

HYPOTHESES TESTING OF FUNCTIONAL PRINCIPAL COMPONENTS

Zening Song¹, Lijian Yang^{2*}, Yuanyuan Zhang³

¹*Nankai University*, ²*Tsinghua University*, ³*Soochow University*

Supplementary Material

This supplement provides technical lemmas and detailed proofs of the main asymptotic results. Throughout this section, \mathcal{O}_p (or o_p) denotes a sequence of random variables of certain order in probability. For instance, $o_p(n^{-1/2})$ means a smaller order than $n^{-1/2}$ in probability, and by $\mathcal{O}_{a.s.}$ (or $o_{a.s.}$) almost surely \mathcal{O} (or o).

For any \mathcal{L}^2 integrable functions $\phi(x)$ and $\varphi(x)$, $x \in [0, 1]$, denote by $\langle \phi, \varphi \rangle = \int_{[0,1]} \phi(x) \varphi(x) dx$ their theoretical inner product, and $\langle \phi, \varphi \rangle_N = N^{-1} \sum_{j=1}^N \phi(j/N) \varphi(j/N)$ the empirical inner product. The related theoretical and empirical norms are $\|\phi\|_2^2 = \langle \phi, \phi \rangle$, $\|\phi\|_{2,N}^2 = \langle \phi, \phi \rangle_N$. For any vector $\mathbf{a} = (a_1, \dots, a_n)^\top \in \mathbb{R}^n$, denote the norm $\|\mathbf{a}\|_r = (|a_1|^r + \dots + |a_n|^r)^{1/r}$, $1 \leq r < +\infty$, $\|\mathbf{a}\|_\infty = \max(|a_1|, \dots, |a_n|)$. For any matrix $\mathbf{A} = (a_{ij})_{i=1, j=1}^{m,n}$, denote its L_r norm as $\|\mathbf{A}\|_r = \max_{\mathbf{a} \in \mathbb{R}^n, \mathbf{a} \neq \mathbf{0}} \|\mathbf{A}\mathbf{a}\|_r \|\mathbf{a}\|_r^{-1}$, for $r < +\infty$ and $\|\mathbf{A}\|_r = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$, for $r = \infty$. For any random variable \mathbf{X} , if it is L_p -integrable, denotes its L_p norm as $\|\mathbf{X}\|_p = (\mathbb{E} |\mathbf{X}|^p)^{1/p}$.

S.1 Technical Lemmas

Lemma S.1. *[Theorem 2.6.7 of Csörgő and Révész (1981)] Suppose that $\{\xi_i, i \in \mathbb{N}_+\}$ are i.i.d. on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(\xi_1) = 0$, $\mathbb{E}(\xi_1^2) = 1$, and $H(x) > 0$ ($x \geq 0$) is an increasing continuous function such that $x^{-2-\gamma}H(x)$ is increasing for some $\gamma > 0$ and $x^{-1} \log H(x)$ is decreasing with $\mathbb{E}H(|\xi_1|) < \infty$. Then there exist constants $C_1, C_2, a > 0$ which depend only on the distribution of ξ_1 , a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ on which are $\{\tilde{\xi}_i, i \in \mathbb{N}_+\} \stackrel{D}{=} \{\xi_i, i \in \mathbb{N}_+\}$ and a Brownian motion $W(t)$, such that for $\{x_n\}_{n=1}^\infty$ satisfying $H^{-1}(n) < x_n < C_1(n \log n)^{1/2}$, $\tilde{\mathbb{P}} \left\{ \max_{1 \leq m \leq n} |\tilde{S}_m - W(m)| > x_n \right\} \leq C_2 n \{H(ax_n)\}^{-1}$ where $\tilde{S}_m = \sum_{i=1}^m \tilde{\xi}_i$.*

For simplicity, we abuse notation à la Csörgő and Révész (1981), by abandoning the precise statement in Lemma S.1 about partial sum \tilde{S}_m on new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and stating instead the following for $S_m = \sum_{i=1}^m \xi_i$ on the original $(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathbb{P} \left\{ \max_{1 \leq m \leq n} |S_m - W(m)| > x_n \right\} \leq C_2 n \{H(ax_n)\}^{-1}.$$

Lemma S.2. *The distribution function of S is continuous and strictly increasing.*

PROOF. As $\sum_{1 \leq k < k' < \infty} \lambda_k \lambda_{k'} < \infty$, Zolotarev (1961) and Hoeffding (1964) ensure that S has a Lebesgue density function over $(0, +\infty)$, so its

distribution function is continuous. In addition, equations (2) and (4) of Christoph et al. (1996) provide that the density function of S is positive on $(0, +\infty)$, sandwiched by constant multiples of the density function of any finite partial sum. Thus the distribution function of S is strictly increasing. The proof is completed. \square

Lemma S.3. *[Lemma A.12 of Wang et al. (2020)] Let $W_i \sim N(0, \sigma_i^2)$, $\sigma_i > 0$, $i = 1, \dots, n$, then for $n > 2, a > 2$,*

$$\mathbb{P}\left(\max_{1 \leq i \leq n} |W_i/\sigma_i| > a\sqrt{\log n}\right) < 2n^{1-a^2/2}.$$

Hence, $(\max_{1 \leq i \leq n} |W_i|) / (\max_{1 \leq i \leq n} \sigma_i) \leq \max_{1 \leq i \leq n} |W_i/\sigma_i| = \mathcal{O}_{a.s.}(\sqrt{\log n})$.

Consequently, for n random variables $\chi_i \sim \chi^2(1)$, $\max_{1 \leq i \leq n} \chi_i = \mathcal{O}_{a.s.}(\log n)$.

Lemma S.4. *[Theorem 1.2 of Bosq (1998)] Let X_1, \dots, X_n be independent zero-mean real-valued random variables and let $S_n = \sum_{i=1}^n X_i$. If there exists $c > 0$ such that*

$$\mathbb{E}|X_i|^k \leq c^{k-2} k! \mathbb{E}X_i^2 < +\infty, i = 1, \dots, n, k = 3, 4, \dots$$

(Cramér's conditions) then

$$\mathbb{P}(|S_n| \geq t) \leq 2 \exp\left(-\frac{t^2}{4 \sum_{i=1}^n \mathbb{E}X_i^2 + 2ct}\right), t > 0$$

(Bernstein's inequality).

Lemma S.5. For κ_n in (2.22), let $R_{kk',n}$, $1 \leq k < k', k' > \kappa_n$ be an array of nonnegative-valued random variables with $\max_{1 \leq k < k', k' > \kappa_n} \mathbb{E}R_{kk',n} \leq M$, then

$$\sum_{1 \leq k < k', k' > \kappa_n} \lambda_k \lambda_{k'} R_{kk',n} = o_p(1).$$

PROOF. One simply takes expectation of the non-negative random variable

$$\sum_{1 \leq k < k', k' > \kappa_n} \lambda_k \lambda_{k'} R_{kk',n}$$

$$\begin{aligned} \mathbb{E} \sum_{1 \leq k < k', k' > \kappa_n} \lambda_k \lambda_{k'} R_{kk',n} &= \sum_{1 \leq k < k', k' > \kappa_n} \lambda_k \lambda_{k'} \mathbb{E}R_{kk',n} \leq \\ &M \sum_{1 \leq k < k', k' > \kappa_n} \lambda_k \lambda_{k'} \leq M \sum_{k=1}^{\infty} \lambda_k \sum_{k'=\kappa_n+1}^{\infty} \lambda_{k'} = o(1). \end{aligned}$$

By (2.22), $\kappa_n \rightarrow \infty$, so $\sum_{k'=\kappa_n+1}^{\infty} \lambda_{k'} \rightarrow 0$, and hence $\sum_{1 \leq k < k', k' > \kappa_n} \lambda_k \lambda_{k'} R_{kk',n} = o_p(1)$. \square

Lemma S.6. Under Assumptions (A1), (B1), there exists a $\mathcal{C}[0, 1]$ -valued mean zero Gaussian random variable \mathcal{N}_ξ such that as $n \rightarrow \infty$,

$$n^{1/2} \left(n^{-1} \sum_{i=1}^n \xi_i \right) \xrightarrow{D} \mathcal{N}_\xi, \left\| n^{-1} \sum_{i=1}^n \xi_i \right\|_\infty = \mathcal{O}_p(n^{-1/2}).$$

Under Assumptions (A1), (B1)–(B2), as $n \rightarrow \infty$

$$\max_{1 \leq i \leq n} \left\| \hat{\xi}_i - \xi_i \right\|_\infty = \mathcal{O}_p(n^{-1/2}). \quad (\text{S.1})$$

PROOF. Assumptions (A1) and (B1) ensure that ξ_1 is a mean zero $\mathcal{C}[0, 1]$ random variable, whose Lipschitz seminorm has finite second moment, so according to Theorem 2.8 of Bosq (2000), $n^{1/2} (n^{-1} \sum_{i=1}^n \xi_i)$ as sample mean of i.i.d. $\mathcal{C}[0, 1]$ -valued mean zero random variables converge in distribution

to a $\mathcal{C}[0, 1]$ -valued mean zero Gaussian random variable \mathcal{N}_ξ . Continuous Mapping Theorem then implies that $\|n^{1/2}(n^{-1}\sum_{i=1}^n \xi_i)\|_\infty \xrightarrow{D} \|\mathcal{N}_\xi\|_\infty$, hence $\|n^{1/2}(n^{-1}\sum_{i=1}^n \xi_i)\|_\infty = \mathcal{O}_p(1)$. This, together with Assumption (B2) lead to (S.39). \square

Lemma S.7. (i) If $\{\kappa_n\}_{n=1}^\infty$ satisfy (2.22) and positive sequences $\{\rho_{n,N}\}_{n=1}^\infty$ $\kappa_n^2 n^{1/2} \rho_{n,N} \log^{1/2} n \rightarrow 0$, then

$$\kappa_n = o(n^{1/4}), \rho_{n,N} = o(n^{-1/2}), \quad (\text{S.2})$$

$$\kappa_n^2 n \rho_{n,N}^2 \rightarrow 0. \quad (\text{S.3})$$

(ii) For positive integers $\{\kappa_n\}_{n=1}^\infty$ with $\kappa_n^2 = o(n^{1/2})$ and $r_1 > 6$, there exist constants $\alpha_1 \in (3/2r_1, 1/2), \alpha_2 \in (3/r_1, 1/2)$, such that for $D_n = n^{\alpha_1}, D'_n = n^{\alpha_2}$, the following are fulfilled:

$$D_n^{-(r_1-1)} n^{1/2} \log^{-1/2} n \rightarrow 0, \quad (\text{S.4})$$

$$\sum_{n=1}^\infty \kappa_n^2 D_n^{-r_1} < \infty, \quad (\text{S.5})$$

$$D_n \sqrt{\log n} n^{-1/2} \rightarrow 0, \quad (\text{S.6})$$

$$D_n^{-(r_1/2-1)} n^{1/2} \log^{-1/2} n \rightarrow 0, \quad (\text{S.7})$$

$$\sum_{n=1}^{\infty} \kappa_n^2 D_n'^{-r_1/2} < \infty, \quad (\text{S.8})$$

$$D_n' \sqrt{\log nn}^{-1/2} \rightarrow 0. \quad (\text{S.9})$$

PROOF. (i) $\kappa_n^2 n^{-1/2} \log^{3/2} n \rightarrow 0$ leads to $\kappa_n^2 n^{-1/2} \rightarrow 0$, hence $\kappa_n = o(n^{1/4})$.

