SUPPLEMENT TO "EFFICIENT ESTIMATION AND INFERENCE FOR THE SIGNED β-MODEL IN DIRECTED SIGNED NETWORKS"

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Supplementary Material

In this supplement, we provide technical proofs for all the theoretical results, as well as an estimation procedure for $\boldsymbol{\kappa}$.

S1 Appendix A: notations and necessary lemmas

Throughout the appendices, we use c to denote a generic positive constant whose value may vary according to context. For two nonnegative sequences a_n and b_n, $a_n \lesssim b_n$ means there exists a positive constant c such that $a_n \leq cb_n$ when n is sufficiently large, and $a_n \lesssim_P b_n$ means there exist positive constants c_1, c_2 such that $\Pr(a_n \geq c_1 b_n) \leq c_2 n^{-1}$ for n large enough.

With slight abuse of notations, we rewrite $\boldsymbol{\theta} = (\boldsymbol{\alpha}^\top, \boldsymbol{\beta}^\top, \beta_n)^\top$ with $\theta_{2n} = \beta_n = 0$, and define $\partial^2 l(\boldsymbol{\theta}; \boldsymbol{\kappa})/\partial\alpha_i\partial\beta_n = l''_{in}(\alpha_i + \beta_n; \kappa_i)$ for $i \in [n-1]$

and $\partial^2 l(\boldsymbol{\theta};\boldsymbol{\kappa})/\partial\alpha_n\partial\beta_n = 0$. Further define

$$\frac{\partial l(\boldsymbol{\theta};\boldsymbol{\kappa})}{\partial\beta_n} = \sum_{i=1}^{n-1} l'_{in}(\alpha_i+\beta_n;\kappa_i) \quad \text{and} \quad \frac{\partial^2 l(\boldsymbol{\theta};\boldsymbol{\kappa})}{\partial\beta_n^2} = \sum_{i=1}^{n-1} l''_{in}(\alpha_i+\beta_n;\kappa_i).$$

Similarly, $\check{\boldsymbol{\theta}}$, $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^*$ are all augmented with an additional 0 entry, and the corresponding partial derivatives of $l(\check{\boldsymbol{\theta}};\boldsymbol{\kappa})$, $l(\widehat{\boldsymbol{\theta}};\boldsymbol{\kappa})$ and $l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})$ are defined as above. Also, we denote

$$\mathcal{I}_{ij} = \frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j} - \mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}\right], \text{ for } i,j\in[n],$$

and $\mathcal{I}_{ii} = 0$ for $i\in[n]$. Define $d_i = \sum_{k=1,k\neq i}^n y_{ik}$ for any $i\in[n]$, $b_j = \sum_{k=1,k\neq j}^n y_{kj}$ for any $j\in[n]$, and $(g_i)_{i=1}^{2n} = (d_1,...,d_n,b_1,...,b_n)^\top$. Further, for any $i,j\in[n]$, define

$$g_{i\backslash j} = g_i - y_{ij} = \sum_{k=1,k\neq i,j}^n y_{il}, \text{ and } g_{n+j\backslash i} = g_{n+j} - y_{ij} = \sum_{k=1,k\neq i,j}^n y_{kj}.$$

Recall the definitions in (??), (??), (??) and (??):

$$v_i = \sum_{k=1,k\neq i}^n \frac{(1+\kappa_i)e^{\alpha_i^*+\beta_k^*}}{(1+e^{\alpha_i^*+\beta_k^*})^2}, \qquad w_i = \sum_{k=1,k\neq i}^n \mathrm{var}(y_{ik}), \qquad \text{for } i\in[n],$$
$$v_{n+j} = \sum_{k=1,k\neq j}^n \frac{(1+\kappa_k)e^{\alpha_k^*+\beta_j^*}}{(1+e^{\alpha_k^*+\beta_j^*})^2}, \quad w_{n+j} = \sum_{k=1,k\neq j}^n \mathrm{var}(y_{kj}), \quad \text{for } j\in[n],$$

$$u_i = -\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\theta_i^2}, \quad \check{u}_i = -\frac{\partial^2 l(\check{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\theta_i^2}, \quad \widehat{u}_i = -\frac{\partial^2 l(\widehat{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\theta_i^2}, \quad \text{for } i\in[2n].$$

and $\widehat{\delta}_{ij}^2 = \widehat{u}_i^{-1} + \widehat{u}_j^{-1}$, $(\delta_{ij}^*)^2 = u_i^{-1} + (u_j^*)^{-1}$. Below we list some necessary lemmas, and their proof are provided in Appendix C.

Lemma 1. *There exists a positive constant c such that*

$$c^{-1}ne^{-2\|\boldsymbol{\theta}^*\|_\infty} \leq \min_{1\leq i\leq 2n} u_i \leq \max_{1\leq i\leq 2n} u_i \leq cn,$$

$$c^{-1}ne^{-2\|\boldsymbol{\theta}^*\|_\infty} \leq \min_{1\leq i\leq 2n} v_i \leq \max_{1\leq i\leq 2n} v_i \leq cn,$$

$$c^{-1}ne^{-2\|\boldsymbol{\theta}^*\|_\infty} \leq \min_{1\leq i\leq 2n} w_i \leq \max_{1\leq i\leq 2n} w_i \leq cn.$$

Further, there exists a constant $\epsilon > 0$ such that for any $\boldsymbol{\theta}, \boldsymbol{\kappa}$ satisfying $\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_\infty \leq \epsilon$ and $\|\boldsymbol{\kappa} - \boldsymbol{\kappa}^*\|_\infty \leq \epsilon$, we have

$$n^{-1} \max_{1\leq i\leq 2n} \left| \frac{\partial^2 l(\boldsymbol{\theta}; \boldsymbol{\kappa})}{\partial \theta_i^2} - \frac{\partial^2 l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \theta_i^2} \right| = O(\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_\infty),$$

$$n^{-1} \max_{1\leq i\leq 2n} \left| \frac{\partial^2 l(\boldsymbol{\theta}; \boldsymbol{\kappa})}{\partial \theta_i^2} - \frac{\partial^2 l(\boldsymbol{\theta}; \boldsymbol{\kappa}^*)}{\partial \theta_i^2} \right| = O(\|\boldsymbol{\kappa} - \boldsymbol{\kappa}^*\|_\infty).$$

Lemma 2. *It holds true that*

$$\Pr\left(\max_{1\leq i\leq 2n} |g_i - \mathbb{E}g_i| \leq \sqrt{4(n-1)\log(n-1)} \right) \geq 1 - \frac{4n}{(n-1)^2}. \quad \text{(S1.1)}$$

Further, there exists a constant $c > 0$ such that for n large enough, we have

$$\Pr\left(\max_{1\leq i\leq 2n} \left| \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \theta_i} \right| \leq \sqrt{16(n-1)\log(n-1)} + cn\|\boldsymbol{\kappa} - \boldsymbol{\kappa}^*\|_\infty \right) \geq 1 - \frac{4n}{(n-1)^2}, \quad \text{(S1.2)}$$

$$\Pr\left(\max_{1\leq i\leq 2n} \left| \frac{\partial^2 l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \theta_i^2} + u_i \right| \leq c\sqrt{(n-1)\log(n-1)} \right) \geq 1 - \frac{4n}{(n-1)^2}. \quad \text{(S1.3)}$$

