|\to 0,$$

where $\bar{\pi}_{n\ell}=n^{-1}\sum_{i=1}^n\pi_{i\ell}$ and $\bar{g}_{n\ell}=n^{-1}\sum_{i=1}^n g_{i\ell}$. To this end, we shall apply Lemma 2. Recall that $\sigma^2(t)=n^{-1}\sum_{i=1}^n\mathrm{var}\{\eta_i(t)\}$ and $\pi_i(t)=\eta_i(t)/\sigma(t)$. Hence

$$\frac{1}{n}\sum_{i=1}^n\mathrm{var}(\pi_{i\ell})=1>0, \tag{S3.1}$$

for all $\ell=1,\ldots,\mathcal{L}$. Applying the Rosenthal inequality (cf. Rosenthal (1970)), we find that

$$\begin{aligned}\mathbb{E}|\pi_{i\ell}|^3&\lesssim\frac{\mathbb{E}\sup_{t\in\mathcal{T}}|\nu(t)|^3\mathbb{E}|\sum_{j=1}^{m_i}K_b(t_{ij}-t_\ell)|^3}{(m_i\vartheta)^3}+\frac{\mathbb{E}|\sum_{j=1}^{m_i}K_b(t_{ij}-t_\ell)\varepsilon_{ij}|^3}{(m_i\vartheta)^3}\\&\lesssim\frac{m_i/b^2+(m_i/b)^{3/2}+m_i^3}{(m_i\vartheta)^3}\lesssim\frac{m_i/b^2+m_i^3}{(m_i\vartheta)^3},\end{aligned}$$

which implies that

$$\max_{\ell\le\mathcal{L}}\frac{1}{n}\sum_{i=1}^n\mathbb{E}|\pi_{i\ell}|^3\lesssim\frac{1}{n}\sum_{i=1}^n\frac{m_i/b^2+m_i^3}{(m_i\vartheta)^3}=\frac{V_2/b^2+1}{\vartheta^3}=:L_\diamond.$$

By Lemma 1, it follows that

$$\begin{aligned}\mathbb{E}\max_{\ell\le\mathcal{L}}|\pi^\star_{i\ell}|^q&\lesssim\frac{\mathbb{E}\sup_{t\in\mathcal{T}}|\nu(t)|^q\mathbb{E}\max_{\ell\le\mathcal{L}}|\sum_{j=1}^{m_i}K_b(t_{ij}-t_\ell)|^q}{(m_i\vartheta)^q}\\&\lesssim\frac{((\log n)/b)^q+m_i^q}{(m_i\vartheta)^q}\wedge(b\vartheta)^{-q}.\end{aligned}\tag{S3.2}$$

Similarly, we have $\mathbb{E}\max_{j\le m_i,\ell\le\mathcal{L}}|K_b(t_{ij}-t_\ell)\varepsilon_{ij}|^q\le(M_K/b)^q\sum_{j=1}^{m_i}\mathbb{E}|\varepsilon_{ij}|^q\le$

$Cm_i b^{-q}$ and

$$\mathbb{E}\max_{\ell\le\mathcal{L}}|\pi_{i\ell}^{\diamond}|^q \lesssim \frac{(m_i\log n/b)^{q/2}+m_i(\log n/b)^q}{(m_i\vartheta)^q}\wedge(b\vartheta)^{-q}.$$

Combined with (S3.2), we obtain

$$\mathbb{E}\max_{\ell\le\mathcal{L}}|\pi_{i\ell}|^q \lesssim \frac{m_i(\log n/b)^q+m_i^q}{(m_i\vartheta)^q}\wedge(b\vartheta)^{-q}. \tag{S3.3}$$

For $\chi\in(0,1)$, let

$$\mathcal{M}=\left[L_\diamond\vee\frac{W^{3/q}\log^{1-3/q}n}{n^{1-3/q}}\vee\frac{\bar{\sigma}^3\log^{5/2}n}{n}\right]^{1-\chi}\left(\frac{n}{\log^7 n}\right)^{\chi/2}$$

and $\kappa=\sqrt{n}/(4\phi_{\mathcal{M}}\log\mathcal{L})$, where $\phi_{\mathcal{M}}=\kappa_2(\mathcal{M}^2\log^4\mathcal{L}/n)^{-1/6}$ is defined in $(S2.3)$ with $\bar{L}_n$ replaced by $\mathcal{M}$. Elementary calculations together with (S3.3) imply that

$$M_{n,\pi}(\phi_{\mathcal{M}})\le\frac{1}{n}\sum_{i=1}^n\frac{\mathbb{E}\max_{\ell\le\mathcal{L}}|\pi_{i\ell}|^q}{\kappa^{q-3}}\asymp W/\kappa^{q-3}. \tag{S3.4}$$

Now we upper bound the Gaussian analog term $M_{n,g}(\phi_{\mathcal{M}})$. Note that $g_{i\ell}$ is a zero-mean normal random variable with variance

$$\operatorname{var}(g_{i\ell})=\operatorname{var}(\pi_{i\ell})\asymp\frac{m_i^{-1}/b+1}{\vartheta^2}\le\frac{m_\diamond^{-1}+b}{b\vartheta^2}=\bar{\sigma}^2. \tag{S3.5}$$

