TIME-VARYING CORRELATION FOR NONCENTERED NONSTATIONARY TIME SERIES: SIMULTANEOUS INFERENCE AND VISUALIZATION

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Supplementary Material

S1 Technical Proofs

Lemma 1. *Assume that* $\mu_x \in \mathcal{C}^3$*,* $\Theta_{0,2}(G) < \infty$*,* $G \in \mathrm{SLC}_2$*,* $b_n \to 0$ *and* $nb_n \to \infty$*. Let* $^\top$ *denote the transpose operator, then the local linear estimator*

$$\{\hat{\mu}_x(t), \hat{\mu}_x'(t)\}^\top = \operatorname*{argmin}_{(\eta,\eta')^\top \in \mathbb{R}^2} \sum_{i=1}^{n} \{X_i - \eta - \eta'(i/n - t)\}^2 K\left(\frac{i/n - t}{b_n}\right)$$

satisfies

$$[\hat{\mu}_x(t) - \mu_x(t), b_n\{\hat{\mu}_x'(t) - \mu_x'(t)\}]^\top = O_p\{(nb_n)^{-1/2} + b_n^2\}$$

for any $t \in \mathcal{T}_n$. *If in addition* $\theta_{k,4}(G) = O(k^{-2})$ *and* $nb_n^2(\log n)^2 \to \infty$, *then*

$$\sup_{t\in\mathcal{T}_n} |[\hat{\mu}_x(t) - \mu_x(t), b_n\{\hat{\mu}'_x(t) - \mu'_x(t)\}]^\top| = O_p\{(nb_n)^{-1/2}(-\log b_n)^{1/2} + b_n^2\}.$$

Proof. Write $R_{x,n,l}(t) = (nb_n)^{-1}\sum_{i=1}^n\{X_i - E(X_i)\}\{(i/n-t)/b_n\}^l K_\mu\{(i/n-t)/b_n\}$ and define the deterministic sums $w_{n,l}(t) = (nb_n)^{-1}\sum_{i=1}^n\{(i/n - t)/b_n\}^l K_\mu\{(i/n-t)/b_n\}$ and $\delta_{x,n,l}(t) = (nb_n)^{-1}\sum_{i=1}^n\{\mu_x(i/n) - \mu_x(t) - \mu'_x(t)(i/n-t)\}\{(i/n-t)/b_n\}^l K_\mu\{(i/n-t)/b_n\}$, then by solving the minimization problem in the local linear estimation (Fan and Gijbels, 1996) we can obtain that

$$\begin{Bmatrix} w_{n,0}(t) & w_{n,1}(t) \\ w_{n,1}(t) & w_{n,2}(t) \end{Bmatrix} \begin{bmatrix} \hat{\mu}_x(t) - \mu_x(t) \\ b_n\{\hat{\mu}'_x(t) - \mu'_x(t)\} \end{bmatrix} = \begin{Bmatrix} R_{x,n,0}(t) \\ R_{x,n,1}(t) \end{Bmatrix} + \begin{Bmatrix} \delta_{x,n,0}(t) \\ \delta_{x,n,1}(t) \end{Bmatrix}.$$

By Lamma A.1 of Zhang and Wu (2012), $R_{x,n,l}(t) = O_p\{(nb_n)^{-1/2}\}$ holds for any $t \in [0,1]$ and $l \in \{0,1\}$, and by the same argument as in Lemma A.3 of Zhang and Wu (2012) we can obtain that $\sup_{t\in\mathcal{T}_n} |R_{x,n,l}(t)| = O_p\{(nb_n)^{-1/2}(-\log b_n)^{1/2}\}$. Note that $w_{n,l}(t) = \int_0^1 v^l K(v)dv + O\{(nb_n)^{-1}\}$ and $\delta_{x,n,l}(t) = O(b_n^2)$ hold uniformly over $t \in \mathcal{T}_n$ for $l \in \{0,1,2\}$, Lemma 1 follows. □

Lemma 2. *Assume that* $\mu_x, \mu_y \in \mathcal{C}^3$, $\theta_{k,4}(G) + \theta_{k,4}(H) = O(k^{-2})$, *and* $G, H \in \mathrm{SLC}_2$. *If* $b_n \to 0$, $nb_n^2(\log n)^2 \to \infty$, *and the trend estimator satisfies*

$$\sup_{t\in\mathcal{T}_n} |[\hat{\mu}_x(t) - \mu_x(t), b_n\{\hat{\mu}'_x(t) - \mu'_x(t)\}]^\top| = O_p\{(nb_n)^{-1/2}(-\log b_n)^{1/2} + b_n^2\},$$

then

$$\sup_{t\in\mathcal{T}_n} |\hat{\gamma}_n(t) - \tilde{\gamma}_n(t)| = O_p[\{(nb_n)^{-1/2}(-\log b_n)^{1/2} + b_n^2\}^2 + n^{-1/2}b_n].$$

Proof. Let $d_{\mu_x}(t,t') = \mu_x(t) + \mu_x'(t)(t'-t) - \mu_x(t')$ and $\hat{d}_{\mu_x}(t,t') = \hat{\mu}_x(t) + \hat{\mu}_x'(t)(t'-t) - \mu_x(t')$. Similarly we define $d_{\mu_y}(t,t')$ and $\hat{d}_{\mu_y}(t,t')$, then we can write

$$\hat{\gamma}_n(t) - \tilde{\gamma}_n(t) = \mathrm{I}_n(t) + \mathrm{II}_n(t) + \mathrm{III}_n(t),$$

where

$$\begin{aligned}
\mathrm{I}_n(t) &= \frac{1}{nb_n}\sum_{i=1}^n \hat{d}_{\mu_x}(t,i/n)\hat{d}_{\mu_y}(t,i/n)K\left(\frac{i/n-t}{b_n}\right);\\
\mathrm{II}_n(t) &= -\frac{1}{nb_n}\sum_{i=1}^n \{X_i - \mu_x(i/n)\}\{\hat{\mu}_y(t) + \hat{\mu}_y'(t)(i/n-t) - \mu_y(i/n)\}K\left(\frac{i/n-t}{b_n}\right);\\
\mathrm{III}_n(t) &= -\frac{1}{nb_n}\sum_{i=1}^n \{\hat{\mu}_x(t) + \hat{\mu}_x'(t)(i/n-t) - \mu_x(i/n)\}\{Y_i - \mu_y(i/n)\}K\left(\frac{i/n-t}{b_n}\right).
\end{aligned}$$

