Statistica Sinica

# Power enhancement for dimension detection of Gaussian signals

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## S1. Further simulations

In the present section, our objective is to provide Monte-Carlo simulation results to corroborate the conclusions drawn from Proposition 1 and in Section 4. In the first simulation exercise, the objective is to illustrate Proposition 1. We generated $M = 10,000$ independent samples of i.i.d. observations $\mathbf{X}_1^{(b)}, \ldots, \mathbf{X}_{10,000}^{(b)}$, for $b = 0, \frac{1}{4}, \frac{1}{2}, 1$. The $\mathbf{X}_i^{(b)}$'s are i.i.d. with a common ($p = 8$)-dimensional Gaussian distribution with mean zero and covariance matrix

$$\mathbf{\Sigma}(b) = \mathrm{diag}((1 + n^{-b})\mathbf{1}_q, \mathbf{1}_{p-q}).$$

For various values of $q$, we computed the value of $T_q^{(n)}$ and performed the test that rejects the null hypothesis $\mathcal{H}_{0q}^{(n)}$ when $T_q^{(n)} > \chi^2_{d(p,q);.95}$. In Figure 1, we provide histograms of the distribution of the values of $T_q^{(n)}$ (obtained from the $M = 10,000$ replications). The histograms have to be compared with (i) the red line which is the chi-square density function with $d(p,q)$ degrees of freedom and (ii) the grey line which is an approximation of the density of $T_q^{(n)}$ obtained in Proposition 1; the approximation has been obtained by computing a kernel density estimator based on $100,000$ replications of the random variable in (3.3). In Figure 2 we provide the empirical rejection frequencies (out of the $M = 10,000$ replications) of the tests rejecting $\mathcal{H}_{0q}^{(n)}$ when $T_q^{(n)} > \chi^2_{d(p,q);.95}$. Inspection of Figures 1 and 2 clearly reveals that the conclusions drawn from Proposition 1 are correct. Provided that $n^{1/2} r_q^{(n)} \to \infty$, the weak limit of $T_q^{(n)}$ is chi-square with $d(p,q)$ degrees of freedom. Now if $n^{1/2} r_q^{(n)}$ does not diverge to $\infty$, the weak limit of $T_q^{(n)}$ is not chi-square and the test $\phi^{(n)}$ is such that $\lim_{n\to\infty} \mathrm{E}[\phi^{(n)}]$ is far below the asymptotic nominal level $\alpha$.

The second simulation study illustrates the results obtained in Section 4. We generated $M = 1,000$ independent samples of i.i.d. observations

$$\mathbf{X}_1^{(b,\tau)}, \ldots, \mathbf{X}_{10,000}^{(b,\tau)},$$

for $\tau = 0, 1, 2, 4, 6, 8$ and $b = 0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2$. The $\mathbf{X}_i^{(b,\tau)}$'s are i.i.d. with a common $(p =)5$-dimensional Gaussian distribution with mean zero and

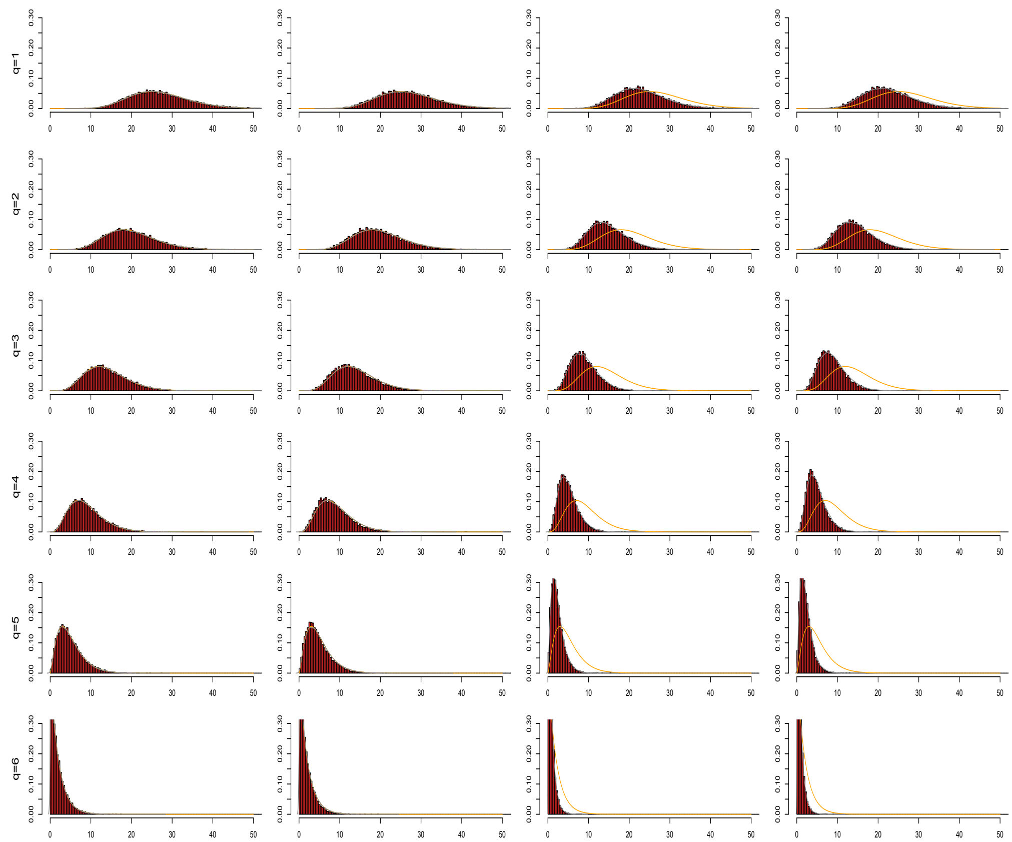


Figure 1: In red, histograms of the distribution of the values of $T_q^{(n)}$ (obtained from the $M = 10,000$ replications) for various values of $q$ and $b$. The histograms have to be compared with (i) the orange line which is the chi-square density function with $d(p, q)$ degrees of freedom and (ii) the grey line which is an approximation of the density of $T_q^{(n)}$ obtained in Proposition 1.

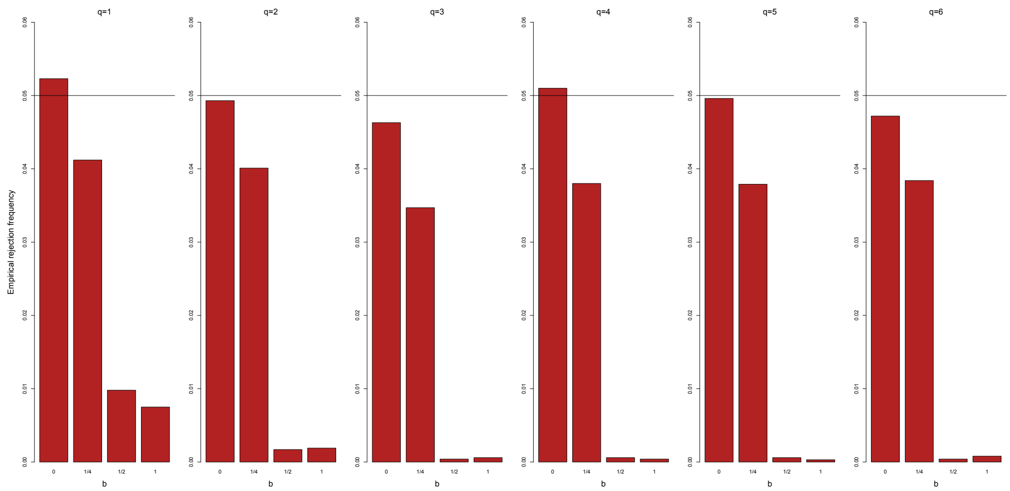


Figure 2: Empirical rejection frequencies of the tests that reject the null hypothesis $\mathcal{H}_{0q}^{(n)}$ when $T_q^{(n)} > \chi^2_{d(p,q);.95}$ for various values of $q$ and various values of $b$.

covariance matrix

$$\mathbf{\Sigma}(b,\tau) = \text{diag}(3, 1+n^{-b}, 1+n^{-b}, 1, 1-\frac{\tau}{n^{1/2}}).$$