Hence, for any $x>0$, we have $\mathbb{P}(|g_{i\ell}|>x)\le 2\exp(-Cx^2/\bar{\sigma}^2)$. Combined with (2.12), we obtain

$$\begin{aligned} M_{n,g}(\phi_{\mathcal{M}}) &\le \frac{1}{n}\sum_{i=1}^n\sum_{\ell=1}^{\mathcal{L}}\left\{\mathbb{P}(|g_{i\ell}|>\kappa)\kappa^3+3\int_\kappa^\infty\mathbb{P}(|g_{i\ell}|>x)x^2dx\right\} \\ &\le C\mathcal{L}(\kappa^3+\kappa\bar{\sigma}+\bar{\sigma}^2)\exp(-C\kappa^2/\bar{\sigma}^2)\le Cn^{-c}, \end{aligned} \tag{S3.6}$$

where the last inequality follows from

$$\frac{\kappa^2/\bar{\sigma}^2}{\log n} \geq \frac{(n\mathcal{M}/\log n)^{2/3}}{\bar{\sigma}^2 \log n} \geq \left(\frac{n}{\bar{\sigma}^2 \log^4 n}\right)^{\chi} \to \infty. \tag{S3.7}$$

By (2.12) and $\chi \in (0,1)$, it follows that

$$\begin{aligned}
\frac{L_\diamond}{\mathcal{M}} &\leq \frac{L_\diamond^\chi}{(n/\log^7 n)^{\chi/2}} = \left[\frac{(V_2/b^2+1)\log^{7/2} n}{n^{1/2}\vartheta^3}\right]^{\chi} \to 0,\\
\frac{L_\diamond^2 \log^7 n}{n} &= \left[\frac{(V_2/b^2+1)\log^{7/2} n}{n^{1/2}\vartheta^3}\right]^{2} \to 0.
\end{aligned}$$

Combined with (S3.1), (S3.4) and (S3.6), it follows by Lemma 2 that

$$\begin{aligned}
\widetilde{\rho}_2 &\lesssim \left(\frac{\mathcal{M}^2 \log^7 n}{n}\right)^{1/6} + \frac{M_{n,\pi}(\phi_{\mathcal{M}}) + M_{n,g}(\phi_{\mathcal{M}})}{\mathcal{M}}\\
&\lesssim \left(\frac{L_\diamond^2 \log^7 n}{n}\right)^{(1-\chi)/6} + \left(\frac{\bar{\sigma}^2 \log^4 n}{n}\right)^{(1-\chi)/2} + \left(\frac{W \log^{3q/2-1} n}{n^{q/2-1}}\right)^{\chi}\\
&+ \left(\frac{W \log^{3q/2-1} n}{n^{q/2-1}}\right)^{(1-\chi)/q} + Cn^{-c} \to 0.
\end{aligned}$$

Step 2. Now we are to show that $\widetilde{\rho}_1 \to 0$. Define $\Delta_\pi = \sup_{|t-s|\leq 1/\mathcal{L}} \sqrt{n}|\bar{\pi}_n(t) - \bar{\pi}_n(s)|$. By Lemma 3 and Assumption 3,

$$\begin{aligned}
\Delta_\pi &= \sup_{|t-s|\leq 1/\mathcal{L}} \left|\frac{\sum_{i=1}^n \sigma(s)\{\eta_i(t)-\eta_i(s)\} + \sum_{i=1}^n \{\sigma(s)-\sigma(t)\}\eta_i(s)}{\sqrt{n}\sigma(t)\sigma(s)}\right|\\
&\lesssim \frac{\sum_{i=1}^n m_i^{-1}\sum_{j=1}^{m_i}|e_{ij}|}{\vartheta\sqrt{n}\mathcal{L}b^2} + \frac{\sum_{i=1}^n m_i^{-1}\sum_{j=1}^{m_i}|e_{ij}|}{\vartheta^2\sqrt{n}\mathcal{L}b^3}.
\end{aligned}$$

Then, by Lemma 2.1 in Chernozhukov et al. (2013) and Assumption 2, it follows that

$$\begin{aligned}\widetilde{\rho}_1 &\le \mathbb{P}\Big\{\Delta_\pi > Cn^{-1/2}\Big\} + \sup_{y\ge 0}\mathbb{P}\Big\{\Big|\max_{\ell\le\mathcal{L}}\sqrt{n}|\bar{\pi}_n(t_\ell)| - y\Big| \le Cn^{-1/2}\Big\} \\ &\le Cn^{1/2}\mathbb{E}(\Delta_\pi) + \sup_{y\ge 0}\mathbb{P}\Big\{\Big|\max_{\ell\le\mathcal{L}}|\mathcal{G}(t_\ell)| - y\Big| \le Cn^{-1/2}\Big\} + \widetilde{\rho}_2 \\ &\le C(nb)^{-3} + C(\log n/n)^{1/2} + \widetilde{\rho}_2 \to 0.\end{aligned}$$

Step 3. It remains to show that $\widetilde{\rho}_3 \to 0$. For the Gaussian process $\{\mathcal{G}(t)\}_{t\in\mathcal{T}}$, we define the semi metric $d^2(s,t) = \mathbb{E}|\mathcal{G}(s) - \mathcal{G}(t)|^2$. By (S2.4) and Assumption 3, it follows that $d^2(s,t) \le C|t-s|/b^2$, which implies $\sup_{|s-t|\le 1/\mathcal{L}} d(s,t) \le C\mathcal{L}^{-1/2}/b =: \delta_\star$. Consequently, by Dudley's entropy inequality (cf. van der Vaart and Wellner (1996)),

$$\mathbb{E}\sup_{|t-s|\le 1/\mathcal{L}}|\mathcal{G}(t) - \mathcal{G}(s)| \le C\int_0^{\delta_\star}\{\log(b^{-2}/\epsilon^2)\}^{1/2}d\epsilon \le C\delta_\star(\log n)^{1/2}.$$