By using the decomposition

$$\hat{d}_{\mu_x}(t,i/n) = \{\hat{\mu}_x(t) - \mu_x(t)\} + \{\hat{\mu}_x'(t) - \mu_x'(t)\}(i/n-t) + d_{\mu_x}(t,i/n),$$

we have

$$\begin{aligned}
\mathrm{III}_n(t) &= -\frac{1}{nb_n}\sum_{i=1}^n \{\hat{\mu}_x(t) - \mu_x(t)\}\{Y_i - \mu_y(i/n)\}K\left(\frac{i/n-t}{b_n}\right)\\
&\quad -\frac{1}{nb_n}\sum_{i=1}^n b_n\{\hat{\mu}_x'(t) - \mu_x'(t)\}\{Y_i - \mu_y(i/n)\}\left(\frac{i/n-t}{b_n}\right)K\left(\frac{i/n-t}{b_n}\right)\\
&\quad -\frac{1}{nb_n}\sum_{i=1}^n d_{\mu_x}(t,i/n)\{Y_i - \mu_y(i/n)\}K\left(\frac{i/n-t}{b_n}\right).
\end{aligned}$$

Since $Y_i - \mu_y(i/n) = Y_i - E(Y_i)$ is a zero-mean process, by the proof of Lemma 1 we have

$$\sup_{t\in\mathcal{T}_n}\left|\frac{1}{nb_n}\sum_{i=1}^{n}\{Y_i-\mu_y(i/n)\}K\left(\frac{i/n-t}{b_n}\right)\right| = O_p\{(nb_n)^{-1/2}(-\log b_n)^{1/2}\}.$$

On the other hand, note that $|d_{\mu_x}(t,i/n)| \leq 2^{-1}b_n^2\sup_{t\in[0,1]}|\mu_x''(t)|$ holds for any $|i/n-t|\leq b_n$ and that

$$\begin{aligned} |d_{\mu_x}\{t,(i+1)/n\}-d_{\mu_x}(t,i/n)| &= |n^{-1}\mu_x'(t)-\mu_x\{(i+1)/n\}+\mu_x(i/n)| \\ &\leq 2n^{-1}b_n\sup_{t\in[0,1]}|\mu_x''(t)|, \end{aligned}$$

by Lemma A.1(ii) of Zhang and Wu (2012) we have

$$\sup_{t\in[0,1]}\left|\frac{1}{nb_n}\sum_{i=1}^{n}d_{\mu_x}(t,i/n)\{Y_i-\mu_y(i/n)\}K\left(\frac{i/n-t}{b_n}\right)\right| = O_p(n^{-1/2}b_n).$$

As a result,

$$\sup_{t\in\mathcal{T}_n}|\mathrm{III}_n(t)| = O_p[n^{-1/2}b_n+(nb_n)^{-1/2}(-\log b_n)^{1/2}\{(nb_n)^{-1/2}(-\log b_n)^{1/2}+b_n^2\}], \tag{S1.1}$$

and similarly we can show that

$$\sup_{t\in\mathcal{T}_n}|\mathrm{II}_n(t)| = O_p[n^{-1/2}b_n+(nb_n)^{-1/2}(-\log b_n)^{1/2}\{(nb_n)^{-1/2}(-\log b_n)^{1/2}+b_n^2\}]. \tag{S1.2}$$

We shall now deal with the term $\mathrm{I}_n(t)$, for which we need to further decompose it into

$$\mathrm{I}_n(t) = \mathrm{I}_{1,n}(t)+\mathrm{I}_{2,n}(t)+\mathrm{I}_{3,n}(t)+\mathrm{I}_{4,n}(t),$$

where

$$
\begin{aligned}
\mathrm{I}_{1,n}(t) &= \frac{1}{nb_n}\sum_{i=1}^{n}\{\hat{d}_{\mu_x}(t,i/n)-d_{\mu_x}(t,i/n)\}\{\hat{d}_{\mu_y}(t,i/n)-d_{\mu_y}(t,i/n)\}K\left(\frac{i/n-t}{b_n}\right);\\
\mathrm{I}_{2,n}(t) &= \frac{1}{nb_n}\sum_{i=1}^{n}d_{\mu_x}(t,i/n)\{\hat{d}_{\mu_y}(t,i/n)-d_{\mu_y}(t,i/n)\}K\left(\frac{i/n-t}{b_n}\right);\\
\mathrm{I}_{3,n}(t) &= \frac{1}{nb_n}\sum_{i=1}^{n}\{\hat{d}_{\mu_x}(t,i/n)-d_{\mu_x}(t,i/n)\}d_{\mu_y}(t,i/n)K\left(\frac{i/n-t}{b_n}\right);\\
\mathrm{I}_{4,n}(t) &= \frac{1}{nb_n}\sum_{i=1}^{n}d_{\mu_x}(t,i/n)d_{\mu_y}(t,i/n)K\left(\frac{i/n-t}{b_n}\right).
\end{aligned}
$$

Note that for any $t \in \mathcal{T}_n$,

$$
\frac{1}{nb_n}\sum_{i=1}^{n}\left(\frac{i/n-t}{b_n}\right)^{l}K\left(\frac{i/n-t}{b_n}\right)=\int v^{l}K(v)dv+O\{(nb_n)^{-1}\},
$$

we have

$$
\sup_{t\in\mathcal{T}_n}|\mathrm{I}_{1,n}(t)|=O_p[\{(nb_n)^{-1/2}(-\log b_n)^{1/2}+b_n^2\}^2],
$$

and

$$
\sup_{t\in\mathcal{T}_n}\{|\mathrm{I}_{2,n}(t)|+|\mathrm{I}_{3,n}(t)|+|\mathrm{I}_{4,n}(t)|\}=O_p[\{(nb_n)^{-1/2}(-\log b_n)^{1/2}+b_n^2\}b_n^2].
$$

The result then follows by combining these with (S1.1) and (S1.2). □

Proof. (Theorem 1) Note that

$$
\tilde{\gamma}_n(t)-E\{\tilde{\gamma}_n(t)\}=\frac{1}{nb_n}\sum_{i=1}^{n}[U(i/n,\boldsymbol{\mathcal{F}}_i)-E\{U(i/n,\boldsymbol{\mathcal{F}}_i)\}]K\left(\frac{i/n-t}{b_n}\right),
$$

then by the proof of Theorem 2.1 in Zhang (2013),

$$
(nb_n)^{1/2}[\tilde{\gamma}_n(t)-E\{\tilde{\gamma}_n(t)\}]\to_d N\{0,\varpi_U(t)\phi_2\}.
$$

On the other hand, by Theorem 2 of Zhou and Wu (2010), on a richer probability space there exist independent standard normal random variables $Z_1, Z_2, \ldots$ and a process (W_k) such that the partial sum process $(\sum_{i=1}^k [U(i/n, \boldsymbol{\mathcal{F}}_i) - E\{U(i/n, \boldsymbol{\mathcal{F}}_i)\}])_{k=1}^n$ has the same joint distribution as $(\sum_{i=1}^k W_i)_{k=1}^n$ and that

$$\max_{1\le k\le n}\left|\sum_{i=1}^k W_i - \sum_{i=1}^k \varpi_U(i/n) Z_i\right| = o_p(n^{3/10}\log n).$$