The values $\tau = 0$ and $b < 1/2$ provide data generating processes that belong to the null hypothesis $\lambda_3 > \lambda_4 = \lambda_5$ while for $\tau = 1, 2, 4, 6, 8$, the corresponding distributions are increasingly under the alternative. The value $b = 0$ provides data generating processes with three eigenvalues (virtually) in block 1 of (3.2), the values $b = \frac{1}{8}, \frac{1}{4}$ provide data generating processess with one eigenvalue (virtually) in block 1 and two eigenvalues (virtually) in block 2 of (3.2), the value $b = \frac{1}{2}$ provides data generating processes with one eigenvalue (virtually) in block 1 and two eigenvalues (virtually) in block 3 of (3.2) while the values $b = 1, 2$ provide data generating procesess with one eigenvalue (virtually) in block 1 and two eigenvalues (virtually) in block 4 of (3.2). For each scenario, we performed at each replication the three tests $\phi^{(n)}_{\boldsymbol{\beta}}$, $\phi^{(n)}_{\mathrm{LRT}}$ and $\phi^{(n)}$ at the nominal level $\alpha = .05$. In Figure 3, we provide the empirical power curves of the various tests as functions of $\tau$. Inspection of Figure 3 clearly confirms our theoretical findings: the three tests $\phi^{(n)}$, $\phi^{(n)}_{\mathrm{LRT}}$ and $\phi^{(n)}_{\boldsymbol{\beta}}$ are asymptotically equivalent provided that $n^{1/2} r^{(n)}_q$ diverges to infinity as $n \to \infty$ and the tests $\phi^{(n)}$ and $\phi^{(n)}_{\mathrm{LRT}}$ (that are asymptotically equivalent in all regimes as stated in Lemma 1) are such that $\lim_{n\to\infty} \mathrm{E}[\phi^{(n)}]$ is far below the asymptotic nominal level $\alpha$ when $n^{1/2} r^{(n)}_q$ does not diverge to infinity as $n \to \infty$.

Now, we conclude this Section by providing some further simulation results for a test similar to $\phi^{(n)}_{\mathrm{new}}$ based on the pseudo-Gaussian tests mentioned in the real data application section. Letting $\hat{\kappa}^{(n)}$ be a consistent estimator of the underlying kurtosis coefficient, define

$$\tilde{T}^{(n)}_{q,q+1} := (1 + \hat{\kappa}^{(n)})^{-1} T^{(n)}_{q,q+1},$$

and

$$\tilde{T}^{(n)}_{q} := (1 + \hat{\kappa}^{(n)})^{-1} T^{(n)}_{q}.$$

Our pseudo-Gaussian test $\tilde{\phi}^{(n)}_{\mathrm{new}}$ rejects $\mathcal{H}^{(n)}_{0q}$ at asymptotic confidence level

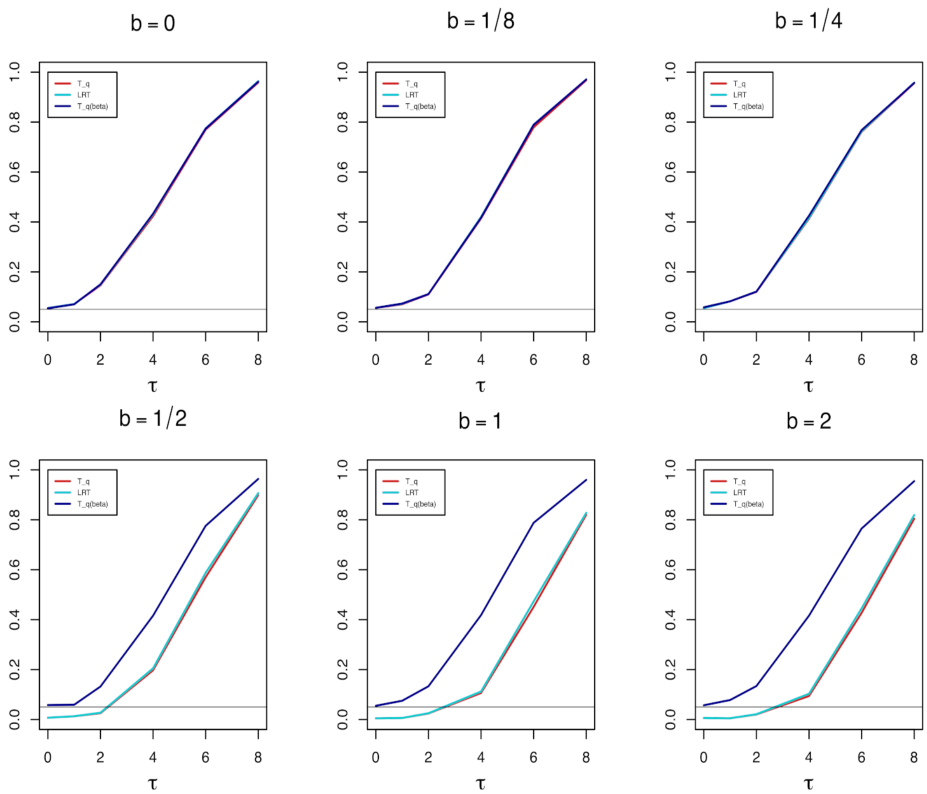


Figure 3: Empirical rejection frequencies of the tests $\phi_{\mathrm{LRT}}^{(n)}$ (based on $L_q^{(n)}$), $\phi^{(n)}$ (based on $T_q^{(n)}$) and $\phi_{\boldsymbol{\beta}}^{(n)}$ (based on $T_q^{(n)}(\boldsymbol{\beta})$) performed at the nominal level $\alpha = .05$.

$\alpha$ when

$$\tilde{\phi}^{(n)}_{\rm new} := \mathbb{I}[\tilde{T}^{(n)}_q > \chi^2_{d(p,q);1-\alpha}]\mathbb{I}[\tilde{T}^{(n)}_{q,q+1} > \chi^2_{2;1-\gamma}] + \mathbb{I}[\tilde{T}^{(n)}_{q,q+1} \leq \chi^2_{2;1-\gamma}] = 1 \tag{S1.1}$$

Letting $\tilde{\phi}^{(n)}$ be the pseudo-Gaussian version of $\phi^{(n)}$, we compare here $\tilde{\phi}^{(n)}_{\rm new}$ and $\tilde{\phi}^{(n)}$ through simulations.

To do so, we generated $M = 2,000$ independent samples of i.i.d. observations

$$\mathbf{X}^{(b,\tau)}_1, \ldots, \mathbf{X}^{(b,\tau)}_n,$$

for $\tau = 0, 1, 2, 4, 6, 8$ and $b = 0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2$. The $\mathbf{X}^{(b,\tau)}_i$'s are i.i.d. having ($p = 5$)-dimensional multivariate student distribution with zero mean, 5 degrees of freedom and scatter matrix

$$\mathbf{\Sigma}(b, \tau) = \mathrm{diag}(3, 1 + n^{-b}, 1 + n^{-b}, 1, 1 - \frac{\tau}{n^{1/2}}).$$

We performed the classical test $\phi^{(n)}$, three versions of the $\phi^{(n)}_{\rm new}$ test ($\gamma = .9$, $\gamma = .5$ and $\gamma = .05$) for $\mathcal{H}^{(n)}_{03}$ ($q = 3$). We also performed the pseudo-Gaussian versions $\tilde{\phi}^{(n)}$ and $\tilde{\phi}^{(n)}_{\rm new}$ of these tests, with same choices of $\gamma$. In Figures 4 and 5, we provide empirical rejection frequencies of the Gaussian tests as functions of $\tau$ for sample sizes $n = 500$ and $n = 10,000$. Clearly, the lack of robustness to non-Gaussian assumptions is shown in Figures 4 and 5.

In Figures 6 and 7, the same empirical rejection frequencies are displayed for the pseudo-Gaussian tests. Pseudo-Gaussian procedures are clearly robusts to non-Gaussianity.