Taking $\delta = 2C\delta_\star(\log n)^{1/2}$, by the Borell-Sudakov-Tsirel'son inequality (cf. van der Vaart and Wellner (1996)) and Lemma 2.1 in Chernozhukov et al. (2017), it follows that

$$\begin{aligned}\widetilde{\rho}_3 &\le \mathbb{P}\left\{\sup_{|t-s|\le 1/\mathcal{L}}|\mathcal{G}(t) - \mathcal{G}(s)| > \delta\right\} + \sup_{y\ge 0}\mathbb{P}\left\{\left|\max_{\ell\le\mathcal{L}}|\mathcal{G}(t_\ell)| - y\right| \le \delta\right\} \\ &\le 2\exp\left(-\frac{4C^2\delta_\star^2\log n}{2\delta_\star^2}\right) + C\delta(\log n)^{1/2} \le 2n^{-2C^2} + \frac{C\log n}{n^2 b} \to 0.\end{aligned}$$

□

Remark 1. Now we shall discuss the condition on the quantity $\bar{\sigma}^2$ in Theorem 1. Recall that (2.12) in Theorem 1 requires $\bar{\sigma}^2$ satisfy

$$\frac{\bar{\sigma}^2(\log n)^4}{n} \to 0, \tag{S3.8}$$

which ensures (S3.7). However, a careful inspection of (S3.5)–(S3.7) reveals that a weaker condition that (S3.8) may be sufficient for (2.13) as long as

$$M_{n,g}(\phi_{\mathcal{M}}) \to 0.$$

For instance, instead of controlling the variance term $\mathrm{var}(g_{i\ell})$ via the uniform upper bound $\bar{\sigma}^2$ in (S3.5), we can bound $\mathrm{var}(g_{i\ell})$ via

$$\mathrm{var}(g_{i\ell}) \asymp \frac{m_i^{-1}/b + 1}{\vartheta^2} =: \widetilde{\sigma}_i^2,$$

uniformly for all $i \in \{1, \ldots, n\}$. Then a weaker condition for $M_{n,g}(\phi_{\mathcal{M}}) \to 0$ would be

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{\ell=1}^{\mathcal{L}}\left\{\exp(-C\kappa^2/\widetilde{\sigma}_i^2) + 3\int_{\kappa}^{\infty}\exp(-Cx^2/\widetilde{\sigma}_i^2)x^2dx\right\} \to 0.$$

Nevertheless, condition (S3.8) on $\bar{\sigma}^2$ is actually quite mild and does not introduce any additional restrictive condition. More specifically, as $m_{\diamond} = \min_{1\le i\le n} m_i \ge 1$, a simple upper bound for $\bar{\sigma}^2$ is

$$\bar{\sigma}^2 = \frac{m_{\diamond}^{-1} + b}{n\vartheta^2} \le \frac{2}{nV_1 + nb}.$$

Consequently, (S3.8) holds as long as

$$\frac{\bar{\sigma}^2(\log n)^4}{n} \le \frac{2(\log n)^4}{nV_1 + nb} \le \frac{2(\log n)^4}{nb} \to 0,$$

which is fairly mild condition on the bandwidth b in the context of non-parametric estimation and inference for longitudinal/functional data.

S4 Proof of Theorem 2

Lemma 4. *Under Assumptions 1, 3 and 6, we have*

$$B_n = \sup_{t\in\mathcal{T}_b} |\mathbb{E}\{\bar{\varphi}_n(t)\} - b^2 r(t)| = o(b^2),$$
$$T_n = \sup_{t\in\mathcal{T}} |\bar{\varphi}_n(t) - \mathbb{E}\{\bar{\varphi}_n(t)\}| = O_{\mathbb{P}}\bigg\{ \bigg(\frac{V_1 b^3 \log n}{n}\bigg)^{1/2} + \frac{b\log n}{m_\diamond n}\bigg\}.$$

Proof of Lemma 4. Under Assumptions 1, 3 and 6, it is straightforward to verify that

$$\sup_{t\in\mathcal{T}_b} |\mathbb{E}\{\varphi_i(t)\} - b^2 r(t)| = o(b^2).$$

Hence $B_n = o(b^2)$. Applying an elementary discretization argument, we find that

$$T_n \le \max_{\ell\le\mathcal{L}} |\bar{\varphi}_n(t_\ell) - \mathbb{E}\{\bar{\varphi}_n(t_\ell)\}| + \sup_{|t-s|\le 1/\mathcal{L}} |\bar{\varphi}_n(t) - \bar{\varphi}_n(s)|$$
$$+ \sup_{|t-s|\le 1/\mathcal{L}} |\mathbb{E}\{\bar{\varphi}_n(t) - \bar{\varphi}_n(s)\}| =: T_1 + T_2 + T_3. \quad \text{(S4.1)}$$

It is straightforward to verify that $\sum_{i=1}^n \mathrm{var}\{\varphi_i(t_\ell)\} \le CnV_1b^3$ and

$$\max_{\ell\le\mathcal{L}} |K_b(t_{ij}-t_\ell)\{\mu(t_{ij}) - \mu(t_\ell) - (t_{ij}-t_\ell)\mu'(t_\ell)\}| \le M_K b \sup_{t\in\mathcal{T}} |\mu''(t)|/2.$$

