Enhanced Structural Break Detection in Functional Means

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Supplementary Material

This online supplementary material contains the technical proofs and additional simulation results. In Section S1, we display the proof of the main theorems in the article and the relevant lemmas. Section S2 presents additional simulation results.

S1 Technical Proofs

Proof of Theorem 2. Define

$$\widehat{C}_{X,h}(t,s) = \frac{1}{N-h}\sum_{n=1}^{N-h}\left\{X_n - \bar{X}_n)\right\}\left\{X_{n+h} - \bar{X}_{n+h})\right\}(t,s),$$
$$\widetilde{C}_{X,h}(t,s) = \frac{1}{N-h}\sum_{n=1}^{N-h}\left\{X_n - E(X_n)\right\}\left\{X_{n+h} - E(X_{n+h})\right\}(t,s).$$

Clearly, given arbitrary $\hat{\delta}(t)$, $\{Y_n(t)\colon n\in\mathbb{N}\}$ are L^4-m approximable, since

$$\begin{aligned}
&E\|Y_n - Y_{n,m}\|^4\\
\leq&\,\text{const.}\left\{E\|X_n - X_{n,m}\|^4 + E\left\langle X_n - X_{n,m}, \frac{\hat{\delta}}{\|\hat{\delta}\|+\kappa}\right\rangle^4 \frac{\|\hat{\delta}\|^4}{(\|\hat{\delta}\|+\kappa)^4}\right\}\\
\leq&\,\text{const.}\, E\|X_n - X_{n,m}\|^4,
\end{aligned}$$

and $\{X_n(t)\colon n\in\mathbb{N}\}$ are L^4-m approximable. Note that the second inequality is due to that fact $\|\hat{\delta}\|/(\|\hat{\delta}\|+\kappa) < 1$.

Under H_0, observe that

$$\begin{aligned}
&\widehat{C}^{(\kappa)}_{Y,h}(t,s)\\
=&\frac{1}{N-h}\sum_{n=1}^{N-h}\left\{Y_n^{(\kappa)} - \bar{Y}_n^{(\kappa)}\right\}\left\{Y_{n+h}^{(\kappa)} - \bar{Y}_{n+h}^{(\kappa)}\right\}(t,s)\\
=&\widehat{C}_{X,h}(t,s) + \frac{1}{N-h}\sum_{n=1}^{N-h}\left\langle X_{n+h} - \bar{X}_{n+h}, \frac{\hat{\delta}}{\|\hat{\delta}\|+\kappa}\right\rangle \frac{(X_n - \bar{X}_n)\hat{\delta}}{\|\hat{\delta}\|+\kappa}(t,s)\\
&+\frac{1}{N-h}\sum_{n=1}^{N-h}\left\langle X_n - \bar{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\|+\kappa}\right\rangle \frac{\hat{\delta}(X_{n+h} - \bar{X}_{n+h})}{\|\hat{\delta}\|+\kappa}(t,s)
\end{aligned}$$

$$+\frac{1}{N-h}\sum_{n=1}^{N-h}\left\langle X_n-\bar{X}_n,\frac{\hat{\delta}}{\|\hat{\delta}\|+\kappa}\right\rangle\left\langle X_{n+h}-\bar{X}_{n+h},\frac{\hat{\delta}}{\|\hat{\delta}\|+\kappa}\right\rangle\frac{\hat{\delta}(t)\hat{\delta}(s)}{(\|\hat{\delta}\|+\kappa)^2}$$

$$\triangleq\widehat{C}_{X,h}(t,s)+A_1(t,s)+A_2(t,s)+A_3(t,s).$$

Since $N^{1/2}\kappa\to\infty$ and $\|\hat{\delta}\|=O_p(N^{-1/2})$, asymptotically almost surely,

$$\begin{aligned}\|A_1\|&=\left\|\frac{1}{N-h}\sum_{n=1}^{N-h}\left\langle X_{n+h}-\bar{X}_{n+h},\frac{\hat{\delta}}{\|\hat{\delta}\|+\kappa}\right\rangle\frac{(X_n-\bar{X}_n)\hat{\delta}}{\|\hat{\delta}\|+\kappa}\right\|\\&\leq\frac{\|\hat{\delta}\|^2}{4\kappa^2(N-h)}\sum_{n=1}^{N-h}\|X_{n+h}-\bar{X}_{n+h}\|^2\\&=\frac{O_p(1)\|\hat{\delta}\|^2}{\kappa^2}=O_p(N^{-1+2\alpha_\kappa+\epsilon}).\end{aligned}\tag{A1}$$

Additionally, we can show that $\|A_2\|_{\mathcal{S}}=O_p(N^{-1+2\alpha_\kappa+\epsilon})$ and $\|A_3\|_{\mathcal{S}}=O_p(N^{-2+4\alpha_\kappa+\epsilon})$.

It can be shown that $E\|\widehat{C}_{X,h}-C_{X,h}\|^2$ has the same convergence rate as $E\|\widetilde{C}_{X,h}-C_{X,h}\|^2$. Define $\mathcal{X}_n(t)=X_n(t)-E\{X_n(t)\}$, we have

$$\begin{aligned}E\|\widetilde{C}_{X,h}-C_{X,h}\|^2&=E\int\Bigg[\frac{1}{N-h}\sum_{n=1}^{N-h}\{X_n-E(X_n)\}\{X_{n+h}-E(X_{n+h})\}(t,s)\\&\qquad-E\{X_n-E(X_n)\}\{X_{n+h}-E(X_{n+h})\}(t,s)\Bigg]^2dtds\\&=E\int\Bigg[\frac{1}{N-h}\sum_{n=1}^{\lfloor N\theta^*\rfloor-h}(\mathcal{X}_n\mathcal{X}_{n+h}-E\mathcal{X}_n\mathcal{X}_{n+h})(t,s)\\&\qquad+\frac{1}{N-h}\sum_{n=\lfloor N\theta^*\rfloor-h+1}^{\lfloor N\theta^*\rfloor}(\mathcal{X}_n\mathcal{X}_{n+h}-E\mathcal{X}_n\mathcal{X}_{n+h})(t,s)\\&\qquad+\frac{1}{N-h}\sum_{n=\lfloor N\theta^*\rfloor+1}^{N-h}(\mathcal{X}_n\mathcal{X}_{n+h}-E\mathcal{X}_n\mathcal{X}_{n+h})(t,s)\Bigg]^2dtds\\&\leq B_1+B_2+B_3,\end{aligned}$$

where

$$(N-h)B_1 \leq \text{const.}\theta^* \int \left\{ \text{Var}(\mathcal{X}_1\mathcal{X}_{1+h}) + 2\sum_{r=1}^{\infty} |\text{Cov}(\mathcal{X}_1\mathcal{X}_{1+h}, \mathcal{X}_{1+r}\mathcal{X}_{1+h+r})| \right\} dt,$$

$$(N-h)B_2 \leq \text{const.}\frac{h^2}{N-h} \to 0,$$

$$(N-h)B_3 \leq \text{const.}(1-\theta^*) \int \left\{ \text{Var}(\mathcal{X}_1\mathcal{X}_{1+h}) + 2\sum_{r=1}^{\infty} |\text{Cov}(\mathcal{X}_1\mathcal{X}_{1+h}, \mathcal{X}_{1+r}\mathcal{X}_{1+h+r})| \right\} dt.$$

By the same argument in Jiao *et al.* (2023) and Assumption 1, due to the fact that $\{Y_n(t) \colon n \in \mathcal{N}\}$ are L^4-m approximable, we deduce that that $(N-h)B_i,\ i = 1, 2, 3$ are bounded by some constant not related to h, and thus $(N-h)E\|\widehat{C}_{X,h} - C_{X,h}\|^2$ is also bounded by some constant not related to h.

