

Functional Horseshoe Smoothing for Functional Trend Estimation

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Supplementary Material

This supplementary material contains detailed forms of posterior distributions of main parameters and full proofs of the propositions in the main text.

S1 Detailed forms of full conditional distributions

The full conditional distribution of \mathbf{b}_t is $N(\boldsymbol{\mu}_t, \Sigma_t)$. When $k = 0$, it holds

that

$$\boldsymbol{\mu}_t = \begin{cases} \Sigma_t (\lambda_t^{-2} \Phi_t^\top \Phi_t \mathbf{b}_{t+1} + \lambda_{t-1}^{-2} \Phi_{t-1}^\top \Phi_{t-1} \mathbf{b}_{t-1} + \tau^2 \Phi_t^\top \mathbf{y}_t) / (\tau^2 \sigma^2) & 2 \leq t \leq T - 1 \\ \Sigma_t (\lambda_t^{-2} \Phi_t^\top \Phi_t \mathbf{b}_{t+1} + \tau^2 \Phi_t^\top \mathbf{y}_t) / (\tau^2 \sigma^2) & t = 1 \\ \Sigma_t (\lambda_{t-1}^{-2} \Phi_{t-1}^\top \Phi_{t-1} \mathbf{b}_{t-1} + \tau^2 \Phi_t^\top \mathbf{y}_t) / (\tau^2 \sigma^2) & t = T \end{cases}$$

and

$$\Sigma_t^{-1} = \begin{cases} \{(\tau^2 + \lambda_t^{-2}) \Phi_t^\top \Phi_t + \lambda_{t-1}^{-2} \Phi_{t-1}^\top \Phi_{t-1}\} / (\tau^2 \sigma^2) & 2 \leq t \leq T - 1 \\ (\tau^2 + \lambda_t^{-2}) \Phi_t^\top \Phi_t / (\tau^2 \sigma^2) & t = 1 \\ \tau^2 \Phi_t^\top \Phi_t + \lambda_{t-1}^{-2} \Phi_{t-1}^\top \Phi_{t-1} / (\tau^2 \sigma^2) & t = T. \end{cases}$$

Also, when $k = 1$, mean vector and precision matrix are

$$\boldsymbol{\mu}_t = \begin{cases} \Sigma_t \left(\frac{\Phi_t^\top \Phi_t (2\mathbf{b}_{t+1} - \mathbf{b}_{t+2})}{\sigma^2 \tau^2 \lambda_t^2} + \frac{\Phi_t^\top \mathbf{y}_t}{\sigma^2} \right) & t = 1 \\ \Sigma_t \left(\frac{\Phi_t^\top \Phi_t (2\mathbf{b}_{t+1} - \mathbf{b}_{t+2})}{\sigma^2 \tau^2 \lambda_t^2} + \frac{2\Phi_{t-1}^\top \Phi_{t-1} (\mathbf{b}_{t+1} + \mathbf{b}_{t-1})}{\sigma^2 \tau^2 \lambda_{t-1}^2} + \frac{\Phi_t^\top \mathbf{y}_t}{\sigma^2} \right) & t = 2 \\ \Sigma_t \left(\frac{\Phi_t^\top \Phi_t (2\mathbf{b}_{t+1} - \mathbf{b}_{t+2})}{\sigma^2 \tau^2 \lambda_t^2} + \frac{2\Phi_{t-1}^\top \Phi_{t-1} (\mathbf{b}_{t+1} + \mathbf{b}_{t-1})}{\sigma^2 \tau^2 \lambda_{t-1}^2} + \frac{\Phi_{t-2}^\top \Phi_{t-2} (2\mathbf{b}_{t-1} - \mathbf{b}_{t-2})}{\sigma^2 \tau^2 \lambda_{t-2}^2} + \frac{\Phi_t^\top \mathbf{y}_t}{\sigma^2} \right) & 3 \leq t \leq T - 2 \\ \Sigma_t \left(\frac{2\Phi_{t-1}^\top \Phi_{t-1} (\mathbf{b}_{t+1} + \mathbf{b}_{t-1})}{\sigma^2 \tau^2 \lambda_{t-1}^2} + \frac{\Phi_{t-2}^\top \Phi_{t-2} (2\mathbf{b}_{t-1} - \mathbf{b}_{t-2})}{\sigma^2 \tau^2 \lambda_{t-2}^2} + \frac{\Phi_t^\top \mathbf{y}_t}{\sigma^2} \right) & t = T - 1 \\ \Sigma_t \left(\frac{\Phi_{t-2}^\top \Phi_{t-2} (2\mathbf{b}_{t-1} - \mathbf{b}_{t-2})}{\sigma^2 \tau^2 \lambda_{t-2}^2} + \frac{\Phi_t^\top \mathbf{y}_t}{\sigma^2} \right) & t = T \end{cases}$$

and

$$\Sigma_t^{-1} = \begin{cases} \frac{\tau^2 \lambda_t^2 + 1}{\sigma^2 \tau^2 \lambda_t^2} \Phi_t^\top \Phi_t & t = 1 \\ \frac{\tau^2 \lambda_t^2 + 1}{\sigma^2 \tau^2 \lambda_t^2} \Phi_t^\top \Phi_t + \frac{4}{\sigma^2 \tau^2 \lambda_{t-1}^2} \Phi_{t-1}^\top \Phi_{t-1} & t = 2 \\ \frac{\tau^2 \lambda_t^2 + 1}{\sigma^2 \tau^2 \lambda_t^2} \Phi_t^\top \Phi_t + \frac{4}{\sigma^2 \tau^2 \lambda_{t-1}^2} \Phi_{t-1}^\top \Phi_{t-1} + \frac{1}{\sigma^2 \tau^2 \lambda_{t-2}^2} \Phi_{t-2}^\top \Phi_{t-2} & 3 \leq t \leq T - 2 \\ \frac{\tau^2 \lambda_t^2}{\sigma^2 \tau^2 \lambda_t^2} \Phi_t^\top \Phi_t + \frac{4}{\sigma^2 \tau^2 \lambda_{t-1}^2} \Phi_{t-1}^\top \Phi_{t-1} + \frac{1}{\sigma^2 \tau^2 \lambda_{t-2}^2} \Phi_{t-2}^\top \Phi_{t-2} & t = T - 1 \\ \frac{\tau^2 \lambda_t^2}{\sigma^2 \tau^2 \lambda_t^2} \Phi_t^\top \Phi_t + \frac{1}{\sigma^2 \tau^2 \lambda_{t-2}^2} \Phi_{t-2}^\top \Phi_{t-2} & t = T \end{cases}$$

Assume data is observed homogeneously, that is, $\Phi_1 = \Phi_2 = \dots = \Phi_T$.