Lemma 3. *Under the conditions of Theorem* **??**, *with probability at least* $1-4n/(n-1)^2$, *it holds true that* (**??**) *has a unique solution* $\check{\boldsymbol{\theta}}$, *and*

$$\|\check{\boldsymbol{\theta}}-\boldsymbol{\theta}^*\|_\infty \lesssim e^{6\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}}.$$

By Lemmas 1-3, we have

$$\max_{1\le i\le 2n}|\check{u}_i-u_i| \le \max_{1\le i\le 2n}\left|\frac{\partial^2 l(\check{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\theta_i^2}-\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\theta_i^2}\right|+\max_{1\le i\le 2n}\left|\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\theta_i^2}+u_i\right| \lesssim_P e^{6\|\boldsymbol{\theta}^*\|_\infty}\sqrt{n\log n},$$

and

$$\max_{1\le i\le 2n}\left|\check{u}_i^{-1}-({u_i}^*)^{-1}\right| = \max_{1\le i\le 2n}\frac{|\check{u}_i-u_i|}{|u_i\check{u}_i|} \lesssim_P e^{10\|\boldsymbol{\theta}^*\|_\infty}\frac{\sqrt{\log n}}{n^{3/2}}. \qquad \text{(S1.4)}$$

Lemma 4. *Under the conditions of Theorem* **??**, *it holds true that*

$$\check{\alpha}_i-\alpha_i^* = v_i^{-1}(g_i-\mathbb{E}g_i)+v_{2n}^{-1}(g_{2n}-\mathbb{E}g_{2n})+\epsilon_i, \qquad i=1,...,n,$$

$$\check{\beta}_j-\beta_j^* = v_{n+j}^{-1}(g_{n+j}-\mathbb{E}g_{n+j})-v_{2n}^{-1}(g_{2n}-\mathbb{E}g_{2n})+\epsilon_{n+j}, \qquad j=1,...,n-1,$$

where ϵ_i *satisfies that*

$$P\Big(\max_{1\le i\le 2n-1}|\epsilon_i| \lesssim \frac{e^{18\|\boldsymbol{\theta}^*\|_\infty}\log n}{n}\Big) \ge 1-\frac{4n}{(n-1)^2}.$$

Lemma 5. *Under the conditions of Theorem* **??**, *it holds true that*

$$\Pr\left(\max_{1\le i\le n}\left|\sum_{l=1}^{n-1}u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}v_{n+l}^{-1}(g_{n+l}-\mathbb{E}g_{n+l})\right| \lesssim e^{4\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}}\right) \ge 1-\frac{2}{n},$$
$$\Pr\left(\max_{1\le j\le n}\left|\sum_{k=1}^{n}u_{n+j}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_j}\right]\sqrt{n}v_k^{-1}(g_k-\mathbb{E}g_k)\right| \lesssim e^{4\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}}\right) \ge 1-\frac{2}{n}, \qquad \text{(S1.5)}$$

and

$$\Pr\left(\max_{1\le i\le n}\left|\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}v_{n+l}^{-1}(g_{n+l\setminus i}-\mathbb{E}g_{n+l\setminus i})\right| \lesssim e^{4\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}}\right) \ge 1-\frac{2}{n},$$
$$\Pr\left(\max_{1\le j\le n}\left|\sum_{k=1}^{n} u_{n+j}^{-1}\mathcal{I}_{kj}\sqrt{n}v_k^{-1}(g_{k\setminus j}-\mathbb{E}g_{k\setminus j})\right| \lesssim e^{4\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}}\right) \ge 1-\frac{2}{n}. \tag{S1.6}$$

Lemma 6. *Under the conditions of Theorem ??, it holds true that*

$$\max_{1\le i\le n}\left|\sum_{k=1}^{n} u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right]\sqrt{n}(\check{\alpha}_k-\alpha_k^*)+\sum_{l=1}^{n-1} u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}(\check{\beta}_l-\beta_l^*)\right| \lesssim_P \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n}{\sqrt{n}},$$

and

$$\max_{1\le j\le n-1}\left|\sum_{k=1}^{n}\sqrt{n}(\check{\alpha}_k-\alpha_k^*)\left\{u_{n+j}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_j}\right]-u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right]\right\}\right| \lesssim_P \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n}{\sqrt{n}}.$$

Lemma 7. *Under the conditions of Theorem ??, it holds true that*

$$\max_{1\le i\le n}\left|\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}(\check{\alpha}_k-\alpha_k^*)+\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}(\check{\beta}_l-\beta_l^*)\right| \lesssim_P \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n}{\sqrt{n}},$$

and

$$\max_{1\le j\le n-1}\left|\sum_{k=1}^{n}\sqrt{n}(\check{\alpha}_k-\alpha_k^*)\left\{u_{n+j}^{-1}\mathcal{I}_{kj}-u_{2n}^{-1}\mathcal{I}_{kn}\right\}\right| \lesssim_P \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n}{\sqrt{n}}.$$

S2 Appendix B: main proofs

Proof of Theorem ??. The asymptotic bound for $\|\check{\boldsymbol{\theta}}-\boldsymbol{\theta}^*\|_\infty$ in (**??**) follows from Lemma 3 immediately. To obtain asymptotic normality, it follows from Lemma 4 that

$$\max_{1\le i\le n}\left|(\check{\alpha}_i-\alpha_i^*)-\left\{v_i^{-1}(g_i-\mathbb{E}g_i)+v_{2n}^{-1}(g_{2n}-\mathbb{E}g_{2n})\right\}\right| \lesssim_P \frac{e^{18\|\boldsymbol{\theta}^*\|_\infty}\log n}{n},$$
$$\max_{1\le j\le n-1}\left|(\check{\beta}_j-\beta_j^*)-\left\{v_{n+j}^{-1}(g_{n+j}-\mathbb{E}g_{n+j})-v_{2n}^{-1}(g_{2n}-\mathbb{E}g_{2n})\right\}\right| \lesssim_P \frac{e^{18\|\boldsymbol{\theta}^*\|_\infty}\log n}{n}. \tag{S2.7}$$