Next we develop the convergence rate of $\|\widehat{LC}_X(t,s) - LC_X(t,s)\|_{\mathcal{S}}$. By the triangle inequality, we deduce that

$$\begin{aligned}\|\widehat{LC}_X(t,s) - LC_X(t,s)\|_{\mathcal{S}} \leq \text{const.} &\left\{ \left\| \sum_{h=-\ell}^{\ell} W\left(\frac{h}{\ell}\right)(\widetilde{C}_{X,h} - C_{X,h}) \right\|_{\mathcal{S}} \right. \\ &+ \left. \left\| \sum_{h=-\ell}^{\ell} \left\{ 1 - W\left(\frac{h}{\ell}\right) \right\} C_{X,h} \right\|_{\mathcal{S}} + \left\| \sum_{h=-\infty}^{-\ell-1} C_{X,h} + \sum_{h=\ell+1}^{\infty} C_{X,h} \right\|_{\mathcal{S}} \right\}.\end{aligned}$$

Define

$$t_r(\ell) = \begin{cases} \ell^{r+1}, & \text{if } r > -1, \\ \log(\ell), & \text{if } r = -1, \\ 1, & \text{if } r < -1, \end{cases}$$

and we deduce that

$$E\left\| \sum_{h=-\ell}^{\ell} W\left(\frac{h}{\ell}\right)(\widetilde{C}_{X,h} - C_{X,h}) \right\|_{\mathcal{S}} \leq \sum_{h=-\ell}^{\ell} W\left(\frac{h}{\ell}\right) E\left\| \widetilde{C}_{X,h} - C_{X,h} \right\|_{\mathcal{S}}$$

$$\leq \text{const.} N^{-1/2} \sum_{h=-\ell}^{\ell} \left\{ 1 - \text{const.} \left(\frac{h}{\ell} \right)^{\alpha} \right\}$$

$$\leq \text{const.} \ell N^{-1/2} = \text{const.} N^{\alpha_\ell - 1/2},$$

$$\left\| \sum_{h=-\ell}^{\ell} \left\{ 1 - W\left(\frac{h}{\ell} \right) \right\} C_{X,h} \right\|_{\mathcal{S}} \leq \sum_{h=-\ell}^{\ell} \left\{ 1 - W\left(\frac{h}{\ell} \right) \right\} \|C_{X,h}\|_{\mathcal{S}}$$

$$\leq \text{const.} \sum_{h=-\ell}^{\ell} \left(\frac{h}{\ell} \right)^{\alpha_w} \cdot h^{-\alpha_c}$$

$$\leq \text{const.} t_{\alpha_w - \alpha_c}(\ell) \ell^{-\alpha_w} = \text{const.} t_{\alpha_w - \alpha_c}(\ell) N^{-\alpha_w \alpha_\ell}$$

$$\leq \begin{cases} \text{const.} N^{-(\alpha_c - 1)\alpha_\ell}, & \text{if } \alpha_w - \alpha_c > -1, \\ \text{const.} \log(N) N^{-(\alpha_c - 1)\alpha_\ell}, & \text{if } \alpha_w - \alpha_c = -1, \\ \text{const.} N^{-\alpha_w \alpha_\ell}, & \text{if } \alpha_w - \alpha_c < -1, \end{cases}$$

$$\left\| \sum_{h=-\infty}^{-\ell-1} C_{X,h} + \sum_{h=\ell+1}^{\infty} C_{X,h} \right\|_{\mathcal{S}} \leq \text{const.} \ell^{-\alpha_c + 1} = \text{const.} N^{-(\alpha_c - 1)\alpha_\ell}.$$

Then we conclude that

$$E \left\| \widehat{LC}_X(t,s) - LC_X(t,s) \right\|_{\mathcal{S}} \leq \text{const.} N^{\max\{\alpha_\ell - 1/2, -(\alpha_c - 1)/\alpha_\ell\}}$$

$$\vee \begin{cases} N^{-(\alpha_c - 1)\alpha_\ell}, & \text{if } \alpha_w - \alpha_c > -1. \\ N^{-(\alpha_c - 1)\alpha_\ell + \epsilon}, & \text{if } \alpha_w - \alpha_c = -1. \\ N^{-\alpha_w \alpha_\ell}, & \text{if } \alpha_w - \alpha_c < -1. \end{cases}$$

Combining (A1) leads to the final results under H_0.

Under H_a, by Lemma 2 and Lemma 3, $\|\widehat{LC}_{Y,\kappa} - \widehat{LC}_Y\|_{\mathcal{S}} \leq O_p(1)\ell\kappa\|\delta\|^{-1}$ and $\|C_{Y,h}\|_{\mathcal{S}} \leq \text{const.} h^{-\alpha_c}$, the convergence rate can be obtained with similar arguments.

□

Proof of Theorem 3. For the simplicity of notations, we assume that

$$\operatorname{sign}\langle \hat{\psi}_d^{(\kappa)}, \phi_d \rangle = 1, \text{for } d \geq 1.$$

It suffices to show that $\sup_{\theta\in(0,1)}|\widetilde{T}_N(\theta) - T_N^o(\theta)| \xrightarrow{p} 0$. Define $\mu_X(t) = EX_n(t)$, then by Theorem 1.1 from Berkes *et al.* (2013),

$$\sup_{0<\theta<1} \frac{1}{\sqrt{N}} \sum_{n=1}^{\lfloor N\theta \rfloor} (X_n - \mu_X)(t) = O_p(1).$$

Thus it is deduced that

$$\begin{aligned}
&\sup_{\theta\in(0,1)} \left|\widetilde{T}_N(\theta) - T_N^o(\theta)\right| \\
&= \sup_{\theta\in(0,1)} \frac{1}{N} \left| \sum_{d=1}^{D} \left\{ \left(\sum_{n=1}^{\lfloor N\theta \rfloor} \langle X_n - \mu_X, \hat{\psi}_d^{(\kappa)} \rangle - \theta \sum_{n=1}^{N} \langle X_n - \mu_X, \hat{\psi}_d^{(\kappa)} \rangle \right)^2 \right.\right. \\
&\qquad \left.\left. - \left(\sum_{n=1}^{\lfloor N\theta \rfloor} \langle X_n - \mu_X, \phi_d \rangle - \theta \sum_{n=1}^{N} \langle X_n - \mu_X, \phi_d \rangle \right)^2 \right\} \right| \\
&\leq O_p(1) \left\{ \sup_{\theta\in(0,1)} \sum_{d=1}^{D} \frac{1}{N} \left(\sum_{n=1}^{\lfloor N\theta \rfloor} \langle X_n - \mu_X, \hat{\psi}_d^{(\kappa)} - \phi_d \rangle \right.\right. \\
&\qquad \left.\left. - \theta \sum_{n\geq 1} \langle X_n - \mu_X, \hat{\psi}_d^{(\kappa)} - \phi_d \rangle \right)^2 \right\}^{1/2}.
\end{aligned}$$

Additionally, since the enhancement term does not influence the convergence rate of $\widehat{K}^{(\kappa)}$ under H_0, we deduce that

$$\sup_{\theta\in(0,1)} \sum_{d=1}^{D} \frac{1}{N} \left(\sum_{n=1}^{\lfloor N\theta \rfloor} \langle X_n - \mu_X, \hat{\psi}_d^{(\kappa)} - \phi_d \rangle - \theta \sum_{n\geq 1} \langle X_n - \mu_X, \hat{\psi}_d^{(\kappa)} - \phi_d \rangle \right)^2$$

$$
\begin{aligned}
&= \sup_{\theta\in(0,1)} \sum_{d=1}^{D} \left(\frac{1}{\sqrt{N}} \sum_{n=1}^{\lfloor N\theta \rfloor} \left\langle X_n - \mu_X, \frac{\hat{\psi}_d^{(\kappa)} - \phi_d}{\|\hat{\psi}_d^{(\kappa)} - \phi_d\|} \right\rangle \|\hat{\psi}_d^{(\kappa)} - \phi_d\| \right. \\
&\qquad \left. -\theta \frac{1}{\sqrt{N}} \sum_{n\geq 1} \left\langle X_n - \mu_X, \frac{\hat{\psi}_d^{(\kappa)} - \phi_d}{\|\hat{\psi}_d^{(\kappa)} - \phi_d\|} \right\rangle \|\hat{\psi}_d^{(\kappa)} - \phi_d\| \right)^2 \\
&= O_p(1) \sum_{d=1}^{D} \|\hat{\psi}_d^{(\kappa)} - \phi_d\|^2 \leq O_p(1) \sum_{d=1}^{D} \|\widehat{LC}_{Y,\kappa} - LC_X\| \\
&\leq O_p(1) N^{-r_0} \sum_{d=1}^{D} \delta_{\tau,d}^{-1} \xrightarrow{p} 0.
\end{aligned}
$$