Likewise, $\rho_{n,N} = o(n^{-1/2})$ since $\kappa_n \rightarrow \infty$ and $\kappa_n^2 \rho_{n,N} n^{1/2} \log^{1/2} n \rightarrow 0$, thus

(S.2) holds. Under the constraints of κ_n and $\rho_{n,N}$,

$$\kappa_n^2 n \rho_{n,N}^2 = \left(\kappa_n^2 \rho_{n,N} n^{1/2} \log^{1/2} n \right)^2 \times \kappa_n^{-2} \times \log^{-1} n \rightarrow 0,$$

thus (S.3) holds.

(ii) Since $\alpha_1 > 3/2r_1$ and $r_1 > 3/2$ by (2.32), $\alpha_1 > 1/(2r_1 - 2)$, then $-\alpha_1(r_1 - 1) + 1/2 < 0$. Thus,

$$D_n^{-(r_1-1)} n^{1/2} \log^{-1/2} n = n^{-\alpha_1(r_1-1)+1/2} \log^{-1/2} n \rightarrow 0,$$

(S.4) holds. Similarly, since $\alpha_2 > 3/r_1$ and $r_1 > 3$ by (2.32), (S.7) can be proved.

Since $\alpha_1 > 3/2r_1$, $D_n^{-r_1} = n^{-\alpha_1 r_1} = o(n^{-3/2})$. Thus $\kappa_n^2 = o(n^{1/2})$ implies $\kappa_n^2 D_n^{-r_1} = o(n^{-1})$, so $\sum_{n=1}^{\infty} \kappa_n^2 D_n^{-r_1} < \infty$, which yields (S.5). (S.8) can be proved in a similar way.

Note that $\alpha_1 < 1/2, \alpha_2 < 1/2$, (S.6) and (S.9) hold immediately. The proof of Lemma S.7 is finished. \square

Lemma S.8. *Under Assumptions (A1), (B2), if $\max_{1 \leq k < \infty} \mathbb{E} |\xi_{1k}|^{r_1} < \infty$ for some $r_1 > 6$, then*

$$\max_{1 \leq k < k' \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{ik} \xi_{ik'} \right| = \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right), \quad (\text{S.10})$$

$$\max_{1 \leq k \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{ik}^2 - 1 \right| = \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right), \quad (\text{S.11})$$

$$\max_{1 \leq k \leq \kappa_n} |\bar{\xi}_{\cdot k}| = \max_{1 \leq k \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{ik} \right| = \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right), \quad (\text{S.12})$$

$$\max_{1 \leq k \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n |\xi_{ik}| - \mu_k \right| = \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right), \quad (\text{S.13})$$

where $\mu_k = \mathbb{E} |\xi_{ik}| \leq 1$.

PROOF. For any $1 \leq k < k' \leq \kappa_n$, denote $\xi_{i,kk'} = \xi_{ik} \xi_{ik'}$ and decompose it into tail part, truncated mean and truncated part $\xi_{i,kk'} = \xi_{i,1,kk'}^{D_n} + \xi_{i,2,kk'}^{D_n} + \mu_{kk'}^{D_n}$,

$$\begin{aligned} \xi_{i,1,kk'}^{D_n} &= \xi_{i,kk'} I_{\{|\xi_{i,kk'}| > D_n\}}, \mu_{kk'}^{D_n} = \mathbb{E} \left[\xi_{i,kk'} I_{\{|\xi_{i,kk'}| \leq D_n\}} \right], \\ \xi_{i,2,kk'}^{D_n} &= \xi_{i,kk'} I_{\{|\xi_{i,kk'}| \leq D_n\}} - \mu_{kk'}^{D_n}, \end{aligned}$$

where $D_n = n^{\alpha_1}$ satisfying (S.4)–(S.6) in Lemma S.7.

It is straightforward to verify that $\mu_{kk'}^{D_n} = -\mathbb{E} \left[\xi_{i,kk'} I_{\{|\xi_{i,kk'}| > D_n\}} \right]$, hence

$$\begin{aligned} \max_{1 \leq k < k' \leq \kappa_n} |\mu_{kk'}^{D_n}| &\leq \max_{1 \leq k < k' \leq \kappa_n} \left| \mathbb{E} \left[\xi_{i,kk'} I_{\{|\xi_{i,kk'}| > D_n\}} \right] \right| \\ &\leq \max_{1 \leq k < k' \leq \kappa_n} \mathbb{E} \xi_{i,kk'}^{r_1} D_n^{-(r_1-1)} \leq \left(\max_{1 \leq k \leq \kappa_n} \mathbb{E} \xi_{ik}^{r_1} \right)^2 D_n^{-(r_1-1)}, \end{aligned}$$

consequently the truncated mean is negligible

$$\max_{1 \leq k < k' \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \mu_{kk'}^{D_n} \right| \leq \max_{1 \leq k < k' \leq \kappa_n} |\mu_{kk'}^{D_n}| \leq \left(\max_{1 \leq k \leq \kappa_n} \mathbb{E} \xi_{ik}^{r_1} \right)^2 D_n^{-(r_1-1)} = o\left(n^{-1/2} \log^{1/2} n\right) \quad (\text{S.14})$$

by (S.4) that $D_n^{-(r_1-1)} = o\left(n^{-1/2} \log^{1/2} n\right)$ and the assumption that $\max_{1 \leq k < \infty} \mathbb{E} |\xi_{1k}|^{r_1} < \infty$.

Next we show that tail part is also negligible. Note that

$$\max_{1 \leq k < k' \leq \kappa_n} \mathbb{P} \{ |\xi_{n,kk'}| > D_n \} \leq \left(\max_{1 \leq k \leq \kappa_n} \mathbb{E} \xi_{nk}^{r_1} \right)^2 D_n^{-r_1},$$

hence by (S.5) that $\sum_{n=1}^{\infty} \kappa_n^2 D_n^{-r_1} < \infty$ and by the assumption that $\max_{1 \leq k < \infty} \mathbb{E} |\xi_{1k}|^{r_1} < \infty$,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P} \left\{ \max_{1 \leq k < k' \leq \kappa_n} |\xi_{n,kk'}| > D_n \right\} &\leq \sum_{n=1}^{\infty} \kappa_n^2 \left(\max_{1 \leq k \leq \kappa_n} \mathbb{E} \xi_{nk}^{r_1} \right)^2 D_n^{-r_1} \\ &\leq \left(\max_{1 \leq k < \infty} \mathbb{E} \xi_{1k}^{r_1} \right)^2 \sum_{n=1}^{\infty} \kappa_n^2 D_n^{-r_1} < \infty. \end{aligned}$$

Borel–Cantelli lemma then implies that

$$\mathbb{P} \left\{ \omega \mid \exists N(\omega), \max_{1 \leq k < k' \leq \kappa_n} |\xi_{n,kk'}(\omega)| \leq D_n \text{ for } n > N(\omega) \right\} = 1,$$

which, because $\{D_n\}_{n=1}^\infty$ is monotone increasing, entails that

$$\mathbb{P} \left\{ \omega \mid \exists N_1(\omega), \max_{1 \leq k < k' \leq \kappa_n} |\xi_{i,kk'}(\omega)| \leq D_n, 1 \leq i \leq n \text{ for } n > N_1(\omega) \right\} = 1,$$

hence

$$\mathbb{P} \left\{ \omega \mid \exists N_1(\omega), \xi_{i,1,kk'}^{D_n}(\omega) = 0, 1 \leq k < k' \leq \kappa_n, 1 \leq i \leq n \text{ for } n > N_1(\omega) \right\} = 1,$$

which further implies that

$$\mathbb{P} \left\{ \omega \mid \exists N_1(\omega), \max_{1 \leq k < k' \leq \kappa_n} |\xi_{i,1,kk'}^{D_n}(\omega)| = 0, 1 \leq i \leq n \text{ for } n > N_1(\omega) \right\} = 1.$$

One therefore has for any $l > 1/2$,

$$\max_{1 \leq k < k' \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{i,1,kk'}^{D_n} \right| \leq n^{-1} \sum_{i=1}^n \max_{1 \leq k < k' \leq \kappa_n} |\xi_{i,1,kk'}^{D_n}| = \mathcal{O}_{a.s.}(n^{-l}) = o_{a.s.}(n^{-1/2} \log^{1/2} n). \quad (\text{S.15})$$

Next, note that $\mathbb{E}(\xi_{i,2,kk'}^{D_n}) = 0$, one has

$$\text{Var}(\xi_{i,2,kk'}^{D_n}) = \mathbb{E}(\xi_{i,2,kk'}^2) - \mathbb{E}(\xi_{i,2,kk'}^2 I_{\{|\xi_{i,2,kk'}| > D_n\}}) - \left(\mathbb{E}[\xi_{i,2,kk'} I_{\{|\xi_{i,2,kk'}| \leq D_n\}}] \right)^2,$$

thus

$$\max_{1 \leq k < k' \leq \kappa_n} |\text{Var}(\xi_{i,2,kk'}^{D_n}) - 1| \leq C(D_n^{-(r_1-2)} + D_n^{-2(r_1-1)}).$$

Notice that

$$\mathbb{E} |\xi_{i,2,kk'}^{D_n}|^l \leq 2^{l-2} D_n^{l-2} \mathbb{E} |\xi_{i,2,kk'}^{D_n}|^2, \quad l \geq 2,$$

which implies that $\{\xi_{i,2,kk'}^{D_n}\}_{i=1}^n$ satisfies Cramér's condition with constant

$2D_n$. By Bernstein inequality in Lemma S.4,

$$\mathbb{P} \left\{ \left| \frac{\sum_{i=1}^n \xi_{i,2,kk'}^{D_n}}{\sqrt{\sum_{i=1}^n \mathbb{E} |\xi_{i,2,kk'}^{D_n}|^2}} \right| > 8\sqrt{\log n} \right\}$$

$$\leq 2 \exp \left\{ -\frac{64 \log n}{4} \frac{1}{1 + 8\sqrt{\log n} D_n \left(n \mathbb{E} |\xi_{i,2,kk'}^{D_n}|^2 \right)^{-1/2}} \right\}$$

which by (S.6) that $D_n \sqrt{\log n} n^{-1/2} \rightarrow 0$, is bounded for large enough n by

$$2 \exp \left\{ -16 \log n \frac{1}{1 + 8C D_n \sqrt{\log n} n^{-1/2}} \right\} \leq 2 \exp \left\{ -\frac{16 \log n}{2} \right\} \leq 2n^{-8}.$$

Hence, for $n > N_0$ where N_0 is a large enough constant,

$$\begin{aligned} & \mathbb{P} \left\{ \max_{1 \leq k < k' \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{i,2,kk'}^{D_n} \right| > 8\sqrt{n^{-1} \log n} \right\} \\ & \leq \sum_{1 \leq k < k' \leq \kappa_n} \mathbb{P} \left\{ \left| n^{-1} \sum_{i=1}^n \xi_{i,2,kk'}^{D_n} \right| > 8\sqrt{n^{-1} \log n} \right\} \leq 2n^{-8} \kappa_n^2 \leq n^{-6}, \end{aligned}$$

by (S.2) in Lemma S.7. Then

$$\sum_{n=N_0}^{\infty} \mathbb{P} \left\{ \max_{1 \leq k < k' \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{i,2,kk'}^{D_n} \right| > 8\sqrt{n^{-1} \log n} \right\} \leq \sum_{n=N_0}^{\infty} n^{-6} < \infty.$$

Borel–Cantelli’s lemma entails that

$$\max_{1 \leq k < k' \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{i,2,kk'}^{D_n} \right| = \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right). \quad (\text{S.16})$$

According to (S.14), (S.15) and (S.16), putting the truncated mean, the

tail and truncated parts together leads to $\max_{1 \leq k < k' \leq \kappa_n} |n^{-1} \sum_{i=1}^n \xi_{ik} \xi_{ik'}| = \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right)$. The proof of (S.10) is complete.