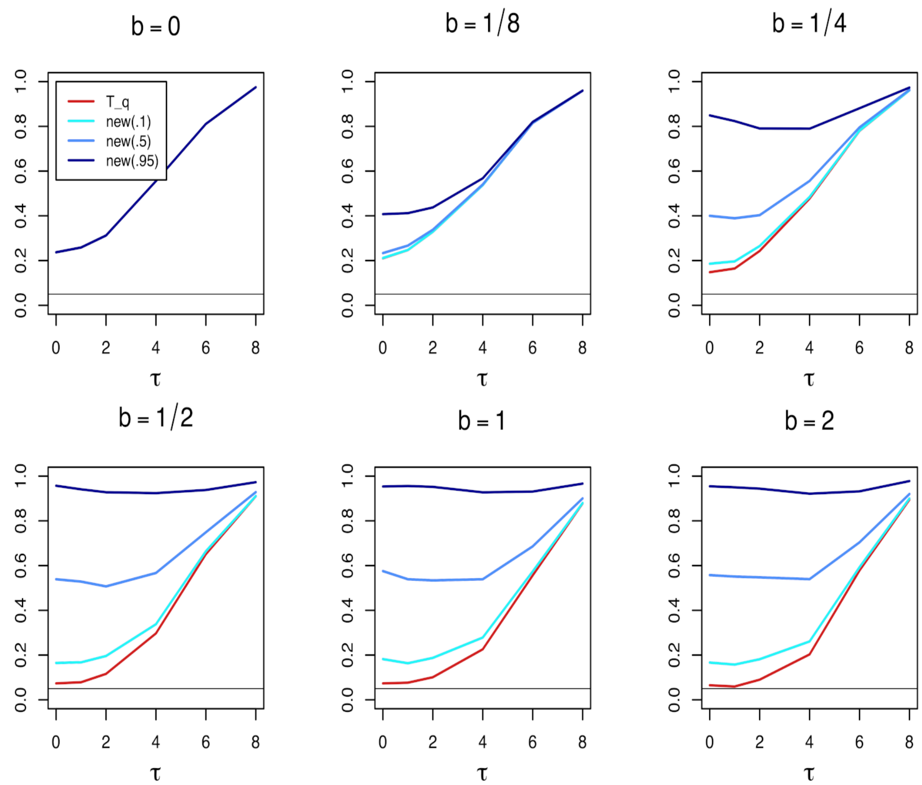


Figure 4: Empirical rejection frequencies of the classical gaussian test $\phi^{(n)}$ performed at the asymptotic nominal level .05 and three versions of the $\phi^{(n)}_{\rm new}$ test (all with $\alpha = .05$) based on three different choices of $\gamma$: $\gamma = .9$ (denoted as new(.1)), $\gamma = .5$ (denoted as new(.5)) and $\gamma = .05$ (denoted as new(.95)). The sample size is $n = 500$.

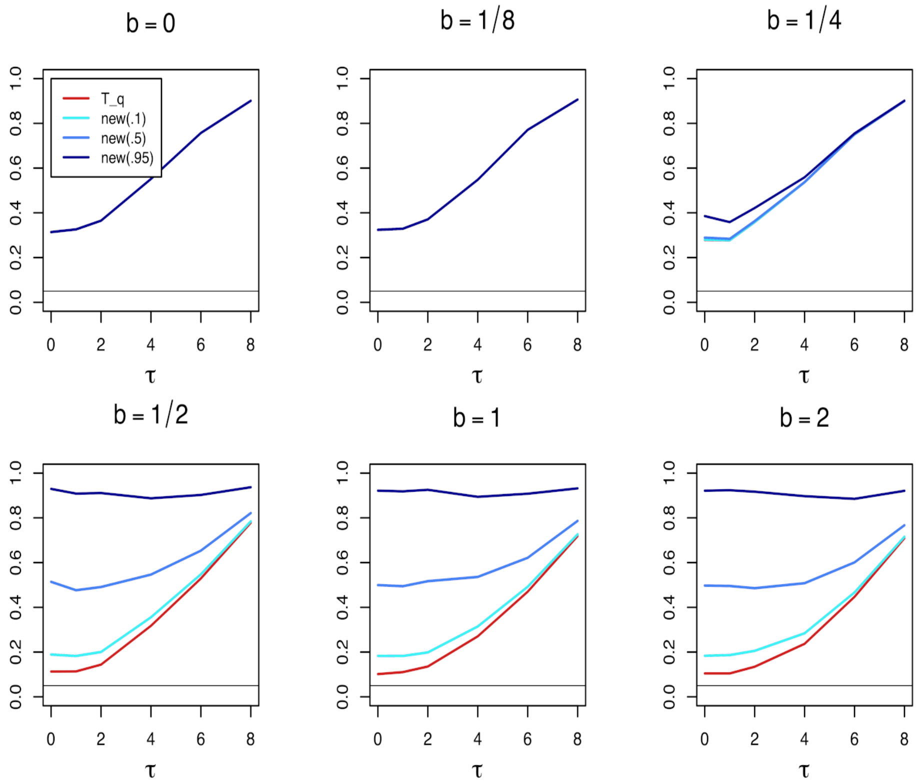


Figure 5: Empirical rejection frequencies of the classical gaussian test $\phi^{(n)}$ performed at the asymptotic nominal level .05 and three versions of the $\phi^{(n)}_{\text{new}}$ test (all with $\alpha = .05$) based on three different choices of $\gamma$: $\gamma = .9$ (denoted as new(.1)), $\gamma = .5$ (denoted as new(.5)) and $\gamma = .05$ (denoted as new(.95)). The sample size is $n = 10,000$.

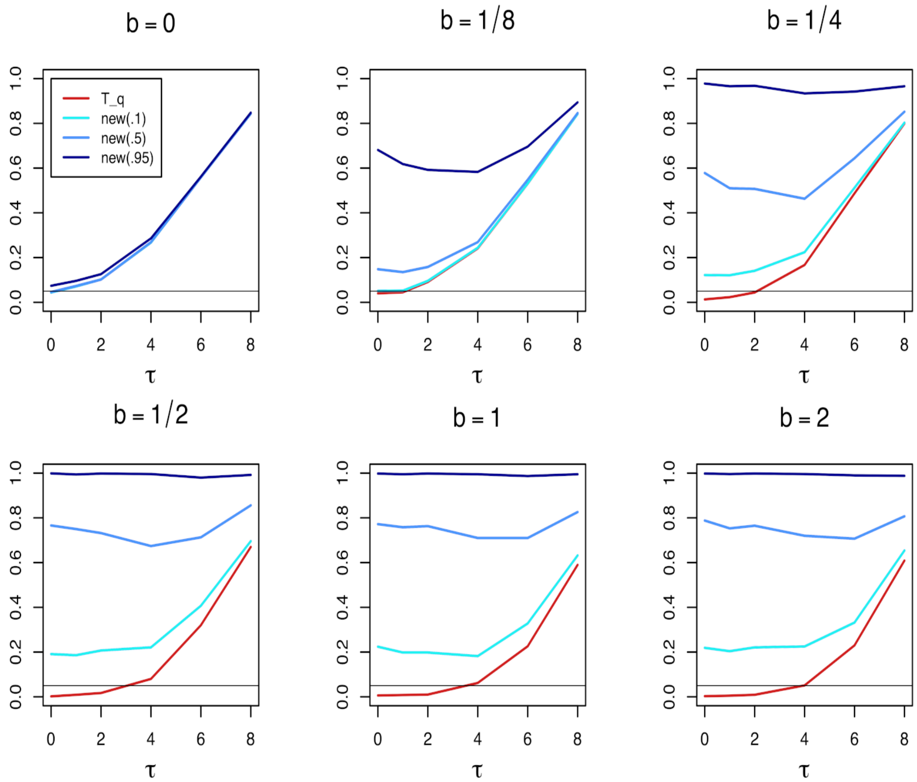


Figure 6: Empirical rejection frequencies of the classical pseudo-gaussian $\tilde{\phi}^{(n)}$ performed at the asymptotic nominal level .05 and three versions of the $\tilde{\phi}^{(n)}_{\text{new}}$ test (all with $\alpha = .05$) based on three different choices of $\gamma$: $\gamma = .9$ (denoted as new(.1)), $\gamma = .5$ (denoted as new(.5)) and $\gamma = .05$ (denoted as new(.95)). The sample size is $n = 500$.