Then, by Bernstein's inequality and $T_2 + T_3 \le C\mathcal{L}^{-1}$, it follows that

$$\mathbb{P}(T_1 > Cy) \le 2\mathcal{L} \exp\left(-\frac{Cn^2y^2}{nV_1b^3 + nby}\right) \le Cn^{-c},$$

where $y = (V_1 b^3 \log n/n)^{1/2} + b \log n/(m_\diamond n)$. □

Lemma 5. *Let Assumptions 1–6 be satisfied with $q \ge 2$. Then we have*

$$\sup_{t\in\mathcal{T}} |S_k(t) - \mathbb{E}\{S_k(t)\}| = O_{\mathbb{P}}\bigg\{ \left(\frac{V_1 \log n}{nb}\right)^{1/2} + \frac{\log n}{m_\diamond nb}\bigg\}, \quad k = 0, 1, 2;$$
$$\sup_{t\in\mathcal{T}} |R_k^*(t)| = O(b^2) + O_{\mathbb{P}}\bigg\{ \left(\frac{\vartheta^2 \log n}{n}\right)^{1/2} + \frac{W^{1/q}\vartheta \log n}{n^{1-1/q}}\bigg\}, \quad k = 0, 1;$$
$$\sup_{t\in\mathcal{T}_b} |R_1^*(t)| = o(b^2) + O_{\mathbb{P}}\bigg\{ \left(\frac{V_1 \log n}{nb} + \frac{b^\alpha \log n}{n}\right)^{1/2} + \frac{W^{1/q}\vartheta \log n}{n^{1-1/q}}\bigg\}.$$

The proof of Lemma 5 is similar with that of (S3.2)–(S3.4) and Lemma 4 and thus omitted.

Proof of Theorem 2. A careful inspection of the proof of Lemma 1 reveals that $\rho_{\mathcal{T}_b} \to 0$. Hence it suffices to show that

$$\rho_\diamond = \sup_{y\ge 0} \left| \mathbb{P}\Big\{ \sup_{t\in\mathcal{T}_b} |Z_b(t)| \le y \Big\} - \mathbb{P}\Big\{ \sup_{t\in\mathcal{T}_b} \sqrt{n}|\bar{\pi}_n(t)| \le y \Big\} \right| \to 0.$$

Define $\Delta_Z(t) = Z_b(t) - \sqrt{n}\bar{\pi}_n(t)$. By (2.15) and Lemma 4, it follows that $\sup_{t\in\mathcal{T}_b} |\Delta_Z(t)| = o_p\{(\log n)^{-1/2}\}$, which implies that there exists a positive sequence $d_n \downarrow 0$ such that $\mathbb{P}\{\sup_{t\in\mathcal{T}_b} |\Delta_Z(t)| > d_n(\log n)^{-1/2}\} \le d_n$. Combined with Lemma 2.1 in Chernozhukov et al. (2013), we conclude that

$$\rho_\diamond \le \sup_{y\in R} \mathbb{P}\Big\{\Big|\sup_{t\in\mathcal{T}_b} |\mathcal{G}(t)| - y\Big| \le d_n(\log n)^{-1/2}\Big\} + \rho_{\mathcal{T}_b}$$

$$+ \mathbb{P}\Big\{\sup_{t\in\mathcal{T}_b} |\Delta_Z(t)| > d_n(\log n)^{-1/2}\Big\}$$

$$\le Cd_n + \rho_{\mathcal{T}_b} + d_n \to 0.$$

□

Proof of Theorem 3. Define $\Delta_\mu = \sup_{t\in\mathcal{T}} |\widehat{\mu}_b(t) - \mu(t)|$. By Lemma 5, we have $\Delta_\mu = O_{\mathbb{P}}(\Delta_0)$. Observe that

$$\sup_{t\in\mathcal{T}} \left|\frac{\widehat{\sigma}(t) - \sigma(t)}{\sigma(t)}\right| \le \sup_{t\in\mathcal{T}} \left|\frac{\widehat{\sigma}^2(t)}{\sigma^2(t)} - 1\right| \asymp \frac{\sup_{t\in\mathcal{T}} |\widehat{\sigma}^2(t) - \sigma^2(t)|}{\vartheta^2}.$$

Hence it suffices to show that $\Delta_\sigma^2 = \sup_{t\in\mathcal{T}} |\widehat{\sigma}^2(t) - \sigma^2(t)| = O_{\mathbb{P}}(\Delta_0\Delta_1 + \Delta_2)$. By the triangle inequality,

$$\Delta_\sigma^2 \le \sup_{t\in\mathcal{T}} \left|\frac{1}{n}\sum_{i=1}^n \{\widehat{\eta}_i^2(t) - \eta_i^2(t)\}\right| + \sup_{t\in\mathcal{T}} \left|\frac{1}{n}\sum_{i=1}^n \{\eta_i^2(t) - \sigma^2(t)\}\right| =: \Xi_1 + \Xi_2.$$

We first show that $\Xi_1 = O_{\mathbb{P}}(\Delta_0\Delta_1)$. For each $1 \le i \le n$, define the event $\mathcal{E}_i = \{\sup_{t\in\mathcal{T}} |\widehat{\eta}_i(t) - \eta_i(t)| \ge \widetilde{\kappa}\upsilon_i\Delta_\mu\}$, where $\upsilon_i = 1 + \log n/(m_i b)$ and $\widetilde{\kappa}$ is a positive constant satisfying

$$\widetilde{\kappa}\upsilon_i \ge \frac{2L_K}{n^2b^2} + 2M_f + 4\left(\frac{M_f\lambda_K \log n}{m_i b}\right)^{1/2} + \frac{3M_K \log n}{m_i b}.$$

Define $\mathcal{E}^c = \cap_{i=1}^n \mathcal{E}_i^c$. By a similar argument as the proof of Lemma 4, it follows that $\mathbb{P}(\mathcal{E}_i) \le 2n^{-2}$ for all $1 \le i \le n$. Consequently,

$$\mathbb{P}(\mathcal{E}^c) \ge 1 - \sum_{i=1}^{n} \mathbb{P}(\mathcal{E}_i) \ge 1 - \frac{2}{n}.$$