Then by applying the summation by parts, we can obtain that

$$\sup_{t\in\mathcal{T}_n}\left|\frac{1}{nb_n}\sum_{i=1}^n\{W_i - \varpi_U(i/n)Z_i\}K\left(\frac{i/n - t}{b_n}\right)\right| = o_p\{(nb_n)^{-1}n^{3/10}\log n\},$$

which is of order $o_p\{(nb_n\log n)^{-1/2}\}$, and thus by the proof of Lemma A.2 in Zhang (2016) we have

$$\operatorname{pr}\left\{\frac{(nb_n)^{1/2}}{\phi_2^{1/2}}\sup_{t\in\mathcal{T}_n}\left|\frac{\tilde\gamma_n(t) - \gamma(t) - 2^{-1}\kappa_2 b_n^2\gamma''(t)}{\varpi_U(t)^{1/2}}\right| - (-2\log b_n)^{1/2} - \frac{C_K}{(-2\log b_n)^{1/2}} \le \frac{z}{(-2\log b_n)^{1/2}}\right\} \to \exp\{-2\exp(-z)\}.$$

Since $(nb_n)^{1/2}n^{-1/2}b_n(-2\log b_n)^l = b_n^{3/2}(-2\log b_n)^l \to 0$ for any $l > 0$, the results follow by Lemmas 1 and 2. □

Proof. (Theorem 2) Note that

$$\begin{aligned}\hat\rho_n(t) &= \frac{\hat\gamma_n(t) - E\{\tilde\gamma_n(t)\}}{\sigma_x(t)\sigma_y(t)} + [\hat\gamma_n(t) - E\{\tilde\gamma_n(t)\}]\left\{\frac{1}{\hat\sigma_{x,n}(t)\hat\sigma_{y,n}(t)} - \frac{1}{\sigma_x(t)\sigma_y(t)}\right\}\\ &\quad + E\{\tilde\gamma_n(t)\}\left\{\frac{1}{\hat\sigma_{x,n}(t)\hat\sigma_{y,n}(t)} - \frac{1}{\sigma_x(t)\sigma_y(t)}\right\} + \frac{E\{\tilde\gamma_n(t)\}}{\sigma_x(t)\sigma_y(t)},\end{aligned}$$

where

$$\frac{1}{\hat{\sigma}_{x,n}(t)\hat{\sigma}_{y,n}(t)} - \frac{1}{\sigma_x(t)\sigma_y(t)} = \frac{1}{\sigma_y(t)}\left\{\frac{1}{\hat{\sigma}_{x,n}(t)} - \frac{1}{\sigma_x(t)}\right\} + \frac{1}{\sigma_x(t)}\left\{\frac{1}{\hat{\sigma}_{y,n}(t)} - \frac{1}{\sigma_y(t)}\right\} + \left\{\frac{1}{\hat{\sigma}_{x,n}(t)} - \frac{1}{\sigma_x(t)}\right\}\left\{\frac{1}{\hat{\sigma}_{y,n}(t)} - \frac{1}{\sigma_y(t)}\right\}.$$

Write

$$\frac{1}{\hat{\sigma}_{x,n}(t)} - \frac{1}{\sigma_x(t)} = -\frac{\hat{\sigma}^2_{x,n}(t) - \sigma^2_x(t)}{\hat{\sigma}_{x,n}(t)\sigma_x(t)\{\hat{\sigma}_{x,n}(t) + \sigma_x(t)\}},$$

then by applying the proof of Theorem 1 to the case when $Y = X$ and using Lemma A.3 of Zhang and Wu (2012) we can obtain that

$$\sup_{t\in\mathcal{T}_n} |\hat{\sigma}^2_{x,n}(t) - \sigma^2_x(t)| = O_p\{(nb_n)^{-1/2}(-\log b_n)^{1/2} + b_n^2\}.$$

Since $\sigma_x(t)$ is bounded away from zero on $[0,1]$, we have

$$\sup_{t\in\mathcal{T}_n}\left|\frac{1}{\hat{\sigma}_{x,n}(t)} - \frac{1}{\sigma_x(t)}\right| = O_p\{(nb_n)^{-1/2}(-\log b_n)^{1/2} + b_n^2\},$$

and

$$\sup_{t\in\mathcal{T}_n}\left|\frac{1}{\hat{\sigma}_{x,n}(t)} - \frac{1}{\sigma_x(t)} + \frac{\hat{\sigma}^2_{x,n}(t) - \sigma^2_x(t)}{2\sigma^3_x(t)}\right| = O_p[\{(nb_n)^{-1/2}(-\log b_n)^{1/2} + b_n^2\}^2].$$

By a similar argument, we can obtain the associated bounds for $\hat{\sigma}_{y,n}(t)$ as well. Combining them together, we can get

$$\sup_{t\in\mathcal{T}_n}\left|\frac{1}{\hat{\sigma}_{x,n}(t)\hat{\sigma}_{y,n}(t)} - \frac{1}{\sigma_x(t)\sigma_y(t)}\right| = O_p\{(nb_n)^{-1/2}(-\log b_n)^{1/2} + b_n^2\},$$

and

$$\sup_{t\in\mathcal{T}_n}\left|\frac{1}{\hat{\sigma}_{x,n}(t)\hat{\sigma}_{y,n}(t)} - \frac{1}{\sigma_x(t)\sigma_y(t)} + \frac{\hat{\sigma}^2_{x,n}(t) - \sigma^2_x(t)}{2\sigma^3_x(t)\sigma_y(t)} + \frac{\hat{\sigma}^2_{y,n}(t) - \sigma^2_y(t)}{2\sigma_x(t)\sigma^3_y(t)}\right| = O_p[\{(nb_n)^{-1/2}(-\log b_n)^{1/2} + b_n^2\}^2].$$

Since $E\{\tilde{\gamma}_n(t)\} = \gamma(t) + 2^{-1}\kappa_2 b_n^2 \gamma''(t) + O(b_n^3)$ uniformly over $t \in \mathcal{T}_n$, we have

$$\sup_{t\in\mathcal{T}_n}\left|\hat{\rho}_n(t) - \frac{\hat{\gamma}_n(t) - E\{\tilde{\gamma}_n(t)\}}{\sigma_x(t)\sigma_y(t)} + \gamma(t)\left\{\frac{\hat{\sigma}^2_{x,n}(t) - \sigma^2_x(t)}{2\sigma^3_x(t)\sigma_y(t)} + \frac{\hat{\sigma}^2_{y,n}(t) - \sigma^2_y(t)}{2\sigma_x(t)\sigma^3_y(t)}\right\} - \frac{E\{\tilde{\gamma}_n(t)\}}{\sigma_x(t)\sigma_y(t)}\right| = O_p[\{(nb_n)^{-1/2}(-\log b_n)^{1/2} + b_n^2\}^2].$$