This finishes the proof. □

Proof of Theorem 4. First we prove that event (1) holds asymptotically almost surely. Based on the selection of D, we conclude $\hat{\lambda}_{D-1}^{(\kappa)} < \rho\|\hat{\delta}\|^2 = \hat{\theta}_{d^*}$, which leads to $\hat{\lambda}_{D-1}^{(\kappa)} < \hat{\theta}_{d^*} < \hat{\lambda}_{d^*-1}$ by the fact that the sequence $\{\hat{\theta}_d \colon d \geq 1\}$ is decreasing and $\{\hat{\theta}_d \colon d \neq d^*\} = \{\hat{\lambda}_d \colon d \geq 1\}$. Then we deduce that

$$
\begin{aligned}
P(\hat{\lambda}_{D-1} - \hat{\lambda}_{d^*-1} \geq 0) &= P(\hat{\lambda}_{D-1}^{(\kappa)} - \hat{\lambda}_{d^*-1} \geq \hat{\lambda}_{D-1}^{(\kappa)} - \hat{\lambda}_{D-1}) \\
&= P\left(\frac{\hat{\lambda}_{D-1}^{(\kappa)} - \hat{\lambda}_{d^*-1}}{\ell\kappa\|\delta\|^{-1}} \geq \frac{\hat{\lambda}_{D-1}^{(\kappa)} - \hat{\lambda}_{D-1}}{\ell\kappa\|\delta\|^{-1}} \right).
\end{aligned}
$$

By Lemma 3 and the fact that $\max_{d\geq 1}\{\hat{\lambda}_d^{(\kappa)} - \hat{\lambda}_d\} \leq \|\widehat{\mathcal{LC}}_{Y,\kappa} - \widehat{\mathcal{LC}}_Y\|$, $\ell^{-1}\kappa^{-1}\|\delta\|(\hat{\lambda}_{D-1}^{(\kappa)} - \hat{\lambda}_{D-1}) = O_p(1)$, and by assumption $\ell^{-1}\kappa^{-1}\|\delta\|(\hat{\lambda}_{D-1}^{(\kappa)} - \hat{\lambda}_{d^*-1}) \xrightarrow{p} -\infty$, thus $P(\hat{\lambda}_{D-1} - \hat{\lambda}_{d^*-1} \geq 0) \to 0$. Then we conclude that asymptotically almost surely, $\hat{\lambda}_{D-1} < \hat{\lambda}_{d^*-1}$, and equivalently, $D > d^*$.

For event (2), it is deduced that

$$
|\hat{\theta}_{d^*}^{(\kappa)} - \hat{\theta}_{d^*-1}^{(\kappa)}| = |\hat{\theta}_{d^*-1}^{(\kappa)} - \hat{\theta}_{d^*+1}^{(\kappa)}|/2
$$

$$\geq |\omega_{d^*-1} - \omega_{d^*+1}|/2 - |\hat{\theta}^{(\kappa)}_{d^*-1} - \omega_{d^*-1}|/2 - |\hat{\theta}^{(\kappa)}_{d^*+1} - \omega_{d^*+1}|/2$$

$$\geq |\omega_{d^*-1} - \omega_{d^*+1}|/2 - \|\Delta_K\|.$$

Consequently, for arbitrary $\epsilon > 0$,

$$|\hat{\theta}^{(\kappa)}_{d^*} - \hat{\theta}^{(\kappa)}_{d^*-1}| \geq \frac{|\omega_{d^*-1} - \omega_{d^*+1}|}{2+\epsilon} + \frac{|\omega_{d^*-1} - \omega_{d^*+1}|}{2(2/\epsilon+1)} - \|\Delta_K\|.$$

Therefore it suffices to show that, asymptotically almost surely,

$$\frac{|\omega_{d^*-1} - \omega_{d^*+1}|}{2(2/\epsilon+1)} - \|\Delta_K\| \geq 0,$$

or equivalently,

$$P\left(\|\Delta_K\| \geq \frac{|\omega_{d^*-1} - \omega_{d^*+1}|}{2(2/\epsilon+1)}\right) \to 0.$$

Since $D > d^*$ asymptotically almost surely, by the Markov inequality, with some $c_0 > 0$ related to ϵ,

$$P\left(\|\Delta_K\| \geq \frac{|\omega_{d^*-1} - \omega_{d^*+1}|}{2(2/\epsilon+1)}\right) \leq P\left(\|\Delta_K\| \geq c_0 (d^*)^{-(\alpha_\lambda+1)}\right)$$

$$\leq c_0^{-1} (d^*)^{\alpha_\lambda+1} E\|\Delta_K\| \to 0.$$

The same arguments can be applied to $|\hat{\theta}^{(\kappa)}_{d^*} - \hat{\theta}^{(\kappa)}_{d^*+1}|$, then result follows. Similar arguments can be applied to the case $d^* = 1$. □

Proof of Theorem 5. For the simplicity of notations, we assume that

$$\mathrm{sign}\langle \hat{\psi}^{(\kappa)}_d, \psi_d \rangle = 1, \mathrm{for}\, d \geq 1.$$

Define

$$\hat{f}_d = \frac{1}{N}\left(\sum_{n=1}^{\lfloor N\theta \rfloor} \hat{\eta}_{nd} - \theta \sum_{n=1}^{N} \hat{\eta}_{nd}\right), \text{ where } \hat{\eta}_{nd} = \langle X_n, \hat{\phi}_d^{(\kappa)} \rangle,$$

$$f_d = 0 \text{ for } d \neq d^*, \ f_{d^*}(\theta) = \begin{cases} \theta(1-\theta^*)\|\delta\|, & 0 < \theta \leq \theta^*, \\ \theta^*(1-\theta)\|\delta\|, & \theta^* < \theta < 1, \end{cases}$$

and

$$R_d(\theta) = N^{-1}\left(\sum_{n=1}^{\lfloor N\theta \rfloor} \int e_n \hat{\psi}_d^{(\kappa)} - \theta \sum_{n=1}^{N} \int e_n \hat{\psi}_d^{(\kappa)}\right).$$

Observe that

$$\hat{f}_d(\theta) = \begin{cases} \theta(1-\theta^*) \int \delta \hat{\psi}_d^{(\kappa)} + R_d(\theta), & 0 < \theta \leq \theta^*, \\ \theta^*(1-\theta) \int \delta \hat{\psi}_d^{(\kappa)} + R_d(\theta), & \theta^* < \theta < 1. \end{cases}$$

Define

$$\hat{g}_d^{(\kappa)} = \left\{\int \delta(t)\hat{\psi}_d^{(\kappa)}(t)dt\right\}^2, \qquad g_d = \left\{\int \delta(t)\psi_d(t)dt\right\}^2.$$

It is deduced that, with some constant related to θ^*, $C_1(\theta^*), C_2(\theta^*)$,

$$\begin{aligned} &\sup_{0<\theta<1} |N^{-1}\widetilde{T}_N(\theta) - \|\delta\|^2 V^2(\theta)| \leq \sup_{0<\theta<1} \sum_{d=1}^{D} |\hat{f}_d^2 - f_d^2| \\ \leq & C_1(\theta^*) \sum_{d=1}^{D} |\hat{g}_d^{(\kappa)} - g_d| + C_2(\theta^*) \sup_{0<\theta<1} \sum_{d=1}^{D} \left| R_d(\theta) \int \delta \hat{\psi}_d^{(\kappa)} \right| + \sup_{0<\theta<1} \sum_{d=1}^{D} R_d^2(\theta) \\ \triangleq & S_1 + S_2 + S_3. \end{aligned}$$

By Berkes *et al.* (2009), $\sup_{0<\theta<1} \int \left\{N^{-1/2} \sum_{n=1}^{\lfloor N\theta \rfloor} e_n(t)\right\}^2 dt = O_p(1)$, and it can be deduced that $S_3 = \sup_{0<\theta<1} \sum_d R_d^2(\theta) = O_p(N^{-1})$ from the assumption $E\|e_n\|^2 < \infty$.