Note that under the event $\mathcal{E}^c$, we have $\Xi_1 \le \Xi_{11} + \Xi_{12}$, where

$$\Xi_{11} = \sup_{t\in\mathcal{T}} \sum_{i=1}^{n} \frac{\widetilde{\kappa}\Delta_\mu^2 \upsilon_i |\sum_{j=1}^{m_i} K_b(t_{ij}-t)|}{nm_i},$$
$$\Xi_{12} = \sup_{t\in\mathcal{T}} \sum_{i=1}^{n} \frac{\widetilde{\kappa}\Delta_\mu \upsilon_i |\sum_{j=1}^{m_i} K_b(t_{ij}-t)e_{ij}|}{nm_i/2}.$$

By Lemma E.3 in Chernozhukov et al. (2017) and Assumption 3,

$$\begin{aligned}\mathbb{E}\max_{\ell\le\mathcal{L}} \sum_{i=1}^{n} \frac{\upsilon_i}{m_i} \sum_{j=1}^{m_i} K_b(t_{ij}-t_\ell) &\lesssim \sum_{i=1}^{n} \upsilon_i + \mathbb{E} \max_{i\le n, j\le m_i, \ell\le\mathcal{L}} \frac{K_b(t_{ij}-t_\ell)\upsilon_i \log n}{m_i} \\ &\lesssim n + \frac{nV_1 \log n}{b} + \frac{\log^2 n}{(m_\diamond b)^2}.\end{aligned} \tag{S4.2}$$

Combining this bound with Lemma E.4 in Chernozhukov et al. (2017) and $m_\diamond^{-2} \le n^{1/q} V_{2q-1}^{1/q} \le n^{1/q} V_q^{1/q}$, we find that

$$\max_{\ell\le\mathcal{L}} \sum_{i=1}^{n} \frac{\upsilon_i}{m_i} \sum_{j=1}^{m_i} K_b(t_{ij}-t_\ell) = O_{\mathbb{P}}\left\{ n + \frac{nV_1 \log n}{b} + \frac{\log^2 n}{(m_\diamond b)^2} \right\} = O_{\mathbb{P}}(n\Delta_1). \tag{S4.3}$$

Consequently, by Assumption 3 and the discretization argument as in (S4.1),

$$\begin{aligned}\Xi_{11} &\le \max_{\ell\le\mathcal{L}} \frac{\Delta_\mu^2}{n} \sum_{i=1}^{n} \frac{\upsilon_i}{m_i} \sum_{j=1}^{m_i} K_b(t_{ij}-t_\ell) + \frac{\Delta_\mu^2}{n} \sum_{i=1}^{n} \frac{\upsilon_i}{m_i} \sum_{j=1}^{m_i} \frac{L_K}{b^2\mathcal{L}} \\ &= \frac{\Delta_\mu^2 O_{\mathbb{P}}(n\Delta_1)}{n} + \frac{L_K \Delta_\mu^2 (1 + V_1 \log n/b)}{n^4 b^2} = O_{\mathbb{P}}(\Delta_0^2 \Delta_1).\end{aligned} \tag{S4.4}$$

Similarly, by (S3.3),

$$\begin{aligned}\mathbb{E}\max_{\ell\le\mathcal{L}}\sum_{i=1}^{n}\frac{\upsilon_i}{m_i}\left|\sum_{j=1}^{m_i}K_b(t_{ij}-t_\ell)e_{ij}\right| &\lesssim \sum_{i=1}^{n}\upsilon_i+\mathbb{E}\max_{i\le n,\ell\le\mathcal{L}}\frac{\upsilon_i\log n}{m_i}\left|\sum_{j=1}^{m_i}K_b(t_{ij}-t_\ell)e_{ij}\right|\\ &\lesssim n+\frac{nV_1\log n}{b}+\left|\sum_{i=1}^{n}\upsilon_i^q\left[\frac{m_i(\log n/b)^q+m_i^q}{m_i^q}\wedge b^{-q}\right]\right|^{1/q}\log n\\ &\lesssim n+\frac{nV_1\log n}{b}+(nW)^{1/q}\vartheta\log n+\frac{\{V_{2q-1}(\log n/b)^q+V_q\}^{1/q}\wedge(V_q^{1/q}/b)}{bn^{-1/q}/(\log n)^2}.\end{aligned}$$

Hence $\Xi_{12}=\Delta_\mu O_{\mathbb{P}}(\Delta_1)=O_{\mathbb{P}}(\Delta_0\Delta_1)$. Together with $\mathbb{P}(\mathcal{E}^c)\to 1$, we conclude that $\Xi_1=O_{\mathbb{P}}(\Delta_0\Delta_1)$.

Now we show that $\Xi_2=O_{\mathbb{P}}(\Delta_2)$. By Lemma 1 and the Rosenthal inequality,

$$\max_{\ell\le\mathcal{L}}\sum_{i=1}^{n}\mathbb{E}|\eta_i(t_\ell)|^4\lesssim\sum_{i=1}^{n}\frac{m_i/b^3+(m_i/b)^2+m_i^4}{m_i^4}\le 2n\left(\frac{V_3}{b^3}+1\right), \quad \text{(S4.5)}$$

$$\mathbb{E}\max_{i\le n,\ell\le\mathcal{L}}|\eta_i(t_\ell)|^4\le\left\{\mathbb{E}\max_{i\le n,\ell\le\mathcal{L}}|\eta_i(t_\ell)|^q\right\}^{4/q}\asymp(nW)^{4/q}\vartheta^4. \quad \text{(S4.6)}$$

Combined with Lemma E.1 in Chernozhukov et al. (2017), we have

$$\mathbb{E}\max_{\ell\le\mathcal{L}}\left|\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_0\{\eta_i^2(t_\ell)\}\right|\lesssim\left(\frac{V_3\log n}{nb^3}+\frac{\log n}{n}\right)^{1/2}+\frac{W^{2/q}\vartheta^2\log n}{n^{1-2/q}}=\Delta_2,$$

which implies $\Xi_2=O_{\mathbb{P}}(\Delta_2)$ in view of Lemma E.2 in Chernozhukov et al. (2017) and a similar discretization argument as in (S4.4). □