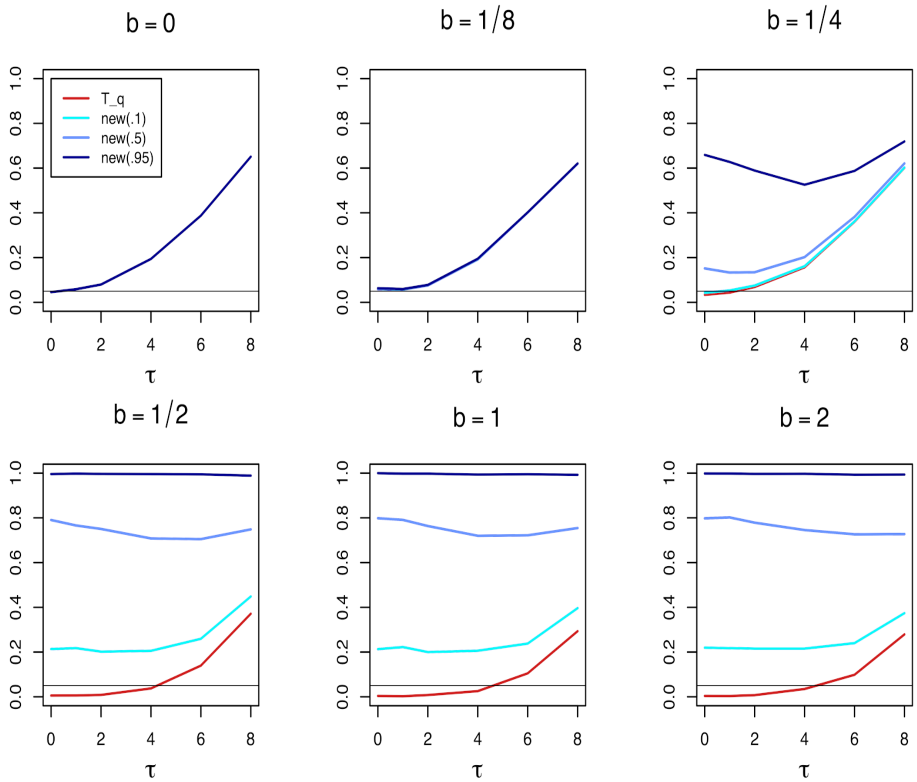


Figure 7: Empirical rejection frequencies of the classical pseudo-gaussian $\tilde{\phi}^{(n)}$ performed at the asymptotic nominal level .05 and three versions of the $\tilde{\phi}^{(n)}_{\text{new}}$ test (all with $\alpha = .05$) based on three different choices of $\gamma$: $\gamma = .9$ (denoted as new(.1)), $\gamma = .5$ (denoted as new(.5)) and $\gamma = .05$ (denoted as new(.95)). The sample size is $n = 10,000$.

## S2. Proofs of the various results.

*Proof of Proposition 1.* The proof of Proposition 1 directly follows from Proposition 1 below. □

**Proposition 6.** *Let $\mathbf{r}^{(n)}$ and $\mathbf{v}$ be such that (3.1) holds and such that for $0 \leq s_1 \leq s_2 \leq s_3 \leq q$, (i) $r_j^{(n)} \equiv 1$ for each $1 \leq j \leq s_1$, (ii) $r_j^{(n)} = o(1)$ with $n^{1/2} r_j^{(n)} \to \infty$, for each $s_1 < j \leq s_2$, (iii) $r_j^{(n)} = n^{-1/2}$, for each $s_2 < j \leq s_3$ and (iv) $r_j^{(n)} = o(n^{-1/2})$, for each $s_3 < j \leq q$. Let*

$$\mathbf{Z}(v_1, \ldots, v_{s_1}) = \begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{21}' \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{pmatrix}$$

*where $\mathbf{Z}_{11}$ is the $s_2 \times s_2$ upper left block of $\mathbf{Z}(v_1, \ldots, v_{s_1})$, $\mathbf{Z}_{22}$ is the $p - s_2 \times p - s_2$ lower right block of $\mathbf{Z}(v_1, \ldots, v_{s_1})$, etc, be such that*

$$\mathrm{vec}(\mathbf{Z}(v_1, \ldots, v_{s_1})) \sim \mathcal{N}_{p^2}(\mathbf{0}, (\mathbf{I}_{p^2} + \mathbf{K}_p)(\mathrm{diag}(1 + v_1, \ldots, 1 + v_{s_1}, \mathbf{1}_{p-s_1}'))^{\otimes 2}).$$

*Then as $n \to \infty$ under $\mathrm{P}_{\boldsymbol{\beta}, \boldsymbol{\lambda}^{(n)}}^{(n)}$,*

$$\boldsymbol{\ell}_{p-q}^{(n)} = (\ell_{q+1}^{(n)}, \ldots, \ell_p^{(n)})' := n^{1/2}(\hat{\lambda}_{q+1} - 1, \ldots, \hat{\lambda}_p - 1)' \tag{S2.1}$$

*converges weakly to the $(p-q)$ smallest roots of $\mathbf{Z}_{22} + \mathrm{diag}(v_{s_2+1}, \ldots, v_{s_3}, \mathbf{0}_{q-s_3}', \mathbf{0}_{p-q}')$.*

*Proof of Proposition 1.* Throughout the proof, we put

$$\mathbf{Z}^{(n)} := n^{1/2} \boldsymbol{\beta}'(\mathbf{S}^{(n)} - \boldsymbol{\Sigma}^{(n)})\boldsymbol{\beta} = \begin{pmatrix} \mathbf{Z}_{11}^{(n)} & (\mathbf{Z}_{21}^{(n)})' \\ \mathbf{Z}_{21}^{(n)} & \mathbf{Z}_{22}^{(n)} \end{pmatrix},$$

where $\mathbf{Z}_{11}^{(n)}$ is the $s_2 \times s_2$ upper left block of $\mathbf{Z}^{(n)}$, $\mathbf{Z}_{22}^{(n)}$ the $p - s_2 \times p - s_2$ lower right block, etc. It follows along the same lines as in Lemma 2.1 of Paindaveine et al. (2020) that $\mathbf{Z}^{(n)}$ converges weakly to $\mathbf{Z}(v_1, \ldots, v_{s_1})$ with

$$\mathrm{vec}(\mathbf{Z}(v_1, \ldots, v_{s_1})) \sim \mathcal{N}_{p^2}(\mathbf{0}, (\mathbf{I}_{p^2} + \mathbf{K}_p)(\mathrm{diag}(1 + v_1, \ldots, 1 + v_{s_1}, \mathbf{1}'_{p-s_1}))^{\otimes 2})$$

under $\mathrm{P}^{(n)}_{\boldsymbol{\beta}, \boldsymbol{\lambda}^{(n)}}$. First note that for every $q + 1 \leq j \leq p$, $\ell_j^{(n)}$ is the $j$th largest root of ($\boldsymbol{\Lambda}^{(n)} = \boldsymbol{\beta}'\boldsymbol{\Sigma}^{(n)}\boldsymbol{\beta}$)

$$\begin{aligned} P_{q+1,n}(h) &= \det(n^{1/2}(\mathbf{S}^{(n)} - \mathbf{I}_p) - h\mathbf{I}_p) \\ &= \det(\mathbf{Z}^{(n)} + n^{1/2}(\boldsymbol{\Lambda}^{(n)} - \mathbf{I}_p) - h\mathbf{I}_p) = \det(\mathbf{Z}^{(n)} + n^{1/2}\ \mathrm{diag}((\mathrm{diag}((\mathbf{r}^{(n)})')\mathbf{v})', \mathbf{0}'_{p-q}) - h\mathbf{I}_p). \end{aligned}$$

It follows that $(\ell_{q+1}^{(n)}, \ldots, \ell_p^{(n)})$ are the smallest roots of

$$\begin{aligned} \frac{P_{q+1,n}(h)}{(n^{1/2} r_{s_2}^{(n)} v_{s_2})^{s_2}} &= \frac{1}{(n^{1/2} r_{s_2}^{(n)} v_{s_2})^{s_2}} \det(\mathbf{Z}^{(n)} + n^{1/2}\ \mathrm{diag}((\mathrm{diag}((\mathbf{r}^{(n)})')\mathbf{v})', \mathbf{0}'_{p-q})) - h\mathbf{I}_p) \\ &= \det\left( \begin{pmatrix} (n^{1/2} r_{s_2}^{(n)} v_{s_2})^{-1}(\mathbf{Z}_{11}^{(n)} + \mathrm{diag}(n^{1/2} r_1^{(n)} v_1 - h, \ldots, n^{1/2} r_{s_2}^{(n)} v_{s_2} - h)) & (\mathbf{Z}_{21}^{(n)})' \\ (n^{1/2} r_{s_2}^{(n)} v_{s_2})^{-1}(\mathbf{Z}_{21}^{(n)}) & \mathbf{Z}_{22}^{(n)} \end{pmatrix} \right. \\ &\quad \left. + \mathrm{diag}(\mathbf{0}'_{s_2}, v_{s_2+1} - h, \ldots, v_{s_3} - h, n^{1/2} r_{s_3+1}^{(n)} v_{s_3+1} - h, \ldots, n^{1/2} r_q^{(n)} v_q - h, -h\mathbf{1}'_{p-q}) \right). \end{aligned}$$