Define $\mathbf{q} \in \mathbb{R}^{2n-1}$ with $q_i = v_i^{-1}(g_i - \mathbb{E}g_i) + v_{2n}^{-1}(g_{2n} - \mathbb{E}g_{2n})$ for $i = 1, ..., n$, and $q_{n+j} = v_{n+j}^{-1}(g_{n+j} - \mathbb{E}g_{n+j}) - v_{2n}^{-1}(g_{2n} - \mathbb{E}g_{2n})$ for $j = 1, ..., n-1$. It is not difficult to verify that for any fixed d, $\mathbf{q}_{[1:d]}$ is asymptotically multivariate normal with mean zero and covariance matrix $\mathbf{\Sigma}_{[1:d,1:d]}$. By (S2.7) and Lemma 1, we have

$$\|(\mathbf{\Sigma}_{[1:d,1:d]})^{-\frac{1}{2}}(\check{\boldsymbol{\theta}} - \boldsymbol{\theta}^* - \mathbf{q})_{[1:d]}\|_\infty \lesssim_P \sqrt{n} e^{\|\boldsymbol{\theta}^*\|_\infty} \frac{e^{18\|\boldsymbol{\theta}^*\|_\infty} \log n}{n} = \frac{e^{19\|\boldsymbol{\theta}^*\|_\infty} \log n}{\sqrt{n}} = o(1),$$

which completes the proof of Theorem ??. □

Proof of Theorem ??. By Taylor's expansion, there exists a $\widetilde{\boldsymbol{\theta}}$ such that

$$\frac{\partial l(\check{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \boldsymbol{\theta}} = \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \boldsymbol{\theta}} + \frac{\partial^2 l(\widetilde{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \boldsymbol{\theta}^2}(\check{\boldsymbol{\theta}} - \boldsymbol{\theta}^*),$$

where $\widetilde{\theta}_i$ lies between θ_i^* and $\check{\theta}_i$ for $i \in [2n-1]$ and $\widetilde{\theta}_{2n} = \theta_{2n}^* = \check{\theta}_{2n} = 0$. It is not difficult to verify that $\partial l(\boldsymbol{\theta}; \boldsymbol{\kappa})/\partial \beta_n = \sum_{k=1}^n \partial l(\boldsymbol{\theta}; \boldsymbol{\kappa})/\partial \alpha_k -$

$\sum_{l=1}^{n-1} \partial l(\boldsymbol{\theta};\boldsymbol{\kappa})/\partial\beta_l$. Then, we have

$$
\begin{aligned}
\widehat{\alpha}_i-\alpha_i^* =&(\check{\alpha}_i-\alpha_i^*)+\check{u}_i^{-1}\frac{\partial l(\check{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\alpha_i}+\check{u}_{2n}^{-1}\frac{\partial l(\check{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\beta_n}\\
=&(\check{\alpha}_i-\alpha_i^*)+\check{u}_i^{-1}\left\{\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i}+\frac{\partial^2 l(\widetilde{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\alpha_i^2}(\check{\alpha}_i-\alpha_i^*)+\sum_{j=1}^{n-1}\frac{\partial^2 l(\widetilde{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}(\check{\beta}_j-\beta_j^*)\right\}\\
&+\check{u}_{2n}^{-1}\left\{\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\beta_n}+\sum_{k=1}^{n}\frac{\partial^2 l(\widetilde{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}(\check{\alpha}_k-\alpha_k^*)\right\}\\
=&\left\{\check{u}_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i}+\check{u}_{2n}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\beta_n}\right\}+(\check{\alpha}_i-\alpha_i^*)\left\{1+\check{u}_i^{-1}\frac{\partial^2 l(\widetilde{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\alpha_i^2}\right\}\\
&+\left\{\check{u}_{2n}^{-1}\sum_{k=1}^{n}\frac{\partial^2 l(\widetilde{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}(\check{\alpha}_k-\alpha_k^*)+\check{u}_i^{-1}\sum_{j=1}^{n-1}\frac{\partial^2 l(\widetilde{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}(\check{\beta}_j-\beta_j^*)\right\}\\
=:&I_1^{(i)}+I_2^{(i)}+I_3^{(i)}.
\end{aligned}
$$

For $I_1^{(i)}$, we have

$$
I_1^{(i)}=\left\{u_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i}+u_{2n}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\beta_n}\right\}+\left[-u_i^{-1}+\check{u}_i^{-1}\right]\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i}+\left[\check{u}_{2n}^{-1}-u_{2n}^{-1}\right]\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\beta_n}.
$$

The first term can be rewritten as

$$
\begin{aligned}
&u_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i}+u_{2n}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\beta_n}\\
=&\left\{u_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i}+u_{2n}^{-1}\sum_{k=1,k\neq i}^{n-1}l'_{kn}(\alpha_k^*+\beta_n^*;\kappa_k)\right\}+u_{2n}^{-1}l'_{in}(\alpha_i^*+\beta_n^*;\kappa_i)\\
=:&I_{11}^{(i)}+I_{12}^{(i)}
\end{aligned}
$$

By central limit theorem, $I_{11}^{(i)}$ is the sum of two independent variables which are asymptotic normal with variances u_i^{-1} and u_{2n}^{-1}, respectively. Therefore, $I_{11}^{(i)}$ is also asymptotic normal with variance $u_i^{-1}+u_{2n}^{-1}$. Also, Lemma 1 implies that $\max_{1\le i\le n}|I_{12}^{(i)}|=O(e^{2\|\boldsymbol{\theta}^*\|_\infty}n^{-1})=o(n^{-1/2})$. Furthermore, it follows from Lemma 2 and (S1.4) that

$$\max_{1\le i\le n}\left|\left[u_i^{-1}-\check{u}_i^{-1}\right]\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \alpha_i}+\left[\check{u}_{2n}^{-1}-u_{2n}^{-1}\right]\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \beta_n}\right|\lesssim_P \frac{e^{10\|\boldsymbol{\theta}^*\|_\infty}\log n}{n},$$

which is of order $o_p(n^{-1/2})$. Therefore, we have $\sqrt{u_i^{-1}+u_{2n}^{-1}}I_1^{(i)}\to N(0,1)$ as n diverges.

For $I_2^{(i)}$, it follows from Lemmas 1 and 3 that

$$\max_{1\le i\le n}|I_2^{(i)}|\le \max_{1\le i\le n}|\check{\alpha}_i-\alpha_i^*|\left|\frac{\check{u}_i+\frac{\partial^2 l(\widetilde{\boldsymbol{\theta}})}{\partial\alpha_i^2}}{\check{u}_i}\right|\lesssim_P e^{6\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}}\frac{e^{6\|\boldsymbol{\theta}^*\|_\infty}\sqrt{n\log n}}{e^{-2\|\boldsymbol{\theta}^*\|_\infty}n}=\frac{e^{14\|\boldsymbol{\theta}^*\|_\infty}\log n}{n},$$

which is of order $o_p(n^{-1/2})$.