S5 Proof of Corollary 2

Proof of Corollary 2. Denote $\widetilde{\Delta}_\sigma = \sup_{t\in\mathcal{T}} |\sigma(t)/\widehat{\sigma}(t)-1|$. Note that $\widehat{Z}_b(t) = \sigma(t)Z_b(t)/\widehat{\sigma}(t)$. Hence

$$\begin{aligned}\sup_{t\in\mathcal{T}_b}|Z_b(t)-\widehat{Z}_b(t)| &\le \sup_{t\in\mathcal{T}_b}|\sigma(t)/\widehat{\sigma}(t)-1|\sup_{t\in\mathcal{T}_b}|Z_b(t)| \\ &\le \widetilde{\Delta}_\sigma \sup_{t\in\mathcal{T}_b}|Z_b(t)|.\end{aligned}$$

By Theorem 3 and (3.24), we have $\widetilde{\Delta}_\sigma = o_p\{(\log n)^{-1}\}$. Consequently, there exists a positive sequence $\widetilde{d}_n \downarrow 0$ such that $\mathbb{P}\{\widetilde{\Delta}_\sigma > \widetilde{d}_n(\log n)^{-1}\} \le \widetilde{d}_n$. In view of (2.16), it suffices to prove

$$\rho_Z = \sup_{y\ge 0}\left|\mathbb{P}\left\{\sup_{t\in\mathcal{T}_b}|Z_b(t)|\le y\right\} - \mathbb{P}\left\{\sup_{t\in\mathcal{T}_b}|\widehat{Z}_b(t)|\le y\right\}\right| \to 0.$$

Notice that $\mathbb{E}\sup_{t\in\mathcal{T}_b}|\mathcal{G}(t)| \le C(\log n)^{1/2}$ and $\mathrm{var}\{\mathcal{G}(t)\} = 1$ for each $t \in \mathcal{T}$. Hence, by the Borell-Sudakov-Tsirel'son inequality, we have

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}_b}|\mathcal{G}(t)| > C(\log n)^{1/2}\right\} \le Cn^{-c}. \tag{S5.1}$$

Taking $\delta = C\widetilde{d}_n(\log n)^{-1/2}$, by the triangle inequality, it follows that

$$\begin{aligned}\rho_Z &\le \sup_{y\in R}\mathbb{P}\left\{\left|\sup_{t\in\mathcal{T}_b}|Z_b(t)|-y\right|\le\delta\right\} + \mathbb{P}\left\{\sup_{t\in\mathcal{T}_b}|Z_b(t)-\widehat{Z}_b(t)|>\delta\right\} \\ &\le \sup_{y\in R}\mathbb{P}\left\{\left|\sup_{t\in\mathcal{T}_b}|\mathcal{G}(t)|-y\right|\le\delta\right\} + \rho + \mathbb{P}\left\{\widetilde{\Delta}_\sigma > \widetilde{d}_n(\log n)^{-1}\right\} \\ &+ \rho + \mathbb{P}\left\{\sup_{t\in\mathcal{T}_b}|\mathcal{G}(t)| > C(\log n)^{1/2}\right\} \\ &\le 2\rho + C\delta(\log n)^{1/2} + \widetilde{d}_n + Cn^{-c} \to 0.\end{aligned}$$

□

S6 Proof of Theorem 4

Proof of Theorem 4. In view of (2.16), it suffices to show that

$$\rho_z = \sup_{y\geq 0}\left|\mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b}|\widehat{G}(t)|\leq y\right\} - \mathbb{P}\left\{\sup_{t\in\mathcal{T}_b}|\mathcal{G}(t)|\leq y\right\}\right| = o_{\mathbb{P}}(1).$$

For any $t\in\mathcal{T}$, define

$$G(t) = \frac{1}{\sqrt{n}\sigma(t)}\sum_{i=1}^n z_i\eta_i(t) \text{ and } \widetilde{G}(t) = \frac{1}{\sqrt{n}\sigma(t)}\sum_{i=1}^n z_i\widehat{\eta}_i(t).$$

Then, by the triangle inequality, we have $\rho_z \leq \rho_{z1}+\rho_{z2}+\rho_{z3}$, where

$$\rho_{z1} = \sup_{y\geq 0}\left|\mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b}|\widehat{G}(t)|\leq y\right\} - \mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b}|\widetilde{G}(t)|\leq y\right\}\right|,$$
$$\rho_{z2} = \sup_{y\geq 0}\left|\mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b}|\widetilde{G}(t)|\leq y\right\} - \mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b}|G(t)|\leq y\right\}\right|,$$
$$\rho_{z3} = \sup_{y\geq 0}\left|\mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b}|G(t)|\leq y\right\} - \mathbb{P}\left\{\sup_{t\in\mathcal{T}_b}|\mathcal{G}(t)|\leq y\right\}\right|.$$

Hence it suffices to show that $\rho_{zk} = o_{\mathbb{P}}(1)$ for all $k = 1,2,3$. In what follows, let $\mathbb{E}_z(\cdot) = \mathbb{E}(\cdot|\mathcal{D})$ and $\mathrm{cov}_z(\cdot,\cdot) = \mathrm{cov}(\cdot,\cdot|\mathcal{D})$ denote the conditional expectation and the conditional covariance, respectively.