Regarding S_2, it is deduced that

$$\sup_{0<\theta<1}\sum_{d=1}^{D}\left|R_d(\theta)\int\delta\hat{\psi}_d^{(\kappa)}\right| \le \left\{\sup_{0<\theta<1}\sum_{d=1}^{D}R_d^2(\theta)\right\}^{1/2}\left[\sum_{d=1}^{D}\left\{\int\delta\hat{\psi}_d^{(\kappa)}\right\}^2\right]^{1/2}$$
$$\le O_p(N^{-1/2})\|\delta\|.$$

Next we study the convergence rate of S_1. Define $\Delta_K(t,s)=\widehat{K}^{(\kappa)}(t,s)-K(t,s)$, then it can be deduced that

$$\begin{aligned}\hat{\psi}_d^{(\kappa)}-\psi_d &= \sum_{k\neq d}\{(\hat{\theta}_d^{(\kappa)}-\omega_k)^{-1}-(\omega_d-\omega_k)^{-1}\}\psi_k\int\Delta_K\hat{\psi}_d^{(\kappa)}\psi_k \\ &+ \sum_{k\neq d}(\omega_d-\omega_k)^{-1}\psi_k\int\Delta_K(\hat{\psi}_d^{(\kappa)}-\psi_d)\psi_k \\ &+ \sum_{k\neq d}(\omega_d-\omega_k)^{-1}\psi_k\left(\int\Delta_K\psi_d\psi_k\right)+\psi_d\int(\hat{\psi}_d^{(\kappa)}-\psi_d)\psi_d\end{aligned}$$

and

$$\begin{aligned}\left|\hat{g}_d^{(\kappa)}-g_d\right| &= \left|\left\{\int\delta(t)\hat{\psi}_d^{(\kappa)}(t)dt\right\}^2-\left\{\int\delta(t)\psi_d(t)dt\right\}^2\right| \\ &\le O_p(1)\left|\int\delta(t)\{\hat{\psi}_d^{(\kappa)}(t)-\psi_d(t)\}dt\right|.\end{aligned}$$

When $d=d^*$, since $\langle\psi_{d^*},\psi_{d'}\rangle=0$ for $d'\neq d^*$,

$$\begin{aligned}\left|\hat{g}_{d^*}^{(\kappa)}-g_{d^*}\right| &\le O_p(1)\left|\int\delta(t)\{\hat{\psi}_{d^*}^{(\kappa)}(t)-\psi_{d^*}(t)\}dt\right| \\ &= O_p(1)\|\delta\|\left|\int(\hat{\psi}_{d^*}^{(k)}-\psi_{d^*})\psi_{d^*}dt\right|.\end{aligned}$$

By Assumption 4, $d^*=O((\rho\|\delta\|^2)^{-1/\alpha_\lambda})$, then

$$\left|\int(\hat{\psi}_{d^*}^{(k)}-\psi_{d^*})\psi_{d^*}dt\right| \le \|\hat{\psi}_{d^*}^{(\kappa)}-\psi_{d^*}\| \le (d^*)^{\alpha_\lambda+1}\|\Delta_K\|$$

$$\leq O_p(N^{-r_a})(\rho\|\delta\|^2)^{-(1+1/\alpha_\lambda)}.$$

Consequently,

$$\left|\hat{g}_{d^*}^{(\kappa)} - g_{d^*}\right| \leq O_p(N^{-r_a})\|\delta\|(\rho\|\delta\|^2)^{-(1+1/\alpha_\lambda)}.$$

When $d \neq d^*$,

$$\left|\hat{g}_d^{(\kappa)} - g_d\right| \leq \text{const.} \left|\int \delta(t)\{\hat{\psi}_d^{(\kappa)}(t) - \psi_d(t)\}dt\right|,$$

and

$$\begin{aligned}
&\int \delta(t)\{\hat{\psi}_d^{(\kappa)}(t) - \psi_d(t)\}dt \\
&\quad = \|\delta\| \left[\{(\hat{\theta}_d^{(\kappa)} - \omega_{d^*})^{-1} - (\omega_d - \omega_{d^*})^{-1}\} \int \Delta_K \hat{\psi}_d^{(\kappa)} \psi_{d^*} \right. \\
&\qquad \left. + (\omega_d - \omega_{d^*})^{-1} \int \Delta_K (\hat{\psi}_d^{(\kappa)} - \psi_d)\psi_{d^*} + (\omega_d - \omega_{d^*})^{-1} \left(\int \Delta_K \psi_d \psi_{d^*}\right)\right].
\end{aligned}$$

We deduce that $|\hat{\theta}_d^{(\kappa)} - \omega_{d^*}|^{-1} \leq 2|\omega_d - \omega_{d^*}|^{-1}$ asymptotically almost surely. By the mean value theorem and Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left|\{(\hat{\theta}_d^{(\kappa)} - \omega_{d^*})^{-1} - (\omega_d - \omega_{d^*})^{-1}\} \left(\int \Delta_K \hat{\psi}_d^{(\kappa)} \psi_{d^*}\right)\right| \\
&\leq \text{const.}|\hat{\theta}_d^{(\kappa)} - \omega_d|(\omega_d - \omega_{d^*})^{-2} \left|\int \Delta_K \hat{\psi}_d^{(\kappa)} \psi_{d^*}\right| \\
&\leq \text{const.}\|\Delta_K\|(\omega_d - \omega_{d^*})^{-2} \left\{\left|\int \Delta_K (\hat{\psi}_d^{(\kappa)} - \psi_d)\psi_{d^*}\right| + \left|\int \Delta_K \psi_d \psi_{d^*}\right|\right\}.
\end{aligned}$$

Under Assumption 4, by simple calculus, as $d < d^*$, $\omega_d - \omega_{d^*} \geq \text{const.}d^{-\alpha_\lambda}$, and as $d > d^*$, $\omega_{d^*} - \omega_d \geq \text{const.}(d^*)^{-\alpha_\lambda}$.

$$\sum_{d \neq d^*} |\hat{g}_d^{(\kappa)} - g_d|$$

$$
\begin{aligned}
&\leq \text{const.}\|\delta\| \sum_{d\neq d^*} \left\{ \left| \{(\hat{\theta}_d^{(\kappa)} - \omega_{d^*})^{-1} - (\omega_d - \omega_{d^*})^{-1}\} \left(\int \Delta_K \hat{\psi}_d^{(\kappa)} \psi_{d^*} \right) \right| \right. \\
&\quad \left. + \left| (\omega_d - \omega_{d^*})^{-1} \int \Delta_K (\hat{\psi}_d^{(\kappa)} - \psi_d)\psi_{d^*} \right| + \left| (\omega_d - \omega_{d^*})^{-1} \left(\int \Delta_K \psi_d \psi_{d^*} \right) \right| \right\} \\
&\triangleq \text{const.}\|\delta\|(A_{d^*,1} + A_{d^*,2} + A_{d^*,3}).
\end{aligned}
$$

Next we study the convergence rate of the three summands $\{A_{d^*,i} \colon i = 1,2,3\}$. By Lemma 1,

$$
\left| \int \Delta_K \psi_d \psi_{d^*} \right| \leq O_p(1)\omega_d^{1/2} N^{-r_\delta}, \qquad \left\| \int \Delta_K \psi_{d^*} \right\| \leq O_p(1) N^{-r_\delta}.
$$

Consequently,

$$
\begin{aligned}
A_{d^*,1} &= \sum_{d\neq d^*} \left| \{(\hat{\theta}_d^{(\kappa)} - \omega_{d^*})^{-1} - (\omega_d - \omega_{d^*})^{-1}\} \left(\int \Delta_K \hat{\psi}_d^{(\kappa)} \psi_{d^*} \right) \right| \\
&\leq \text{const.}\|\Delta_K\| \sum_{d\neq d^*} (\omega_d - \omega_{d^*})^{-2} \left\{ \left| \int \Delta_K (\hat{\psi}_d^{(\kappa)} - \psi_d)\psi_{d^*} \right| + \left| \int \Delta_K \psi_d \psi_{d^*} \right| \right\} \\
&\leq \text{const.}\|\Delta_K\| \left\{ \sum_{d\neq d^*} (\omega_d - \omega_{d^*})^{-4} \right\}^{1/2} \\
&\qquad \times \left\{ \sum_{d\neq d^*} \left| \int \Delta_K (\hat{\psi}_d^{(\kappa)} - \psi_d)\psi_{d^*} \right|^2 + \sum_{d\neq d^*} \left| \int \Delta_K \psi_d \psi_{d^*} \right|^2 \right\}^{1/2} \\
&\leq \text{const.}\|\Delta_K\| \left(\sum_{d<d^*} d^{4\alpha_\lambda} + \sum_{d>d^*} (d^*)^{4\alpha_\lambda} \right)^{1/2} \\
&\qquad \times \left\{ \sum_{d\neq d^*} \left| \int \Delta_K (\hat{\psi}_d^{(\kappa)} - \psi_d)\psi_{d^*} \right|^2 + \sum_{d\neq d^*} \left| \int \Delta_K \psi_d \psi_{d^*} \right|^2 \right\}^{1/2} \\
&\leq \text{const.}\|\Delta_K\| D^{1/2} (d^*)^{2\alpha_\lambda}
\end{aligned}
$$

$$\times\left\{\sum_{d\neq d^*}\left\|\int \Delta_K\psi_{d^*}\right\|^2\|\hat{\psi}_d^{(\kappa)}-\psi_d\|^2+\sum_{d\neq d^*}\left|\int \Delta_K\psi_d\psi_{d^*}\right|^2\right\}^{1/2}$$
$$\le O_p(1)\|\Delta_K\|D^{1/2}(d^*)^{2\alpha_\lambda}\left\{N^{-2(r_a+r_\delta)}\sum_{d\neq d^*}d^{2(\alpha_\lambda+1)}+N^{-2r_\delta}\sum_{d\neq d^*}d^{-\alpha_\lambda}\right\}^{1/2}.$$