For $I_3^{(i)}$, we have

$$\begin{aligned}I_3^{(i)}=&\left\{\sum_{k=1}^{n}\check{u}_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right](\check{\alpha}_k-\alpha_k^*)+\sum_{j=1}^{n-1}\check{u}_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}\right](\check{\beta}_j-\beta_j^*)\right\}\\&+\left\{\sum_{k=1}^{n}\check{u}_{2n}^{-1}\mathcal{I}_{kn}(\check{\alpha}_k-\alpha_k^*)+\sum_{j=1}^{n-1}\check{u}_i^{-1}\mathcal{I}_{ij}(\check{\beta}_j-\beta_j^*)\right\}\\&+\left\{\sum_{k=1}^{n}\check{u}_{2n}^{-1}\left(\frac{\partial^2 l(\widetilde{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}-\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right)(\check{\alpha}_k-\alpha_k^*)+\sum_{j=1}^{n-1}\check{u}_i^{-1}\left(\frac{\partial^2 l(\widetilde{\boldsymbol{\theta}};\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}-\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}\right)(\check{\beta}_j-\beta_j^*)\right\}\\&=:I_{31}^{(i)}+I_{32}^{(i)}+I_{33}^{(i)}.\end{aligned}$$

Note that $|I_{31}^{(i)}|$ can be further bounded as

$$\begin{aligned}|I_{31}^{(i)}|\le&\left|\sum_{k=1}^{n}u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right](\check{\alpha}_k-\alpha_k^*)+\sum_{j=1}^{n-1}u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}\right](\check{\beta}_j-\beta_j^*)\right|\\&+\left|\sum_{k=1}^{n}\left[\check{u}_{2n}^{-1}-u_{2n}^{-1}\right]\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right](\check{\alpha}_k-\alpha_k^*)+\sum_{j=1}^{n-1}\left[\check{u}_i^{-1}-u_i^{-1}\right]\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}\right](\check{\beta}_j-\beta_j^*)\right|.\end{aligned}$$

By Lemma 6, the first term is upper bounded by $e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n/n$ in probability, and by Lemma 3 and (S1.4), the second term is upper bounded by $e^{16\|\boldsymbol{\theta}^*\|_\infty}\log n/n$ in probability. Therefore, we have $\max_{1\le i\le n}|I_{31}^{(i)}|\lesssim_P e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n/n$.

For $I_{32}^{(i)}$, we have

$$
\begin{aligned}
|I_{32}^{(i)}| \le & \left|\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}(\check{\alpha}_k-\alpha_k^*)+\sum_{j=1}^{n-1}u_i^{-1}\mathcal{I}_{ij}(\check{\beta}_j-\beta_j^*)\right| \\
& +\left|\sum_{k=1}^{n}\left[\check{u}_{2n}^{-1}-u_{2n}^{-1}\right]\mathcal{I}_{kn}(\check{\alpha}_k-\alpha_k^*)+\sum_{j=1}^{n-1}\left[\check{u}_i^{-1}-u_i^{-1}\right]\mathcal{I}_{ij}(\check{\beta}_j-\beta_j^*)\right|.
\end{aligned}
$$

By Lemma 7, the first term is upper bounded by $e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n/n$ in probability, and by Lemma 3 and (S1.4),the second term is upper bounded by $e^{16\|\boldsymbol{\theta}^*\|_\infty}\log n/n$ in probability. Therefore, we have $\max_{1\le i\le n}|I_{32}^{(i)}|\lesssim_P e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n/n$.

For $I_{33}^{(i)}$, it follows from Lemma 1, Lemma 3 and (S1.4) that $\max_{1\le i\le n}|I_{33}^{(i)}|\lesssim_P e^{14\|\boldsymbol{\theta}^*\|_\infty}\log n/n$. Then, combining the upper bounds for each term and Lemma 2 yields that

$$
\max_{1\le i\le n}\left|(\widehat{\alpha}_i-\alpha_i^*)-\left\{u_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i}+u_{2n}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\beta_n}\right\}\right|\lesssim_P \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n}{n}, \tag{S2.8}
$$

where $u_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i}+u_{2n}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\beta_n}$ is asymptotic normal with variance $u_i^{-1}+u_{2n}^{-1}$.

Next, we turn to bound $\widehat{\beta}_j - \beta_j^*$, where

$$\begin{aligned}
\widehat{\beta}_j - \beta_j^* =& (\check{\beta}_j - \beta_j^*) + \check{u}_{n+j}^{-1} \frac{\partial l(\check{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \beta_j} - \check{u}_{2n}^{-1} \frac{\partial l(\check{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \beta_n} \\
=& (\check{\beta}_j - \beta_j^*) + \check{u}_{n+j}^{-1} \left\{ \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_j} + \frac{\partial^2 l(\widetilde{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \beta_j^2} (\check{\beta}_j - \beta_j^*) + \sum_{k=1}^{n} \frac{\partial^2 l(\widetilde{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \alpha_k \partial \beta_j} (\check{\alpha}_k - \alpha_k^*) \right\} \\
& - \check{u}_{2n}^{-1} \left\{ \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_n} + \sum_{k=1}^{n} \frac{\partial^2 l(\widetilde{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \alpha_k \partial \beta_n} (\check{\alpha}_k - \alpha_k^*) \right\} \\
=& \left\{ \check{u}_{n+j}^{-1} \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_j} - \check{u}_{2n}^{-1} \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_n} \right\} + (\check{\beta}_j - \beta_j^*) \left\{ 1 + \check{u}_{n+j}^{-1} \frac{\partial^2 l(\widetilde{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \beta_j^2} \right\} \\
& + \left\{ \sum_{k=1}^{n} (\check{\alpha}_k - \alpha_k^*) \left[-\check{u}_{2n}^{-1} \frac{\partial^2 l(\widetilde{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \alpha_k \partial \beta_n} + \check{u}_{n+j}^{-1} \frac{\partial^2 l(\widetilde{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \alpha_k \partial \beta_j} \right] \right\} \\
=:& J_1^{(j)} + J_2^{(j)} + J_3^{(j)}.
\end{aligned}$$

Similarly as the case of $\widehat{\alpha}_i - \alpha_i^*$, we have

$$J_1^{(j)} = \left\{ u_{n+j}^{-1} \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_j} - u_{2n}^{-1} \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_n} \right\} + \left[-u_{n+j}^{-1} + \check{u}_{n+j}^{-1} \right] \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_j} + \left[-\check{u}_{2n}^{-1} + u_{2n}^{-1} \right] \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_n},$$

where it follows from Lemma 2 and (S1.4) that

$$\max_{1 \le j \le n-1} \left| \left[-u_{n+j}^{-1} + \check{u}_{n+j}^{-1} \right] \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_j} + \left[-\check{u}_{2n}^{-1} + u_{2n}^{-1} \right] \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_n} \right| \lesssim_P \frac{e^{10\|\boldsymbol{\theta}^*\|_\infty} \log n}{n}.$$

Further, we also have

$$\max_{1 \le j \le n-1} |J_2^{(j)}| \lesssim_P \frac{e^{14\|\boldsymbol{\theta}^*\|_\infty} \log n}{n}, \text{ and } \max_{1 \le j \le n-1} |J_3^{(j)}| \lesssim_P \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty} \log n}{n}.$$

Combing all the results together and Lemma 2, we have

$$\left| (\widehat{\beta}_j - \beta_j^*) - \left\{ u_{n+j}^{-1} \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_j} - u_{2n}^{-1} \frac{\partial l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \beta_n} \right\} \right| \lesssim_P \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty} \log n}{n} \quad \text{(S2.9)}$$

where central limit theorem implies that $u_{n+j}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \beta_j} - u_{2n}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \beta_n}$ is asymptotic normal with variance $u_{n+j}^{-1} + u_{2n}^{-1}$.