Step 1. First we show that $\rho_{z3} = o_{\mathbb{P}}(1)$. Observe that conditional on $\mathcal{D}$, $\{G(t)\}_{t\in\mathcal{T}_b}$ is a zero-mean Gaussian process with covariance function

$$\mathrm{cov}_z\{G(t),G(s)\} = \frac{1}{n}\sum_{i=1}^n \pi_i(t)\pi_i(s).$$

Define the semi metric

$$d_z^2(t,s) = \mathbb{E}_z|G(t) - G(s)|^2 = \frac{1}{n}\sum_{i=1}^{n}|\pi_i(t) - \pi_i(s)|^2.$$

By Lemma 3 and Assumption 4,

$$\begin{aligned} d_z^2(t,s) &= \frac{\sum_{i=1}^{n}|\sigma(s)\{\eta_i(t) - \eta_i(s)\} + \eta_i(s)\{\sigma(s) - \sigma(t)\}|^2}{n\sigma^2(t)\sigma^2(s)} \\ &\lesssim \frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i}|e_{ij}|^2|t-s|^2/m_i}{n\sigma^2(t)b^4} + \frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i}|e_{ij}|^2|t-s|^2/m_i}{n\sigma^2(s)\sigma^2(t)b^6} =: \Theta_n^2|t-s|^2. \end{aligned}$$

Consequently, by Dudley's entropy integral, we have

$$\begin{aligned} \mathbb{E}_z \sup_{|t-s|\le 1/\mathcal{L}}|G(t) - G(s)| &\le C\int_0^{\Theta_n/\mathcal{L}}\{\log(\Theta_n/\epsilon)\}^{1/2}d\epsilon \\ &\le \frac{C\Theta_n(\log n)^{1/2}}{\mathcal{L}} =: \theta_n, \end{aligned}$$

combined with which the Borell-Sudakov-Tsirel'son inequality imply that

$$\mathbb{P}_z\left\{\sup_{|t-s|\le 1/\mathcal{L}}|G(t) - G(s)| > C\theta_n\right\} \le 2\exp\left(-\frac{C\mathcal{L}^2\theta_n^2}{2\Theta_n^2}\right) \le Cn^{-c}. \quad \text{(S6.1)}$$

Define $\Delta_\Sigma = \max_{\jmath,\ell\le\mathcal{L}}|\mathrm{cov}_z\{G(t_\ell), G(t_\jmath)\} - \mathrm{cov}\{\mathcal{G}(t_\ell), \mathcal{G}(t_\jmath)\}|$. Note that

$$\max_{\jmath,\ell\le\mathcal{L}}\sum_{i=1}^{n}\mathbb{E}|\pi_i(t_\jmath)\pi_i(t_\ell)|^2 \le \max_{\ell\le\mathcal{L}}\sum_{i=1}^{n}\mathbb{E}|\pi_i(t_\ell)|^4 \lesssim n\left(\frac{V_3}{b^3}+1\right)/\vartheta^4$$

and

$$\mathbb{E}\max_{i\le n,\jmath,\ell\le\mathcal{L}}|\pi_i(t_\jmath)\pi_i(t_\ell)|^2 \le \mathbb{E}\max_{i\le n,\ell\le\mathcal{L}}|\pi_i(t_\ell)|^4 \lesssim (nW)^{4/q}.$$

Hence, by (S4.5), (S4.6) and Lemma E.1 in Chernozhukov et al. (2017), it follows that $\mathbb{E}(\Delta_\Sigma) \lesssim \Delta_2/\vartheta^2$, which further implies that

$$\Delta_\Sigma = \max_{\jmath,\ell\le\mathcal{L}}\frac{1}{n}\left|\sum_{i=1}^{n}\mathbb{E}_0\{\pi_i(t_\jmath)\pi_i(t_\ell)\}\right| = O_{\mathbb{P}}\left(\frac{\Delta_2}{\vartheta^2}\right) = o_{\mathbb{P}}\left\{\frac{1}{(\log n)^2}\right\} \quad \text{(S6.2)}$$

in view of (3.26). By Assumption 2, it follows that $\mathbb{E}(\Theta_n^2) \le Cb^{-6}$. Hence

$$\theta_n(\log n)^{1/2} = o_{\mathbb{P}}\left(\frac{\log n}{b^3\mathcal{L}}\right) = o_{\mathbb{P}}\left(\frac{\log n}{n}\right).$$

Combined with (S6.1) and (S6.2), by Lemma 2.1 and Lemma 3.1 in Chernozhukov et al. (2013), we conclude that

$$\begin{aligned}
\rho_{z3} &\le \mathbb{P}_z\left\{\sup_{|t-s|\le 1/\mathcal{L}} |G(t)-G(s)| > C\theta_n\right\} + \sup_{y\in R}\mathbb{P}\left\{\left|\max_{\ell\le\mathcal{L}} |\mathcal{G}(t_\ell)| - y\right| \le C\theta_n\right\} \\
&+ 2\sup_{y\in R}\left|\mathbb{P}_z\left\{\max_{\ell\le\mathcal{L}} |G(t_\ell)| \le y\right\} - \mathbb{P}\left\{\max_{\ell\le\mathcal{L}} |\mathcal{G}(t_\ell)| \le y\right\}\right| + o_{\mathbb{P}}(1) \\
&\lesssim n^{-c} + \theta_n(\log n)^{1/2} + \Delta_\Sigma^{1/3}(\log n)^{2/3} + o_{\mathbb{P}}(1) = o_{\mathbb{P}}(1).
\end{aligned}$$

Step 2. Now we prove that $\rho_{z2} = o_{\mathbb{P}}(1)$. Define $\Delta_G(t) = \sqrt{n}\sigma(t)\{G(t) - \widetilde{G}(t)\}$ and the semi-metric

$$d_\diamond^2(t,s) = \mathbb{E}_z|\Delta_G(t) - \Delta_G(s)|^2.$$

Conditional on $\mathcal{D}$, we note that $\{\Delta_G(t)\}_{t\in\mathcal{T}}$ is a zero-mean Gaussian process with covariance function

$$\mathrm{cov}_z\{\Delta_G(t), \Delta_G(s)\} = \sum_{i=1}^n \{\eta_i(t) - \widehat{\eta}_i(t)\}\{\eta_i(s) - \widehat{\eta}_i(s)\}.$$