For any $1 \leq k \leq \kappa_n$, decompose ξ_{ik}^2 as $\xi_{ik}^2 = \xi_{ik,1}^{D'_n} + \xi_{ik,2}^{D'_n} + \mu_k^{D'_n}$,

$$\xi_{ik,1}^{D'_n} = \xi_{ik}^2 I_{\{|\xi_{ik}^2| > D'_n\}}, \mu_k^{D'_n} = \mathbb{E} \left[\xi_{ik}^2 I_{\{|\xi_{ik}^2| \leq D'_n\}} \right],$$

$$\xi_{ik,2}^{D'_n} = \xi_{ik}^2 I_{\{|\xi_{ik}^2| \leq D'_n\}} - \mu_k^{D'_n},$$

where $D'_n = n^{\alpha_2}$ satisfying (S.7)-(S.9) in Lemma S.7.

Following a similar procedure as shown in (S.14),

$$\max_{1 \leq k \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \mu_k^{D'_n} \right| \leq \max_{1 \leq k \leq \kappa_n} \left| \mu_k^{D'_n} \right| \leq \max_{1 \leq k \leq \kappa_n} \mathbb{E} \xi_{ik}^{r_1} D_n'^{-(r_1/2-1)} = o\left(n^{-1/2} \log^{1/2} n\right) \quad (\text{S.17})$$

by (S.7) that $D_n'^{-(r_1/2-1)} = o\left(n^{-1/2} \log^{1/2} n\right)$ and the assumption that $\max_{1 \leq k < \infty} \mathbb{E} |\xi_{1k}|^{r_1} < \infty$.

For the tail part, note that

$$\max_{1 \leq k \leq \kappa_n} \mathbb{P} \left\{ |\xi_{nk}^2| > D'_n \right\} \leq \max_{1 \leq k \leq \kappa_n} \mathbb{E} \xi_{nk}^{r_1} D_n'^{-r_1/2},$$

hence

$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \max_{1 \leq k \leq \kappa_n} |\xi_{nk}^2| > D'_n \right\} \leq \sum_{n=1}^{\infty} \kappa_n^2 \max_{1 \leq k \leq \kappa_n} \mathbb{E} \xi_{nk}^{r_1} D_n'^{-r_1/2} < \infty,$$

by (S.8) that $\sum_{n=1}^{\infty} \kappa_n^2 D_n'^{-r_1/2} < \infty$ and by the assumption that $\max_{1 \leq k < \infty} \mathbb{E} |\xi_{1k}|^{r_1} < \infty$. Borel–Cantelli lemma and D'_n 's monotone increasing property entails that

$$\mathbb{P} \left\{ \omega \mid \exists N(\omega), \max_{1 \leq k \leq \kappa_n} |\xi_{nk}^2(\omega)| \leq D'_n \text{ for } n > N(\omega) \right\} = 1,$$

$$\mathbb{P} \left\{ \omega \mid \exists N_1(\omega), \max_{1 \leq k \leq \kappa_n} |\xi_{nk}^2(\omega)| \leq D'_n, 1 \leq i \leq n \text{ for } n > N_1(\omega) \right\} = 1.$$

Then similar to the proof of (S.15), one obtains that for any $l > 1/2$,

$$\max_{1 \leq k \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{ik,1}^{D'_n} \right| \leq n^{-1} \sum_{i=1}^n \max_{1 \leq k \leq \kappa_n} \left| \xi_{ik,1}^{D'_n} \right| = \mathcal{O}_{a.s.}(n^{-l}) = o_{a.s.}\left(n^{-1/2} \log^{1/2} n\right). \quad (\text{S.18})$$

As for the truncated part, note that $\mathbb{E} \left(\xi_{ik,2}^{D'_n} \right) = 0$,

$$\text{Var} \left(\xi_{ik,2}^{D'_n} \right) = \mathbb{E} \left(\xi_{ik}^4 \right) - \mathbb{E} \left(\xi_{ik}^4 I_{\{|\xi_{ik}^2| > D'_n\}} \right) - \left(\mathbb{E} \left[\xi_{ik}^2 I_{\{|\xi_{ik}^2| \leq D'_n\}} \right] \right)^2,$$

thus

$$\max_{1 \leq k \leq \kappa_n} \left| \text{Var} \left(\xi_{ik,2}^{D'_n} \right) - \mathbb{E} \xi_{ik}^4 \right| \leq C \left(D_n^{-(r_1/2-2)} + D_n^{-(r_1-2)} \right).$$

Notice that

$$\mathbb{E} \left| u_{i,2,kk'}^{D_n} \right|^l \leq 2^{l-2} D_n^{l-2} \mathbb{E} \left| u_{i,2,kk'}^{D_n} \right|^2, \quad l \geq 2,$$

which implies that $\left\{ \xi_{ik,2}^{D'_n} \right\}_{i=1}^n$ satisfies Cramér's condition with constant

$2D'_n$. By Bernstein inequality in Lemma S.4,

$$\begin{aligned} & \mathbb{P} \left\{ \left| \frac{\sum_{i=1}^n \xi_{ik,2}^{D'_n}}{\sqrt{\sum_{i=1}^n \mathbb{E} \left| \xi_{ik,2}^{D'_n} \right|^2}} \right| > 8\sqrt{\log n} \right\} \\ & \leq 2 \exp \left\{ -\frac{64 \log n}{4} \frac{1}{1 + 8\sqrt{\log n} D'_n \left(n \mathbb{E} \left| \xi_{ik,2}^{D'_n} \right|^2 \right)^{-1/2}} \right\} \end{aligned}$$

which by (S.9) that $D'_n \sqrt{\log n} n^{-1/2} \rightarrow 0$, is bounded by

$$2 \exp \left\{ -16 \log n \frac{1}{1 + 8C \left(\mathbb{E} \xi_{ik}^4 \right)^{-1/2} D'_n \sqrt{\log n} n^{-1/2}} \right\} \leq 2 \exp \left\{ -\frac{16 \log n}{2} \right\} \leq 2n^{-8}$$

for $n > N_0$ where N_0 is a large enough constant. Hence, according to (S.2),

$$\begin{aligned} & \sum_{n=N_0}^{\infty} \mathbb{P} \left\{ \max_{1 \leq k \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{ik,2}^{D'_n} \right| > 8\sqrt{n^{-1} \log n} \right\} \\ & \leq \sum_{n=N_0}^{\infty} \sum_{1 \leq k \leq \kappa_n} \mathbb{P} \left\{ \left| n^{-1} \sum_{i=1}^n \xi_{ik,2}^{D'_n} \right| > 8\sqrt{n^{-1} \log n} \right\} \leq \sum_{n=N_0}^{\infty} n^{-7} < \infty. \end{aligned}$$

Borel–Cantelli’s lemma entails that

$$\max_{1 \leq k \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{ik,2}^{D'_n} \right| = \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right). \quad (\text{S.19})$$

The above (S.17), (S.18) and (S.19) together imply that $\max_{1 \leq k < k' \leq \kappa_n} |n^{-1} \sum_{i=1}^n \xi_{ik} \xi_{ik'}| = \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right)$. The proof of (S.11) is complete.

(S.12) and (S.13) can be proved in a similar way. □

Proposition 1. *Under Assumptions (A1), (B1)–(B2) and under H_0 in (2.3), as $n \rightarrow \infty$,*

$$\max_{1 \leq i \leq n, 1 \leq k < \infty} \left| \hat{\zeta}_{ik} - \zeta_{ik} + \bar{\zeta}_{\cdot k} \right| = \mathcal{O}_{a.s.} \left(\rho_{n,N} \right), \quad (\text{S.20})$$

$$\max_{1 \leq k < \infty} \left| \bar{\zeta}_{\cdot k} \right| = \mathcal{O}_p \left(n^{-1/2} \right), \quad (\text{S.21})$$

$$\max_{1 \leq i \leq n, 1 \leq k < \infty} \left| \hat{\zeta}_{ik} - \zeta_{ik} \right| = \mathcal{O}_p \left(n^{-1/2} \right), \quad (\text{S.22})$$

where $\bar{\zeta}_{\cdot k} = n^{-1} \sum_{i=1}^n \zeta_{ik}$, and $\rho_{n,N}$ in (2.23) of Assumption (B2).

PROOF. By definition of $\hat{\zeta}_{ik}$ in (1.6), under H_0

$$\max_{1 \leq i \leq n, 1 \leq k < \infty} \left| \hat{\zeta}_{ik} - \zeta_{ik} + \bar{\zeta}_{\cdot k} \right| = \max_{1 \leq i \leq n, 1 \leq k < \infty} \left| \int \left\{ \hat{\xi}_i(x) - \xi_i(x) + n^{-1} \sum_{i'=1}^n \xi_{i'}(x) \right\} \psi_{0,k}(x) dx \right|$$

$$\begin{aligned}
&\leq \max_{1 \leq i \leq n} \left\| \hat{\xi}_i - \xi_i + n^{-1} \sum_{i'=1}^n \xi_{i'} \right\|_{\infty} \max_{1 \leq k < \infty} \|\psi_{0,k}\|_1 \\
&\leq \max_{1 \leq i \leq n} \left\| \hat{\xi}_i - \xi_i + n^{-1} \sum_{i'=1}^n \xi_{i'} \right\|_{\infty} \max_{1 \leq k < \infty} \|\psi_{0,k}\|_2 \\
&= \max_{1 \leq i \leq n} \left\| \hat{\xi}_i - \xi_i + n^{-1} \sum_{i'=1}^n \xi_{i'} \right\|_{\infty} = \mathcal{O}_{a.s.}(\rho_{n,N}) = o_{a.s.}\left(n^{-1/2} \kappa_n^{-2} \log^{-1/2} n\right),
\end{aligned}$$

according to Assumption (B2), so (S.20) is proved.