When $k = 0$, mean vector and precision matrix can be written as

$$\boldsymbol{\mu}_t = \begin{cases} \{c_t (\lambda_t^{-2} \mathbf{b}_{t+1} + \lambda_{t-1}^{-2} \mathbf{b}_{t-1}) + \tau^2 \Sigma_t \Phi_t^\top \mathbf{y}_t\} / (\tau^2 \sigma^2) & 2 \leq t \leq T - 1 \\ (c_t \lambda_t^{-2} \mathbf{b}_{t+1} + \tau^2 \Sigma_t \Phi_t^\top \mathbf{y}_t) / (\tau^2 \sigma^2) & t = 1 \\ (c_t \lambda_{t-1}^{-2} \mathbf{b}_{t-1}) + \tau^2 \Sigma_t \Phi_t^\top \mathbf{y}_t / (\tau^2 \sigma^2) & t = T \end{cases}$$

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and $\Sigma_t = c_t(\Phi^\top \Phi)^{-1}$ with

$$c_t = \begin{cases} \tau^2 \sigma^2 / (\tau^2 + \lambda_t^{-2} + \lambda_{t-1}^{-2}) & 2 \leq t \leq T-1 \\ \tau^2 \sigma^2 / (\tau^2 + \lambda_t^{-2}) & t = 1 \\ \tau^2 \sigma^2 / (\tau^2 + \lambda_{t-1}^{-2}) & t = T. \end{cases}$$

Furthermore, when $k = 1$, the forms of $\boldsymbol{\mu}_t$ and c_t are

$$\boldsymbol{\mu}_t = \begin{cases} [c_t \{ (2\lambda_t^{-2} + 2\lambda_{t-1}^{-2}) \mathbf{b}_{t+1} + (2\lambda_{t-1}^{-2} + 2\lambda_{t-2}^{-2}) \mathbf{b}_{t-1} - \lambda_t^{-2} \mathbf{b}_{t+2} - \lambda_{t-2}^{-2} \mathbf{b}_{t-2} \} \\ \quad + \tau^2 \Sigma_t \Phi^\top \mathbf{y}_t] / (\tau^2 \sigma^2) & 3 \leq t \leq T-2 \\ [c_t (2\lambda_t^{-2} \mathbf{b}_{t+1} - \lambda_t^{-2} \mathbf{b}_{t+2}) + \tau^2 \Sigma_t \Phi^\top \mathbf{y}_t] / (\tau^2 \sigma^2) & t = 1 \\ [c_t \{ (2\lambda_t^{-2} + 2\lambda_{t-1}^{-2}) \mathbf{b}_{t+1} + 2\lambda_{t-1}^{-2} \mathbf{b}_{t-1} - \lambda_t^{-2} \mathbf{b}_{t+2} \} + \tau^2 \Sigma_t \Phi^\top \mathbf{y}_t] / (\tau^2 \sigma^2) & t = 2 \\ [c_t \{ 2\lambda_{t-1}^{-2} \mathbf{b}_{t+1} + (2\lambda_{t-1}^{-2} + 2\lambda_{t-2}^{-2}) \mathbf{b}_{t-1} - \lambda_{t-2}^{-2} \mathbf{b}_{t-2} \} + \tau^2 \Sigma_t \Phi^\top \mathbf{y}_t] / (\tau^2 \sigma^2) & t = T-1 \\ [c_t (2\lambda_{t-2}^{-2} \mathbf{b}_{t-1} - \lambda_{t-2}^{-2} \mathbf{b}_{t-2}) + \tau^2 \Sigma_t \Phi^\top \mathbf{y}_t] / (\tau^2 \sigma^2) & t = T \end{cases}$$

and

$$c_t = \begin{cases} \tau^2 \sigma^2 / (\tau^2 + \lambda_t^{-2} + 4\lambda_{t-1}^{-2} + \lambda_{t-2}^{-2}) & 3 \leq t \leq T-2 \\ \tau^2 \sigma^2 / (\tau^2 + \lambda_t^{-2} + 4\lambda_{t-1}^{-2}) & t = 2 \\ \tau^2 \sigma^2 / (\tau^2 + \lambda_t^{-2}) & t = 1 \\ \tau^2 \sigma^2 / (\tau^2 + 4\lambda_{t-1}^{-2} + \lambda_{t-2}^{-2}) & t = T-1 \\ \tau^2 \sigma^2 / (\tau^2 + \lambda_{t-2}^{-2}) & t = T. \end{cases}$$

S2 Proofs of the propositions

This section provides proofs of Proposition 1 and Proposition 2. Before proving them, we introduce the important notations. If $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$, we denote $f \approx g$. \otimes denotes Kronecker product. $\|M\|$ is the operator norm of any matrix M . Let $\text{blockdiag}(M_1, \dots, M_K)$ be the block diagonal matrix, whose k th diagonal matrix is M_k , and 1_E be indicator function, which is 1 if E is true and 0 otherwise. \mathbf{e}_i denotes the vector with a 1 in the i th coordinate and 0 other coordinates. I denotes identity matrix and its dimension is not written when it is trivial.