Let

$$Q_{i,n}(t) = \frac{U(i/n, \boldsymbol{\mathcal{F}}_i)}{\sigma_x(t)\sigma_y(t)} - \gamma(t)\left[\frac{\{X_i - \mu_x(i/n)\}^2}{2\sigma^3_x(t)\sigma_y(t)} + \frac{\{Y_i - \mu_y(i/n)\}^2}{2\sigma_x(t)\sigma^3_y(t)}\right], \quad \text{(S1.3)}$$

then by Lemma 2 and its application to the case when $Y = X$ we can obtain that

$$\begin{aligned} &\sup_{t\in\mathcal{T}_n}\left|\hat{\rho}_n(t) - \frac{1}{nb_n}\sum_{i=1}^n Q_{i,n}(t) K\left(\frac{i/n - t}{b_n}\right) - \frac{E\{\tilde{\gamma}_n(t)\}}{\sigma_x(t)\sigma_y(t)}\right| \\ = \; & O_p[\{(nb_n)^{-1/2}(-\log b_n)^{1/2} + b_n^2\}^2 + n^{-1/2}b_n]. \end{aligned}$$

Note that $nb_n^7 = o(nb_n^7 \log n) \to 0$ and $(nb_n)^{1/2} n^{-1/2} b_n (-2\log b_n)^l = b_n^{3/2}(-2\log b_n)^l \to 0$ for any $l > 0$, by the proof of Theorem 1 it suffices to show that

$$\sup_{t\in\mathcal{T}_n}\left|\frac{1}{nb_n}\sum_{i=1}^n \{Q_{i,n}(t) - V(i/n, \boldsymbol{\mathcal{F}}_i)\} K\left(\frac{i/n - t}{b_n}\right)\right| = o_p\{(-nb_n \log b_n)^{-1/2}\}.$$

We first deal with the first term in (S1.3) and consider

$$\frac{1}{nb_n}\sum_{i=1}^n U(i/n, \boldsymbol{\mathcal{F}}_i)\left\{\frac{1}{\sigma_x(t)\sigma_y(t)} - \frac{1}{\sigma_x(i/n)\sigma_y(i/n)}\right\} K\left(\frac{i/n - t}{b_n}\right),$$

which by Lemma A.1(ii) of Zhang and Wu (2012) satisfies

$$\left\|\frac{1}{nb_n}\sum_{i=1}^n U(i/n, \boldsymbol{\mathcal{F}}_i)\left\{\frac{1}{\sigma_x(t)\sigma_y(t)} - \frac{1}{\sigma_x(i/n)\sigma_y(i/n)}\right\} K\left(\frac{i/n - t}{b_n}\right)\right\| = O(n^{-1/2}).$$

The remaining two terms in (S1.3) can be similarly dealt with, and we can show that

$$\sup_{t\in\mathcal{T}_n}\left|\frac{1}{nb_n}\sum_{i=1}^{n}\{Q_{i,n}(t)-V(i/n,\boldsymbol{\mathcal{F}}_i)\}K\left(\frac{i/n-t}{b_n}\right)\right| = O_p(n^{-1/2}),$$

which is of order $o_p\{(-nb_n\log b_n)^{-1/2}\}$ if $b_n\log b_n\to 0$. The latter is automatically satisfied when $b_n\to 0$, and Theorem 2 follows. □

S2 Additional Simulation Results

S2.1 Simulation-Assistance with Other Distributions

We shall here examine the possibility of using distributions other than the normal to perform the simulation-assisted procedure described in Section 4.1. In particular, we consider generating the simulation-assisted samples $X_i^\diamond$ and $Y_i^\diamond$ in step (v) of the algorithm from the following distributions (in addition to the standard normal reported in Section 4.2):

(a) uniform distribution on $(-1,1)$;

(b) centered exponential distribution with rate parameter 1;

(c) Rademacher distribution;

(d) t-distribution with degree of freedom 4;

(e) t-distribution with degree of freedom 9.

The inclusion of the t_4 distribution is to examine the sensitivity of the simulation-assisted procedure to some of the moment conditions in finite-sample performance. To compare with the baseline results using the standard normal distribution reported in Section 4.2, we shall here follow the same simulation setting as that in Section 4.2. In particular, let $(\epsilon_{i,1})$ be a sequence of independent standard normal random variables and $(\epsilon_{i,2})$ be a sequence of independent Rademacher random variables that is also independent of $(\epsilon_{i,1})$, we generate the data (X_i, Y_i), $i = 1, \ldots, n$, by

$$\begin{aligned} X_i &= \mu_x(i/n) + 3\sin(1.5\pi i/n)\{|\epsilon_{i,1}| - (2/\pi)^{1/2}\} + 2\cos(1.5\pi i/n)\epsilon_{i,2} + \sum_{j=1}^{\infty} j^{-2}\epsilon_{i-j,2}; \\ Y_i &= \mu_y(i/n) + \{1.5 - (i/n)^2\}\epsilon_{i,1} + (i/n)\epsilon_{i,2} + \sum_{j=1}^{\infty} 2^{-j}\epsilon_{i-j,1}, \end{aligned}$$

where $\mu_x(t) = 2t^2 + 2t$ and $\mu_y(t) = 2\{\sin(1.5\pi t) + t\}$. Following Section 4.2, we consider making simultaneous inference on (1) the time-varying covariance and correlation between the two time series; and (2) the time-varying first-order autocovariance and autocorrelation of (X_i). By elementary calculations, the true underlying time-varying covariance function between (X_i) and (Y_i) in this case has the explicit form

$$\gamma(t) = 2t\cos(1.5\pi t),$$

while the corresponding time-varying correlation function is given by

$$\rho(t) = \frac{2t\cos(1.5\pi t)}{\{(9 - 18/\pi)\sin^2(1.5\pi t) + 4\cos^2(1.5\pi t) + \pi^4/90\}^{1/2}\{(1.5 - t^2)^2 + t^2 + 1/3\}^{1/2}}.$$

Similarly, the time-varying first-order autocovariance and autocorrelation of (X_i) can be calculated as

$$\begin{aligned}\gamma_{x,1}(t) &= 2\cos(1.5\pi t) + \pi^2/3 - 3;\\ \rho_{x,1}(t) &= \frac{2\cos(1.5\pi t) + \pi^2/3 - 3}{(9 - 18/\pi)\sin^2(1.5\pi t) + 4\cos^2(1.5\pi t) + \pi^4/90}.\end{aligned}$$

The results are summarized in Tables 1–5, from which we can see that the performance of the simulation-assisted procedure is reasonably robust to different choices of the distribution used to generate $X_i^\diamond$ and $Y_i^\diamond$.

We shall here further examine the situation when different distributions are used to generate $X_i^\diamond$ and $Y_i^\diamond$ in step (v) of the algorithm. For this, we set the distribution of $X_i^\diamond$ as the standard normal (baseline) in the simulation-assisted procedure and generate $Y_i^\diamond$ from different distributions listed in (a)–(e) to investigate the effect when $X_i^\diamond$ and $Y_i^\diamond$ are not generated from the same distribution. The results are summarized in Tables 6–10, from which we can see that the simulation-assisted procedure is reasonably robust to the situation when the practitioner decides to use different distributions to generate $X_i^\diamond$ and $Y_i^\diamond$.