Since $(n^{1/2} r_{s_2}^{(n)} v_{s_2})^{-1} = o(1)$, $(n^{1/2} r_{s_2}^{(n)} v_{s_2})^{-1}\mathbf{Z}_{21}^{(n)} = o_{\mathrm{P}}(1)$ and $(n^{1/2} r_{s_2}^{(n)} v_{s_2})^{-1}\mathbf{Z}_{11}^{(n)} = o_{\mathrm{P}}(1)$ as $n \to \infty$. It follows that the $s_2$ largest roots of $(n^{1/2} r_{s_2}^{(n)} v_{s_2})^{-s_2} P_{q+1,n}(h)$ converge to $+\infty$ in probability while the $p - s_2$ smallest roots converge weakly to the roots (in the decreasing order) of the weak limit of the polynomial

$$\det(\mathbf{Z}_{22}^{(n)} - h\mathbf{I}_{p-s_2} + \mathrm{diag}(v_{s_2+1}, \ldots, v_{s_3}, n^{1/2} r_{s_3+1}^{(n)} v_{s_3+1}, \ldots, n^{1/2} r_q^{(n)} v_q, \mathbf{0}'_{p-q}));$$

that is the vector of the ordered eigenvalues of

$$\mathbf{Z}_{22} + \mathrm{diag}(v_{s_2+1}, \ldots, v_{s_3}, \mathbf{0}'_{q-s_3}, \mathbf{0}'_{p-q}).$$

This implies that the vector $(\ell_{q+1}^{(n)}, \ldots, \ell_p^{(n)})$ converges weakly to the $(p-q)$ smallest eigenvalues of

$$\mathbf{Z}_{22} + \mathrm{diag}(v_{s_2+1}, \ldots, v_{s_3}, \mathbf{0}'_{q-m}, \mathbf{0}'_{p-q})$$

which is the desired result. □

In the proof of Proposition 2, we will use the following preliminary Lemma that follows along the same lines as Lemma S.1.1 in Paindaveine et al. (2020).

**Lemma 1.** *Let* $\mathbf{A}$ *be a* $p \times p$ *matrix. Assume that* $\lambda$ *is an eigenvalue of* $\mathbf{A}$ *and that the corresponding eigenspace* $V_\lambda$ *has dimension one. Denoting as* $C = (C_{ij})$ *the cofactor matrix of* $\mathbf{A} - \lambda \mathbf{I}_p$*, assume that for* $1 \leq j \leq p$ $\mathbf{v} := (C_{j1}, \ldots, C_{jp})' \neq \mathbf{0}$*. Then* $V_\lambda = \{t\mathbf{v} : t \in \mathbb{R}\}$*.*

*Proof of Proposition 2.* First, note that using the fact that

$$\mathbf{E}^{(n)} = \begin{pmatrix} \mathbf{E}_{11}^{(n)} & \mathbf{E}_{12}^{(n)} \\ \mathbf{E}_{21}^{(n)} & \mathbf{E}_{22}^{(n)} \end{pmatrix}$$

is an orthogonal matrix, we have that

$$\mathbf{E}_{21}^{(n)}(\mathbf{E}_{11}^{(n)})' = -\mathbf{E}_{22}^{(n)}(\mathbf{E}_{12}^{(n)})' \quad \text{and} \quad \mathbf{E}_{22}^{(n)}(\mathbf{E}_{22}^{(n)})' = \mathbf{I}_{p-q} - \mathbf{E}_{21}^{(n)}(\mathbf{E}_{21}^{(n)})'. \text{(S2.2)}$$

We first will prove $(a)$. Following the notations of the proof of Proposition 1, we put

$$\mathbf{Z}^{(n)} := n^{1/2}\boldsymbol{\beta}'(\mathbf{S}^{(n)} - \boldsymbol{\Sigma}^{(n)})\boldsymbol{\beta} = \begin{pmatrix} \mathbf{Z}_{11}^{(n)} & (\mathbf{Z}_{21}^{(n)})' \\ \mathbf{Z}_{21}^{(n)} & \mathbf{Z}_{22}^{(n)} \end{pmatrix},$$

where $\mathbf{Z}_{11}^{(n)}$ the $q \times q$ upper-left block and $\mathbf{Z}_{22}^{(n)}$ the $p-q \times p-q$ lower-right block of $\mathbf{Z}^{(n)}$. For $1 \leq j \leq q$, we put

$$\hat{\boldsymbol{\theta}}_j^{(n)} := \boldsymbol{\beta}'\hat{\boldsymbol{\beta}}_j = (\mathbf{b}_j^{(n)}, \mathbf{e}_j^{(n)})',$$

where $\mathbf{b}_j^{(n)}$ is the $j$th line of $\mathbf{E}_{11}^{(n)}$ and $\mathbf{e}_j^{(n)}$ the $j$th line of $\mathbf{E}_{12}^{(n)}$. We also use the notation

$$\boldsymbol{\ell}_q^{(n)} = (\ell_1^{(n)}, \ldots, \ell_q^{(n)}) := n^{1/2}(\hat{\lambda}_1 - (1 + r_1^{(n)}v_1), \ldots, \hat{\lambda}_q - (1 + r_q^{(n)}v_q)) \quad \text{(S2.3)}$$

as in Proposition 1. For any $1 \leq j \leq q$, it is easy to verify that $\ell_j^{(n)}$ is the $j$th (largest) eigenvalue of

$$n^{1/2}\boldsymbol{\beta}'\mathbf{S}^{(n)}\boldsymbol{\beta} - n^{1/2}(1 + r_j^{(n)}v_j)\mathbf{I}_p \;=\; \mathbf{Z}^{(n)} - \text{diag}(k_{j1}^{(n)}, \ldots, k_{jq}^{(n)}, n^{1/2}r_j^{(n)}v_j\mathbf{1}'_{p-q}),$$

where $k_{j\ell}^{(n)} := n^{1/2}(r_j^{(n)}v_j - r_\ell^{(n)}v_\ell)$; obviously $k_{jj}^{(n)} = 0$. The eigenvector associated with $\ell_j^{(n)}$ is $\hat{\boldsymbol{\theta}}_j^{(n)}$. Therefore, using Lemma 1, we have that $\hat{\boldsymbol{\theta}}_j^{(n)}$ is proportional to the vector of the cofactors obtained with respect to the $j$th row of ($\mathbf{K}^{(n)} := \mathrm{diag}(k_{j1}^{(n)}, \ldots, k_{jq}^{(n)})$)

$$\begin{pmatrix} \mathbf{Z}_{11}^{(n)} - \mathbf{K}^{(n)} - \ell_j^{(n)} \mathbf{I}_q & (\mathbf{Z}_{21}^{(n)})' \\ \mathbf{Z}_{21}^{(n)} & \mathbf{Z}_{22}^{(n)} - n^{1/2} r_j^{(n)} v_j \mathbf{I}_{p-q} - \ell_j^{(n)} \mathbf{I}_{p-q} \end{pmatrix},$$

or equivalently, proportional to the vector of the cofactors obtained with respect to the $j$th row of

$$(\prod_{\ell=1}^{q} c_{j\ell}^{(n)})(n^{1/2} r_j^{(n)} v_j)^{-(p-q)} \begin{pmatrix} \mathbf{Z}_{11}^{(n)} - \mathbf{K}^{(n)} - \ell_j^{(n)} \mathbf{I}_q & (\mathbf{Z}_{21}^{(n)})' \\ \mathbf{Z}_{21}^{(n)} & \mathbf{Z}_{22}^{(n)} - n^{1/2} r_j^{(n)} v_j \mathbf{I}_{p-q} - \ell_j^{(n)} \mathbf{I}_{p-q} \end{pmatrix}, \tag{S2.4}$$

where

$$c_{j\ell}^{(n)} := \begin{cases} (k_{j\ell}^{(n)})^{-1} & \text{if } \lim_{n\to\infty} k_{j\ell}^{(n)} \neq 0 \\ 1 & \text{else.} \end{cases}$$

Note that since $n^{1/2} r_q^{(n)}$ diverges to $\infty$, provided that $\lim_{n\to\infty} r_j^{(n)} v_j - r_\ell^{(n)} v_\ell \neq 0$, $(k_{j\ell}^{(n)})^{-1} = c_{j\ell}^{(n)}$ converges to zero as $n \to \infty$ while obviously $c_{jj}^{(n)} = 1$. Following (S2.4), letting $\mathbf{C}^{(n)} := \mathrm{diag}(c_{j1}^{(n)}, \ldots, c_{jq}^{(n)})$, classical algebra yields to the fact that $\hat{\boldsymbol{\theta}}_j^{(n)}$ is also proportional to the vector of the cofactors obtained with respect to the $j$th row of

$$\begin{pmatrix} (\mathbf{C}^{(n)}(\mathbf{Z}_{11}^{(n)} - \ell_j^{(n)} \mathbf{I}_q) - \mathbf{C}^{(n)} \mathbf{K}^{(n)} & \mathbf{C}^{(n)} (\mathbf{Z}_{21}^{(n)})' \\ (n^{1/2} r_j^{(n)} v_j)^{-1} \mathbf{Z}_{21}^{(n)} & (n^{1/2} r_j^{(n)} v_j)^{-1} \mathbf{Z}_{22}^{(n)} - \mathbf{I}_{p-q} - \frac{\ell_j^{(n)}}{n^{1/2} r_j^{(n)} v_j} \mathbf{I}_{p-q} \end{pmatrix}. \tag{S2.5}$$