Proof of Proposition 1 (i). For $L = 1$, the divergence of $\pi(\boldsymbol{\delta}_t \mid \tau, \sigma)$ follows from the proof of Theorem 1 in Carvalho et al. (2010). We restrict our investigation to $L \geq 2$. For simplicity, we fix $\sigma = \tau = 1$. We have

$$\begin{aligned} \pi(\boldsymbol{\delta}_t) &\propto \int_0^\infty \frac{1}{\lambda_t^L(1 + \lambda_t^2)} \exp \left\{ -\frac{1}{2\lambda_t^2} \boldsymbol{\delta}_t^\top \Phi_t^\top \Phi_t \boldsymbol{\delta}_t \right\} d\lambda_t \\ &\geq \int_0^1 \frac{1}{\lambda_t^L(1 + \lambda_t^2)} \exp \left\{ -\frac{1}{2\lambda_t^2} \boldsymbol{\delta}_t^\top \Phi_t^\top \Phi_t \boldsymbol{\delta}_t \right\} d\lambda_t. \end{aligned}$$

Let $u = \lambda_t^{-2}$. Then, we obtain

$$\begin{aligned} \pi(\boldsymbol{\delta}_t) &\geq 2 \int_1^\infty \frac{u^{\frac{L-1}{2}}}{1+u} \exp \left\{ -\frac{u}{2} \boldsymbol{\delta}_t^\top \Phi_t^\top \Phi_t \boldsymbol{\delta}_t \right\} du \\ &\geq \int_1^\infty u^{\frac{L-3}{2}} \exp \left\{ -\frac{u}{2} \boldsymbol{\delta}_t^\top \Phi_t^\top \Phi_t \boldsymbol{\delta}_t \right\} du. \end{aligned}$$

Also, we transform $\tilde{u} = \frac{\boldsymbol{\delta}_t^\top \Phi_t^\top \Phi_t \boldsymbol{\delta}_t}{2} u$. It leads to

$$\pi(\boldsymbol{\delta}_t) \geq \left(\frac{\sqrt{2}}{\|\Phi \boldsymbol{\delta}_t\|_2} \right)^{L-1} \int_{\frac{\|\Phi \boldsymbol{\delta}_t\|_2^2}{2}}^{\infty} \tilde{u}^{\frac{L-3}{2}} \exp\{-\tilde{u}\} d\tilde{u}.$$

The right-hand equals to

$$\left(\frac{\sqrt{2}}{\|\Phi \boldsymbol{\delta}_t\|_2} \right)^{L-1} \Gamma\left(\frac{L-1}{2}, \frac{\|\Phi \boldsymbol{\delta}_t\|_2^2}{2}\right),$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function. If $\boldsymbol{\delta}_t \rightarrow 0$, it holds that

$$\frac{\sqrt{2}}{\|\Phi \boldsymbol{\delta}_t\|_2} \rightarrow \infty, \quad \Gamma\left(\frac{L-1}{2}, \frac{\|\Phi \boldsymbol{\delta}_t\|_2^2}{2}\right) \rightarrow \Gamma\left(\frac{L-1}{2}\right)$$

Hence, $\pi(\boldsymbol{\delta}_t) \rightarrow \infty$ as $\boldsymbol{\delta}_t \rightarrow 0$. □

Before proving Proposition 1 (ii), we introduce an important result.

First, slight modification with Lemma S5 in Hamura et al. (2020) leads to the following lemma, which gives rise to Corollary 1.

Lemma 1. *Let $f(\cdot)$ and $g(\cdot)$ be positive and continuous functions such that $e^{-\frac{1}{x}} f(x)$ and $e^{-\frac{1}{x}} g(x)$ are integrable on the real line. Then, it holds that*

$$\frac{\lim_{x \rightarrow \infty} \int_0^\infty \phi(x; 0, v) f(v) dv}{\lim_{x \rightarrow \infty} \int_0^\infty \phi(x; 0, v) g(v) dv} = \lim_{v \rightarrow \infty} \frac{f(v)}{g(v)}, \quad (\text{S2.1})$$

where $\phi(x; 0, v)$ is p.d.f. of $N(0, v)$.

Corollary 1. *Let $f(v) = \frac{1}{v^{L/2}(1+v)}$ and $g(v) = \frac{1}{v^{L/2+1}}$ for fixed $L \in \mathbb{N}$. Then,*

$\int_0^\infty \phi(x; 0, v) f(v) dv \approx \int_0^\infty \phi(x; 0, v) g(v) dv$ holds.

Proof of Corollary 1. These are obviously positive and continuous. Also

$e^{-\frac{1}{x}}f(x)$ and $e^{-\frac{1}{x}}g(x)$ are integrable on the real line, since

$$\begin{aligned} \int_0^\infty f(v)e^{-\frac{1}{v}}dv &\leq \int_0^\infty \frac{1}{v^{\frac{L}{2}+1}}e^{-\frac{1}{v}}dv + \int_0^\infty \frac{1}{v^{\frac{L}{2}}}e^{-\frac{1}{v}}dv \\ &\leq \Gamma\left(\frac{L+1}{2}\right) + \Gamma\left(\frac{L-1}{2}\right) \\ &< \infty, \end{aligned}$$

and similarly $\int_0^\infty g(v)\phi(x; 0, v)dv < \infty$. Then, Lemma 1 and $\lim_{v \rightarrow \infty} f(v)/g(v) = 1$ lead to the result. \square

Based on the above results, we prove Proposition 1 (ii) as follows.

Proof of Proposition 1 (ii). For simplicity, we fix $\sigma = \tau = 1$. With transformation $\bar{u} = \lambda_t^2$,

$$\begin{aligned} \pi(\boldsymbol{\delta}_t \mid \tau, \sigma) &\propto \int_0^\infty \frac{1}{\lambda_t^L(1 + \lambda_t^2)} \exp\left\{-\frac{1}{2\sigma^2\tau^2\lambda_t^2}\boldsymbol{\delta}_t^\top \Phi_t^\top \Phi_t \boldsymbol{\delta}_t\right\} d\lambda_t \\ &= \frac{1}{2} \int_0^\infty \frac{1}{\bar{u}^{\frac{L}{2}}(1 + \bar{u})} \frac{1}{\bar{u}^{\frac{1}{2}}} \exp\left(-\frac{z^2}{2\bar{u}}\right) d\bar{u}, \end{aligned}$$

where $z = \|\Phi_t \boldsymbol{\delta}_t\|_2$. From Corollary 1,

$$\begin{aligned} \frac{1}{2} \int_0^\infty \frac{1}{\bar{u}^{\frac{L}{2}}(1 + \bar{u})} \frac{1}{\bar{u}^{\frac{1}{2}}} \exp\left(-\frac{z^2}{2\bar{u}}\right) d\bar{u} &\approx \frac{1}{2} \int_0^\infty \frac{1}{\bar{u}^{\frac{L}{2}+1}} \frac{1}{\bar{u}^{\frac{1}{2}}} \exp\left(-\frac{z^2}{2\bar{u}}\right) d\bar{u} \\ &= \frac{1}{2} \frac{\Gamma\left(\frac{L+1}{2}\right)}{\left(\frac{z^2}{2}\right)^{\frac{L+1}{2}}} \\ &= O(z^{-L-1}). \end{aligned}$$

\square

Next, we prove the weak tail robustness of the posterior mean.