S2.2 Simulation-Assistance with an Artificial Trend

We in this section consider the situation when the simulation-assisted samples $X_i^\diamond$ and $Y_i^\diamond$ in step (v) of the algorithm are not both generated from

Table 1: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where the uniform distribution in (a) is used to generate $X_i^\diamond$ and $Y_i^\diamond$ in the simulation-assisted procedure.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.841	0.906	0.952	0.849	0.915	0.952
	0.15	0.870	0.927	0.983	0.895	0.943	0.982
	0.2	0.884	0.950	0.983	0.911	0.958	0.986
	0.25	0.902	0.946	0.988	0.916	0.958	0.987
	0.3	0.911	0.951	0.991	0.917	0.957	0.993
1000	0.1	0.890	0.938	0.988	0.894	0.938	0.991
	0.15	0.891	0.938	0.985	0.904	0.941	0.991
	0.2	0.891	0.947	0.984	0.903	0.955	0.986
	0.25	0.894	0.949	0.986	0.902	0.955	0.989
	0.3	0.904	0.948	0.988	0.893	0.957	0.989
				correlation			
500	0.1	0.877	0.933	0.979	0.874	0.935	0.981
	0.15	0.834	0.907	0.984	0.836	0.918	0.988
	0.2	0.844	0.919	0.983	0.861	0.924	0.990
	0.25	0.855	0.916	0.982	0.876	0.941	0.988
	0.3	0.863	0.931	0.987	0.882	0.939	0.988
1000	0.1	0.848	0.921	0.988	0.857	0.927	0.987
	0.15	0.841	0.913	0.973	0.855	0.924	0.972
	0.2	0.860	0.913	0.976	0.870	0.930	0.980
	0.25	0.862	0.913	0.980	0.878	0.930	0.982
	0.3	0.869	0.929	0.981	0.882	0.933	0.982

centered distributions. For this, we follow the simulation-assisted algorithm described in Section 4.1 but added an artificial trend $\mu^\diamond(i/n)$ to $X_i^\diamond$,

Table 2: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where the exponential distribution in (b) is used to generate $X_i^\diamond$ and $Y_i^\diamond$ in the simulation-assisted procedure.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.843	0.913	0.959	0.854	0.918	0.959
	0.15	0.863	0.919	0.983	0.878	0.935	0.982
	0.2	0.880	0.937	0.981	0.901	0.951	0.983
	0.25	0.895	0.942	0.985	0.909	0.951	0.983
	0.3	0.911	0.946	0.987	0.910	0.952	0.988
1000	0.1	0.890	0.942	0.989	0.894	0.946	0.992
	0.15	0.883	0.937	0.984	0.893	0.935	0.988
	0.2	0.879	0.935	0.982	0.896	0.949	0.983
	0.25	0.887	0.937	0.986	0.890	0.952	0.988
	0.3	0.896	0.944	0.987	0.885	0.952	0.989
				correlation			
500	0.1	0.905	0.952	0.987	0.902	0.950	0.989
	0.15	0.896	0.949	0.984	0.908	0.949	0.988
	0.2	0.898	0.943	0.992	0.913	0.954	0.994
	0.25	0.899	0.952	0.988	0.920	0.958	0.991
	0.3	0.913	0.958	0.989	0.920	0.959	0.990
1000	0.1	0.922	0.963	0.990	0.930	0.959	0.991
	0.15	0.899	0.938	0.981	0.907	0.954	0.987
	0.2	0.886	0.935	0.981	0.892	0.945	0.983
	0.25	0.879	0.925	0.985	0.893	0.946	0.984
	0.3	0.884	0.937	0.983	0.896	0.943	0.988

$i = 1, \ldots, n$. For the artificial trend function, we consider a linear trend $\mu^\diamond(t) = t$ and a quadratic trend $\mu^\diamond(t) = (t - 0.5)^2$. The results are sum-

Table 3: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where the Rademacher distribution in (c) is used to generate $X_i^\diamond$ and $Y_i^\diamond$ in the simulation-assisted procedure.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.856	0.924	0.964	0.864	0.929	0.965
	0.15	0.882	0.931	0.988	0.899	0.949	0.986
	0.2	0.892	0.951	0.982	0.915	0.958	0.986
	0.25	0.905	0.945	0.987	0.917	0.961	0.986
	0.3	0.911	0.952	0.988	0.917	0.960	0.990
1000	0.1	0.890	0.941	0.988	0.895	0.945	0.991
	0.15	0.893	0.938	0.984	0.904	0.941	0.990
	0.2	0.894	0.946	0.984	0.905	0.955	0.986
	0.25	0.894	0.949	0.986	0.902	0.955	0.989
	0.3	0.904	0.945	0.989	0.893	0.957	0.989
				correlation			
500	0.1	0.888	0.938	0.983	0.883	0.944	0.983
	0.15	0.825	0.920	0.986	0.836	0.923	0.988
	0.2	0.843	0.918	0.987	0.858	0.924	0.992
	0.25	0.851	0.915	0.977	0.870	0.936	0.985
	0.3	0.854	0.927	0.981	0.869	0.938	0.983
1000	0.1	0.841	0.921	0.987	0.848	0.927	0.987
	0.15	0.838	0.910	0.970	0.851	0.921	0.972
	0.2	0.854	0.910	0.973	0.868	0.926	0.976
	0.25	0.856	0.911	0.982	0.876	0.925	0.982
	0.3	0.867	0.924	0.981	0.878	0.933	0.982

marized in Tables 11 and 12, from which we can see that the results are reasonably robust to the presence of an artificial trend in the simulation-

Table 4: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where the t_4 distribution in (d) is used to generate $X_i^\diamond$ and $Y_i^\diamond$ in the simulation-assisted procedure.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.841	0.918	0.970	0.852	0.922	0.969
	0.15	0.856	0.911	0.981	0.874	0.932	0.982
	0.2	0.876	0.935	0.981	0.900	0.947	0.983
	0.25	0.882	0.933	0.982	0.904	0.944	0.981
	0.3	0.902	0.942	0.983	0.898	0.943	0.988
1000	0.1	0.883	0.938	0.987	0.891	0.938	0.991
	0.15	0.876	0.925	0.978	0.890	0.929	0.983
	0.2	0.873	0.930	0.979	0.890	0.947	0.982
	0.25	0.880	0.930	0.985	0.886	0.946	0.988
	0.3	0.883	0.937	0.984	0.878	0.940	0.987
				correlation			
500	0.1	0.914	0.957	0.989	0.912	0.957	0.991
	0.15	0.925	0.970	0.992	0.923	0.968	0.992
	0.2	0.907	0.951	0.996	0.916	0.961	0.995
	0.25	0.899	0.955	0.992	0.920	0.958	0.996
	0.3	0.903	0.958	0.991	0.917	0.960	0.991
1000	0.1	0.951	0.986	0.998	0.955	0.983	0.998
	0.15	0.910	0.958	0.998	0.921	0.961	0.997
	0.2	0.893	0.937	0.989	0.899	0.953	0.988
	0.25	0.883	0.932	0.986	0.899	0.951	0.988
	0.3	0.883	0.937	0.986	0.893	0.944	0.988

assisted samples. This is largely because the simulation-assisted samples $X_i^\diamond$ and $Y_i^\diamond$ are used to calculate the test statistic in the same way as the