Assumptions (A1) and (B1) lead to Lemma S.6, which, together with the following arguments establish (S.21)

$$\begin{aligned}
\max_{1 \leq k < \infty} |\bar{\zeta}_{\cdot,k}| &= \max_{1 \leq k < \infty} \left| \int \left\{ n^{-1} \sum_{i'=1}^n \xi_{i'}(x) \right\} \psi_{0,k}(x) dx \right| \\
&\leq \left\| n^{-1} \sum_{i=1}^n \xi_i \right\|_{\infty} \max_{1 \leq k < \infty} \|\psi_{0,k}\|_1 \\
&\leq \left\| n^{-1} \sum_{i=1}^n \xi_i \right\|_{\infty} \max_{1 \leq k < \infty} \|\psi_{0,k}\|_2 = \left\| n^{-1} \sum_{i=1}^n \xi_i \right\|_{\infty} = \mathcal{O}_p(n^{-1/2}).
\end{aligned}$$

Putting together (S.20) and (S.21) yields (S.22), namely

$$\max_{1 \leq i \leq n, 1 \leq k < \infty} \left| \hat{\zeta}_{ik} - \zeta_{ik} \right| = \mathcal{O}_p(\rho_{n,N} + n^{-1/2}) = \mathcal{O}_p(n^{-1/2}).$$

The proof is completed. \square

Proposition 2. *Under Assumptions (A1), (B1)–(B2), as $n \rightarrow \infty$,*

$$\sup_{k \in \mathbb{N}_+} \hat{\lambda}_k \leq \Lambda_n = 2n^{-1} \sum_{i=1}^n \int \xi_i^2(x) dx + 2 \left(\max_{1 \leq i \leq n} \left\| \hat{\xi}_i - \xi_i \right\|_{\infty} \right)^2 = \mathcal{O}_p(1). \tag{S.23}$$

$\{\hat{Z}_{kk'}\}_{1 \leq k, k' < \infty}$ approximate $\{Z_{kk'}\}_{1 \leq k, k' < \infty}$ uniformly:

$$\max_{1 \leq k, k' < \infty} \left| \hat{Z}_{kk'} - Z_{kk'} \right| = \mathcal{O}_p(\rho_{n,N} + n^{-1}) = o_p\left(n^{-1/2} \kappa_n^{-2} \log^{-1/2} n\right). \quad (\text{S.24})$$

Under H_0 in (2.3), let $\bar{\lambda}_k = Z_{kk} = n^{-1} \sum_{i=1}^n \zeta_{ik}^2$, as $n \rightarrow \infty$,

$$\max_{1 \leq k < \infty} \left| \hat{\lambda}_k - \bar{\lambda}_k \right| = \mathcal{O}_p(\rho_{n,N} + n^{-1}), \quad (\text{S.25})$$

$$\max_{1 \leq k \leq \kappa_n} \left| \bar{\lambda}_k - \lambda_k \right| = \mathcal{O}_{a.s.}\left(n^{-1/2} \log^{1/2} n\right), \quad (\text{S.26})$$

$$\max_{1 \leq k \leq \kappa_n} \left| \hat{\lambda}_k - \lambda_k \right| = \mathcal{O}_p\left(n^{-1/2} \log^{1/2} n\right). \quad (\text{S.27})$$

PROOF. By definition in (1.7), one can write

$$\begin{aligned} \hat{\lambda}_k &= n^{-1} \sum_{i=1}^n \hat{\zeta}_{ik}^2 = n^{-1} \sum_{i=1}^n \left\{ \int \hat{\xi}_i(x) \psi_{0,k}(x) dx \right\}^2 \\ &= n^{-1} \sum_{i=1}^n \left\{ \int \left\{ \xi_i(x) + \hat{\xi}_i(x) - \xi_i(x) \right\} \psi_{0,k}(x) dx \right\}^2 \\ &\leq 2n^{-1} \sum_{i=1}^n \left\{ \int \xi_i(x) \psi_{0,k}(x) dx \right\}^2 + 2n^{-1} \sum_{i=1}^n \left\{ \int \left\{ \hat{\xi}_i(x) - \xi_i(x) \right\} \psi_{0,k}(x) dx \right\}^2 \\ &\leq 2n^{-1} \sum_{i=1}^n \|\xi_i\|_2^2 \|\psi_{0,k}(x)\|_2^2 + 2n^{-1} \sum_{i=1}^n \|\hat{\xi}_i - \xi_i\|_2^2 \|\psi_{0,k}(x)\|_2^2 \\ &\leq 2n^{-1} \sum_{i=1}^n \int \xi_i^2(x) dx + 2n^{-1} \sum_{i=1}^n \|\hat{\xi}_i - \xi_i\|_\infty^2. \end{aligned}$$

Thus,

$$\hat{\lambda}_k \leq \Lambda_n = 2n^{-1} \sum_{i=1}^n \int \xi_i^2(x) dx + 2 \left(\max_{1 \leq i \leq n} \|\hat{\xi}_i - \xi_i\|_\infty \right)^2$$

since $\max_{1 \leq i \leq n} \|\hat{\xi}_i - \xi_i\|_\infty = \mathcal{O}_p(n^{-1/2})$ by (S.1) while $n^{-1} \sum_{i=1}^n \int \xi_i^2(x) dx \xrightarrow{a.s.} \int G(x, x) dx < \infty$ by strong law of large numbers, thus $\Lambda_n = \mathcal{O}_p(1)$ and (S.23) is proved.

Note that

$$\hat{Z}_{kk'} = n^{-1} \sum_{i=1}^n \int \hat{\xi}_i(x) \psi_{0,k}(x) dx \int \hat{\xi}_i(x') \psi_{0,k'}(x') dx',$$

thus $\hat{Z}_{kk'} - Z_{kk'}$ is decomposed as

$$\hat{Z}_{kk'} - Z_{kk'} = I_1 + I_2 + I_3,$$

$$I_1 = n^{-1} \sum_{i=1}^n \int \left\{ \hat{\xi}_i(x) - \xi_i(x) \right\} \psi_{0,k}(x) dx \int \xi_i(x) \psi_{0,k'}(x) dx,$$

$$I_2 = n^{-1} \sum_{i=1}^n \int \left\{ \hat{\xi}_i(x) - \xi_i(x) \right\} \psi_{0,k'}(x) dx \int \xi_i(x) \psi_{0,k}(x) dx,$$

$$I_3 = n^{-1} \sum_{i=1}^n \int \left\{ \hat{\xi}_i(x) - \xi_i(x) \right\} \psi_{0,k}(x) dx \int \left\{ \hat{\xi}_i(x) - \xi_i(x) \right\} \psi_{0,k'}(x) dx.$$

Note that maximum of the third term is $\max_{1 \leq k, k' < \infty} |I_3|$ is bounded as

$$\begin{aligned} & \max_{1 \leq k, k' < \infty} \left| n^{-1} \sum_{i=1}^n \int \left\{ \hat{\xi}_i(x) - \xi_i(x) \right\} \psi_{0,k}(x) dx \int \left\{ \hat{\xi}_i(x) - \xi_i(x) \right\} \psi_{0,k'}(x) dx \right| \\ & \leq \left(\max_{1 \leq i \leq n} \|\hat{\xi}_i - \xi_i\|_\infty \right)^2 \max_{1 \leq k, k' < \infty} \int |\psi_{0,k}(x)| dx \int |\psi_{0,k'}(x)| dx \\ & \leq \left(\max_{1 \leq i \leq n} \|\hat{\xi}_i - \xi_i\|_\infty \right)^2 \max_{1 \leq k < \infty} \int \psi_{0,k}^2(x) dx \\ & = \left(\max_{1 \leq i \leq n} \|\hat{\xi}_i - \xi_i\|_\infty \right)^2 = \mathcal{O}_p(n^{-1}). \end{aligned}$$

Maximum of the first term $\max_{1 \leq k, k' < \infty} |I_1|$ is bounded as

$$\begin{aligned}
& \max_{1 \leq k, k' < \infty} \left| n^{-1} \sum_{i=1}^n \int \left\{ \hat{\xi}_i(x) - \xi_i(x) \right\} \psi_{0,k}(x) dx \int \xi_i(x) \psi_{0,k'}(x) dx \right| \\
&= \max_{1 \leq k, k' < \infty} \left| n^{-1} \sum_{i=1}^n \int \left\{ \hat{\xi}_i(x) - \xi_i(x) + n^{-1} \sum_{i'=1}^n \xi_{i'}(x) \right\} \psi_{0,k}(x) dx \int \xi_i(x) \psi_{0,k'}(x) dx \right. \\
&\quad \left. - \int n^{-1} \sum_{i'=1}^n \xi_{i'}(x) \psi_{0,k}(x) dx \int n^{-1} \sum_{i=1}^n \xi_i(x) \psi_{0,k'}(x) dx \right| \\
&\leq \max_{1 \leq k, k' < \infty} \left\| \hat{\xi}_i - \xi_i + n^{-1} \sum_{i'=1}^n \xi_{i'} \right\|_{\infty} \int |\psi_{0,k}(x)| dx \int n^{-1} \sum_{i=1}^n |\xi_i(x)| |\psi_{0,k'}(x)| dx \\
&\quad + \max_{1 \leq k, k' < \infty} \int \left| n^{-1} \sum_{i'=1}^n \xi_{i'}(x) \right| |\psi_{0,k}(x)| dx \int \left| n^{-1} \sum_{i=1}^n \xi_i(x) \right| |\psi_{0,k'}(x)| dx \\
&= \mathcal{O}_{a.s.}(\rho_{n,N}) \times \mathcal{O}_p(1) + \mathcal{O}_p(n^{-1/2} \times n^{-1/2}) = \mathcal{O}_p(\rho_{n,N} + n^{-1}).
\end{aligned}$$

The same bound is good for $\max_{1 \leq k, k' < \infty} |I_2|$, thus one has established that

$$\max_{1 \leq k, k' < \infty} \left| \hat{Z}_{kk'} - Z_{kk'} \right| = \mathcal{O}_p(\rho_{n,N} + n^{-1}).$$

According to (1.5), $\lambda_k = \mathbb{E}\zeta_{ik}^2$. Denote $\bar{\lambda}_k = Z_{kk} = n^{-1} \sum_{i=1}^n \zeta_{ik}^2$.

Applying (S.11) in Lemma S.8, one easily obtains that

$$\begin{aligned}
\max_{1 \leq k \leq \kappa_n} |\bar{\lambda}_k - \lambda_k| &\leq \max_{1 \leq k \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \zeta_{ik}^2 - \lambda_k \right| \leq \lambda_1 \max_{1 \leq k \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \zeta_{ik}^2 - 1 \right| \\
&= \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right),
\end{aligned}$$

which is (S.26). (S.25) follows as a corollary of (S.24), and (S.27) follows from (S.25) and (S.26). The proof is completed. \square

S.2 Results on B-spline Estimates

Matrix algebra represents the B-spline estimator $\hat{\eta}_i(\cdot)$ in (2.26) as

$$\hat{\eta}_i(x) = \{B_{1,p}(x), \dots, B_{J_s+p,p}(x)\} (\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{Y}_i,$$

where $\mathbf{Y}_i = (Y_{i1}, \dots, Y_{iN})^\top$ and the design matrix \mathbf{B} is

$$\mathbf{B} = \begin{pmatrix} B_{1,p}(1/N) & \cdots & B_{J_s+p,p}(1/N) \\ \vdots & \cdots & \vdots \\ B_{1,p}(N/N) & \cdots & B_{J_s+p,p}(N/N) \end{pmatrix}_{N \times (J_s+p)} = \{\mathbf{B}(1/N), \dots, \mathbf{B}(N/N)\}^\top,$$

in which $\mathbf{B}(x)^\top = \{B_{1,p}(x), \dots, B_{J_s+p,p}(x)\}$. Denote the theoretical and empirical inner product matrixes of B-spline basis $\{B_{J,p}(x)\}_{J=1}^{J_s+p}$ as

$$\mathbf{V}_p = \{\langle B_{J,p}, B_{J',p} \rangle\}_{J,J'=1}^{J_s+p}, \quad (\text{S.28})$$

$$\mathbf{V}_{n,p} = \{\langle B_{J,p}, B_{J',p} \rangle_N\}_{J,J'=1}^{J_s+p} = N^{-1} \mathbf{B}^\top \mathbf{B}. \quad (\text{S.29})$$