Proof of Proposition 2. We define

$$\begin{aligned} \mathbf{z} &\equiv (\mathbf{z}_1^\top, \dots, \mathbf{z}_{T-1}^\top)^\top \sim N(\boldsymbol{\eta}, \Sigma_z), \\ \boldsymbol{\eta} &\equiv ((\Phi\boldsymbol{\delta}_1)^\top, \dots, (\Phi\boldsymbol{\delta}_{T-1})^\top)^\top \sim \prod_{t=1}^{T-1} N(\mathbf{0}, \lambda_t^2 \Sigma_\delta), \\ \lambda_t^2 &\sim \prod_{t=1}^{T-1} \pi(\lambda_t) = \prod_{t=1}^{T-1} \frac{1}{1 + \lambda_t^2}, \end{aligned}$$

where $\Sigma_z \equiv 2\sigma^2 I_n \otimes I_L$ and $\Sigma_\delta \equiv \sigma^2 \tau^2 (\Phi_t^\top \Phi_t)^{-1}$.

Then we rewrite the posterior mean by score function. When we denote

$$\begin{aligned} m(\mathbf{z}) &= \int N(\mathbf{z} \mid \boldsymbol{\eta}, \Sigma_z) p(\boldsymbol{\eta}) d\boldsymbol{\eta}, \\ \frac{\partial m(\mathbf{z})}{\partial \mathbf{z}} &= \int -\Sigma_z (\mathbf{z} - \boldsymbol{\eta}) N(\mathbf{z} \mid \boldsymbol{\eta}, \Sigma_z) p(\boldsymbol{\eta}) d\boldsymbol{\eta}, \end{aligned}$$

the following representation is available,

$$\mathbb{E}[\boldsymbol{\eta} \mid \mathbf{z}] - \mathbf{z} = \Sigma_z \frac{1}{m(\mathbf{z})} \frac{\partial m(\mathbf{z})}{\partial \mathbf{z}}.$$

The goal is to find how this behaves as $\|\mathbf{z}_{t^*}\|_2 \rightarrow \infty$. Hence, we analyze

$m(\mathbf{z})$ first.

Step 1: Analysis of $m(\mathbf{z})$. With $T = \text{blockdiag}(\lambda_1 \Sigma_\delta^{1/2}, \dots, \lambda_{T-1} \Sigma_\delta^{1/2})$,

$$\begin{aligned}
 m(\mathbf{z}) &\propto \int \int \left(\prod_{t=1}^{T-1} \frac{\pi(\lambda_t)}{\lambda_t^L} \right) \exp \left\{ -\frac{1}{2} (\boldsymbol{\eta} - \mathbf{z})^\top \Sigma_z^{-1} (\boldsymbol{\eta} - \mathbf{z}) - \frac{1}{2} \boldsymbol{\eta}^\top T^{-2} \boldsymbol{\eta} \right\} d\boldsymbol{\eta} d\boldsymbol{\lambda} \\
 &= \int \int \exp \left\{ -\frac{1}{2} \{ \boldsymbol{\eta} - (\Sigma_z^{-1} + T^{-2})^{-1} \Sigma_z^{-1} \mathbf{z} \}^\top (\Sigma_z^{-1} + T^{-2}) \{ \boldsymbol{\eta} - (\Sigma_z^{-1} + T^{-2})^{-1} \Sigma_z^{-1} \mathbf{z} \} \right\} d\boldsymbol{\eta} \\
 &\quad \times \left(\prod_{t=1}^{T-1} \frac{\pi(\lambda_t)}{\lambda_t^L} \right) \exp \left\{ -\frac{1}{2} \mathbf{z}^\top \Sigma_z^{-1} \{ \Sigma_z - (\Sigma_z^{-1} + T^{-2})^{-1} \} \Sigma_z^{-1} \mathbf{z} \right\} d\boldsymbol{\lambda} \\
 &\propto \int \int N(\boldsymbol{\eta} \mid (\Sigma_z^{-1} + T^{-2})^{-1} \Sigma_z^{-1} \mathbf{z}, (\Sigma_z^{-1} + T^{-2})^{-1}) d\boldsymbol{\eta} \\
 &\quad \times \frac{1}{|\Sigma_z^{-1} + T^{-2}|^{1/2}} \left(\prod_{t=1}^{T-1} \frac{\pi(\lambda_t)}{\lambda_t^L} \right) \exp \left\{ -\frac{1}{2} \mathbf{z}^\top \Sigma_z^{-1} \{ \Sigma_z - (\Sigma_z^{-1} + T^{-2})^{-1} \} \Sigma_z^{-1} \mathbf{z} \right\} d\boldsymbol{\lambda} \\
 &= \int \frac{1}{|\Sigma_z^{-1} + T^{-2}|^{1/2}} \left(\prod_{t=1}^{T-1} \frac{\pi(\lambda_t)}{\lambda_t^L} \right) \exp \left\{ -\frac{1}{2} \mathbf{z}^\top \Sigma_z^{-1} \{ \Sigma_z - (\Sigma_z^{-1} + T^{-2})^{-1} \} \Sigma_z^{-1} \mathbf{z} \right\} d\boldsymbol{\lambda}.
 \end{aligned}$$

From Woodbury matrix identity, we see

$$\begin{aligned}
 m(\mathbf{z}) &\propto \int \frac{\prod_{t=1}^{T-1} \lambda_t^L |\Sigma_\delta|}{|I + T^{-1} \Sigma_z^{-1} T^{-1}|^{1/2}} \left(\prod_{t=1}^{T-1} \frac{\pi(\lambda_t)}{\lambda_t^L} \right) \exp \left\{ -\frac{1}{2} \mathbf{z}^\top T^{-1} \{ I + T^{-1} \Sigma_z T^{-1} \}^{-1} T^{-1} \mathbf{z} \right\} d\boldsymbol{\lambda} \\
 &\propto \int \frac{\prod_{t=1}^{T-1} \pi(\lambda_t)}{|I + T^{-1} \Sigma_z^{-1} T^{-1}|^{1/2}} \exp \left\{ -\frac{1}{2} \mathbf{z}^\top T^{-1} \{ I + T^{-1} \Sigma_z T^{-1} \}^{-1} T^{-1} \mathbf{z} \right\} d\boldsymbol{\lambda}.
 \end{aligned}$$