It is easy to check that $(\mathbf{b}_j^{(n)})' = O_{\mathrm{P}}(1)$ (and not $o_{\mathrm{P}}(1)$) while the cofactors associated with the $j$th line and the $\ell$th column of the above matrix with $q < \ell \leq p$ are all $O_{\mathrm{P}}((n^{1/2}r_j^{(n)})^{-1})$; indeed these cofactors are obtained by computing the determinants of matrices containing a line with only $O_{\mathrm{P}}((n^{1/2}r_j^{(n)})^{-1})$ elements or zeros while the other entries are $O_{\mathrm{P}}(1)$ as $n \to \infty$. It follows that $\mathbf{e}_j^{(n)} = O_{\mathrm{P}}((n^{1/2}r_j^{(n)})^{-1})$. Point (a) then follows easily from (S2.2). We now turn to point (b). For point (b), we will show the result for the $q$th line of $\mathbf{E}_{12}^{(n)}$ which is obviously enough. First note that since $n^{1/2}r_q^{(n)} \to c < \infty$,

$$n^{1/2}(\hat{\lambda}_q - 1) = \ell_q^{(n)} + n^{1/2}r_q^{(n)}v_q$$

is (i) $O_{\mathrm{P}}(1)$ and (ii) the $q$th (largest) eigenvalue of

$$\mathbf{Z}^{(n)}+\mathrm{diag}(n^{1/2}r_1^{(n)}v_1, \ldots, n^{1/2}r_q^{(n)}v_q, \mathbf{0}'_{p-q}) = \mathbf{Z}^{(n)}+\mathrm{diag}(n^{1/2}(\mathrm{diag}((\mathbf{r}^{(n)})')\mathbf{v})', \mathbf{0}'_{p-q}).$$

The associated eigenvector is $\hat{\boldsymbol{\theta}}_q^{(n)} = \boldsymbol{\beta}'\hat{\boldsymbol{\beta}}_q = (\mathbf{b}_q^{(n)}, \mathbf{e}_q^{(n)})'$, using the same notations as in point (a) above. Using again the same technique as for point (a) above, Lemma 1 entails that $\hat{\boldsymbol{\theta}}_q^{(n)}$ is proportional to the vector of the cofactors obtained with respect to the first row of

$$\Big(\prod_{\ell=1}^{s^*}(n^{1/2}r_\ell^{(n)}v_\ell)^{-1}\Big) \begin{pmatrix} \mathbf{Z}_{11}^{(n)} + n^{1/2}\mathrm{diag}((\mathrm{diag}((\mathbf{r}^{(n)})')\mathbf{v})') - n^{1/2}(\hat{\lambda}_q - 1)\mathbf{I}_q & (\mathbf{Z}_{21}^{(n)})' \\ \mathbf{Z}_{21}^{(n)} & \mathbf{Z}_{22}^{(n)} - n^{1/2}(\hat{\lambda}_q - 1)\mathbf{I}_{p-q} \end{pmatrix}, \tag{S2.6}$$

where $s^* = \max\{j \in \{1,\ldots,q\}, n^{1/2} r_j^{(n)} \to \infty\}$ ($s^* \equiv 0$ if the maximum does not exist). Letting

$$\mathbf{R}^{(n)} := \mathrm{diag}((n^{1/2} r_1^{(n)} v_1)^{-1}, \ldots, (n^{1/2} r_{s^*}^{(n)} v_{s^*})^{-1}, \mathbf{1}'_{q-s^*}),$$

it follows from (S2.6) that $\hat{\boldsymbol{\theta}}_q^{(n)}$ is proportional to the vector of the cofactors obtained with respect to the $q$th row of

$$\mathbf{M}^{(n)} := \begin{pmatrix} \mathbf{R}^{(n)}\mathbf{Z}_{11}^{(n)} + n^{1/2}\mathbf{R}^{(n)}\mathrm{diag}((\mathrm{diag}((\mathbf{r}^{(n)})')\mathbf{v})') - n^{1/2}(\hat{\lambda}_q - 1)\mathbf{R}^{(n)} & \mathbf{R}^{(n)}(\mathbf{Z}_{21}^{(n)})' \\ \mathbf{Z}_{21}^{(n)} & \mathbf{Z}_{22}^{(n)} - n^{1/2}(\hat{\lambda}_q - 1)\mathbf{I}_{p-q} \end{pmatrix}.$$

Since $\mathbf{Z}^{(n)}$ and $n^{1/2}(\hat{\lambda}_q - 1)$ are both $O_{\mathrm{P}}(1)$ (and not $o_{\mathrm{P}}(1)$) as $n \to \infty$, we readily have that

$$\mathbf{M}^{(n)} := \begin{pmatrix} \mathbf{I}_{s^*} & \mathbf{0} \\ \mathbf{Y}_{11}^{(n)} & \mathbf{Y}_{12}^{(n)} \\ \mathbf{Z}_{21}^{(n)} & \mathbf{Z}_{22}^{(n)} - n^{1/2}(\hat{\lambda}_q - 1)\mathbf{I}_{p-q} \end{pmatrix} + o_{\mathrm{P}}(1)$$

as $n \to \infty$, where $(\mathbf{Y}_{11}^{(n)} \vdots \mathbf{Y}_{12}^{(n)})$ is a $(q - s^*) \times p$ matrix whose elements are $O_{\mathrm{P}}(1)$ (and not $o_{\mathrm{P}}(1)$). It is then easy to see that the cofactor associated with the $q$th line and the $(q+1)$th column of the above matrix is not $o_{\mathrm{P}}(1)$. Point (b) follows. $\square$

Before providing the proof of Proposition 3, the following Lemma can be obtained along the same lines as Proposition 1.

**Lemma 2.** *Let* $\mathbf{r}^{(n)}$, $\mathbf{v}$ *and* $\mathbf{Z}$ *be as in Proposition 1. Then we have that as* $n \to \infty$ *under* $\mathrm{P}^{(n)}_{\boldsymbol{\beta},\boldsymbol{\lambda}^{(n)}}$ *with* $\boldsymbol{\lambda}^{(n)}$ *as in* (3.1), $((\boldsymbol{\ell}^{(n)}_q)', (\boldsymbol{\ell}^{(n)}_{p-q})')'$ *defined through* (S2.1) *and* (S2.3) *is* $O_{\mathrm{P}}(1)$ *as* $n \to \infty$.

*Proof of Proposition 3.* In this proof, we put $\hat{\boldsymbol{\Lambda}}^{(n)} = \mathrm{diag}(\hat{\boldsymbol{\Lambda}}^{(n)}_q, \hat{\boldsymbol{\Lambda}}^{(n)}_{p-q}) = \hat{\boldsymbol{\beta}}'\mathbf{S}^{(n)}\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\Lambda}}^{(n)}_q := \mathrm{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_q)$ and $\hat{\boldsymbol{\Lambda}}^{(n)}_{p-q} := \mathrm{diag}(\hat{\lambda}_{q+1}, \ldots, \hat{\lambda}_p)$ and $\mathbf{R}^{(n)} := \mathrm{diag}((\mathbf{r}^{(n)})')$. We have that

$$\mathbf{S}^{(n)}_{\mathbf{Y}} = (\mathbf{0}_{(p-q)\times q} \vdots \mathbf{I}_{p-q})(\mathbf{E}^{(n)})'\hat{\boldsymbol{\Lambda}}^{(n)}\mathbf{E}^{(n)}(\mathbf{0}_{(p-q)\times q} \vdots \mathbf{I}_{p-q})' = ((\mathbf{E}^{(n)}_{12})' \vdots (\mathbf{E}^{(n)}_{22})')\hat{\boldsymbol{\Lambda}}^{(n)}((\mathbf{E}^{(n)}_{12})' \vdots (\mathbf{E}^{(n)}_{22})')'. \tag{S2.7}$$