By the assumption $D^{\alpha_\delta+3/2}/N^{r_a}\to 0$, we have $A_{d^*,1}\le O_p(1)N^{-(r_a+r_\delta)}D^{1/2}(d^*)^{2\alpha_\lambda}$. Regarding $A_{d^*,2}$, we deduce that

$$\begin{aligned}A_{d^*,2}=&\sum_{d\neq d^*}\left|(\omega_d-\omega_{d^*})^{-1}\left\{\int \Delta_K(\hat{\psi}_d^{(\kappa)}-\psi_d)\psi_{d^*}\right\}\right|\\ \le&\left\{\sum_{d\neq d^*}(\omega_d-\omega_{d^*})^{-2}\right\}^{1/2}\left\{\left\|\int \Delta_K\psi_{d^*}\right\|^2\sum_{d\neq d^*}\|\hat{\psi}_d^{(\kappa)}-\psi_d\|^2\right\}^{1/2}\\ =&O_p(1)D^{1/2}(d^*)^{\alpha_\lambda}N^{-r_\delta}\|\Delta_K\|\left\{\sum_{d\neq d^*}d^{2(\alpha_\lambda+1)}\right\}^{1/2}\\ \le&O_p(1)D^{\alpha_\lambda+2}(d^*)^{\alpha_\lambda}N^{-(r_a+r_\delta)}\end{aligned}$$

and

$$\begin{aligned}A_{d^*,3}=&\sum_{d\neq d^*}\left|(\omega_d-\omega_{d^*})^{-1}\left(\int \Delta_K\psi_d\psi_{d^*}\right)\right|\\ \le&\left\{\sum_{d\neq d^*}(\omega_d-\omega_{d^*})^{-2}\right\}^{1/2}\left\{\sum_{d\neq d^*}\left(\int \Delta_K\psi_d\psi_{d^*}\right)^2\right\}^{1/2}\\ \le&O_p(1)\left\{\sum_{d\neq d^*}(\omega_d-\omega_{d^*})^{-2}\right\}^{1/2}\left\{\sum_{d\neq d^*}N^{-2r_\delta}\omega_d\right\}^{1/2}\\ \le&O_p(1)D^{1/2}(d^*)^{\alpha_\lambda}N^{-r_\delta}.\end{aligned}$$

By the assumption $D^{\alpha_\lambda+3/2}N^{-r_a} \to 0$, we have $A_{d^*,2}/A_{d^*,3} \xrightarrow{p} 0$. Therefore,

$$\sum_{d\neq d^*}\left|\int \delta(t)\{\hat{\psi}_d^{(\kappa)}(t)-\psi_d(t)\}dt\right| \leq O_p(N^{-r_\delta})D^{1/2}(d^*)^{\alpha_\lambda},$$

and that leads to,

$$\begin{aligned}\sum_{d\neq d^*}|\hat{g}_d^{(\kappa)}-g_d| &\leq O_p(N^{-r_\delta})D^{1/2}(d^*)^{\alpha_\lambda}\|\delta\| \\ &\leq O_p(N^{-r_\delta})D^{1/2}(\rho\|\delta\|^2)^{-1}\|\delta\|.\end{aligned}$$

Consequently,

$$\begin{aligned}&\sup_{0<\theta<1}\sum_{d=1}^{D}|\hat{f}_d^2-f_d^2| \leq S_1+S_2+S_3 \\ &\quad\leq O_p(N^{-r_\delta})\|\delta\|\max\{D^{1/2}(\rho\|\delta\|^2)^{-1}, N^{r_\delta-r_a}(\rho\|\delta\|^2)^{-(1+1/\alpha_\lambda)}, N^{r_\delta-1/2}\},\end{aligned}$$

and the result follows. □

Lemma 1. *Under Assumption 4,*

$$\left|\int \Delta_K\psi_d\psi_{d^*}\right| \leq O_p(1)\omega_d^{1/2}N^{\max\{\alpha_\ell-\alpha_\kappa-\alpha_\delta-1/2,\alpha_\delta+\beta-1/2+\epsilon\}},$$

$$\left\|\int \Delta_K\psi_{d^*}\right\| \leq O_p(1)N^{\max\{\alpha_\ell-\alpha_\kappa-\alpha_\delta-1/2,\alpha_\delta+\beta-1/2+\epsilon\}},$$

where ϵ is some arbitrary small positive value.

Proof. We only show the first inequality, and the second one can be deduced with the same argument. Since the bias of $\hat{\delta}(t)$ does not lead to loss of information, we assume $E\hat{\delta}(t)=\delta(t)$. Define $\Delta_C(t,s)=\widehat{LC}_{Y,\kappa}(t,s)-LC_Y(t,s)$, and observe that

$$\int \Delta_K\psi_d\psi_{d^*} = \int \rho\{\hat{\delta}(t)\hat{\delta}(s)-\delta(t)\delta(s)\}\psi_d(t)\psi_{d^*}(s)dtds$$

$$+\int \Delta_C(t,s)\psi_d(t)\psi_{d^*}(s)dtds. \tag{A2}$$

For the first part,

$$\left|\int \rho\{\hat{\delta}(t)\hat{\delta}(s)-\delta(t)\delta(s)\}\psi_d\psi_{d^*}dtds\right| = \rho\left|\langle\hat{\delta},\psi_d\rangle\langle\hat{\delta},\psi_{d^*}\rangle\right|.$$

We deduce that $|\langle\hat{\delta},\psi_d\rangle| = |\langle\hat{\delta}-\delta,\psi_d\rangle| \leq \omega_d^{1/2}O_p(N^{-1/2+\epsilon})$, and $|\langle\hat{\delta},\psi_{d^*}\rangle| \leq O_p(N^{\alpha_\delta})$.

Therefore, the first part in (A2) is bounded by

$$\omega_d^{1/2}O_p(N^{\alpha_\delta+\beta-1/2+\epsilon}).$$

For the second part, since $\langle Y,\psi_{d^*}\rangle = 0$,

$$\begin{aligned}|\langle\Delta_C(\psi_d),\psi_{d^*}\rangle| &= \left|\left\langle\left\{\sum_{h=-\ell}^{\ell} W\left(\frac{h}{\ell}\right)\widehat{C}_{Y,h}^{(\kappa)} - \sum_{h=-\infty}^{\infty} C_{Y,h}\right\}(\psi_d),\psi_{d^*}\right\rangle\right| \\ &= \left|\left\langle\left\{\sum_{h=-\ell}^{\ell} W\left(\frac{h}{\ell}\right)\widehat{C}_{Y,h}^{(\kappa)}\right\}(\psi_d),\psi_{d^*}\right\rangle\right|.\end{aligned}$$

It can be shown that $E\|\widehat{C}_{Y,h}^{(\kappa)} - C_{Y,h}\|^2$ has the same convergence rate as $E\|\widetilde{C}_{Y,h}^{(\kappa)} - C_{Y,h}\|^2$, where

$$\widetilde{C}_{Y,h}^{(\kappa)}(t,s) = \frac{1}{N-h}\sum_{n=1}^{N-h}\left\{Y_n^{(\kappa)} - E(Y_n^{(\kappa)})\right\}\left\{Y_{n+h}^{(\kappa)} - E(Y_{n+h}^{(\kappa)})\right\}(t,s).$$