Hence, we obtain

$$\begin{aligned}
 \frac{1}{m(\mathbf{z})} \frac{\partial m(\mathbf{z})}{\partial \mathbf{z}} &= \int_{\mathbb{R}_+^{T-1}} \frac{\prod_{t=1}^{T-1} \pi(\lambda_t)}{|I + T^{-1} \Sigma_z^{-1} T^{-1}|^{1/2}} \exp \left\{ -\frac{1}{2} \mathbf{z}^\top T^{-1} \{ I + T^{-1} \Sigma_z T^{-1} \}^{-1} T^{-1} \mathbf{z} \right\} \\
 &\quad \{ -T^{-1} \{ I + T^{-1} \Sigma_z T^{-1} \}^{-1} T^{-1} \mathbf{z} \} d\boldsymbol{\lambda} \\
 &\quad / \int_{\mathbb{R}_+^{T-1}} \frac{\prod_{t=1}^{T-1} \pi(\lambda_t)}{|I + T^{-1} \Sigma_z^{-1} T^{-1}|^{1/2}} \exp \left\{ -\frac{1}{2} \mathbf{z}^\top T^{-1} \{ I + T^{-1} \Sigma_z T^{-1} \}^{-1} T^{-1} \mathbf{z} \right\} d\boldsymbol{\lambda}.
 \end{aligned} \tag{S2.2}$$

Define $v_t \equiv \frac{\lambda_t}{\|\mathbf{z}_t\|_2^2}$, $W \equiv \text{diag}(\|\mathbf{z}_1\|_2^2, \dots, \|\mathbf{z}_{T-1}\|_2^2)$ and $\boldsymbol{\omega} \equiv (W^{-1} \otimes I_L) \mathbf{z}$.

With $V \equiv \text{diag}(v_1, \dots, v_{T-1})$, remark that $T = (VW) \otimes \Sigma_\delta^{1/2}$ holds. Using these new variables, we calculate (S2.2) as

$$\begin{aligned}
 & \int_{\mathbb{R}_+^{T-1}} \frac{\prod_{t=1}^{T-1} \pi(\|\mathbf{z}_t\|_2^2 v_t)}{|I + \{(VW) \otimes \Sigma_\delta^{1/2}\} \Sigma_z^{-1} \{(VW) \otimes \Sigma_\delta^{1/2}\}|^{1/2}} \\
 & \quad \exp \left\{ -\frac{1}{2} \boldsymbol{\omega}^\top \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} \\
 & \quad \left\{ -\{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} d\mathbf{v} \\
 & / \int_{\mathbb{R}_+^{T-1}} \frac{\prod_{t=1}^{T-1} \pi(\|\mathbf{z}_t\|_2^2 v_t)}{|I + \{(VW) \otimes \Sigma_\delta^{1/2}\} \Sigma_z^{-1} \{(VW) \otimes \Sigma_\delta^{1/2}\}|^{1/2}} \\
 & \quad \exp \left\{ -\frac{1}{2} \boldsymbol{\omega}^\top \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} d\mathbf{v}.
 \end{aligned} \tag{S2.3}$$

Step 2: Limit of the integrand. To evaluate (S2.3) by the dominated convergence theorem, we need the integrable dominating functions and limits of the integrands. We calculate the limits in this step and find the dominating function in the next step. $t^* \in \{1, \dots, T-1\}$ is arbitrarily chosen index and we fix \mathbf{z}_t for $t \neq t^*$. We investigate (S2.3) when $\|\mathbf{z}_{t^*}\|_2 \rightarrow \infty$. Since we are considering asymptotics regarding $\|\mathbf{z}_{t^*}\|_2$, if $\lim_{\|\mathbf{z}_{t^*}\|_2 \rightarrow \infty} f(\mathbf{z})/g(\mathbf{z}) = 1$, we simply denote $f \approx g$.

Then, we provide the following four approximations.

$$\text{E1: } \prod_{t=1}^{T-1} \pi(\|\mathbf{z}_t\|_2^2 v_t) \approx \prod_{t \neq t^*} \pi(\|\mathbf{z}_t\|_2^2 v_t) \pi(\|\mathbf{z}_{t^*}\|_2^2) / v_{t^*}^2$$

$$\text{E2: } |I + \{(VW) \otimes \Sigma_\delta^{1/2}\} \Sigma_z^{-1} \{(VW) \otimes \Sigma_\delta^{1/2}\}| \approx |W \otimes I_L| |\tilde{\Sigma} + V^2 \otimes \Sigma_\delta| / |\Sigma_z|$$

$$\text{E3: } \exp \left\{ -\frac{1}{2} \boldsymbol{\omega}^\top \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} \approx \exp \left(-\frac{1}{2} \boldsymbol{\omega}^\top (V^2 \otimes \Sigma_\delta + \tilde{\Sigma})^{-1} \boldsymbol{\omega} \right)$$

$$\text{E4: } \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \approx A(v_{t^*}) (I + A(v_{t^*}) \Sigma A(v_{t^*}))^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega},$$

where

$$A(v_{t^*}) \equiv \lim_{\|\mathbf{z}_{t^*}\|_2 \rightarrow \infty} \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1},$$

$$\tilde{\Sigma} \equiv \lim_{\|\mathbf{z}_{t^*}\|_2 \rightarrow \infty} (W \otimes I_L)^{-1} \Sigma_z (W \otimes I_L)^{-1}.$$

E1, E3 and E4 are obvious, and E2 is immediately obtained from the following relationship

$$\begin{aligned} & | -\Sigma_z \{I - \{(VW) \otimes \Sigma_\delta^{1/2}\} (-\Sigma_z^{-1}) \{(VW) \otimes \Sigma_\delta^{1/2}\}\} | \\ &= \left| \begin{array}{cc} I & (VW) \otimes \Sigma_\delta^{1/2} \\ (VW) \otimes \Sigma_\delta^{1/2} & -\Sigma_z \end{array} \right| \\ &= |I| | -\Sigma_z - \{(VW) \otimes \Sigma_\delta^{1/2}\} I \{(VW) \otimes \Sigma_\delta^{1/2}\} |. \end{aligned}$$

Then, the limit of the integrand of (S2.3) is a combination of E1~E4.