According to model (1.2), one obtains

$$\boldsymbol{\eta}_i = \mathbf{m} + \boldsymbol{\xi}_i,$$

where $\boldsymbol{\eta}_i = \{\eta_i(1/N), \dots, \eta_i(N/N)\}^\top$, $\mathbf{m} = \{m(1/N), \dots, m(N/N)\}^\top$

and

$\boldsymbol{\xi}_i = \{\xi_i(1/N), \dots, \xi_i(N/N)\}^\top$. Then, denote $\boldsymbol{\varepsilon}_i = (\sigma_i(1/N) \varepsilon_{i1}, \dots, \sigma_i(N/N) \varepsilon_{iN})^\top$,

$\widehat{\eta}_i(x)$ can be decomposed as:

$$\widehat{\eta}_i(x) = \widetilde{\eta}_i(x) + \widetilde{\varepsilon}_i(x),$$

where

$$\begin{aligned} \widetilde{\eta}_i(x) &= N^{-1} \mathbf{B}(x)^T \mathbf{V}_{n,p}^{-1} \mathbf{B}^T \boldsymbol{\eta}_i = \widetilde{m}(x) + \widetilde{\xi}_i(x), \\ \widetilde{m}(x) &= N^{-1} \mathbf{B}(x)^T \mathbf{V}_{n,p}^{-1} \mathbf{B}^T \mathbf{m}, \\ \widetilde{\xi}_i(x) &= N^{-1} \mathbf{B}(x)^T \mathbf{V}_{n,p}^{-1} \mathbf{B}^T \boldsymbol{\xi}_i, \end{aligned} \tag{S.30}$$

$$\widetilde{\varepsilon}_i(x) = N^{-1} \mathbf{B}(x)^T \mathbf{V}_{n,p}^{-1} \mathbf{B}^T \boldsymbol{\varepsilon}_i. \tag{S.31}$$

So $\widehat{\eta}_i(x) = \widetilde{\xi}_i(x) + \widetilde{m}(x) + \widetilde{\varepsilon}_i(x)$. By the definition of $\widehat{\xi}_i(x), \widehat{m}(x)$ in (2.28) and (2.27)

$$\begin{aligned} \widehat{\xi}_i(x) - \xi_i(x) &= \widehat{\eta}_i(x) - \widehat{m}(x) - \xi_i(x) \\ &= \widetilde{\xi}_i(x) + \widetilde{m}(x) + \widetilde{\varepsilon}_i(x) - n^{-1} \sum_{i'=1}^n \left\{ \widetilde{\xi}_i(x) + \widetilde{m}(x) + \widetilde{\varepsilon}_i(x) \right\} - \xi_i(x) \\ &= \widetilde{\xi}_i(x) - \xi_i(x) + \widetilde{\varepsilon}_i(x) - n^{-1} \sum_{i'=1}^n \left\{ \widetilde{\xi}_{i'}(x) - \xi_{i'}(x) + \widetilde{\varepsilon}_{i'}(x) \right\} - n^{-1} \sum_{i'=1}^n \xi_{i'}(x). \end{aligned} \tag{S.32}$$

Lemma S.9. *There exist $0 < c_p < C_p < \infty$ such that*

$$c_p \|\boldsymbol{\alpha}\|_2 J_s^{-1/2} \leq \left\| \sum_{J=1}^{J_s+p} \alpha_J B_{J,p} \right\|_2 \leq C_p \|\boldsymbol{\alpha}\|_2 J_s^{-1/2}, \forall \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{J_s+p}) \in \mathbb{R}^{J_s+p}, \tag{S.33}$$

and the theoretical inner product matrix \mathbf{V}_p in (S.28) satisfies

$$c_p J_s \|\boldsymbol{\alpha}\|_\infty \leq \|\mathbf{V}_p^{-1} \boldsymbol{\alpha}\|_\infty \leq C_p J_s \|\boldsymbol{\alpha}\|_\infty, \forall \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{J_s+p}) \in \mathbb{R}^{J_s+p}. \quad (\text{S.34})$$

Under Assumption (C5), for n large enough, the empirical inner product matrix $\mathbf{V}_{n,p}$ in (S.29) satisfies

$$c_p J_s \|\boldsymbol{\alpha}\|_\infty \leq \|\mathbf{V}_{n,p}^{-1} \boldsymbol{\alpha}\|_\infty \leq C_p J_s \|\boldsymbol{\alpha}\|_\infty, \forall \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{J_s+p}) \in \mathbb{R}^{J_s+p}. \quad (\text{S.35})$$

PROOF. (S.33) follows from Theorem 5.4.2 of DeVore and Lorentz (1993), while (S.34), (S.35) follow from Lemma A.3 and Lemma A.4 of Cao et al. (2012). \square

Lemma S.10. *Assumptions (C3) and (C4') imply Assumption (C4).*

PROOF. Taking $H(x) = x^{r_2}$, Lemma S.1 entails that there exist constants c and a depending on the distribution of ε_{ij} , such that for $x_N = N^{\beta_2}$, $N/H(ax_N) = a^{-r_2} N^{1-r_2\beta_2}$ and i.i.d. standard normal random variables $\{U_{ij,\varepsilon}\}_{i=1,j=1}^{n,N}$ such that

$$\max_{1 \leq i \leq n} \mathbb{P} \left\{ \max_{1 \leq t \leq N} \left| \sum_{j=1}^t \varepsilon_{ij} - \sum_{j=1}^t U_{ij,\varepsilon} \right| > N^{\beta_2} \right\} < ca^{-r_2} N^{1-r_2\beta_2}.$$

By Assumption (C3) that $n = \mathcal{O}(N^\theta)$, there exists a constant $C_\varepsilon > 0$ such

that

$$\mathbb{P} \left\{ \max_{1 \leq i \leq n} \max_{1 \leq t \leq N} \left| \sum_{j=1}^t \varepsilon_{ij} - \sum_{j=1}^t U_{ij,\varepsilon} \right| > N^{\beta_2} \right\} < ca^{-r_2} n \times N^{1-r_2\beta_2} \leq C_\varepsilon N^{\theta+1-r_2\beta_2}.$$

Since $r_2 > (2 + \theta) / \beta_2$ according to Assumption (C4'), there is $\gamma_2 = r_2\beta_2 - 1 - \theta > 1$ and Assumption (C4) follows. \square

Under Assumption (C5) and (2.33), $J_s = N^\gamma d_N$ with $d_N + d_N^{-1} = \mathcal{O}(\log^\tau N)$ and $\gamma > 1 - \nu$. Thus, $J_s N^{-(1-\nu)} = N^{\gamma-(1-\nu)} d_N \rightarrow \infty$ as $N \rightarrow \infty$, which leads to

$$J_s N^{-1} \gg N^{-\nu}. \quad (\text{S.36})$$

Lemma S.11. *Under Assumptions (C1), (C4) and (C5), as $n \rightarrow \infty$*

$$\begin{aligned} \max_{1 \leq i \leq n} \|N^{-1} \mathbf{B}^T \varepsilon_i\|_\infty &= \mathcal{O}_{a.s.} \left(N^{-1/2} J_s^{-1/2} \log^{1/2} N + N^{\beta_2-1} \right), \\ \max_{1 \leq i \leq n} \|\tilde{\varepsilon}_i\|_\infty &= \mathcal{O}_{a.s.} \left(N^{-1/2} J_s^{1/2} \log^{1/2} N + J_s N^{\beta_2-1} \right). \end{aligned} \quad (\text{S.37})$$

PROOF. According to Assumption (C4), it is easy to see

$$\max_{1 \leq i \leq n} \max_{1 \leq t \leq N} \left| N^{-1} \sum_{j=1}^t (\varepsilon_{ij} - U_{ij,\varepsilon}) \right| = \mathcal{O}_{a.s.}(N^{\beta_2-1}).$$

Note that the B-spline basis satisfies

$$\left| B_{J,p} \left(\frac{j}{N} \right) - B_{J,p} \left(\frac{j+1}{N} \right) \right| \leq N^{-1} \|B_{J,p}\|_{0,1} \leq C J_s N^{-1}$$

uniformly over $1 \leq j \leq N$ and $1 \leq J \leq J_s + p$, while (S.36) provides that

$J_s N^{-1} \gg N^{-\nu}$, hence

$$\max_{1 \leq i \leq n} \left| \sigma_i \left(\frac{j}{N} \right) - \sigma_i \left(\frac{j+1}{N} \right) \right| \leq N^{-\nu} \max_{1 \leq i \leq n} \|\sigma_i\|_{0,\nu} \leq C J_s N^{-1}$$

uniformly over $1 \leq j \leq N$ by Assumption (C1). Since for $1 \leq J \leq J_s + p$,

both $B_{J,p}(\cdot)$ and $\sigma_i(\cdot)$ are bounded on $[0, 1]$, then

$$\begin{aligned} & \max_{1 \leq i \leq n} \left| B_{J,p} \left(\frac{j}{N} \right) \sigma_i \left(\frac{j}{N} \right) - B_{J,p} \left(\frac{j+1}{N} \right) \sigma_i \left(\frac{j+1}{N} \right) \right| \\ &= \max_{1 \leq i \leq n} \left| \left\{ B_{J,p} \left(\frac{j}{N} \right) - B_{J,p} \left(\frac{j+1}{N} \right) + B_{J,p} \left(\frac{j+1}{N} \right) \right\} \sigma_i \left(\frac{j}{N} \right) - B_{J,p} \left(\frac{j+1}{N} \right) \sigma_i \left(\frac{j+1}{N} \right) \right| \\ &\leq \left| B_{J,p} \left(\frac{j}{N} \right) - B_{J,p} \left(\frac{j+1}{N} \right) \right| \max_{1 \leq i \leq n} \sigma_i \left(\frac{j}{N} \right) + \max_{1 \leq i \leq n} \left| \sigma_i \left(\frac{j}{N} \right) - \sigma_i \left(\frac{j+1}{N} \right) \right| B_{J,p} \left(\frac{j+1}{N} \right) \\ &\leq C J_s N^{-1}. \end{aligned}$$

Since the support of $B_{J,p}(\cdot)$ has length at most $p/(J_s + 1)$, one obtains that

$$\max_{1 \leq i \leq n} \sum_{j=1}^{N-1} \left| B_{\ell,p} \left(\frac{j}{N} \right) \sigma_i \left(\frac{j}{N} \right) - B_{\ell,p} \left(\frac{j+1}{N} \right) \sigma_i \left(\frac{j+1}{N} \right) \right| \leq C.$$

Hence,

$$\begin{aligned} & \max_{1 \leq i \leq n} \left| N^{-1} \sum_{j=1}^N B_{J,p}(j/N) \sigma_i(j/N) (\varepsilon_{ij} - U_{ij,\varepsilon}) \right| \\ &= \max_{1 \leq i \leq n} \sum_{j=1}^{N-1} \left\{ B_{J,p} \left(\frac{j}{N} \right) \sigma_i \left(\frac{j}{N} \right) - B_{J,p} \left(\frac{j+1}{N} \right) \sigma_i \left(\frac{j+1}{N} \right) \right\} N^{-1} \sum_{t=1}^j (\varepsilon_{it} - U_{it,\varepsilon}) \\ & \quad + \max_{1 \leq i \leq n} B_{J,p}(1) \sigma_i(1) N^{-1} \sum_{t=1}^N (\varepsilon_{it} - U_{it,\varepsilon}) \end{aligned}$$

$$\begin{aligned}
&\leq \left\{ \max_{1 \leq i \leq n} \max_{1 \leq j \leq N} \left| N^{-1} \sum_{t=1}^j (\varepsilon_{it} - U_{it,\varepsilon}) \right| \right\} \times \\
&\quad \left\{ \sum_{j=1}^{N-1} \left| B_{J,p} \left(\frac{j}{N} \right) \sigma_i \left(\frac{j}{N} \right) - B_{J,p} \left(\frac{j+1}{N} \right) \sigma_i \left(\frac{j+1}{N} \right) \right| + C \right\} \\
&= \mathcal{O}_{a.s.} (N^{\beta_2-1}).
\end{aligned}$$