This entails that

$$\begin{aligned} n^{1/2}(\mathbf{S}^{(n)}_{\mathbf{Y}} - \mathbf{I}_{p-q}) &= n^{1/2}(((\mathbf{E}^{(n)}_{12})' \vdots (\mathbf{E}^{(n)}_{22})')\hat{\boldsymbol{\Lambda}}^{(n)}((\mathbf{E}^{(n)}_{12})' \vdots (\mathbf{E}^{(n)}_{22})')' - \mathbf{I}_{p-q}) \\ &= n^{1/2}((\mathbf{E}^{(n)}_{12})'\hat{\boldsymbol{\Lambda}}^{(n)}_q \vdots (\mathbf{E}^{(n)}_{22})'\hat{\boldsymbol{\Lambda}}^{(n)}_{p-q})((\mathbf{E}^{(n)}_{12})' \vdots (\mathbf{E}^{(n)}_{22})')' - \mathbf{I}_{p-q}). \end{aligned}$$

Now, since $\mathbf{E}^{(n)}$ is orthogonal, $(\mathbf{E}^{(n)}_{12})'\mathbf{E}^{(n)}_{12} + (\mathbf{E}^{(n)}_{22})'\mathbf{E}^{(n)}_{22} = \mathbf{I}_{p-q}$, which implies, that letting

$$\begin{aligned} n^{1/2}(\mathbf{S}^{(n)}_{\mathbf{Y}} - \mathbf{I}_{p-q}) &= n^{1/2}((\mathbf{E}^{(n)}_{12})'(\hat{\boldsymbol{\Lambda}}^{(n)}_q - \mathbf{I}_q)\mathbf{E}^{(n)}_{12} + (\mathbf{E}^{(n)}_{22})'(\hat{\boldsymbol{\Lambda}}^{(n)}_{p-q} - \mathbf{I}_{p-q})\mathbf{E}^{(n)}_{22}) \\ &= (\mathbf{E}^{(n)}_{12})'(\mathbf{R}^{(n)})^{-1}(\hat{\boldsymbol{\Lambda}}^{(n)}_q - \mathbf{I}_q)n^{1/2}\mathbf{R}^{(n)}\mathbf{E}^{(n)}_{12} + +(\mathbf{E}^{(n)}_{22})'(\hat{\boldsymbol{\Lambda}}^{(n)}_{p-q} - \mathbf{I}_{p-q})\mathbf{E}^{(n)}_{22}) \end{aligned}$$

Note that for each $1 \leq j \leq q$, $(r^{(n)}_j)^{-1}(\hat{\lambda}_j - 1) = (r^{(n)}_j)^{-1}(\hat{\lambda}_j - 1 - r^{(n)}_j v_j) + v_j$. Since $n^{1/2}r^{(n)}_j \to \infty$, Lemma 2 entails that

$$(r^{(n)}_j)^{-1}(\hat{\lambda}_j - 1 - r^{(n)}_j v_j) = (n^{1/2}r^{(n)}_j)^{-1}n^{1/2}(\hat{\lambda}_j - 1 - r^{(n)}_j v_j)$$

is $o_{\mathrm{P}}(1)$ as $n \to \infty$ so that since from Proposition 2 (point $(a)$), $n^{1/2}\mathbf{R}^{(n)}\mathbf{E}_{12}^{(n)}$ is $O_{\mathrm{P}}(1)$ as $n \to \infty$, we have that

$$n^{1/2}(\mathbf{S}_{\mathbf{Y}}^{(n)} - \mathbf{I}_{p-q}) \;=\; n^{1/2}(\mathbf{E}_{22}^{(n)})'(\hat{\boldsymbol{\Lambda}}_{p-q}^{(n)} - \mathbf{I}_{p-q})\mathbf{E}_{22}^{(n)} + o_{\mathrm{P}}(1) \quad \text{(S2.8)}$$

as $n \to \infty$. Now, using (S2.8),

$$\begin{aligned} T_q^{(n)}(\boldsymbol{\beta}) &= \frac{n}{2}\left(\frac{(p-q)}{\mathrm{tr}(\mathbf{S}_{\mathbf{Y}}^{(n)})}\right)^2 \left(\mathrm{tr}((\mathbf{S}_{\mathbf{Y}}^{(n)})^2) - \frac{1}{p-q}\mathrm{tr}^2(\mathbf{S}_{\mathbf{Y}}^{(n)})\right) \\ &= \frac{n}{2}\left(\frac{(p-q)}{\mathrm{tr}(\mathbf{S}_{\mathbf{Y}}^{(n)})}\right)^2 \left(\mathrm{tr}((\mathbf{S}_{\mathbf{Y}}^{(n)} - \mathbf{I}_{p-q})^2) - \frac{1}{p-q}\mathrm{tr}^2(\mathbf{S}_{\mathbf{Y}}^{(n)} - \mathbf{I}_{p-q})\right) \\ &= \frac{1}{2}\left(\frac{(p-q)}{\sum_{j=q+1}^{p}\hat{\lambda}_j}\right)^2 \Big(\mathrm{tr}((n^{1/2}(\mathbf{E}_{22}^{(n)})'(\hat{\boldsymbol{\Lambda}}_{p-q}^{(n)} - \mathbf{I}_{p-q})\mathbf{E}_{22}^{(n)})^2) \\ &\quad - \frac{1}{p-q}\mathrm{tr}^2(n^{1/2}(\mathbf{E}_{22}^{(n)})'(\hat{\boldsymbol{\Lambda}}_{p-q}^{(n)} - \mathbf{I}_{p-q})\mathbf{E}_{22}^{(n)})\Big) + o_{\mathrm{P}}(1), \qquad \text{(S2.9)} \end{aligned}$$

as $n \to \infty$. Using the fact that Proposition 2 yields $\mathbf{E}_{22}^{(n)}(\mathbf{E}_{22}^{(n)})' = \mathbf{I}_{p-q} + o_{\mathrm{P}}(1)$, (S2.9) and the Slutsky Lemma yield

$$\begin{aligned} T_q^{(n)}(\boldsymbol{\beta}) &= \frac{1}{2}\left(\frac{(p-q)}{\sum_{j=q+1}^{p}\hat{\lambda}_j}\right)^2 \left(\sum_{j=q+1}^{p}(n^{1/2}(\hat{\lambda}_j - 1))^2 - \frac{1}{p-q}(\sum_{j=q+1}^{p} n^{1/2}(\hat{\lambda}_j - 1))^2\right) + o_{\mathrm{P}}(1) \\ &= T_q^{(n)} + o_{\mathrm{P}}(1) \end{aligned}$$

under $\mathrm{P}_{\boldsymbol{\beta},\boldsymbol{\lambda}^{(n)}}^{(n)}$ as $n \to \infty$, which ends the proof. □

*Proof of Proposition 4.* Fix $0 < q \leq p-2$ (the case $q = 0$ is trivial). Letting

$$\boldsymbol{\ell}_{2,q}^{(n)} := (\ell_q^{(n)}, \ell_{q+1}^{(n)}) = n^{1/2}((\hat{\lambda}_q - (1 + r_q^{(n)} v_q), (\hat{\lambda}_{q+1} - 1)),$$

we have that (with $\mathbf{e}_1 = (1,0) \in \mathbb{R}^2$)

$$
\begin{aligned}
T^{(n)}_{q,q+1} &= \frac{n(\sum_{j=q}^{q+1} \hat{\lambda}_j^2 - \frac{1}{2}(\sum_{j=q}^{q+1} \hat{\lambda}_j)^2)}{\frac{1}{2}(\sum_{j=q}^{q+1} \hat{\lambda}_j)^2} \\
&= \frac{n(\sum_{j=q}^{q+1} (\hat{\lambda}_j - 1)^2 - \frac{1}{2}(\sum_{j=q}^{q+1} (\hat{\lambda}_j - 1))^2)}{\frac{1}{2}(\sum_{j=q}^{q+1} \hat{\lambda}_j)^2} \\
&= \frac{(\boldsymbol{\ell}^{(n)}_{2,q} + n^{1/2} r^{(n)}_q v_q \mathbf{e}_1)'(\mathbf{I}_2 - \mathbf{1}_2(\mathbf{1}_2'\mathbf{1}_2)^{-1}\mathbf{1}_2')(\boldsymbol{\ell}^{(n)}_{2,q} + n^{1/2} r^{(n)}_q v_q \mathbf{e}_1)}{\frac{1}{2}(\sum_{j=q}^{q+1} \hat{\lambda}_j)^2}
\end{aligned}
$$