Thus we assume $E\{Y_n^{(\kappa)}\}$ is known and define $Z_n(t) = Y_n^{(\kappa)}(t) - E\{Y_n^{(\kappa)}\}$ and $\hat{\delta}_\kappa(t) = \hat{\delta}(t)/(\|\hat{\delta}\|+\kappa)$, we deduce that

$$\begin{aligned}&\langle\widetilde{C}_{Y,h}^{(\kappa)}(\psi_d),\psi_{d^*}\rangle \\ &= \frac{1}{N}\sum_{n=1}^{N-h}\Big(\langle Z_n,\psi_d\rangle - \langle Z_n,\hat{\delta}_\kappa\rangle\langle\hat{\delta}_\kappa,\psi_d\rangle\Big)\end{aligned}$$

$$\times \left(\langle Z_{n+h}, \psi_{d^*} \rangle - \langle Z_{n+h}, \hat{\delta}_\kappa \rangle \langle \hat{\delta}_\kappa, \psi_{d^*} \rangle \right)$$

$$= \|\psi_d - \langle \hat{\delta}_\kappa, \psi_d \rangle \hat{\delta}_\kappa\| \|\psi_{d^*} - \langle \hat{\delta}_\kappa, \psi_{d^*} \rangle \hat{\delta}_\kappa\|$$

$$\times \frac{1}{N} \sum_{n=1}^{N-h} \left\langle Z_n, \frac{\psi_d - \langle \hat{\delta}_\kappa, \psi_d \rangle \hat{\delta}_\kappa}{\|\psi_d - \langle \hat{\delta}_\kappa, \psi_d \rangle \hat{\delta}_\kappa\|} \right\rangle \left\langle Z_{n+h}, \frac{\psi_{d^*} - \langle \hat{\delta}_\kappa, \psi_{d^*} \rangle \hat{\delta}_\kappa}{\|\psi_{d^*} - \langle \hat{\delta}_\kappa, \psi_{d^*} \rangle \hat{\delta}_\kappa\|} \right\rangle,$$

and by the Cauchy-Schwarz inequality, we have that, asymptotically almost surely,

$$E|\langle \widehat{C}^{(\kappa)}_{Y,h}(\psi_d), \psi_{d^*} \rangle| \le \text{const.} \{E\|\psi_{d^*} - \langle \hat{\delta}_\kappa, \psi_{d^*} \rangle \hat{\delta}_\kappa\|^2\}^{1/2}$$

$$\times \left\{ E \left(\frac{1}{N} \sum_{n=1}^{N-h} \left\langle Z_n, \frac{\psi_d - \langle \hat{\psi}_{d^*}, \psi_d \rangle \hat{\psi}_{d^*}}{\|\psi_d - \langle \hat{\psi}_{d^*}, \psi_d \rangle \hat{\psi}_{d^*}\|} \right\rangle \left\langle Z_{n+h}, \frac{\psi_{d^*} - \langle \hat{\psi}_{d^*}, \psi_{d^*} \rangle \hat{\psi}_{d^*}}{\|\psi_{d^*} - \langle \hat{\psi}_{d^*}, \psi_{d^*} \rangle \hat{\psi}_{d^*}\|} \right\rangle \right)^2 \right\}^{1/2}$$

$$= B_1 \times B_2.$$

As for B_1, clearly,

$$\psi_{d^*} - \langle \hat{\delta}_\kappa, \psi_{d^*} \rangle \hat{\delta}_\kappa = -\langle \hat{\delta}_\kappa - \psi_{d^*}, \psi_{d^*} \rangle (\hat{\delta}_\kappa - \psi_{d^*}) - \langle \hat{\delta}_\kappa - \psi_{d^*}, \psi_{d^*} \rangle \psi_{d^*} - (\hat{\delta}_\kappa - \psi_{d^*}),$$

and we deduce that $B_1 \le \{E\|\hat{\delta}_\kappa - \psi_{d^*}\|^2\}^{1/2} \le \text{const.} \kappa \|\delta\|^{-1}$.

As for B_2,

$$B_2^2 = \left\{ \text{var} \left(\left\langle Z_n, \frac{\psi_d - \langle \hat{\psi}_{d^*}, \psi_d \rangle \hat{\psi}_{d^*}}{\|\psi_d - \langle \hat{\psi}_{d^*}, \psi_d \rangle \hat{\psi}_{d^*}\|} \right\rangle \right) \right\}$$

$$\times \left\{ \text{var} \left(\left\langle Z_{n+h}, \frac{\psi_{d^*} - \langle \hat{\psi}_{d^*}, \psi_{d^*} \rangle \hat{\psi}_{d^*}}{\|\psi_{d^*} - \langle \hat{\psi}_{d^*}, \psi_{d^*} \rangle \hat{\psi}_{d^*}\|} \right\rangle \right) \right\} \times E \left(\frac{1}{N} \sum_{n=1}^{N-h} \xi_{n,d} \xi_{n+h,d^*} \right)^2,$$

where

$$\xi_{n,d} = \left\langle Z_n, \frac{\psi_d - \langle \hat{\psi}_{d^*}, \psi_d \rangle \hat{\psi}_{d^*}}{\|\psi_d - \langle \hat{\psi}_{d^*}, \psi_d \rangle \hat{\psi}_{d^*}\|} \right\rangle \Big/ \left\{ \text{var} \left(\left\langle Z_n, \frac{\psi_d - \langle \hat{\psi}_{d^*}, \psi_d \rangle \hat{\psi}_{d^*}}{\|\psi_d - \langle \hat{\psi}_{d^*}, \psi_d \rangle \hat{\psi}_{d^*}\|} \right\rangle \right) \right\}^{1/2}.$$

By Lemma 4 in Jiao *et al.* (2023),

$$E\left(\frac{1}{N}\sum_{n=1}^{N-h}\xi_{n,d}\xi_{n+h,d^*}\right)^2 \le \text{const.}N^{-1}.$$

In addition, since $\psi_d - \langle\hat{\delta}_\kappa, \psi_d\rangle\hat{\delta}_\kappa \xrightarrow{p} \psi_d$, it can be deduced that

$$\text{var}\left(\left\langle Z_n, \frac{\psi_d - \langle\hat{\psi}_{d^*}, \psi_d\rangle\hat{\psi}_{d^*}}{\|\psi_d - \langle\hat{\psi}_{d^*}, \psi_d\rangle\hat{\psi}_{d^*}\|}\right\rangle\right) \le \text{const.}\omega_d.$$

as $N \to \infty$. Therefore, we conclude $B_2^2 \le \text{const.}\omega_d N^{-1}$. Combining the previous results, $|\langle\widehat{C}_h^{(\kappa)}(\psi_d), \psi_{d^*}\rangle| \le O_p(1)\omega_d^{1/2}\kappa\|\delta\|^{-1}N^{-1/2}$, and

$$\begin{aligned}\left|\left\langle\sum_{h=-\ell}^{\ell} W\left(\frac{h}{\ell}\right)\widehat{C}_h^{(\kappa)}(\psi_d), \psi_{d^*}\right\rangle\right| &\le O_p(1)\omega_d^{1/2}\ell\kappa\|\delta\|^{-1}N^{-1/2} \\ &= O_p(1)\omega_d^{1/2}N^{\alpha_\ell-\alpha_\kappa-\alpha_\delta-1/2}.\end{aligned}$$

Then the result follows. □

Lemma 2. *Under Assumption 3 and H_a, $\|C_{Y,h}\|_2 \le$ const.$h^{-\alpha_c}$ for $h = 1, \ldots, \ell$.*

Proof. It suffices to show that $\|C_{Y,h}\|_2 \le \|C_{X,h}\|_2$ for $h = 1, \ldots, \ell$. Under H_a, assume that $Z_n(t) = X_n(t) - EX_n(t) = \sum_{d\ge1}\zeta_{nd}v_d(t)$, where $\{v_d\colon d \ge 1\}$ are orthonormal basis functions and $v_1 = \delta/\|\delta\|$.

$$\begin{aligned}C_{Y,h}(t,s) &= E\left\{Z_n(t) - \left\langle Z_n, \frac{\delta}{\|\delta\|}\right\rangle\frac{\delta(t)}{\|\delta\|}\right\}\left\{Z_{n+h}(s) - \left\langle Z_{n+h}, \frac{\delta}{\|\delta\|}\right\rangle\frac{\delta(s)}{\|\delta\|}\right\} \\ &= \sum_{d\ge1}\sum_{d'\ge1} E\{\zeta_{nd}\zeta_{n+h,d'}\}v_d(t)v_{d'}(s) - \sum_{d\ge1} E\{\zeta_{nd}\zeta_{n+h,1}\}v_d(t)v_1(s) \\ &\quad - \sum_{d'\ge1} E\{\zeta_{n1}\zeta_{n+h,d'}\}v_1(t)v_{d'}(s) + E\{\zeta_{n1}\zeta_{n+h,1}\}v_1(t)v_1(s)\end{aligned}$$

$$= \sum_{d\geq 2}\sum_{d'\geq 2} E\{\zeta_{nd}\zeta_{n+h,d'}\} v_d(t) v_{d'}(s).$$