Thus,

$$\max_{1 \leq i \leq n} \max_{1 \leq J \leq J_s+p} \left| N^{-1} \sum_{j=1}^N B_{J,p}(j/N) \sigma_i(j/N) (\varepsilon_{ij} - U_{ij,\varepsilon}) \right| = \mathcal{O}_{a.s.} (N^{\beta_2-1}).$$

According to Lemma A.6 of Wang et al. (2020),

$$\max_{1 \leq i \leq n} \max_{1 \leq J \leq J_s+p} \left| N^{-1} \sum_{j=1}^N B_{J,p}(j/N) \sigma_i(j/N) U_{ij,\varepsilon} \right| = \mathcal{O}_{a.s.} (N^{-1/2} J_s^{-1/2} \log^{1/2} N).$$

Then, The above inequalities together imply that

$$\begin{aligned}
\max_{1 \leq i \leq n} \|N^{-1} \mathbf{B}^T \boldsymbol{\varepsilon}_i\|_\infty &= \max_{1 \leq i \leq n} \max_{1 \leq J \leq J_s+p} \left| N^{-1} \sum_{j=1}^N B_{J,p}(j/N) \sigma_i(j/N) \varepsilon_{ij} \right| \\
&= \mathcal{O}_{a.s.} (N^{-1/2} J_s^{-1/2} \log^{1/2} N + N^{\beta_2-1}).
\end{aligned}$$

Note next that according to (S.31) and (S.35) in Lemma S.9,

$$\begin{aligned}
\max_{1 \leq i \leq n} \|\tilde{\varepsilon}_i\|_\infty &= \max_{1 \leq i \leq n} \sup_{x \in [0,1]} |\tilde{\varepsilon}_i(x)| = \max_{1 \leq i \leq n} \sup_{x \in [0,1]} |N^{-1} \mathbf{B}(x)^T \mathbf{V}_{n,p}^{-1} \mathbf{B}^T \boldsymbol{\varepsilon}_i| \\
&= \max_{1 \leq i \leq n} \sup_{x \in [0,1]} |\mathbf{B}(x)^T \mathbf{V}_{n,p}^{-1} N^{-1} \mathbf{B}^T \boldsymbol{\varepsilon}_i| \\
&\leq C_p J_s \max_{1 \leq i \leq n} \|N^{-1} \mathbf{B}^T \boldsymbol{\varepsilon}_i\|_\infty = \mathcal{O}_{a.s.} (N^{-1/2} J_s^{1/2} \log^{1/2} N + J_s N^{\beta_2-1}),
\end{aligned}$$

as for any $x \in [0, 1]$, the vector $\mathbf{B}(x)$ has at most p non-zero elements, each taking value in $[0, 1]$. So (S.37) is proved, and the proof is completed. \square

Lemma S.12. *Under Assumptions (C2) and (C4), as $n \rightarrow \infty$,*

$$\max_{1 \leq i \leq n} \left\| \tilde{\xi}_i - \xi_i \right\|_{\infty} = \mathcal{O}_{a.s.} \left(J_s^{-p^*} (n \log n)^{2/r_1} \right), \quad (\text{S.38})$$

Under Assumptions (C1)-(C2), (C4)-(C5)

$$\max_{1 \leq i \leq n} \left\| \hat{\xi}_i - \xi_i + n^{-1} \sum_{i'=1}^n \xi_{i'} \right\|_{\infty} = \mathcal{O}_{a.s.} \left(J_s^{-p^*} (n \log n)^{2/r_1} + N^{-1/2} J_s^{1/2} \log^{1/2} N + J_s N^{\beta_2 - 1} \right). \quad (\text{S.39})$$

PROOF. For any $k = 1, 2, \dots$, let $\phi_k = (\phi_k(1/N), \dots, \phi_k(N/N))^T$, and denote $\tilde{\phi}_k(x) = N^{-1} \mathbf{B}(x)^T \mathbf{V}_{n,p}^{-1} \mathbf{B}^T \phi_k$. According to (S.30), $\tilde{\xi}_i(x) = \sum_{k=1}^{\infty} \xi_{ik} \tilde{\phi}_k(x)$, therefore,

$$\tilde{\xi}_i(x) - \xi_i(x) = \sum_{k=1}^{\infty} \xi_{ik} \left\{ \tilde{\phi}_k(x) - \phi_k(x) \right\}.$$

By Lemma A.4 of Cao et al. (2012), there exists a constant $C_{q,\mu} > 0$, such that

$$\left\| \tilde{\phi}_k - \phi_k \right\|_{\infty} \leq C_{q,\mu} \|\phi_k\|_{q,\mu} J_s^{-p^*}, k \geq 1.$$

Thus, one obtains

$$\left\| \tilde{\xi}_i - \xi_i \right\|_{\infty} \leq \sum_{k=1}^{\infty} |\xi_{ik}| \left\| \tilde{\phi}_k - \phi_k \right\|_{\infty} \leq C_{q,\mu} W_i J_s^{-p^*},$$

where $W_i = \sum_{k=1}^{\infty} |\xi_{ik}| \|\phi_k\|_{q,\mu}$, $i = 1, \dots, n$, are i.i.d. nonnegative random variables with finite r_1 -th absolute moment. By Assumptions (C2) and (C4), one has

$$\mathbb{P} \left\{ \max_{1 \leq i \leq n} W_i > (n \log n)^{2/r_1} \right\} \leq n \frac{\mathbb{E} W_i^{r_1}}{(n \log n)^2} = \mathbb{E} W_i^{r_1} n^{-1} (\log n)^{-2},$$

thus,

$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \max_{1 \leq i \leq n} W_i > (n \log n)^{2/r_1} \right\} \leq \mathbb{E} W_i^{r_1} \sum_{n=1}^{\infty} n^{-1} (\log n)^{-2} < +\infty,$$

so $\max_{1 \leq i \leq n} W_i = \mathcal{O}_{a.s.} \left\{ (n \log n)^{2/r_1} \right\}$ by Borel-Cantelli Lemma. Thus,

$$\max_{1 \leq i \leq n} \left\| \tilde{\xi}_i - \xi_i \right\|_{\infty} = \mathcal{O}_{a.s.} \left(J_s^{-p^*} (n \log n)^{2/r_1} \right),$$

which establishes (S.38).

Putting together Lemma S.6, (S.38), (S.37), and (S.32) yields (S.39).

The proof is completed. \square

S.3 Proof of Theorem 1

Assumption (A1) and (2.5), (2.6) ensure that

$$\begin{aligned} \mathbb{E} \|\mathbf{X}_i\|_{\mathcal{H}}^2 &= \sum_{k,k'=1}^{\infty} \mathbb{E} \left\{ \sum_{k_1,k_2=1}^{\infty} \zeta_{ik_1} \zeta_{ik_2} u_{k_1 k_2, k k'} \right\}^2 \\ &= \sum_{k,k'=1}^{\infty} \sum_{k_1,k_2,k_3,k_4=1}^{\infty} \left(\mathbb{E} \zeta_{ik_1} \zeta_{ik_2} \zeta_{ik_3} \zeta_{ik_4} \right) u_{k_1 k_2, k k'} u_{k_3 k_4, k k'} \\ &= \sum_{k_1,k_2,k_3,k_4=1}^{\infty} \left(\mathbb{E} \zeta_{ik_1} \zeta_{ik_2} \zeta_{ik_3} \zeta_{ik_4} \right) \delta_{k_1 k_3} \delta_{k_2 k_4} \\ &= \sum_{k_1=1}^{\infty} \mathbb{E} \zeta_{ik_1}^4 + 2 \sum_{1 \leq k_1 < k_2 < \infty} \mathbb{E} \zeta_{ik_1}^2 \zeta_{ik_2}^2 \\ &= \sum_{k=1}^{\infty} \lambda_k^2 \mathbb{E} \xi_k^4 + 2 \sum_{1 \leq k_1 < k_2 < \infty} \lambda_{k_1} \lambda_{k_2}, \\ \mathbb{E} \|\mathbf{X}_i\|_{\mathcal{H}}^2 &= \sum_{k=1}^{\infty} \lambda_k^2 \left(\mathbb{E} \xi_k^4 - 1 \right) + \left(\sum_{k=1}^{\infty} \lambda_k \right)^2 < \infty, \end{aligned}$$

which establishes (2.15) and that $\|\mathbf{X}_i\|_{\mathcal{H}}^2 < \infty$ almost surely, hence $\{\mathbf{X}_i\}_{i=1}^n$ is an i.i.d. sequence of \mathcal{H} -valued random variables according to Bosq (2000), and (2.14) follows also:

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{X}} &= \mathbb{E}\mathbf{X}_i = \left(\sum_{k_1, k_2=1}^{\infty} \mathbb{E}\zeta_{ik_1} \zeta_{ik_2} u_{k_1 k_2, k k'} \right)_{1 \leq k, k' < \infty} \\ &= \left(\sum_{k_1=1}^{\infty} \lambda_{k_1} u_{k_1 k_1, k k'} \right)_{1 \leq k, k' < \infty}. \end{aligned}$$

Hilbert space Central Limit Theorem (Theorem 2.7 in Bosq (2000)) then implies (2.16):

$$n^{1/2} \left\{ (Z_{kk'})_{1 \leq k, k' < \infty} - \boldsymbol{\mu}_{\mathbf{X}} \right\} = n^{-1/2} \sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\mu}_{\mathbf{X}}) \xrightarrow{D} N(\mathbf{0}, \mathbf{C}_{\mathbf{X}}).$$

Under Assumption (A1), the definition of \mathbf{Y}_1 in (2.11), $\mathbb{E}\mathbf{Y}_1 = \mathbf{0}$, and

$$\begin{aligned} \mathbb{E} \|\mathbf{Y}_1\|_{\mathcal{H}}^2 &= \sum_{k_1, k_2=1}^{\infty} \mathbb{E} (\zeta_{ik_1} \zeta_{ik_2} - \lambda_{k_1} \delta_{k_1 k_2})^2 \\ &= \sum_{k_1=1}^{\infty} \mathbb{E} (\zeta_{ik_1}^2 - \lambda_{k_1})^2 + \sum_{1 \leq k_1 \neq k_2 < \infty} \mathbb{E} \zeta_{ik_1}^2 \zeta_{ik_2}^2 \\ &= \sum_{k_1=1}^{\infty} \lambda_{k_1}^2 (\mathbb{E} \zeta_{ik_1}^4 - 1) + \sum_{1 \leq k_1 \neq k_2 < \infty} \lambda_{k_1} \lambda_{k_2} \\ &= \sum_{k_1=1}^{\infty} \lambda_{k_1}^2 (\mathbb{E} \zeta_{ik_1}^4 - 2) + \left(\sum_{1 \leq k < \infty} \lambda_k \right)^2, \end{aligned}$$

which equals $\mathbb{E} \|\mathbf{X}_1 - \boldsymbol{\mu}_{\mathbf{X}}\|_{\mathcal{H}}^2$ since operator \mathbf{U} is unitary. According to Bosq (2000), the covariance operator of \mathbf{Y}_1 is $\mathbf{C}_{\mathbf{Y}}(\cdot) = \mathbb{E}(\langle \mathbf{Y}_1, \cdot \rangle \mathbf{Y}_1)$, a positive, symmetric and nuclear operator which satisfies (2.18) because

$$\begin{aligned} \mathbf{C}_{\mathbf{Y}}(\mathbf{e}_{kk}) &= \mathbb{E}(\langle \mathbf{Y}_1, \mathbf{e}_{kk} \rangle \mathbf{Y}_1) = \mathbb{E}(\zeta_{1k}^2 - \lambda_k) \mathbf{Y}_1 \\ &= \mathbb{E}(\zeta_{1k}^2 - \lambda_k)^2 \mathbf{e}_{kk} = \lambda_k^2 (\mathbb{E} \zeta_{1k}^4 - 1) \mathbf{e}_{kk}, 1 \leq k < \infty \end{aligned}$$

$$\mathbf{C}_Y(\mathbf{e}_{kk'}) = \mathbb{E}(\langle \mathbf{Y}_1, \mathbf{e}_{kk'} \rangle \mathbf{Y}_1) = \mathbb{E} \zeta_{1k} \zeta_{1k'} \mathbf{Y}_1 = \lambda_k \lambda_{k'} (\mathbf{e}_{kk'} + \mathbf{e}_{k'k}), 1 \leq k \neq k' < \infty.$$