Define $\mathbf{r} \in \mathbb{R}^{2n-1}$, where $r_i = u_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \alpha_i} + u_{2n}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \beta_n}$ for $i \in [n]$, and $r_{n+j} = u_{n+j}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \beta_j} - u_{2n}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \beta_n}$ for $j \in [n-1]$. It is not difficult to verify that for any fixed d, $\mathbf{r}_{[1:d]}$ is asymptotically multivariate normal with mean zero and covariance matrix $\mathbf{H}_{[1:d,1:d]}$. By (S2.8) and (S2.9), we obtain $\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^* - \mathbf{r}\|_\infty = o_p(n^{-1/2})$, and thus complete the proof of asymptotical normality.

By Lemma 1 and (S1.2) in Lemma 2, we have

$$\max_{1\le i\le 2n}\left|u_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \theta_i}\right| \lesssim_P e^{2\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}}.$$

Then, (**??**) is obtained from (S2.8) and (S2.9). □

Proof of Proposition ??. Let $\bar{\alpha}_i$ be the solution of the estimation equation

$$d_i - \sum_{k=1,k\neq i}^{n} g(\beta_j^* - \alpha_i) = 0.$$

It follows from Theorem **??** that the asymptotic variance of $\bar{\alpha}_i$ is $w_i(v_i^*)^{-2}$. But by the Cramer-Rao bound, we also have

$$\mathrm{var}(\bar{\alpha}_i) \ge \left(-\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*)}{\partial \alpha_i^2}\right]\right)^{-1} = -(u_i^*)^{-1}.$$

By letting $n \to \infty$, we get $w_i(v_i^*)^{-2} \ge |u_i^*|^{-1}$ for any $i \in [n]$. Similarly, we can also get $w_i(v_i^*)^{-2} \ge |u_i^*|^{-1}$ for any $i = n+1, ..., 2n$. Therefore, for any

$i \in [2n]$, we have

$$\frac{\mathrm{avar}(\check{\theta}_i)}{\mathrm{avar}(\widehat{\theta}_i)} = \frac{w_i(v_i^*)^{-2} + w_{2n}(v_{2n}^*)^{-2}}{|u_i^*|^{-1} + |u_{2n}^*|^{-1}} \geq \min\left\{\frac{w_i(v_i^*)^{-2}}{|u_i^*|^{-1}}, \frac{w_{2n}(v_{2n}^*)^{-2}}{|u_{2n}^*|^{-1}}\right\} \geq 1.$$

This completes the proof of Proposition **??**. □

Proof of Theorem ??. By (S2.8) and (S2.9), we have

$$(\widehat{\alpha}_i - \widehat{\alpha}_j) - (\alpha_i^* - \alpha_j^*) = \left\{u_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \alpha_i} - (u_j^*)^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \alpha_j}\right\} + o_p(n^{-\frac{1}{2}}),$$

$$(\widehat{\beta}_i - \widehat{\beta}_j) - (\beta_i^* - \beta_j^*) = \left\{(u_{n+i}^*)^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \beta_i} - u_{n+j}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \beta_j}\right\} + o_p(n^{-\frac{1}{2}}),$$

which immediately imply (**??**).

By (**??**), similar to the derivation of (S1.4), we have

$$\max_{1\leq i\leq 2n}\left|\widehat{u}_i^{-1} - u_i^{-1}\right| \lesssim_P e^{6\|\boldsymbol{\theta}^*\|_\infty}\frac{\sqrt{\log n}}{n^{3/2}},$$

which, together with Lemma 2, implies that

$$\begin{aligned}\max_{i\neq j}|(\delta_{ij}^*)^{-1} - \widehat{\delta}_{ij}^{-1}| &= \max_{i\neq j}\left|\frac{\widehat{\delta}_{ij}^2 - (\delta_{ij}^*)^2}{\delta_{ij}^*\widehat{\delta}_{ij}(\widehat{\delta}_{ij} + \delta_{ij}^*)}\right| \\ =\max_{i\neq j}\left|\frac{[u_i^{-1} - \widehat{u}_i^{-1}] + [(u_j^*)^{-1} - \widehat{u}_j^{-1}]}{\delta_{ij}^*\widehat{\delta}_{ij}(\widehat{\delta}_{ij} + \delta_{ij}^*)}\right| &\lesssim_P \frac{e^{6\|\boldsymbol{\theta}^*\|_\infty}n^{-3/2}\sqrt{\log n}}{n^{-3/2}} = e^{6\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\log n}.\end{aligned} \tag{S2.10}$$

Simple algebra yields that

$$\begin{aligned}&\widehat{\delta}_{ij}^{-1}\left[(\widehat{\alpha}_i - \widehat{\alpha}_j) - (\alpha_i^* - \alpha_j^*)\right] \\ =&(\delta_{ij}^*)^{-1}\left[(\widehat{\alpha}_i - \widehat{\alpha}_j) - (\alpha_i^* - \alpha_j^*)\right] + (\widehat{\delta}_{ij}^{-1} - (\delta_{ij}^*)^{-1})\left[(\widehat{\alpha}_i - \widehat{\alpha}_j) - (\alpha_i^* - \alpha_j^*)\right],\end{aligned}$$

where the first term converges to $N(0,1)$ in distribution following (**??**), and the second one is bounded by $O_p(e^{8\|\boldsymbol{\theta}^*\|_\infty}\log n/\sqrt{n}) = o_p(1)$ following (**??**)

and (S2.10). The case for $\widehat{\delta}_{n+i,n+j}^{-1}\big[(\widehat{\beta}_i - \widehat{\beta}_j) - (\beta_i^* - \beta_j^*)\big]$ is similar. This completes the proof of Theorem **??**. □

Proof of Theorem ??. First, by the definition of FDR, we have

$$
\begin{aligned}
\mathrm{FDR} = \mathbb{E}\left[\frac{1}{r}\sum_{k\in\mathcal{S}_0} 1_{\{p_k\le \frac{\alpha r}{KL}\}}1_{\{r>0\}}\right] = \sum_{k\in\mathcal{S}_0}\mathbb{E}\left[\frac{1}{r}1_{\{p_k\le \frac{\alpha r}{KL}\}}1_{\{r>0\}}\right] \\
=\sum_{l=1}^{\infty}\frac{1}{l(l+1)}\sum_{k\in\mathcal{S}_0}\mathbb{E}\left[1_{\{p_k\le \frac{\alpha r}{KL}\}}1_{\{0<r\le l\}}\right] \le \sum_{l=1}^{\infty}\frac{1}{l(l+1)}\sum_{k\in\mathcal{S}_0}\Pr\left(p_k\le\frac{\alpha\min(l,K)}{KL}\right),
\end{aligned}
\tag{S2.11}
$$

where the third equality is due to that for each $r_0>0$,

$$
\begin{aligned}
\frac{1}{r_0}1_{\{p_k\le\frac{\alpha r_0}{KL}\}}1_{\{r_0>0\}} =&1_{\{p_k\le\frac{\alpha r_0}{KL}\}}1_{\{r_0>0\}}\sum_{l\ge r_0}\left(\frac{1}{l}-\frac{1}{l+1}\right) = 1_{\{p_k\le\frac{\alpha r_0}{KL}\}}1_{\{r_0>0\}}\sum_{l=r_0}^{\infty}\frac{1}{l(l+1)} \\
=&1_{\{p_k\le\frac{\alpha r_0}{KL}\}}\sum_{l=1}^{\infty}\frac{1}{l(l+1)}1_{\{0<r_0\le l\}}.
\end{aligned}
$$

It thus suffices to establish an upper bound for $\max_{k\in\mathcal{S}_0}\Pr\left(p_k\le\frac{\alpha\min(l,K)}{KL}\right)$.