Table 5: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where the t_9 distribution in (d) is used to generate $X_i^\diamond$ and $Y_i^\diamond$ in the simulation-assisted procedure.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.846	0.924	0.966	0.860	0.928	0.967
	0.15	0.867	0.923	0.984	0.884	0.939	0.982
	0.2	0.882	0.940	0.981	0.904	0.953	0.983
	0.25	0.897	0.938	0.985	0.911	0.950	0.983
	0.3	0.911	0.943	0.983	0.907	0.952	0.988
1000	0.1	0.890	0.941	0.987	0.894	0.941	0.991
	0.15	0.890	0.933	0.985	0.897	0.935	0.992
	0.2	0.885	0.942	0.983	0.897	0.949	0.986
	0.25	0.887	0.937	0.986	0.890	0.952	0.989
	0.3	0.889	0.941	0.987	0.882	0.950	0.989
				correlation			
500	0.1	0.877	0.924	0.979	0.874	0.925	0.978
	0.15	0.858	0.924	0.983	0.869	0.924	0.985
	0.2	0.867	0.923	0.982	0.885	0.934	0.987
	0.25	0.871	0.929	0.982	0.892	0.942	0.986
	0.3	0.887	0.936	0.983	0.893	0.943	0.985
1000	0.1	0.880	0.939	0.986	0.884	0.944	0.986
	0.15	0.864	0.920	0.979	0.874	0.933	0.979
	0.2	0.871	0.916	0.976	0.877	0.934	0.981
	0.25	0.866	0.916	0.984	0.881	0.934	0.982
	0.3	0.870	0.928	0.982	0.882	0.933	0.982

Table 6: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where $X_i^\diamond$ is standard normal and $Y_i^\diamond$ is generated from the uniform distribution in (a) in the simulation-assisted procedure.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.845	0.915	0.958	0.856	0.921	0.958
	0.15	0.869	0.927	0.984	0.891	0.944	0.984
	0.2	0.890	0.945	0.982	0.915	0.954	0.986
	0.25	0.902	0.947	0.987	0.916	0.959	0.986
	0.3	0.911	0.952	0.989	0.917	0.959	0.990
1000	0.1	0.890	0.940	0.988	0.896	0.941	0.991
	0.15	0.888	0.937	0.984	0.898	0.938	0.988
	0.2	0.887	0.939	0.983	0.901	0.949	0.984
	0.25	0.894	0.950	0.985	0.901	0.955	0.988
	0.3	0.904	0.946	0.987	0.894	0.957	0.989
				correlation			
500	0.1	0.875	0.930	0.977	0.874	0.930	0.977
	0.15	0.834	0.918	0.983	0.843	0.923	0.986
	0.2	0.858	0.921	0.981	0.874	0.933	0.989
	0.25	0.863	0.924	0.980	0.885	0.941	0.985
	0.3	0.880	0.937	0.983	0.892	0.943	0.985
1000	0.1	0.862	0.929	0.987	0.874	0.939	0.987
	0.15	0.848	0.915	0.973	0.859	0.926	0.976
	0.2	0.866	0.912	0.976	0.871	0.930	0.980
	0.25	0.870	0.917	0.980	0.881	0.935	0.982
	0.3	0.872	0.929	0.981	0.887	0.933	0.982

observed data, where the trend function can be handled by the proposed locally homogenized centering scheme.

Table 7: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where $X_i^\diamond$ is standard normal and $Y_i^\diamond$ is generated from the exponential distribution in (b) in the simulation-assisted procedure.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.844	0.912	0.958	0.854	0.918	0.958
	0.15	0.864	0.918	0.981	0.881	0.934	0.982
	0.2	0.879	0.935	0.979	0.901	0.947	0.98
	0.25	0.882	0.934	0.982	0.906	0.945	0.981
	0.3	0.906	0.943	0.983	0.905	0.946	0.988
1000	0.1	0.882	0.934	0.987	0.891	0.938	0.991
	0.15	0.883	0.933	0.982	0.893	0.936	0.987
	0.2	0.889	0.942	0.981	0.899	0.949	0.983
	0.25	0.892	0.943	0.985	0.896	0.952	0.988
	0.3	0.903	0.944	0.987	0.891	0.953	0.989
				correlation			
500	0.1	0.892	0.936	0.979	0.892	0.94	0.987
	0.15	0.878	0.937	0.983	0.882	0.935	0.985
	0.2	0.874	0.928	0.983	0.886	0.939	0.99
	0.25	0.877	0.933	0.981	0.896	0.944	0.986
	0.3	0.89	0.943	0.987	0.905	0.949	0.99
1000	0.1	0.895	0.947	0.988	0.905	0.951	0.987
	0.15	0.87	0.922	0.977	0.879	0.934	0.978
	0.2	0.881	0.923	0.976	0.885	0.939	0.98
	0.25	0.879	0.924	0.984	0.897	0.942	0.982
	0.3	0.884	0.936	0.984	0.895	0.943	0.988

Table 8: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where $X_i^\diamond$ is standard normal and $Y_i^\diamond$ is generated from the Rademacher distribution in (c) in the simulation-assisted procedure.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.860	0.929	0.963	0.868	0.930	0.961
	0.15	0.877	0.929	0.985	0.898	0.945	0.985
	0.2	0.890	0.947	0.982	0.915	0.957	0.986
	0.25	0.906	0.945	0.988	0.919	0.958	0.986
	0.3	0.914	0.950	0.988	0.919	0.958	0.990
1000	0.1	0.883	0.939	0.988	0.894	0.940	0.991
	0.15	0.888	0.937	0.984	0.897	0.939	0.989
	0.2	0.887	0.941	0.983	0.902	0.949	0.983
	0.25	0.892	0.948	0.986	0.896	0.954	0.988
	0.3	0.901	0.944	0.987	0.889	0.952	0.989
				correlation			
500	0.1	0.878	0.933	0.978	0.876	0.937	0.979
	0.15	0.840	0.913	0.985	0.847	0.920	0.988
	0.2	0.846	0.919	0.986	0.862	0.924	0.993
	0.25	0.861	0.915	0.981	0.878	0.937	0.986
	0.3	0.877	0.934	0.982	0.889	0.940	0.984
1000	0.1	0.853	0.924	0.986	0.867	0.934	0.986
	0.15	0.842	0.914	0.974	0.855	0.926	0.976
	0.2	0.863	0.912	0.974	0.871	0.930	0.977
	0.25	0.864	0.913	0.978	0.879	0.929	0.982
	0.3	0.868	0.924	0.981	0.882	0.933	0.982