Immediately, (2.17) follows from (2.16) and (2.12) as

$$n^{1/2} (Z_{kk} - \lambda_k, Z_{kk'})_{1 \leq k \neq k' < \infty} = n^{-1/2} \sum_{i=1}^n \mathbf{Y}_i \xrightarrow{D} \mathcal{N} \sim N(\mathbf{0}, \mathbf{C}_Y).$$

Define next a function $f : \mathcal{H} \rightarrow (0, +\infty)$ as norm square of projection

$$f \left\{ (a_{kk'})_{1 \leq k, k' < \infty} \right\} = \sum_{1 \leq k < k' < \infty} a_{kk'}^2 = \left\| \mathcal{P}_{\text{UT}} (a_{kk'})_{1 \leq k, k' < \infty} \right\|_{\mathcal{H}}^2, \quad (\text{S.40})$$

which is continuous over \mathcal{H} . Applying Banach space Continuous Mapping Theorem (equation (2.11) of Bosq (2000)) to (2.17) and (S.40), one obtains

$$\tilde{S}_n = f \left(n^{-1/2} \sum_{i=1}^n \mathbf{Y}_i \right) \xrightarrow{D} f(\mathcal{N}) = \left\| \mathcal{P}_{\text{UT}}(\mathcal{N}) \right\|_{\mathcal{H}}^2 = \sum_{1 \leq k < k' < \infty} \lambda_k \lambda_{k'} \chi_{kk'}^2(1),$$

where the sum $\sum_{1 \leq k < k' < \infty} \lambda_k \lambda_{k'} \chi_{kk'}^2(1)$ is finite with probability 1 since its expectation $\sum_{1 \leq k < k' < \infty} \lambda_k \lambda_{k'} < \infty$. The proof is completed. \square

S.4 Proof of Theorem 2

Denote $\check{S}_n = n \sum_{1 \leq k < k' \leq \kappa_n} Z_{kk'}^2$, then

$$\left| \hat{S}_n - \check{S}_n \right| = n \left| \sum_{1 \leq k < k' \leq \kappa_n} \left(\hat{Z}_{kk'}^2 - Z_{kk'}^2 \right) \right| \leq n (\Delta_1 + \Delta_2),$$

$$\Delta_1 = \left| \sum_{1 \leq k < k' \leq \kappa_n} \left(\hat{Z}_{kk'} - Z_{kk'} \right)^2 \right|, \Delta_2 = \left| \sum_{1 \leq k < k' \leq \kappa_n} 2Z_{kk'} \left(\hat{Z}_{kk'} - Z_{kk'} \right) \right|. \quad (\text{S.41})$$

According to (S.24),

$$\Delta_1 \leq \sum_{1 \leq k < k' \leq \kappa_n} \max_{1 \leq k, k' \leq \kappa_n} \left| \hat{Z}_{kk'} - Z_{kk'} \right|^2 = \mathcal{O}_p \left(\kappa_n^2 \rho_{n,N}^2 + \kappa_n^2 n^{-2} \right). \quad (\text{S.42})$$

Notice that

$$\begin{aligned} \max_{1 \leq k < k' \leq \kappa_n} |Z_{kk'}| &= \max_{1 \leq k < k' \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \zeta_{ik} \zeta_{ik'} \right| = \max_{1 \leq k < k' \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \sqrt{\lambda_k \lambda_{k'}} \xi_{ik} \xi_{ik'} \right| \\ &\leq \lambda_1 \max_{1 \leq k < k' \leq \kappa_n} \left| n^{-1} \sum_{i=1}^n \xi_{ik} \xi_{ik'} \right| = \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right) \end{aligned}$$

by (S.10) in Lemma S.8. Then

$$\begin{aligned} \Delta_2 &\leq 2 \sum_{1 \leq k < k' \leq \kappa_n} \max_{1 \leq k < k' \leq \kappa_n} |Z_{kk'}| \max_{1 \leq k, k' \leq \kappa_n} \left| \hat{Z}_{kk'} - Z_{kk'} \right| \\ &= \kappa_n^2 \times \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right) \times \mathcal{O}_p \left(\rho_{n,N} + n^{-1} \right) \\ &= \mathcal{O}_p \left(\kappa_n^2 n^{-1/2} \rho_{n,N} \log^{1/2} n + \kappa_n^2 n^{-3/2} \log^{1/2} n \right). \quad (\text{S.43}) \end{aligned}$$

Now (S.41), (S.42) and (S.43) imply that

$$\begin{aligned} \left| \hat{S}_n - \check{S}_n \right| &= n \times \mathcal{O}_p \left(\kappa_n^2 \rho_{n,N}^2 + \kappa_n^2 n^{-2} \right) + \\ &\quad n \times \mathcal{O}_p \left(\kappa_n^2 n^{-1/2} \rho_{n,N} \log^{1/2} n + \kappa_n^2 n^{-3/2} \log^{1/2} n \right) \\ &= \mathcal{O}_p \left(\kappa_n^2 n \rho_{n,N}^2 + \kappa_n^2 \rho_{n,N} n^{1/2} \log^{1/2} n + \kappa_n^2 n^{-1/2} \log^{1/2} n \right). \end{aligned}$$

By recalling Assumption (B2) and (2.22) that $\kappa_n^2 \rho_{n,N} n^{1/2} \log^{1/2} n \rightarrow 0$, $\kappa_n^2 n^{-1/2} \log^{3/2} n \rightarrow 0$ and (S.3) in Lemma S.7, $\left| \hat{S}_n - \check{S}_n \right| = o_p(1)$. On the other hand,

$$\tilde{S}_n - \check{S}_n = \sum_{1 \leq k < k', k' > \kappa_n} n Z_{kk'}^2 = n \sum_{1 \leq k < k', k' > \kappa_n} \left(n^{-1} \sum_{i=1}^n \zeta_{ik} \zeta_{ik'} \right)^2$$

$$= \sum_{1 \leq k < k', k' > \kappa_n} \lambda_k \lambda_{k'} \times n^{-1} \left(\sum_{i=1}^n \xi_{ik} \xi_{ik'} \right)^2.$$

Note that under Assumption (A1) for any $1 \leq k < k'$,

$$\mathbb{E} n^{-1} \left(\sum_{i=1}^n \xi_{ik} \xi_{ik'} \right)^2 = n^{-1} \sum_{i, i'=1}^n \mathbb{E} \xi_{ik} \xi_{ik'} \xi_{i'k} \xi_{i'k'} = n^{-1} \sum_{i=1}^n \mathbb{E} \xi_{ik}^2 \xi_{ik'}^2 = 1,$$

hence by taking $R_{kk',n} = n^{-1} (\sum_{i=1}^n \xi_{ik} \xi_{ik'})^2$, $M = 1$ in Lemma S.5, one obtains that $\tilde{S}_n - \check{S}_n = o_p(1)$. Then,

$$\left| \hat{S}_n - \tilde{S}_n \right| \leq \left| \hat{S}_n - \check{S}_n \right| + \left| \check{S}_n - \tilde{S}_n \right| = o_p(1).$$

Slutsky's Theorem then implies that $\hat{S}_n \xrightarrow{D} S$.

Since S has continuous distribution by Lemma S.2, Pólya's Theorem (Theorem 11.2.9, Lehmann et al. (2005)) implies that as $n \rightarrow \infty$

$$\sup_{q \in \mathbb{R}} \left| \mathbb{P} \left[\tilde{S}_n \leq q \right] - \mathbb{P} [S \leq q] \right| \rightarrow 0, \sup_{q \in \mathbb{R}} \left| \mathbb{P} \left[\hat{S}_n \leq q \right] - \mathbb{P} [S \leq q] \right| \rightarrow 0,$$

hence

$$\sup_{q \in \mathbb{R}} \left| \mathbb{P} \left[\tilde{S}_n > q \right] - \mathbb{P} [S > q] \right| \rightarrow 0, \sup_{q \in \mathbb{R}} \left| \mathbb{P} \left[\hat{S}_n > q \right] - \mathbb{P} [S > q] \right| \rightarrow 0,$$

so Theorem 2 follows immediately by plugging in $q = Q_{1-\alpha}$ and $\mathbb{P} [S > Q_{1-\alpha}] = \alpha$. □

S.5 Proof of Theorem 3

By Lemma S.3 and Proposition 2, under H_0

$$\begin{aligned}
& \left| \sum_{1 \leq k < k' \leq \kappa_n} \hat{\lambda}_k \hat{\lambda}_{k'} \chi_{kk'}^2(1) - \sum_{1 \leq k < k' \leq \kappa_n} \lambda_k \lambda_{k'} \chi_{kk'}^2(1) \right| \\
& \leq \kappa_n^2 \max_{1 \leq k < k' \leq \kappa_n} \chi_{kk'}^2(1) \max_{1 \leq k < k' \leq \kappa_n} \left| \hat{\lambda}_k \hat{\lambda}_{k'} - \lambda_k \lambda_{k'} \right| \\
& \leq \kappa_n^2 \times \mathcal{O}_{a.s.}(\log n) \times \max_{1 \leq k < k' \leq \kappa_n} \left(\left| \hat{\lambda}_k - \lambda_k \right| \left| \hat{\lambda}_{k'} - \lambda_{k'} \right| + \lambda_k \left| \hat{\lambda}_{k'} - \lambda_{k'} \right| + \lambda_{k'} \left| \hat{\lambda}_k - \lambda_k \right| \right) \\
& \leq \kappa_n^2 \times \mathcal{O}_{a.s.}(\log n) \times \left(\max_{1 \leq k \leq \kappa_n} \left| \hat{\lambda}_k - \lambda_k \right|^2 + \lambda_1 \max_{1 \leq k \leq \kappa_n} \left| \hat{\lambda}_k - \lambda_k \right| \right) \\
& \leq \kappa_n^2 \times \mathcal{O}_{a.s.}(\log n) \times \mathcal{O}_{a.s.} \left(n^{-1/2} \log^{1/2} n \right) \\
& = \mathcal{O}_{a.s.} \left(\kappa_n^2 n^{-1/2} \log^{3/2} n \right).
\end{aligned}$$