Note that for any $k\in\mathcal{S}_0$, it follows from (S2.8) that

$$
\widehat{\alpha}_i-\widehat{\alpha}_k = V_{ik}+(\omega_i-\omega_k) \quad \text{with} \quad \max_{k\in\mathcal{S}_0}|\omega_i-\omega_k| \lesssim_P e^{20\|\boldsymbol{\theta}^*\|_\infty}\frac{\log n}{n},
$$

where $V_{ik}=u_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i}-u_k^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k}$ and $\omega_i=(\widehat{\alpha}_i-\alpha_i^*)-\left\{u_i^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i}+u_{2n}^{-1}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\beta_n}\right\}$. Then, we have

$$
\begin{aligned}
&\max_{k\in\mathcal{S}_0}\Pr\left(p_k\le\frac{\alpha\min(l,K)}{KL}\right)\\
=&\max_{k\in\mathcal{S}_0}\Pr\left(2\left[1-\Phi\left(\widehat{\delta}_{ik}^{-1}|\widehat{\alpha}_i-\widehat{\alpha}_k|\right)\right]\le\frac{\alpha\min(l,K)}{KL}\right)\\
=&\max_{k\in\mathcal{S}_0}\Pr\left(\widehat{\delta}_{ik}^{-1}\big|V_{ik}+(\omega_i-\omega_k)\big|\ge\Phi^{-1}\left(1-\frac{\alpha\min(l,K)}{2KL}\right)\right)\\
\le&\max_{k\in\mathcal{S}_0}\Pr\left((\delta_{ik}^*)^{-1}|V_{ik}|\ge\Phi^{-1}\left(1-\frac{\alpha\min(l,K)}{2KL}\right)-\mu_{ik}\right)\\
\le&\max_{k\in\mathcal{S}_0}\Pr\left((\delta_{ik}^*)^{-1}|V_{ik}|\ge\Phi^{-1}\left(1-\frac{\alpha\min(l,K)}{2KL}\right)-\frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}(\log n)^2}{\sqrt{n}}\right)+\max_{k\in\mathcal{S}_0}\Pr\left(\mu_{ik}\ge\frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}(\log n)^2}{\sqrt{n}}\right),
\end{aligned}
$$

where $\mu_{ik} = \left| \left\{ (\delta_{ik}^*)^{-1} - \widehat{\delta}_{ik}^{-1} \right\} V_{ik} \right| + \left| \widehat{\delta}_{ik}^{-1}(\omega_i - \omega_k) \right|$.

Note that $(\delta_{ik}^*)^{-1} V_{ik} \to N(0,1)$ in distribution following (**??**), then it follows from the Berry–Esseen theorem that

$$\max_{k\in\mathcal{S}_0} \sup_{t\in\mathbb{R}} \left| \Pr\left((\delta_{ik}^*)^{-1} V_{ik} \le t \right) - \Phi(t) \right| = O\left(\frac{1}{\sqrt{n}} \right).$$

It further implies that

$$\begin{aligned} &\max_{k\in\mathcal{S}_0} \Pr\left((\delta_{ik}^*)^{-1} |V_{ik}| \ge \Phi^{-1}\left(1 - \frac{\alpha \min(l,K)}{2KL} \right) - \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty} (\log n)^2}{\sqrt{n}} \right) \\ \le& 2\left\{ 1 - \Phi\left[\Phi^{-1}\left(1 - \frac{\alpha \min(l,K)}{2KL} \right) - \frac{(\log n)^2}{\sqrt{n}} \right] \right\} + O\left(\frac{1}{\sqrt{n}} \right) \\ =& \frac{\alpha \min(l,K)}{KL} + O\left(\frac{e^{20\|\boldsymbol{\theta}^*\|_\infty} (\log n)^2}{\sqrt{n}} \right). \end{aligned} \tag{S2.12}$$

Also, it follows from (S2.10) that

$$\begin{aligned} &\max_{k\in\mathcal{S}_0} \Pr\left(\mu_{ik} \ge \frac{(\log n)^2}{\sqrt{n}} \right) \\ \le& \max_{k\in\mathcal{S}_0} \Pr\left(\left| (\delta_{ik}^*)^{-1} - \widehat{\delta}_{ik}^{-1} \right| \{ |V_{ik}| + |\omega_i - \omega_k| \} + \left| (\delta_{ik}^*)^{-1}(\omega_i - \omega_k) \right| \ge \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty} (\log n)^2}{\sqrt{n}} \right). \end{aligned}$$

But it follows from (S2.8) and (S2.10) that

$$\max_{k\in\mathcal{S}_0} \left| (\delta_{ik}^*)^{-1} - \widehat{\delta}_{ik}^{-1} \right| \{ |V_{ik}| + |\omega_i - \omega_k| \} + \left| (\delta_{ik}^*)^{-1}(\omega_i - \omega_k) \right| \lesssim_P \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty} \log n}{\sqrt{n}}.$$

Therefore, $\max_{k\in\mathcal{S}_0} \Pr\left(\mu_{ik} \ge e^{20\|\boldsymbol{\theta}^*\|_\infty} (\log n)^2 / \sqrt{n} \right) = O\left(\frac{1}{n} \right)$ following Lemma 2.