Therefore,

$$\|C_{Y,h}\|_2^2 = \sum_{d\geq 2}\sum_{d'\geq 2} E^2\{\zeta_{nd}\zeta_{n+h,d'}\} \leq \sum_{d\geq 1}\sum_{d'\geq 1} E^2\{\zeta_{nd}\zeta_{n+h,d'}\} = \|C_{X,h}\|_2^2.$$

□

Lemma 3. *Under H_a, if the conditions in Theorem 2 hold, then $\|\widehat{LC}_{Y,\kappa} - \widehat{LC}_Y\|_{\mathcal{S}} \leq O_p(1)\ell\kappa\|\delta\|^{-1}$ asymptotically almost surely.*

Proof. Define $Z_n = Y_n - \bar{Y}_n$, $Z_n^{(\kappa)} = Y_n^{(\kappa)} - \bar{Y}_n^{(\kappa)}$, $\mathcal{X}_n = X_n - \bar{X}_n$, and it is deduced that

$$\begin{aligned}
&\|\widehat{C}_{Y,0} - \widehat{C}_{Y,0}^{(\kappa)}\|^2 \\
&= \int \left\{ N^{-1}\sum_{n=1}^{N} Z_n(t)Z_n(s) - N^{-1}\sum_{n=1}^{N} Z_n^{(\kappa)}(t)Z_n^{(\kappa)}(s) \right\}^2 dtds \\
&= \int \left[N^{-1}\sum_{n=1}^{N} \left\{ \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\|} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{\|\hat{\delta}\|^2} - \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{(\|\hat{\delta}\| + \kappa)^2} \right.\right. \\
&\quad - \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\|} \right\rangle \frac{\mathcal{X}_n(t)\hat{\delta}(s) + \hat{\delta}(t)\mathcal{X}_n(s)}{\|\hat{\delta}\|} \\
&\quad \left.\left. + \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle \frac{\mathcal{X}_n(t)\hat{\delta}(s) + \hat{\delta}(t)\mathcal{X}_n(s)}{\|\hat{\delta}\| + \kappa} \right\} \right]^2 dtds \\
&\leq U_N + V_N,
\end{aligned}$$

where

$$
\begin{aligned}
U_N &= \int \left[N^{-1} \sum_{n=1}^{N} \left\{ \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\|} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{\|\hat{\delta}\|^2} - \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{(\|\hat{\delta}\| + \kappa)^2} \right\} \right]^2 dtds \\
&= \int \left[N^{-1} \sum_{n=1}^{N} \left\{ \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\|} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{\|\hat{\delta}\|^2} - \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{\|\hat{\delta}\|^2} \right.\right. \\
&\quad \left.\left. + \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{\|\hat{\delta}\|^2} - \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{(\|\hat{\delta}\| + \kappa)^2} \right\} \right]^2 dtds \\
&\le U_{N1} + U_{N2},
\end{aligned}
$$

and

$$
\begin{aligned}
U_{N1} &= \int \left[N^{-1} \sum_{n=1}^{N} \left\{ \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\|} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{\|\hat{\delta}\|^2} - \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{\|\hat{\delta}\|^2} \right\} \right]^2 dtds \\
&= \int \left[N^{-1} \sum_{n=1}^{N} \left\{ \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\|} - \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\|} + \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle \frac{\hat{\delta}(t)\hat{\delta}(s)}{\|\hat{\delta}\|^2} \right\} \right]^2 dtds \\
&= \left[N^{-1} \sum_{n=1}^{N} \left\{ \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\|} - \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\|} + \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle \right\} \right]^2 \\
&\le O_p(1) \left\| \frac{\hat{\delta}}{\|\hat{\delta}\|} - \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\|^2 \\
&= O_p(1) \left(\frac{\kappa}{\|\hat{\delta}\| + \kappa} \right)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
U_{N2} &= \int \left[N^{-1} \sum_{n=1}^{N} \left\{ \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{\|\hat{\delta}\|^2} - \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle^2 \frac{\hat{\delta}(t)\hat{\delta}(s)}{(\|\hat{\delta}\| + \kappa)^2} \right\} \right]^2 dtds \\
&\le \left\{ N^{-1} \sum_{n=1}^{N} \left\langle \mathcal{X}_n, \frac{\hat{\delta}}{\|\hat{\delta}\| + \kappa} \right\rangle^2 \right\}^2 \int \left\{ \frac{\hat{\delta}(t)\hat{\delta}(s)}{\|\hat{\delta}\|^2} - \frac{\hat{\delta}(t)\hat{\delta}(s)}{(\|\hat{\delta}\| + \kappa)^2} \right\}^2 dtds
\end{aligned}
$$

$$= O_p(1)\left\{\frac{1}{\|\hat{\delta}\|^2} - \frac{1}{(\|\hat{\delta}\|+\kappa)^2}\right\}^2 \|\hat{\delta}\|^4$$
$$\leq O_p(1)\left(\frac{\kappa}{\|\hat{\delta}\|+\kappa}\right)^2.$$

The same argument can be applied to V_N, leading to $V_N \leq O_p(1)\{\kappa/(\kappa+\|\hat{\delta}\|)\}^2$. Similar results also hold for any lagged covariance functions. By Theorem 2, asymptotically almost surely,

$$\frac{\kappa}{\|\hat{\delta}\|+\kappa} \leq \text{const.}\frac{\kappa}{\|\delta\|+\kappa}.$$

Finally we deduce that, asymptotically almost surely,

$$\begin{aligned}
\|\widehat{LC}_{Y,\kappa} - \widehat{LC}_Y\|_{\mathcal{S}} &\leq \left\| \sum_{h=-\ell}^{\ell} W\left(\frac{h}{\ell}\right)(\widehat{C}_{Y,h}^{(\kappa)} - \widehat{C}_{Y,h}) \right\|_{\mathcal{S}} \\
&\leq \sum_{h=-\ell}^{\ell} W\left(\frac{h}{\ell}\right) \left\|\widetilde{C}_{X,h} - C_{X,h}\right\|_{\mathcal{S}} \\
&\leq O_p(1)\|\delta\|^{-1}\kappa \sum_{h=-\ell}^{\ell}\left\{1 - \text{const.}\left(\frac{h}{\ell}\right)^{\alpha}\right\} \\
&\leq O_p(1)\ell\|\delta\|^{-1}\kappa,
\end{aligned}$$

and the proof is finished. □

S2 Box-plots and Additional Tables

The following table is for the results when $\kappa = \kappa_2$ or κ_3.

Table 1: Empirical sizes and powers of the new method (i.i.d., $\kappa = \kappa_2$ and κ_3).

a	N	s	CA ($\kappa = \kappa_2$)				CA ($\kappa = \kappa_3$)			
			ρ_2	ρ_2	ρ_3	ρ_4	ρ_1	ρ_2	ρ_3	ρ_4
0.00	120	0.5	0.056	0.057	0.057	0.057	0.058	0.058	0.058	0.058
		1.5	0.055	0.056	0.056	0.056	0.056	0.056	0.056	0.056
		2.5	0.059	0.058	0.060	0.059	0.059	0.059	0.059	0.059
	200	0.5	0.055	0.055	0.054	0.054	0.056	0.056	0.055	0.055
		1.5	0.059	0.057	0.059	0.058	0.059	0.059	0.059	0.059
		2.5	0.065	0.064	0.064	0.064	0.065	0.064	0.064	0.064
0.25	120	0.5	0.149	0.149	0.149	0.150	0.149	0.149	0.149	0.150
		1.5	0.145	0.146	0.145	0.147	0.149	0.146	0.148	0.146
		2.5	0.148	0.148	0.149	0.149	0.149	0.149	0.148	0.148
	200	0.5	0.279	0.274	0.269	0.270	0.280	0.272	0.271	0.272
		1.5	0.298	0.294	0.291	0.292	0.298	0.295	0.292	0.293
		2.5	0.301	0.298	0.299	0.299	0.302	0.300	0.298	0.299
0.30	120	0.5	0.226	0.225	0.224	0.224	0.227	0.227	0.225	0.226
		1.5	0.226	0.226	0.225	0.226	0.226	0.226	0.226	0.224
		2.5	0.233	0.229	0.230	0.228	0.231	0.231	0.231	0.230
	200	0.5	0.538	0.524	0.520	0.525	0.535	0.523	0.519	0.524
		1.5	0.550	0.545	0.541	0.538	0.551	0.546	0.541	0.540
		2.5	0.527	0.529	0.539	0.530	0.529	0.529	0.530	0.528
0.35	120	0.5	0.375	0.371	0.371	0.369	0.374	0.373	0.372	0.371
		1.5	0.370	0.368	0.368	0.370	0.369	0.367	0.370	0.368
		2.5	0.353	0.352	0.352	0.352	0.354	0.355	0.352	0.353
	200	0.5	0.928	0.927	0.929	0.926	0.929	0.927	0.930	0.927
		1.5	0.882	0.872	0.870	0.870	0.881	0.873	0.871	0.870
		2.5	0.788	0.788	0.789	0.788	0.786	0.788	0.788	0.788

In Figure 1 and 3, subfigures (a1)–(a3) correspond to the cases of $a = 0.55$, subfigures (b1)–(b3) correspond to the cases of $a = 0.65$, and subfigures (c1)–(c3) correspond to the cases of $a = 0.75$. The first column corresponds to the cases of $s = 2$, the second column corresponds to the cases of $s = 3$, and the third column corresponds to the cases of $s = 4$.