By recalling (2.22) that $\kappa_n^2 n^{-1/2} \log^{3/2} n \rightarrow 0$,

$$\left| \sum_{1 \leq k < k' \leq \kappa_n} \hat{\lambda}_k \hat{\lambda}_{k'} \chi_{kk'}^2(1) - \sum_{1 \leq k < k' \leq \kappa_n} \lambda_k \lambda_{k'} \chi_{kk'}^2(1) \right| = o_{a.s.}(1).$$

Note next that for $1 \leq k < k'$, $\mathbb{E} \chi_{kk'}^2(1) = 1$, hence by taking $R_{kk',n} = \chi_{kk'}^2(1)$, $M = 1$ in Lemma S.5, one obtains that

$$\sum_{1 \leq k < k', k' > \kappa_n} \lambda_k \lambda_{k'} \chi_{kk'}^2(1) = o_p(1).$$

Combining the above, one obtains that

$$\begin{aligned}
& \left| \bar{S}_n - S \right| = \left| \sum_{1 \leq k < k' \leq \kappa_n} \hat{\lambda}_k \hat{\lambda}_{k'} \chi_{kk'}^2(1) - \sum_{1 \leq k < k' < \infty} \lambda_k \lambda_{k'} \chi_{kk'}^2(1) \right| \\
& \leq \left| \sum_{1 \leq k < k' \leq \kappa_n} \hat{\lambda}_k \hat{\lambda}_{k'} \chi_{kk'}^2(1) - \sum_{1 \leq k < k' \leq \kappa_n} \lambda_k \lambda_{k'} \chi_{kk'}^2(1) \right| + \sum_{1 \leq k < k', k' > \kappa_n} \lambda_k \lambda_{k'} \chi_{kk'}^2(1)
\end{aligned}$$

$$= o_{a.s.}(1) + o_p(1) = o_p(1).$$

Since the distribution function F_S of S is strictly increasing and continuous and $\bar{S}_n - S = o_p(1)$, Lemma 11.2.1(ii) of Lehmann et al. (2005) ensures that the $100(1 - \alpha)$ -th percentile $\hat{Q}_{1-\alpha}$ of \bar{S}_n converges to that of S , i.e., $|\hat{Q}_{1-\alpha} - Q_{1-\alpha}| = o_p(1)$. Applying Slutsky's Theorem to

$$\hat{S}_n \xrightarrow{D} S, Q_{1-\alpha} - \hat{Q}_{1-\alpha} \rightarrow_p 0,$$

one obtains that

$$\hat{S}_n + Q_{1-\alpha} - \hat{Q}_{1-\alpha} \xrightarrow{D} S.$$

This means that as $n \rightarrow \infty$,

$$\mathbb{P} \left[\hat{S}_n + Q_{1-\alpha} - \hat{Q}_{1-\alpha} \leq Q_{1-\alpha} \right] \rightarrow 1 - \alpha$$

or equivalently

$$\mathbb{P} \left[\hat{S}_n > \hat{Q}_{1-\alpha} \right] \rightarrow \alpha.$$

Likewise, $\mathbb{P} \left(\tilde{S}_n > \hat{Q}_{1-\alpha} \right) \rightarrow \alpha$. This has completed the proof. \square

S.6 Proof of Theorem 4

Under H_1 in (2.3), there exist $k_1 < k_2 \in \mathbb{N}_+$ such that

$$C_{k_1 k_2} = \int G(x, x') \psi_{0, k_1}(x) \psi_{0, k_2}(x') dx dx' \neq 0. \quad (\text{S.44})$$

Applying Hilbert space Central Limit Theorem of (2.16), one concludes that $n^{1/2}(Z_{k_1 k_2} - C_{k_1 k_2})$ converges weakly to a Gaussian distribution of mean zero and finite variance, thus

$$Z_{k_1 k_2} = C_{k_1 k_2} + \mathcal{O}_p(n^{-1/2}), Z_{k_1 k_2}^2 = C_{k_1 k_2}^2 + \mathcal{O}_p(n^{-1/2}). \quad (\text{S.45})$$

According to (2.8), $\tilde{S}_n = n \sum_{1 \leq k < k' < \infty} Z_{kk'}^2 \geq n Z_{k_1 k_2}^2$, so applying (S.45), as $n \rightarrow \infty$

$$\begin{aligned} \mathbb{P}[\tilde{S}_n \geq Q_{1-\alpha}] &\geq \mathbb{P}[n Z_{k_1 k_2}^2 \geq Q_{1-\alpha}] = \mathbb{P}[Z_{k_1 k_2}^2 \geq Q_{1-\alpha}/n] \\ &= \mathbb{P}[C_{k_1 k_2}^2 \geq Q_{1-\alpha}/n + \mathcal{O}_p(n^{-1/2})] = \mathbb{P}[C_{k_1 k_2}^2 \geq \mathcal{O}_p(n^{-1/2})] \rightarrow 1, \end{aligned}$$

because $C_{k_1 k_2}^2 > 0$ according to (S.44).

Putting together (S.24) in Proposition 2 and (S.45), one obtains

$$\hat{Z}_{k_1 k_2} = C_{k_1 k_2} + \mathcal{O}_p(n^{-1/2}), \hat{Z}_{k_1 k_2}^2 = C_{k_1 k_2}^2 + \mathcal{O}_p(n^{-1/2}). \quad (\text{S.46})$$

Since $\kappa_n \rightarrow \infty$ by (2.22), so for large enough n , $\kappa_n > \max(k_1, k_2)$, and according to (2.21) $\hat{S}_n = n \sum_{1 \leq k < k' \leq \kappa_n} \hat{Z}_{kk'}^2$, so $\hat{S}_n \geq n \hat{Z}_{k_1 k_2}^2$. Consequently, applying (S.46), as $n \rightarrow \infty$

$$\begin{aligned} \mathbb{P}[\hat{S}_n \geq Q_{1-\alpha}] &\geq \mathbb{P}[n \hat{Z}_{k_1 k_2}^2 \geq Q_{1-\alpha}] = \mathbb{P}[\hat{Z}_{k_1 k_2}^2 \geq Q_{1-\alpha}/n] \\ &= \mathbb{P}[C_{k_1 k_2}^2 \geq Q_{1-\alpha}/n + \mathcal{O}_p(n^{-1/2})] = \mathbb{P}[C_{k_1 k_2}^2 \geq \mathcal{O}_p(n^{-1/2})] \rightarrow 1, \end{aligned}$$

again making use of $C_{k_1 k_2}^2 > 0$ according to (S.44).

Making use of (S.23), one obtains

$$\sum_{1 \leq k < k' \leq \kappa_n} \hat{\lambda}_k \hat{\lambda}_{k'} \chi_{kk'}^2(1) \leq \Lambda_n^2 \sum_{1 \leq k < k' \leq \kappa_n} \chi_{kk'}^2(1) = \Lambda_n^2 \chi^2 \left(\frac{\kappa_n(\kappa_n - 1)}{2} \right).$$

Note that for the standard χ^2 distribution with degree of freedom $\frac{\kappa_n(\kappa_n - 1)}{2} \rightarrow$

∞ , its $(1 - \alpha)$ -th quantile is asymptotically

$$\frac{\kappa_n(\kappa_n - 1)}{2} + \kappa_n^{1/2} (\kappa_n - 1)^{1/2} z_{1-\alpha/2} + o(\kappa_n) \leq 3\kappa_n^2$$

for large enough n . Thus for large enough n , $\hat{Q}_{1-\alpha} \leq 3\kappa_n^2 \Lambda_n^2$, which implies

that for large enough n ,

$$\begin{aligned} \mathbb{P} \left[\hat{S}_n \geq \hat{Q}_{1-\alpha} \right] &\geq \mathbb{P} \left[n \hat{Z}_{k_1 k_2}^2 \geq 3\kappa_n^2 \Lambda_n^2 \right] = \mathbb{P} \left[\hat{Z}_{k_1 k_2}^2 \geq 3\kappa_n^2 \Lambda_n^2 / n \right] \\ &= \mathbb{P} \left[C_{k_1 k_2}^2 \geq 3\kappa_n^2 \Lambda_n^2 / n + \mathcal{O}_p(n^{-1/2}) \right] = \mathbb{P} \left[C_{k_1 k_2}^2 \geq \mathcal{O}_p(n^{-1/2}) \right] \rightarrow 1, \end{aligned}$$

again making use of $C_{k_1 k_2}^2 > 0$ according to (S.44). The arguments for

$\mathbb{P} \left[\tilde{S}_n \geq \hat{Q}_{1-\alpha} \right]$ is similar.

The proof is completed. \square

S.7 Proof of Theorem 5

For B-spline trajectory estimates, (S.39) in Lemma S.12 provides that

$$\max_{1 \leq i \leq n} \left\| \hat{\xi}_i - \xi_i + n^{-1} \sum_{i'=1}^n \xi_{i'} \right\|_{\infty} = \mathcal{O}_{a.s.} \left(J_s^{-p^*} (n \log n)^{2/r_1} + N^{-1/2} J_s^{1/2} \log^{1/2} N + J_s N^{\beta_2 - 1} \right),$$

hence Assumption (B2) is verified if one shows that

$$\kappa_n^2 n^{1/2} \left\{ J_s^{-p^*} (n \log n)^{2/r_1} + N^{-1/2} J_s^{1/2} \log^{1/2} N + J_s N^{\beta_2 - 1} \right\} \log^{1/2} n \rightarrow 0,$$

for some $\{\kappa_n\}_{n=1}^\infty$ that satisfy (2.22). One needs to show only that

$$n^{1/2} \log^{1/2} n \times J_s^{-p^*} (n \log n)^{2/r_1} \rightarrow 0,$$

$$n^{1/2} \log^{1/2} n \times N^{-1/2} J_s^{1/2} \log^{1/2} N \rightarrow 0,$$

$$n^{1/2} \log^{1/2} n \times J_s N^{\beta_2-1} \rightarrow 0,$$

which follow from Assumptions (C3)-(C5), with all constraints (2.30), (2.31), (2.32), (2.33) on $\theta, p^*, \beta_2, r_1, \gamma$ included.

The proof is completed. □

References

- Bosq, D. (1998). *Nonparametric Statistics for Stochastic Processes*. Springer-Verlag, New York.
- Bosq, D. (2000). *Linear Processes in Function Spaces: Theory and Applications*. Springer-Verlag, New York.
- Cao, G., Yang L. and Todem, D. (2012). Simultaneous inference for the mean function based on dense functional data. *Journal of Nonparametric Statistics* **24**, 359-377.
- Csörgő, M. and Révész, P. (1981). *Strong Approximations in Probability and Statistics*. Academic Press, New York-London.
- Christoph, G., Prokhorov, Y. V. and Ulyanov, V. V. (1996). On distribution of quadratic forms in Gaussian random variables. *Theory of Probability & Its Applications* **40**, 250-260.
- DeVore, R. and Lorentz, G. (1993). *Constructive Approximation: Polynomials and Splines*

REFERENCES

Approximation. Springer-Verlag, Berlin.

Hoeffding, W. (1964). On a theorem of VM Zolotarev. *Theory of Probability & Its Applications* **9**, 89-91.

Lehmann, E. L., Romano, J. P. and Casella, G. (2005). *Testing Statistical Hypotheses*. Springer-Verlag, New York.

Wang, J., Cao, G., Wang, L. and Yang, L. (2020). Simultaneous confidence band for stationary covariance function of dense functional data. *Journal of Multivariate Analysis* **176**, 104584.

Zolotarev, V. M. (1961). Concerning a certain probability problem. *Theory of Probability & Its Applications* **6**, 201-204.