so that since from Lemma 2, $\boldsymbol{\ell}^{(n)}_{2,q}$ is $O_{\mathrm{P}}(1)$ as $n \to \infty$ under $\mathrm{P}^{(n)}_{\boldsymbol{\beta},\boldsymbol{\lambda}^{(n)}}$,

$$
T^{(n)}_{q,q+1} = \frac{(\boldsymbol{\ell}^{(n)}_{2,q} + n^{1/2} r^{(n)}_q v_q \mathbf{e}_1)'(\mathbf{I}_2 - \mathbf{1}_2(\mathbf{1}_2'\mathbf{1}_2)^{-1}\mathbf{1}_2')(\boldsymbol{\ell}^{(n)}_{2,q} + n^{1/2} r^{(n)}_q v_q \mathbf{e}_1)}{\frac{1}{2}(\sum_{j=q}^{q+1} \hat{\lambda}_j)^2} \tag{S2.10}
$$

converges to $+\infty$ in probability when $n^{1/2} r^{(n)}_q \to \infty$. We therefore have that, for any $\gamma \in (0,1)$,

$$
\begin{aligned}
\mathrm{E}[|\phi^{(n)}_{\mathrm{new}} - \phi^{(n)}|] &= \mathrm{P}[(T^{(n)}_q < \chi^2_{d(p,q);1-\alpha}) \cap (T^{(n)}_{q,q+1} \leq \chi^2_{2;1-\gamma})] \\
&\leq \mathrm{P}[T^{(n)}_{q,q+1} \leq \chi^2_{2;1-\gamma}],
\end{aligned}
$$

so that $\mathrm{E}[|\phi^{(n)}_{\mathrm{new}} - \phi^{(n)}|]$ converges to zero when $n^{1/2} r^{(n)}_q \to \infty$. □

## S3. Consistency of $\hat{k}_{\mathrm{new}}$

In this Section, we provide the consistency (under some conditions) of the estimator $\hat{k}_{\mathrm{new}}$.

**Proposition 7.** *Let $c^{(n)}$ and $b_0^{(n)}, \ldots, b_{p-2}^{(n)}$ be positive sequences that diverge to $\infty$ and are such that (i) $c^{(n)} = o(n)$ and $b_q^{(n)} = o(n)$ as $n \to \infty$ for $q = 0, \ldots, p-2$ and (ii) $(\max(c^{(n)}, b_0^{(n)}, \ldots, b_{p-2}^{(n)}))^{-1/2} n^{1/2} (\lambda_k^{(n)} - \lambda_{k+1}^{(n)})$ diverges to $\infty$ as $n \to \infty$. Then $\lim_{n\to\infty} \mathrm{P}(\hat{k}_{\rm new} = k) = 1$.*

To simplify the interpretation of Proposition 2, assume that the sequences $c^{(n)}$ and $b_0^{(n)} = \ldots = b_{p-2}^{(n)} \equiv c^{(n)}$ are all the same. Proposition 2 then shows that provided that $c^{(n)}$ does not diverge too quickly to $\infty$ in the sense that $c^{(n)} = o(n)$ and $(c^{(n)})^{-1/2} n^{1/2} (\lambda_k^{(n)} - \lambda_{k+1}^{(n)})$ diverges to $\infty$ as $n \to \infty$, the resulting estimator $\hat{k}_{\rm new}$ is consistent.

*Proof of Proposition 2.* Fix $k < p-1$ (the case $k = p-1$ is considered at the end of the proof). In the proof, we assume without loss of generality that $\boldsymbol{\lambda}^{(n)}$ is as in (3.1) and work under $\mathrm{P}^{(n)}_{\boldsymbol{\beta},\boldsymbol{\lambda}^{(n)}}$ with $\boldsymbol{\lambda}^{(n)}$ such that $\mathcal{H}_{0k}^{(n)}$ holds. Therefore, since

$$(\max(c^{(n)}, b_0^{(n)}, \ldots, b_{p-2}^{(n)}))^{-1/2} n^{1/2} (\lambda_k^{(n)} - \lambda_{k+1}^{(n)}) \to \infty$$

as $n \to \infty$, we have that $(c^{(n)})^{-1/2} n^{1/2} (\lambda_k^{(n)} - \lambda_{k+1}^{(n)}) \to \infty$ as $n \to \infty$. We then have using the same notations as in (S2.10) that

$$\frac{T_{k,k+1}^{(n)}}{c^{(n)}} = \frac{(c^{(n)})^{-1/2}(\boldsymbol{\ell}_{2,k}^{(n)} + n^{1/2} r_k^{(n)} v_k \mathbf{e}_1)'(\mathbf{I}_2 - \mathbf{1}_2(\mathbf{1}_2'\mathbf{1}_2)^{-1}\mathbf{1}_2')(c^{(n)})^{-1/2}(\boldsymbol{\ell}_{2,k}^{(n)} + n^{1/2} r_k^{(n)} v_k \mathbf{e}_1)}{\frac{1}{2}(\sum_{q=k}^{k+1} \hat{\lambda}_q)^2} \tag{S3.11}$$

converges to $+\infty$ in probability for $k > 0$. When $k = 0$, $T_{0,1}^{(n)}$ is arbitrarily taken such that $T_{0,1}^{(n)} > c^{(n)}$ ($T_{0,1}^{(n)}$ can be arbitrarily fixed). Now, since $\mathcal{H}_{0k}^{(n)}$ holds, we also have by (1) that

$$(\ell_{k+1}^{(n)}, \ldots, \ell_p^{(n)}) = n^{1/2}(\hat{\lambda}_{k+1} - 1, \ldots, \hat{\lambda}_p - 1) = O_{\mathrm{P}}(1)$$

as $n \to \infty$. Following the same lines as in S3.11 then yields that

$$(b_k^{(n)})^{-1} T_k^{(n)} = o_{\mathrm{P}}(1) \tag{S3.12}$$

as $n \to \infty$. Combining S3.11 and S3.12, we obtain that under $\mathrm{P}_{\boldsymbol{\beta},\boldsymbol{\lambda}^{(n)}}^{(n)}$ with $\boldsymbol{\lambda}^{(n)}$ such that $\mathcal{H}_{0k}^{(n)}$ holds,

$$\lim_{n\to\infty} \mathrm{P}(\mathbb{I}[T_k^{(n)} > b_k^{(n)}]\mathbb{I}[T_{k,k+1}^{(n)} > c^{(n)}] + \mathbb{I}[T_{k,k+1}^{(n)} \le c^{(n)}] = 0) = 1. \tag{S3.13}$$

Now, for $0 \le j < k$, since $(b_j^{(n)})^{-1/2} n^{1/2}(\lambda_k^{(n)} - \lambda_{k+1}^{(n)}) \to \infty$, working along the same line as S3.11, we obtain that $T_j^{(n)}/b_j^{(n)}$ converges to $+\infty$ in probability. Therefore,

$$\lim_{n\to\infty} \mathrm{P}(\mathbb{I}[T_j^{(n)} > b_j^{(n)}]\mathbb{I}[T_{j,j+1}^{(n)} > c^{(n)}] + \mathbb{I}[T_{j,j+1}^{(n)} \le c^{(n)}] = 1) = 1. \tag{S3.14}$$

as $n \to \infty$ under $\mathrm{P}_{\boldsymbol{\beta},\boldsymbol{\lambda}^{(n)}}^{(n)}$ with $\boldsymbol{\lambda}^{(n)}$ such that $\mathcal{H}_{0k}^{(n)}$ holds. Since

$$\hat{k}_{\mathrm{new}} = \min_{j\in\{0,\ldots,p-2\}} \{\mathbb{I}[T_j^{(n)} > b_j^{(n)}]\mathbb{I}[T_{j,j+1}^{(n)} > c^{(n)}] + \mathbb{I}[T_{j,j+1}^{(n)} \le c^{(n)}] = 0\},$$

it follows from S3.13 and S3.14 that $\hat{k}_{\mathrm{new}} - k$ is $o_{\mathrm{P}}(1)$ as $n \to \infty$. Finally, for $k = p - 1$, since

$$(\max(c^{(n)}, b_0^{(n)}, \ldots, b_{p-2}^{(n)}))^{-1/2} n^{1/2}(\lambda_{p-1}^{(n)} - \lambda_p^{(n)}) \to \infty$$

as $n \to \infty$ we have that

$$\lim_{n\to\infty} \mathrm{P}(T_j^{(n)} \geq b_j^{(n)}) = 1 \qquad \text{(S3.15)}$$

for all $0 \leq j \leq p-2$. The result follows. □

## References

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