Putting the above results together, we obtain that

$$\begin{aligned} \max_{k\in\mathcal{S}_0} \Pr\left(p_k \le \frac{\alpha \min(l,K)}{KL} \right) &= \frac{\alpha \min(l,K)}{KL} + O\left(\frac{e^{20\|\boldsymbol{\theta}^*\|_\infty} (\log n)^2}{\sqrt{n}} \right) + O\left(\frac{1}{n} \right) \\ &= \frac{\alpha \min(l,K)}{KL} + O\left(\frac{e^{20\|\boldsymbol{\theta}^*\|_\infty} (\log n)^2}{\sqrt{n}} \right). \end{aligned}$$

It then implies that

$$
\begin{aligned}
\mathrm{FDR} &\leq \sum_{l=1}^{\infty} \frac{1}{l(l+1)} \sum_{k\in\mathcal{S}_0} \Pr\left(p_k \leq \frac{\alpha\min(l,K)}{KL}\right) \\
&= \sum_{k\in\mathcal{S}_0} \left\{ \sum_{l=1}^{K} \frac{1}{l(l+1)} \Pr\left(p_k \leq \frac{\alpha l}{KL}\right) + \sum_{l=K+1}^{\infty} \frac{1}{l(l+1)} \Pr\left(p_k \leq \frac{\alpha}{L}\right) \right\} \\
&= \sum_{k\in\mathcal{S}_0} \left\{ \sum_{l=1}^{K} \frac{1}{l(l+1)} \left[\frac{\alpha l}{KL} + O\left(\frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}(\log n)^2}{\sqrt{n}}\right)\right] + \sum_{l=K+1}^{\infty} \frac{1}{l(l+1)} \left[\frac{\alpha}{L} + O\left(\frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}(\log n)^2}{\sqrt{n}}\right)\right] \right\} \\
&\leq \frac{\alpha K_0}{K} + \frac{\alpha K_0}{KL} + O\left(\frac{e^{20\|\boldsymbol{\theta}^*\|_\infty} K_0 (\log n)^2}{\sqrt{n}}\right).
\end{aligned}
$$

As $e^{20\|\boldsymbol{\theta}^*\|_\infty} K_0 n^{-1/2}(\log n)^2 = o(1)$, the desired upper bound on FDR follows immediately. □

S3 Appendix C: proof of lemmas

Proof of Lemma 1. First, simple algebra yields that

$$
\begin{aligned}
u_i &= -\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i^2}\right] = -\sum_{j=1,j\neq i}^{n} \mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}\right], \text{ for } i\in[n], \\
u_{n+j} &= -\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\beta_j^2}\right] = -\sum_{i=1,i\neq j}^{n} \mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}\right], \text{ for } j\in[n].
\end{aligned} \tag{S3.13}
$$

It is shown that $\mathbb{E}[-\frac{\partial^2 l(\boldsymbol{\theta};\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}] = \frac{e^{\alpha_i+\beta_j}}{(1+e^{\alpha_i+\beta_j})^2}\left[2 - \frac{(1-\kappa_i)e^{\alpha_i+\beta_j}}{e^{\alpha_i+\beta_j}+(1-\kappa_i)} - \frac{1-\kappa_i}{e^{\alpha_i+\beta_j}+\frac{1-\kappa_i}{1+\kappa_i}}\right]$ for any $i\neq j$. Also, we have $\max_{i,j}|\alpha_i^*+\beta_j^*| \leq 2\|\boldsymbol{\theta}^*\|_\infty$. It can be verified that there exists a positive constant c such that

$$
c^{-1}e^{-2\|\boldsymbol{\theta}^*\|_\infty} \leq \frac{e^{\alpha_i+\beta_j}}{(1+e^{\alpha_i+\beta_j})^2}\left[2 - \frac{(1-\kappa_i)e^{\alpha_i+\beta_j}}{e^{\alpha_i+\beta_j}+(1-\kappa_i)} - \frac{1-\kappa_i}{e^{\alpha_i+\beta_j}+\frac{1-\kappa_i}{1+\kappa_i}}\right] \leq c
$$

for any $1\leq i,j\leq n$. It then follows from (S3.13) that $c^{-1}ne^{-2\|\boldsymbol{\theta}^*\|_\infty} \leq \min_{1\leq i\leq 2n} u_i \leq \max_{1\leq i\leq 2n} u_i \leq cn$. Further, we have $c^{-1}e^{-2\|\boldsymbol{\theta}^*\|_\infty} \leq \frac{(1+\kappa_i)e^{\alpha_i^*+\beta_j^*}}{(1+e^{\alpha_i^*+\beta_j^*})^2} \leq$

c for any $1 \leq i, j \leq n$, and then it follows from the definition of v_i that $c^{-1}ne^{-2\|\boldsymbol{\theta}^*\|_\infty} \leq \min_{1\leq i\leq 2n} v_i \leq \max_{1\leq i\leq 2n} v_i \leq cn$. Similarly, note that $c^{-1}e^{-2\|\boldsymbol{\theta}^*\|_\infty} \leq \frac{(1+\kappa_i)e^{\alpha_i^*+\beta_j^*}+\kappa_i(1-\kappa_i)}{(1+e^{\alpha_i^*+\beta_j^*})^2} \leq c$ for any $1 \leq i, j \leq n$, which implies that $c^{-1}ne^{-2\|\boldsymbol{\theta}^*\|_\infty} \leq \min_{1\leq i\leq 2n} w_i \leq \max_{1\leq i\leq 2n} w_i \leq cn$.

Let $\mathbf{M} = (m_{ij})_{n\times n}$ with $m_{ij} = \beta_j + \alpha_i$. As $l(\boldsymbol{\theta}; \boldsymbol{\kappa})$ depends on α_i and β_j only through m_{ij}, we may write $l(\boldsymbol{\theta}; \boldsymbol{\kappa})$ as $l(\mathbf{M}; \boldsymbol{\kappa})$ with $\mathbf{M} = (m_{ij})_{i,j=1}^n$ without causing any confusion, and it holds that $\frac{\partial^2 l(\boldsymbol{\theta};\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j} = \frac{\partial^2 l(\mathbf{M};\boldsymbol{\kappa})}{\partial m_{ij}^2}$. Next, it also follows from (S3.13) that

$$\max_{1\leq i\leq n}\left|\frac{\partial^2 l(\boldsymbol{\theta};\boldsymbol{\kappa})}{\partial\theta_i^2} - \frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\theta_i^2}\right| \leq \max_{1\leq i\leq n}\sum_{j=1,j\neq i}^{n}\left|\frac{\partial^2 l(\boldsymbol{\theta};\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j} - \frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_j}\right|$$
$$= \max_{1\leq i\leq n}\sum_{j=1,j\neq i}^{n}\left|\frac{\partial^2 l(\mathbf{M};\boldsymbol{\kappa})}{\partial m_{ij}^2} - \frac{\partial^2 l(\mathbf{M}^*;\boldsymbol{\kappa})}{\partial m_{ij}^2}\right| \lesssim \sum_{j=1,j\neq i}^{n}|m_{ij} - m_{ij}^*| \lesssim n\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_\infty,$$

where the second inequality is due to the fact that there exists a constant $\epsilon > 0$ such that

$$\sup_{\boldsymbol{\theta}:\|\boldsymbol{\theta}-\boldsymbol{\theta}^*\|_\infty\leq\epsilon}\max_{i\neq j}\left|\frac{\partial^3 l(\mathbf{M};\boldsymbol{\kappa})}{\partial m_{ij}^3}\right| = O(1).$$

Similarly, we also have

$$\max_{n+1\leq i\leq 2n}\left|\frac{\partial^2 l(\boldsymbol{\theta};\boldsymbol{\kappa})}{\partial\theta_i^2} - \frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\theta_i^2}\right| \lesssim n\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_\infty.$$