S2.3 Time-Varying Vector Autoregressive Model

We shall here supplement the simulation setting in Section 4.2 by considering an additional data generating mechanism to further examine the

Table 9: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where $X_i^\diamond$ is standard normal and $Y_i^\diamond$ is generated from the t_4 distribution in (d) in the simulation-assisted procedure.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.845	0.919	0.966	0.858	0.926	0.969
	0.15	0.866	0.923	0.982	0.883	0.940	0.982
	0.2	0.882	0.937	0.981	0.905	0.952	0.983
	0.25	0.897	0.938	0.983	0.911	0.949	0.982
	0.3	0.911	0.948	0.985	0.910	0.953	0.988
1000	0.1	0.890	0.941	0.989	0.896	0.942	0.992
	0.15	0.889	0.937	0.985	0.899	0.939	0.990
	0.2	0.885	0.942	0.983	0.900	0.949	0.983
	0.25	0.888	0.947	0.985	0.893	0.952	0.988
	0.3	0.896	0.942	0.987	0.883	0.950	0.989
				correlation			
500	0.1	0.901	0.942	0.979	0.900	0.950	0.987
	0.15	0.895	0.956	0.990	0.907	0.959	0.991
	0.2	0.888	0.940	0.996	0.903	0.952	0.995
	0.25	0.885	0.943	0.987	0.906	0.951	0.989
	0.3	0.896	0.949	0.989	0.910	0.950	0.990
1000	0.1	0.919	0.967	0.998	0.926	0.964	0.998
	0.15	0.894	0.941	0.991	0.901	0.955	0.992
	0.2	0.883	0.935	0.985	0.890	0.946	0.985
	0.25	0.879	0.925	0.985	0.894	0.950	0.986
	0.3	0.878	0.936	0.984	0.891	0.938	0.988

finite-sample performance of the proposed method. For this, we consider

Table 10: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where $X_i^\diamond$ is standard normal and $Y_i^\diamond$ is generated from the t_9 distribution in (c) in the simulation-assisted procedure.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.843	0.915	0.964	0.854	0.921	0.965
	0.15	0.865	0.919	0.981	0.882	0.935	0.982
	0.2	0.879	0.937	0.981	0.900	0.948	0.980
	0.25	0.887	0.938	0.985	0.908	0.950	0.983
	0.3	0.911	0.946	0.983	0.908	0.952	0.988
1000	0.1	0.890	0.939	0.987	0.896	0.938	0.991
	0.15	0.885	0.935	0.982	0.893	0.937	0.987
	0.2	0.883	0.942	0.982	0.897	0.949	0.983
	0.25	0.887	0.941	0.986	0.892	0.952	0.988
	0.3	0.898	0.944	0.987	0.888	0.952	0.989
				correlation			
500	0.1	0.871	0.923	0.977	0.871	0.927	0.977
	0.15	0.848	0.914	0.983	0.854	0.918	0.985
	0.2	0.853	0.918	0.981	0.867	0.922	0.987
	0.25	0.862	0.919	0.980	0.882	0.939	0.985
	0.3	0.878	0.936	0.981	0.891	0.941	0.985
1000	0.1	0.871	0.936	0.989	0.878	0.944	0.988
	0.15	0.859	0.917	0.973	0.869	0.930	0.975
	0.2	0.867	0.915	0.976	0.874	0.930	0.980
	0.25	0.866	0.914	0.981	0.880	0.932	0.982
	0.3	0.869	0.924	0.982	0.884	0.933	0.982

the time-varying vector autoregressive model

$$\begin{pmatrix} \tilde{X}_i \\ \tilde{Y}_i \end{pmatrix} = \begin{Bmatrix} 0.5 & 0.5\cos(\pi i/n) \\ 0 & 0.3 \end{Bmatrix} \begin{pmatrix} \tilde{X}_{i-1} \\ \tilde{Y}_{i-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{i,1} \\ \epsilon_{i,2} \end{pmatrix},$$

Table 11: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where a linear trend $\mu^\diamond(i/n) = i/n$ is added to $X_i^\diamond$ when generating the simulation-assisted samples in the algorithm described in Section 4.1.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.878	0.942	0.972	0.891	0.943	0.974
	0.15	0.885	0.935	0.990	0.891	0.946	0.989
	0.2	0.899	0.944	0.987	0.902	0.949	0.992
	0.25	0.895	0.946	0.986	0.909	0.941	0.990
	0.3	0.904	0.939	0.985	0.901	0.947	0.987
1000	0.1	0.896	0.943	0.987	0.908	0.950	0.987
	0.15	0.903	0.960	0.986	0.915	0.962	0.987
	0.2	0.897	0.955	0.988	0.911	0.961	0.991
	0.25	0.907	0.949	0.992	0.905	0.957	0.991
	0.3	0.913	0.955	0.993	0.897	0.955	0.995
				correlation			
500	0.1	0.881	0.954	0.992	0.893	0.950	0.990
	0.15	0.835	0.919	0.988	0.849	0.920	0.988
	0.2	0.850	0.924	0.982	0.865	0.928	0.981
	0.25	0.856	0.922	0.978	0.869	0.927	0.980
	0.3	0.868	0.920	0.975	0.880	0.924	0.973
1000	0.1	0.853	0.919	0.987	0.857	0.921	0.986
	0.15	0.862	0.925	0.984	0.872	0.932	0.990
	0.2	0.867	0.929	0.984	0.884	0.940	0.985
	0.25	0.884	0.929	0.986	0.882	0.939	0.987
	0.3	0.871	0.930	0.983	0.883	0.934	0.989

for which the correlation between the two time series $(\tilde{X}_i)$ and $(\tilde{Y}_i)$ is changing over time. We then let

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{Bmatrix} \mu_x(i/n) \\ \mu_y(i/n) \end{Bmatrix} + \begin{pmatrix} \tilde{X}_i \\ \tilde{Y}_i \end{pmatrix}, \tag{S2.4}$$