Elementary calculation shows that

$$\begin{aligned}
d_\diamond^2(t,s) &= \sum_{i=1}^n |\{\eta_i(t) - \widehat{\eta}_i(t)\} - \{\eta_i(s) - \widehat{\eta}_i(s)\}|^2 \\
&\lesssim \frac{n\Delta_\mu^2 |t-s|^2}{b^4} =: \Theta_\diamond^2 |t-s|^2.
\end{aligned}$$

Define

$$\sigma_G^2 = \sup_{t\in\mathcal{T}_b} \mathbb{E}_z|\Delta_G(t)|^2 = \sup_{t\in\mathcal{T}_b} \sum_{i=1}^n |\eta_i(t) - \widehat{\eta}_i(t)|^2.$$

Then, by Dudley's entropy integral,

$$\begin{aligned}\mathbb{E}_z \sup_{t\in\mathcal{T}_b} |\Delta_G(t)| &\le C\int_0^{\sigma_G} \{1+\log(\Theta_\diamond/\epsilon)\}^{1/2} d\epsilon \\ &\le C\sigma_G\{\log(\Theta_\diamond/\sigma_G)\}^{1/2} =: \theta_\diamond.\end{aligned}$$

Consequently, applying the Borell-Sudakov-Tsirel'son inequality leads to

$$\mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b} |\Delta_G(t)| > C\theta_\diamond\right\} \le 2\exp\left(-\frac{C\theta_\diamond^2}{2\sigma_G^2}\right) \le C(\sigma_G/\Theta_\diamond)^c.$$

Notice that under the event $\mathcal{E}^c$, we have $\sigma_G^2 \le n\Xi_{11}$. Hence $\sigma_G^2 = n\Delta_\mu^2 O_{\mathbb{P}}(\Delta_1) = \Theta_\diamond^2 b^4 O_{\mathbb{P}}(\Delta_1)$, which further implies that $\sigma_G^2/\Theta_\diamond^2 = O_{\mathbb{P}}(b^4\Delta_1) = o_{\mathbb{P}}(1)$. Taking $\delta = Cn^{-1/2}\theta_\diamond/\vartheta$, by Lemma 2.1 in Chernozhukov et al. (2013) and (3.26), it follows that

$$\begin{aligned}\rho_{z2} &\le \mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b} |G(t) - \widetilde{G}(t)| > \delta\right\} + \sup_{y\in R} \mathbb{P}_z\left\{\left|\sup_{t\in\mathcal{T}_b} |G(t)| - y\right| \le \delta\right\} \\ &\le \mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b} |\Delta_G(t)| > C\theta_\diamond\right\} + \rho_{z3} + \frac{C\theta_\diamond(\log n)^{1/2}}{\vartheta n^{1/2}} \\ &\le C(\sigma_G/\Theta_\diamond)^c + o_{\mathbb{P}}(1) + O_{\mathbb{P}}\left(\frac{\Delta_0\Delta_1^{1/2}\log n}{\vartheta}\right) = o_{\mathbb{P}}(1).\end{aligned} \tag{S6.3}$$

Step 3. It remains to prove that $\rho_{z3} = o_{\mathbb{P}}(1)$. Let $\delta = C\widetilde{\Delta}_\sigma(\log n)^{1/2}$. By Theorem 3 and (3.26), we have $\delta(\log n)^{1/2} = C\widetilde{\Delta}_\sigma \log n = o_{\mathbb{P}}(1)$. Notice that

$$\sup_{t\in\mathcal{T}_b}|\widehat{G}(t)-\widetilde{G}(t)|\leq\sup_{t\in\mathcal{T}_b}|\sigma(t)/\widehat{\sigma}(t)-1|\sup_{t\in\mathcal{T}_b}|\widetilde{G}(t)|=\widetilde{\Delta}_\sigma\sup_{t\in\mathcal{T}_b}|\widetilde{G}(t)|.$$

Hence, by Lemma 2.1 in Chernozhukov et al. (2013) and (S5.1), it follows that

$$\begin{aligned}\rho_{z1}&\leq\mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b}|\widehat{G}(t)-\widetilde{G}(t)|>\delta\right\}+\sup_{y\in R}\mathbb{P}_z\left\{\left|\sup_{t\in\mathcal{T}_b}|\widetilde{G}(t)|-y\right|\leq\delta\right\}\\&\leq\mathbb{P}_z\left\{\sup_{t\in\mathcal{T}_b}|\mathcal{G}(t)|>C(\log n)^{1/2}\right\}+2\rho_{z2}+2\rho_{z3}+C\delta(\log n)^{1/2}\\&\leq 2n^{-C}+o_{\mathbb{P}}(1).\end{aligned}$$

□

S7 Gumbel SCBs

For any $\alpha\in(0,1)$, the SCBs constructed based on the Gumbel approximation is given by

$$\left[\widehat{\mu}_b(t)-\frac{b^2r(t)}{\widetilde{f}_b(t)}-\frac{\mathcal{Q}_{1-\alpha}\widehat{\sigma}(t)}{\sqrt{n}},\widehat{\mu}_b(t)-\frac{b^2r(t)}{\widetilde{f}_b(t)}+\frac{\mathcal{Q}_{1-\alpha}\widehat{\sigma}(t)}{\sqrt{n}}\right],\qquad\text{(S7.1)}$$

where

$$\mathcal{Q}_{1-\alpha}=\frac{-\log(-\log(1-\alpha)/2)}{\sqrt{2\log b}}+g_n.$$

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