In Figure 2 and 4, subfigures (a1)–(a3) correspond to the cases of $a = 0.25$, subfigures (b1)–(b3) correspond to the cases of $a = 0.3$, and subfigures (c1)–(c3) correspond to the cases of $a = 0.35$. The first column corresponds to the cases of $s = 0.5$, the second column corresponds to the cases of $s = 1.5$, and the third column corresponds to the cases of $s = 2.5$.

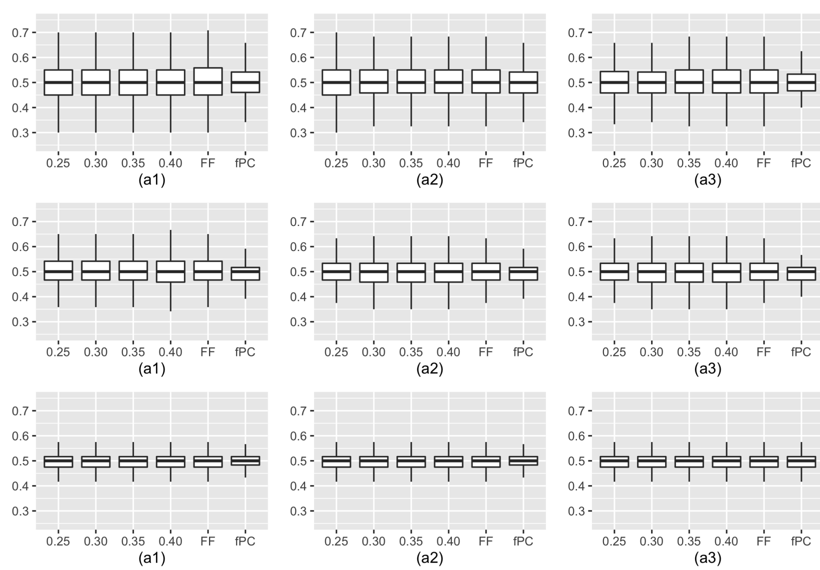

Figure 1: Box-plots of the detected change-points (N=120, FMA).

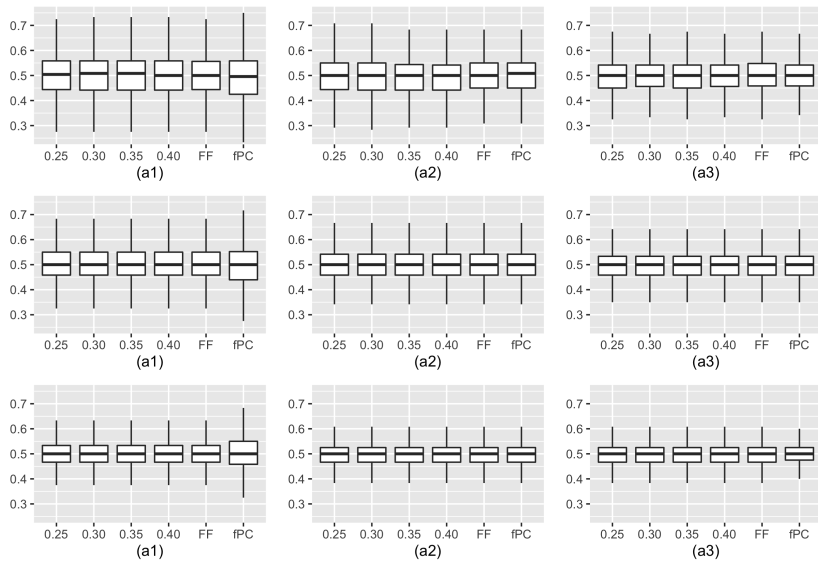

Figure 2: Box-plots of the detected change-points (N=120, i.i.d.).

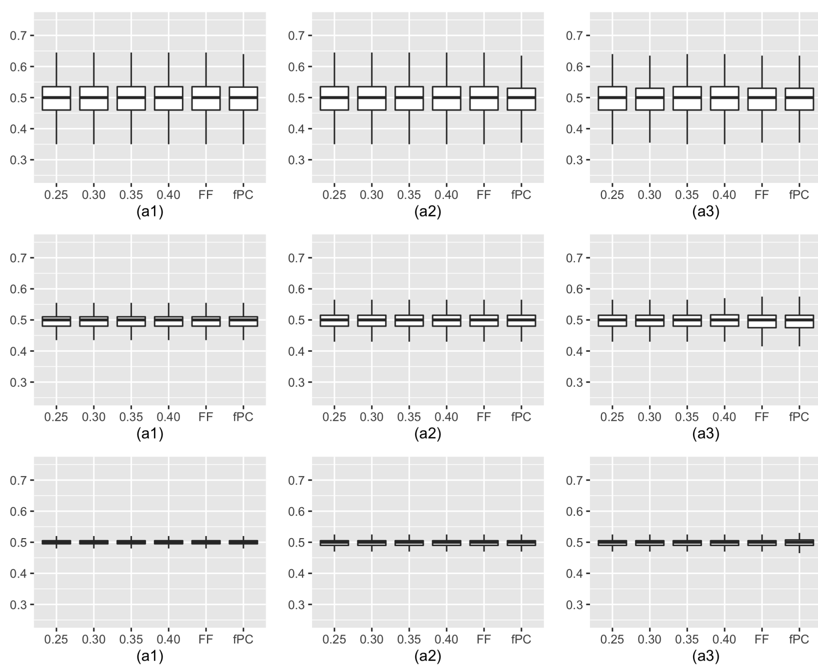

Figure 3: Box-plots of the detected change-points (N200, FMA).

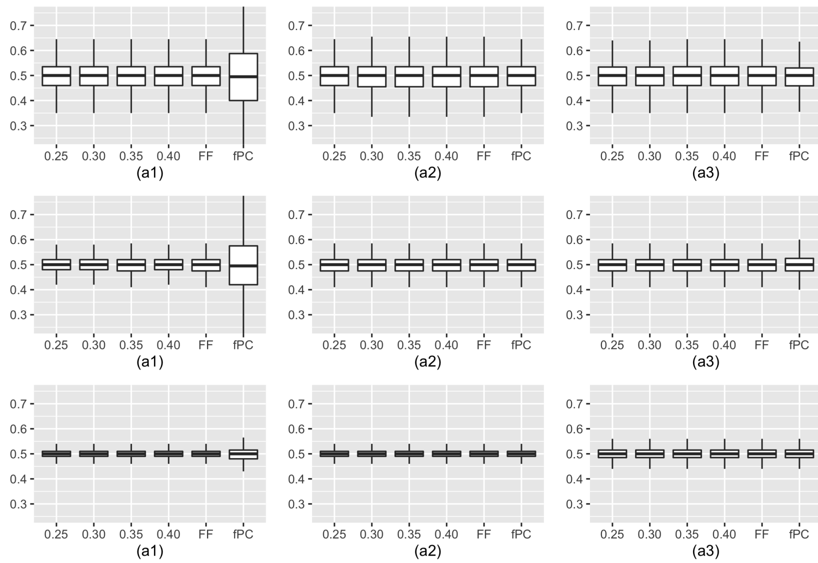

Figure 4: Box-plots of the detected change-points (N=200, i.i.d.).

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