Step 3: Dominating function of the integrand. Assume $\|\mathbf{z}_{t^*}\|_2 \gg$

1. We continue (S2.3) as

$$\begin{aligned}
& \int_{\mathbb{R}_+^{T-1}} \frac{\prod_{t=1}^{T-1} \pi(\|\mathbf{z}_t\|_2^2 v_t)}{|I + \{(VW) \otimes \Sigma_\delta^{1/2}\} \Sigma_z^{-1} \{(VW) \otimes \Sigma_\delta^{1/2}\}|^{1/2}} \\
& \quad \exp \left\{ -\frac{1}{2} \boldsymbol{\omega}^\top \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} \\
& \quad \left\{ -\{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} d\mathbf{v} \\
& / \int_{\mathbb{R}_+^{T-1}} \frac{\prod_{t=1}^{T-1} \pi(\|\mathbf{z}_t\|_2^2 v_t)}{|I + \{(VW) \otimes \Sigma_\delta^{1/2}\} \Sigma_z^{-1} \{(VW) \otimes \Sigma_\delta^{1/2}\}|^{1/2}} \\
& \quad \exp \left\{ -\frac{1}{2} \boldsymbol{\omega}^\top \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} d\mathbf{v} \\
& = \int_{\mathbb{R}_+^{T-1}} \frac{\prod_{t=1}^{T-1} \pi(\|\mathbf{z}_t\|_2^2 v_t) / \pi(\|\mathbf{z}_t\|_2^2)}{|(W^{-1} \otimes I_L) \Sigma_z (W^{-1} \otimes I_L) + \{V^2 \otimes \Sigma_\delta\}|^{1/2}} \\
& \quad \exp \left\{ -\frac{1}{2} \boldsymbol{\omega}^\top \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} \\
& \quad \left\{ -\{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} d\mathbf{v} \\
& / \int_{\mathbb{R}_+^{T-1}} \frac{\prod_{t=1}^{T-1} \pi(\|\mathbf{z}_t\|_2^2 v_t) / \pi(\|\mathbf{z}_t\|_2^2)}{|(W^{-1} \otimes I_L) \Sigma_z (W^{-1} \otimes I_L) + \{V^2 \otimes \Sigma_\delta\}|^{1/2}} \\
& \quad \exp \left\{ -\frac{1}{2} \boldsymbol{\omega}^\top \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} d\mathbf{v} \\
& = \int_{\mathbb{R}_+^{T-1}} h(\mathbf{v}, \mathbf{z}_{t^*}) \left\{ -\{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\} \\
& \quad d\mathbf{v} / \int_{\mathbb{R}_+^{T-1}} h(\mathbf{v}, \mathbf{z}_{t^*}) d\mathbf{v},
\end{aligned}$$

where

$$\begin{aligned}
h(\mathbf{v}, \mathbf{z}_{t^*}) & \equiv \frac{\prod_{t=1}^{T-1} \pi(\|\mathbf{z}_t\|_2^2 v_t) / \pi(\|\mathbf{z}_t\|_2^2)}{|(W^{-1} \otimes I_L) \Sigma_z (W^{-1} \otimes I_L) + \{V^2 \otimes \Sigma_\delta\}|^{1/2}} \\
& \quad \exp \left\{ -\frac{1}{2} \boldsymbol{\omega}^\top \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\}.
\end{aligned}$$

Since there exists $\varepsilon_1 < 1, \varepsilon_2 < 1, M_1$ and M_2 such that $\varepsilon_1 I < \Sigma_z < M_1 I$ and $\varepsilon_2 I < \Sigma_\delta < M_2 I$, we have

$$\prod_{t=1}^{T-1} \frac{\pi(\|\mathbf{z}_t\|_2^2 v_t)}{\pi(\|\mathbf{z}_t\|_2^2)} = \prod_{t=1}^{T-1} \frac{1 + \|\mathbf{z}_t\|_2^4}{1 + \|\mathbf{z}_t\|_2^4 v_t^2} = \prod_{t \neq t^*} \frac{1 + \|\mathbf{z}_t\|_2^4}{1 + \|\mathbf{z}_t\|_2^4 v_t^2} \times \frac{2\|\mathbf{z}_{t^*}\|_2^4}{1 + \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2},$$

$$\begin{aligned} \exp \left\{ -\frac{1}{2} \boldsymbol{\omega}^\top \{M_2(V^2 \otimes I_L) + M_1(W^{-2} \otimes I_L)\}^{-1} \boldsymbol{\omega} \right\} &= \exp \left\{ -\frac{1}{2} \sum_{t=1}^{T-1} \{M_2 v_t^{2L} + M_1 \|\mathbf{z}_t\|_2^{-2L}\}^{-1} \boldsymbol{\omega} \right\} \\ &= \prod_{t=1}^{T-1} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_t\|_2^{4L}}{M_2 v_t^{4L} \|\mathbf{z}_t\|_2^{2L} + M_1} \right\}, \end{aligned}$$

and

$$\begin{aligned} |(W^{-1} \otimes I_L) \Sigma (W^{-1} \otimes I_L) + V^2 \otimes \Sigma_\delta|^{-1/2} &\leq |(\varepsilon_1 W^{-2} + \varepsilon_2 V^2) \otimes I_L|^{-1/2} \\ &= \prod_{t=1}^{T-1} \left(\frac{\|\mathbf{z}_t\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_t\|_2^4 v_t^2} \right)^{L/2}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} h(\mathbf{v}, \mathbf{z}_{t^*}) &\leq \left(\prod_{t \neq t^*} \left(\frac{\|\mathbf{z}_t\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_t\|_2^4 v_t^2} \right)^{L/2} \frac{1 + \|\mathbf{z}_t\|_2^4}{1 + \|\mathbf{z}_t\|_2^4 v_t^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_t\|_2^{4L}}{M_2 v_t^{2L} \|\mathbf{z}_t\|_2^{4L} + M_1} \right\} \right) \\ &\quad \times \left(\frac{\|\mathbf{z}_{t^*}\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \right)^{L/2} \frac{2\|\mathbf{z}_{t^*}\|_2^4}{1 + \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_{t^*}\|_2^{4L}}{M_2 v_{t^*}^{2L} \|\mathbf{z}_{t^*}\|_2^{4L} + M_1} \right\}. \end{aligned}$$