Table 12: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) as functions of time, where a quadratic trend $\mu^\diamond(i/n) = (i/n - 0.5)^2$ is added to $X_i^\diamond$ when generating the simulation-assisted samples in the algorithm described in Section 4.1.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.853	0.921	0.964	0.862	0.928	0.962
	0.15	0.869	0.928	0.985	0.888	0.943	0.986
	0.2	0.887	0.945	0.982	0.913	0.954	0.985
	0.25	0.904	0.945	0.987	0.916	0.957	0.986
	0.3	0.914	0.951	0.988	0.920	0.957	0.988
1000	0.1	0.895	0.946	0.989	0.899	0.953	0.992
	0.15	0.889	0.938	0.985	0.899	0.940	0.992
	0.2	0.889	0.944	0.985	0.902	0.955	0.987
	0.25	0.892	0.949	0.986	0.900	0.955	0.989
	0.3	0.901	0.946	0.988	0.889	0.957	0.989
				correlation			
500	0.1	0.873	0.930	0.979	0.874	0.931	0.981
	0.15	0.851	0.918	0.983	0.854	0.920	0.984
	0.2	0.860	0.922	0.983	0.874	0.934	0.990
	0.25	0.871	0.926	0.984	0.890	0.942	0.988
	0.3	0.889	0.936	0.986	0.897	0.943	0.988
1000	0.1	0.871	0.936	0.987	0.879	0.944	0.987
	0.15	0.859	0.919	0.977	0.869	0.930	0.978
	0.2	0.870	0.916	0.976	0.874	0.933	0.981
	0.25	0.870	0.917	0.984	0.883	0.935	0.982
	0.3	0.872	0.929	0.983	0.884	0.933	0.987

where the trend functions $\mu_x(t) = \log(1 + t)$ and $\mu_y(t) = \sin(t)$. For the innovations, we consider cases when $(\epsilon_{i,1})$ and $(\epsilon_{i,2})$ are independent standard normal, t_4, or t_9. The results are summarized in Tables 13–15, from which we can see that the proposed LHC-DA method in general performs reasonably well as the empirical coverage probabilities are close to their nominal levels for most of the cases considered. However, for the correlation case with t_4 innovations, a certain degree of size distortions can be observed from Table 14. This is possibly due to the fact that a finite eighth moment is assumed in Theorem 2 for simultaneous inference of the time-varying correlation, which is not satisfied by the t_4 distribution. The same moment condition is used in Theorems 1 and 2 of Zhao (2015) for processes with a known zero mean and geometrically decaying dependence. If we consider the t-distribution with degree of freedom 9 as in Table 15 that has a finite eighth moment, then the size distortion diminishes and the empirical coverage probabilities become reasonably close to their nominal levels.

Table 13: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) from the time-varying vector autoregressive model (S2.4) with standard normal innovations.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.878	0.944	0.985	0.882	0.945	0.986
	0.15	0.892	0.944	0.994	0.896	0.945	0.994
	0.2	0.891	0.953	0.994	0.901	0.958	0.992
	0.25	0.911	0.962	0.995	0.918	0.967	0.996
	0.3	0.909	0.955	0.992	0.922	0.964	0.992
1000	0.1	0.911	0.954	0.991	0.921	0.959	0.991
	0.15	0.910	0.955	0.990	0.907	0.957	0.991
	0.2	0.904	0.961	0.991	0.911	0.964	0.991
	0.25	0.904	0.953	0.992	0.910	0.964	0.993
	0.3	0.905	0.953	0.993	0.911	0.962	0.994
				correlation			
500	0.1	0.889	0.961	0.996	0.889	0.956	0.994
	0.15	0.872	0.936	0.988	0.879	0.934	0.988
	0.2	0.865	0.922	0.985	0.869	0.921	0.984
	0.25	0.867	0.942	0.988	0.875	0.946	0.988
	0.3	0.867	0.926	0.986	0.880	0.927	0.988
1000	0.1	0.862	0.939	0.991	0.874	0.940	0.990
	0.15	0.871	0.931	0.992	0.878	0.939	0.992
	0.2	0.877	0.936	0.989	0.888	0.946	0.990
	0.25	0.884	0.939	0.988	0.890	0.948	0.990
	0.3	0.882	0.931	0.988	0.883	0.943	0.989

Table 14: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) from the time-varying vector autoregressive model (S2.4) with t_4 innovations.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.866	0.938	0.981	0.875	0.941	0.980
	0.15	0.885	0.947	0.990	0.885	0.949	0.990
	0.2	0.896	0.943	0.992	0.910	0.946	0.992
	0.25	0.903	0.960	0.994	0.912	0.960	0.993
	0.3	0.915	0.960	0.990	0.921	0.965	0.990
1000	0.1	0.896	0.953	0.989	0.901	0.958	0.991
	0.15	0.907	0.956	0.994	0.911	0.958	0.996
	0.2	0.920	0.953	0.991	0.919	0.957	0.993
	0.25	0.922	0.965	0.993	0.920	0.968	0.995
	0.3	0.913	0.961	0.990	0.921	0.965	0.992
				correlation			
500	0.1	0.798	0.922	0.979	0.828	0.920	0.976
	0.15	0.792	0.866	0.979	0.784	0.864	0.978
	0.2	0.829	0.891	0.964	0.833	0.903	0.970
	0.25	0.834	0.909	0.978	0.849	0.917	0.974
	0.3	0.849	0.903	0.978	0.849	0.910	0.979
1000	0.1	0.758	0.861	0.957	0.770	0.865	0.958
	0.15	0.816	0.898	0.967	0.823	0.910	0.971
	0.2	0.838	0.895	0.971	0.840	0.898	0.972
	0.25	0.849	0.916	0.978	0.859	0.920	0.976
	0.3	0.848	0.911	0.978	0.862	0.921	0.978

Table 15: Empirical coverage probabilities of simultaneous confidence bands for the time-varying covariance and time-varying correlation between (X_i) and (Y_i) from the time-varying vector autoregressive model (S2.4) with t_9 innovations.

		LHC			LHC-DA		
n	b_n	90%	95%	99%	90%	95%	99%
				covariance			
500	0.1	0.864	0.931	0.981	0.870	0.936	0.981
	0.15	0.883	0.937	0.986	0.882	0.943	0.986
	0.2	0.894	0.944	0.981	0.907	0.950	0.982
	0.25	0.909	0.953	0.989	0.908	0.956	0.991
	0.3	0.917	0.959	0.990	0.922	0.962	0.993
1000	0.1	0.901	0.954	0.991	0.907	0.958	0.993
	0.15	0.913	0.960	0.990	0.917	0.966	0.990
	0.2	0.895	0.954	0.994	0.898	0.957	0.993
	0.25	0.907	0.952	0.995	0.909	0.959	0.994
	0.3	0.896	0.954	0.991	0.900	0.952	0.991
				correlation			
500	0.1	0.888	0.950	0.990	0.895	0.953	0.989
	0.15	0.863	0.932	0.995	0.870	0.943	0.994
	0.2	0.853	0.922	0.989	0.863	0.923	0.988
	0.25	0.857	0.920	0.989	0.858	0.929	0.991
	0.3	0.856	0.925	0.984	0.878	0.931	0.989
1000	0.1	0.866	0.934	0.992	0.873	0.928	0.993
	0.15	0.893	0.938	0.988	0.894	0.939	0.988
	0.2	0.880	0.931	0.985	0.890	0.938	0.986
	0.25	0.883	0.946	0.982	0.887	0.948	0.981
	0.3	0.868	0.932	0.983	0.877	0.938	0.984

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