If $v_{t^*} > 1$, it holds that

$$\left(\frac{\|\mathbf{z}_{t^*}\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \right)^{L/2} \frac{2\|\mathbf{z}_{t^*}\|_2^4}{1 + \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_{t^*}\|_2^{4L}}{M_2 v_{t^*}^{2L} \|\mathbf{z}_{t^*}\|_2^{4L} + M_1} \right\} \leq \left(\frac{1}{\varepsilon_2 v_{t^*}^2} \right)^{L/2} \left(\frac{2}{v_{t^*}^2} \right),$$

otherwise, it holds that

$$\begin{aligned}
& \left(\frac{\|\mathbf{z}_{t^*}\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \right)^{L/2} \frac{2\|\mathbf{z}_{t^*}\|_2^4}{1 + \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_{t^*}\|_2^{4L}}{M_2 v_{t^*}^{2L} \|\mathbf{z}_{t^*}\|_2^{4L} + M_1} \right\} \\
& \leq \left(\frac{M_1}{\varepsilon_1} \frac{\|\mathbf{z}_{t^*}\|_2^4}{M_1 + \varepsilon_2 \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \right)^{L/2} \left(\frac{2M_1 \|\mathbf{z}_{t^*}\|_2^4}{M_1 + \varepsilon_2 \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \right) \exp \left\{ -\frac{1}{2} \frac{\varepsilon_2}{M_2} \frac{\|\mathbf{z}_{t^*}\|_2^4}{\varepsilon_2 v_{t^*}^2 \|\mathbf{z}_{t^*}\|_2^4 + M_1} \right\} \\
& \leq \left(\frac{M_1}{\varepsilon_1} \right)^{L/2} 2M_1 \sup_{u \in (0, \infty)} u^{(L+2)/2} \exp \left(-\frac{\varepsilon_2}{M_2} \frac{-u}{2} \right) \equiv M_3 < \infty
\end{aligned}$$

Thus, we get

$$\begin{aligned}
h(\mathbf{v}, \mathbf{z}_{t^*}) & \leq \left(\prod_{t \neq t^*} \frac{\|\mathbf{z}_t\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_t\|_2^4 v_t^2} \frac{1 + \|\mathbf{z}_t\|_2^4}{1 + \|\mathbf{z}_t\|_2^4 v_t^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_t\|_2^{4L}}{M_2 v_t^{2L} \|\mathbf{z}_t\|_2^{4L} + M_1} \right\} \right) \\
& \quad \times \left(1_{\{v_{t^*} \leq 1\}} M_3 + 1_{\{v_{t^*} > 1\}} \left(\frac{1}{\varepsilon_2 v_{t^*}^2} \right)^{L/2} \left(\frac{2}{v_{t^*}^2} \right) \right). \tag{S2.4}
\end{aligned}$$

The right-hand side is integrable and does not depend on \mathbf{z}_{t^*} .

Next, we consider

$$h(\mathbf{v}, \mathbf{z}_{t^*}) \left\{ -\{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1} \Sigma_z \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1} \{V \otimes \Sigma_\delta^{1/2}\}^{-1} \boldsymbol{\omega} \right\}. \tag{S2.5}$$

For large M_4 , it holds that

$$\begin{aligned}
 & \|\{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\{I + \{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\Sigma_z\{(VW) \otimes \Sigma_\delta^{1/2}\}^{-1}\}^{-1}\{V \otimes \Sigma_\delta^{1/2}\}^{-1}\boldsymbol{\omega}\| \\
 & \leq \|W^{-1} \otimes I_L\| \|\{V^2 \otimes \Sigma_\delta + \{W \otimes I_L\}^{-1}\Sigma_z\{W \otimes I_L\}^{-1}\}^{-1}\| \|\boldsymbol{\omega}\| \\
 & \leq \sqrt{\sum_{t=1}^{T-1} \frac{L(T-1)}{\|\mathbf{z}_t\|_2^4}} \sum_{i=1}^{L(T-1)} \sum_{j=1}^{L(T-1)} |\mathbf{e}_i^\top \{V^2 \otimes \Sigma_\delta + \{W \otimes I_L\}^{-1}\Sigma_z\{W \otimes I_L\}^{-1}\}^{-1} \mathbf{e}_j| \\
 & \leq \sqrt{\sum_{t=1}^{T-1} \frac{L(T-1)}{\|\mathbf{z}_t\|_2^4}} \sum_{i=1}^{L(T-1)} \sum_{j=1}^{L(T-1)} \frac{|\mathbf{e}_i^\top \{(\varepsilon_1 W^{-2} + \varepsilon_2 V^2) \otimes I_L\}^{-1} \mathbf{e}_i| + |\mathbf{e}_j^\top \{(\varepsilon_1 W^{-2} + \varepsilon_2 V^2) \otimes I_L\}^{-1} \mathbf{e}_j|}{2} \\
 & \leq \sqrt{\sum_{t=1}^{T-1} \frac{L(T-1)}{\|\mathbf{z}_t\|_2^4}} \frac{L}{2} \sum_{i=1}^{T-1} \sum_{j=1}^{T-1} \left\{ \left(\frac{\varepsilon_1}{\|\mathbf{z}_i\|_2^4} + \varepsilon_2 v_i^2 \right)^{-1} + \left(\frac{\varepsilon_1}{\|\mathbf{z}_j\|_2^4} + \varepsilon_2 v_j^2 \right)^{-1} \right\} \\
 & \leq \sqrt{\sum_{t=1}^{T-1} \frac{L(T-1)}{\|\mathbf{z}_t\|_2^4}} L(T-1) \sum_{t=1}^{T-1} \frac{\|\mathbf{z}_t\|_2^4}{\varepsilon_1 + \varepsilon_2 v_t^2 \|\mathbf{z}_t\|_2^4} \\
 & \leq M_4 \sum_{t=1}^{T-1} \frac{\|\mathbf{z}_t\|_2^4}{\varepsilon_1 + \varepsilon_2 v_t^2 \|\mathbf{z}_t\|_2^4}.
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
& \|h(\mathbf{v}, \mathbf{z}_{t^*}) \{ (VW) \otimes \Sigma_\delta^{1/2} \}^{-1} \{ I + \{ (VW) \otimes \Sigma_\delta^{1/2} \}^{-1} \Sigma_z \{ (VW) \otimes \Sigma_\delta^{1/2} \}^{-1} \}^{-1} \{ V \otimes \Sigma_\delta^{1/2} \}^{-1} \boldsymbol{\omega} \| \\
& \leq M_4 \left(\sum_{t' \neq t^*} \frac{\|\mathbf{z}_{t'}\|_2^4}{\varepsilon_1 + \varepsilon_2 v_{t'}^2 \|\mathbf{z}_{t'}\|_2^4} + \frac{\|\mathbf{z}_{t^*}\|_2^4}{\varepsilon_1 + \varepsilon_2 v_{t^*}^2 \|\mathbf{z}_{t^*}\|_2^4} \right) \\
& \quad \times \left(\prod_{t \neq t^*} \left(\frac{\|\mathbf{z}_t\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_t\|_2^4 v_t^2} \right)^{L/2} \frac{1 + \|\mathbf{z}_t\|_2^4}{1 + \|\mathbf{z}_t\|_2^4 v_t^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_t\|_2^{4L}}{M_2 v_t^{2L} \|\mathbf{z}_t\|_2^{4L} + M_1} \right\} \right) \\
& \quad \times \left(\frac{\|\mathbf{z}_{t^*}\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \right)^{L/2} \frac{2\|\mathbf{z}_{t^*}\|_2^4}{1 + \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_{t^*}\|_2^{4L}}{M_2 v_{t^*}^{2L} \|\mathbf{z}_{t^*}\|_2^{4L} + M_1} \right\} \\
& \leq M_4 \sum_{t' \neq t^*} \frac{\|\mathbf{z}_{t'}\|_2^4}{\varepsilon_1 + \varepsilon_2 v_{t'}^2 \|\mathbf{z}_{t'}\|_2^4} \left(\prod_{t \neq t^*} \left(\frac{\|\mathbf{z}_t\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_t\|_2^4 v_t^2} \right)^{L/2+1_{\{t=t'\}}} \frac{1 + \|\mathbf{z}_t\|_2^4}{1 + \|\mathbf{z}_t\|_2^4 v_t^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_t\|_2^{4L}}{M_2 v_t^{2L} \|\mathbf{z}_t\|_2^{4L} + M_1} \right\} \right) \\
& \quad \times \left(\frac{\|\mathbf{z}_{t^*}\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \right)^{L/2} \frac{2\|\mathbf{z}_{t^*}\|_2^4}{1 + \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_{t^*}\|_2^{4L}}{M_2 v_{t^*}^{2L} \|\mathbf{z}_{t^*}\|_2^{4L} + M_1} \right\} \\
& \quad + M_4 \left(\prod_{t \neq t^*} \left(\frac{\|\mathbf{z}_t\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_t\|_2^4 v_t^2} \right)^{L/2} \frac{1 + \|\mathbf{z}_t\|_2^4}{1 + \|\mathbf{z}_t\|_2^4 v_t^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_t\|_2^{4L}}{M_2 v_t^{2L} \|\mathbf{z}_t\|_2^{4L} + M_1} \right\} \right) \\
& \quad \times \left(\frac{\|\mathbf{z}_{t^*}\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \right)^{L/2+1} \frac{2\|\mathbf{z}_{t^*}\|_2^4}{1 + \|\mathbf{z}_{t^*}\|_2^4 v_{t^*}^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_{t^*}\|_2^{4L}}{M_2 v_{t^*}^{2L} \|\mathbf{z}_{t^*}\|_2^{4L} + M_1} \right\} \\
& \leq M_4 \sum_{t' \neq t^*} \frac{\|\mathbf{z}_{t'}\|_2^4}{\varepsilon_1 + \varepsilon_2 v_{t'}^2 \|\mathbf{z}_{t'}\|_2^4} \left(\prod_{t \neq t^*} \left(\frac{\|\mathbf{z}_t\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_t\|_2^4 v_t^2} \right)^{L/2+1_{\{t=t'\}}} \frac{1 + \|\mathbf{z}_t\|_2^4}{1 + \|\mathbf{z}_t\|_2^4 v_t^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_t\|_2^{4L}}{M_2 v_t^{2L} \|\mathbf{z}_t\|_2^{4L} + M_1} \right\} \right) \\
& \quad \times \left(1_{\{v_{t^*} \leq 1\}} M_3 + 1_{\{v_{t^*} > 1\}} \left(\frac{1}{\varepsilon_2 v_{t^*}^2} \right)^{L/2} \left(\frac{2}{v_{t^*}^2} \right)^{1/2} \right) \\
& \quad + M_4 \left(\prod_{t \neq t^*} \left(\frac{\|\mathbf{z}_t\|_2^4}{\varepsilon_1 + \varepsilon_2 \|\mathbf{z}_t\|_2^4 v_t^2} \right)^{L/2} \frac{1 + \|\mathbf{z}_t\|_2^4}{1 + \|\mathbf{z}_t\|_2^4 v_t^2} \exp \left\{ -\frac{1}{2} \frac{\|\mathbf{z}_t\|_2^{4L}}{M_2 v_t^{2L} \|\mathbf{z}_t\|_2^{4L} + M_1} \right\} \right) \\
& \quad \times \left(1_{\{v_{t^*} \leq 1\}} M_5 + 1_{\{v_{t^*} > 1\}} \left(\frac{1}{\varepsilon_2 v_{t^*}^2} \right)^{L/2} \left(\frac{2}{v_{t^*}^2} \right)^{1/2} \right),
\end{aligned}$$

for some $M_5 > 0$. The last one is integrable and independent of \mathbf{z}_{t^*} . Com-

binning this and (S2.4) shows the existence of integrable functions that dom-

inate the integrands in (S2.3).

Step 4: Conclusion. Step 3 shows the dominated convergence theorem is applicable to (S2.3). Then, from the dominated convergence theorem and Step 2, we obtain

$$\frac{1}{m(\mathbf{z})} \frac{\partial m(\mathbf{z})}{\partial \mathbf{z}} \approx -\Sigma_{\mathbf{z}} \frac{\int h(\mathbf{v}, \mathbf{z}_{t^*}) \{-A(v_{t^*})(I + A(v_{t^*})\Sigma A(v_{t^*}))^{-1}\{V \otimes \Sigma_{\delta}^{1/2}\}^{-1}\boldsymbol{\omega}\} d\mathbf{v}}{\int h(\mathbf{v}, \mathbf{z}_{t^*}) d\mathbf{v}},$$

as $\mathbf{z}_{t^*} \rightarrow \infty$. Therefore $E[\boldsymbol{\eta}|\mathbf{z}] - \mathbf{z}$ is finite and this concludes the proof.

□

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