The proof is similarly for

$$n^{-1}\max_{1\leq i\leq 2n}\left|\frac{\partial^2 l(\boldsymbol{\theta};\boldsymbol{\kappa})}{\partial\theta_i^2} - \frac{\partial^2 l(\boldsymbol{\theta};\boldsymbol{\kappa}^*)}{\partial\theta_i^2}\right| = O(\|\boldsymbol{\kappa} - \boldsymbol{\kappa}^*\|_\infty),$$

which completes the proof of Lemma 1. $\square$

Proof of Lemma 2. Note that $|y_{ij}| \leq 1$ for any $i, j \in [n]$, and it can also be verified that $\mathbb{E}\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa}^*)}{\partial \theta_i} = 0$ and $n^{-1}\left|\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa}^*)}{\partial \theta_i}\right| \leq 2$ for any $i \in [2n]$. It then follows from Hoeffding's inequality that

$$\begin{aligned}
\Pr\left(\max_{1\leq i\leq 2n} |g_i - \mathbb{E}g_i| \geq \sqrt{4(n-1)\log(n-1)}\right) &\leq \sum_{i=1}^{2n} \Pr\left(|g_i - \mathbb{E}g_i| \geq \sqrt{4(n-1)\log(n-1)}\right) \\
&\leq 4n\exp\left(-\frac{8(n-1)\log(n-1)}{4(n-1)}\right) = \frac{4n}{(n-1)^2},
\end{aligned}$$

and

$$\begin{aligned}
\Pr\left(\max_{1\leq i\leq 2n} \left|\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa}^*)}{\partial \theta_i}\right| \geq \sqrt{16(n-1)\log(n-1)}\right) &\leq \sum_{i=1}^{2n} \Pr\left(\left|\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa}^*)}{\partial \theta_i}\right| \geq \sqrt{16(n-1)\log(n-1)}\right) \\
&\leq 4n\exp\left(-\frac{32(n-1)\log(n-1)}{16(n-1)}\right) = \frac{4n}{(n-1)^2}.
\end{aligned}$$

Then, (S1.2) holds since by (1), we have

$$\left|\max_{1\leq i\leq 2n}\left|\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \theta_i}\right| - \max_{1\leq i\leq 2n}\left|\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa}^*)}{\partial \theta_i}\right|\right| \leq \max_{1\leq i\leq 2n}\left|\frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \theta_i} - \frac{\partial l(\boldsymbol{\theta}^*;\boldsymbol{\kappa}^*)}{\partial \theta_i}\right| \lesssim n\|\boldsymbol{\kappa} - \boldsymbol{\kappa}^*\|_\infty.$$

Also, there exists a positive constant c such that $|\mathcal{I}_{ij}| \leq c/2$ for any $i, j \in [n]$. Again, by Hoeffding's inequality, we have

$$\begin{aligned}
&\Pr\left(\max_{1\leq i\leq 2n}\left|\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \theta_i^2} + u_i\right| \geq c\sqrt{(n-1)\log(n-1)}\right) \\
\leq &\sum_{i=1}^{2n} \Pr\left(\left|\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \theta_i^2} + u_i\right| \geq c\sqrt{(n-1)\log(n-1)}\right) \\
\leq &4n\exp\left(-\frac{2c^2(n-1)\log(n-1)}{c^2(n-1)}\right) = \frac{4n}{(n-1)^2}.
\end{aligned}$$

This completes the proof of Lemma 2. □

Proof of Lemma 3. The proof mainly follows from the results in Yan et al. (2016), except that we have $\boldsymbol{\kappa}$ here which approximates true values

$\boldsymbol{\kappa}^*$. Note that

$$\frac{\partial F_i(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \alpha_i} = -\sum_{k=1,k\neq i}^{n} \frac{(1+\kappa_i)e^{\alpha_i^*+\beta_k^*}}{(1+e^{\alpha_i^*+\beta_k^*})^2} = -v_i, \qquad \text{for } i \in [n],$$

$$\frac{\partial F_{n+j}(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial \beta_j} = -\sum_{k=1,k\neq j}^{n} \frac{(1+\kappa_k)e^{\alpha_k^*+\beta_j^*}}{(1+e^{\alpha_k^*+\beta_j^*})^2} = -v_{n+j}, \quad \text{for } j \in [n-1].$$

By Lemma 1, there exist positive constants c_1 and c_2 such that $-\partial F(\boldsymbol{\theta};\boldsymbol{\kappa})/\partial\boldsymbol{\theta} \in \mathcal{L}(c_1 e^{-2\|\boldsymbol{\theta}^*\|_\infty}, c_2)$, where $\mathcal{L}(c_1 e^{-2\|\boldsymbol{\theta}^*\|_\infty}, c_2)$ is defined as in Section 2.1 of Yan et al. (2016).

By Lemma 2, with probability at least $1-4n/(n-1)^2$, we have $\max\big\{\max_{1\le i\le n}|d_i - \mathbb{E}d_i|, \max_{1\le j\le n}|b_j-\mathbb{E}b_j|\big\} \le \sqrt{4(n-1)\log(n-1)}$. Then,

$$\begin{aligned}\max_{1\le i\le n}|F_i(\boldsymbol{\theta};\boldsymbol{\kappa})| \le& \max_{1\le i\le n}|g_i-\mathbb{E}g_i| + \max_{1\le i\le n}\sum_{k=1,k\neq i}^{n}\frac{|\kappa_i-\kappa_i^*|}{1+e^{\alpha_i^*+\beta_k^*}}\\ \le&\sqrt{4(n-1)\log(n-1)} + (n-1)\|\boldsymbol{\kappa}-\boldsymbol{\kappa}^*\|_\infty\\ =&[1+o(1)]\sqrt{4(n-1)\log(n-1)}.\end{aligned}$$

Similarly, $\max_{1\le j\le n}|F_{n+j}(\boldsymbol{\theta};\boldsymbol{\kappa})| \le [1+o(1)]\sqrt{4(n-1)\log(n-1)}$. According to Theorem 7 in Yan et al. (2016) with $m = c_1 e^{-2\|\boldsymbol{\theta}^*\|_\infty}$, $M = c_2$, $K_1 = (1+\kappa)(n-1)$, $K_2 = \frac{(1+\kappa)(n-1)}{2}$, $\rho \asymp e^{6\|\boldsymbol{\theta}^*\|_\infty}$ and $r \lesssim e^{6\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\log n/n}$, it holds that with probability approaching 1, $F(\boldsymbol{\theta};\boldsymbol{\kappa})=0$ has a solution $\check{\boldsymbol{\theta}}$, and it satisfies

$$\|\check{\boldsymbol{\theta}}-\boldsymbol{\theta}^*\|_\infty \le \frac{r}{1-\rho r} \le 2r \lesssim e^{6\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}}.$$

Next, we prove the uniqueness of $\check{\boldsymbol{\theta}}$ by contradiction. If there exists $\widetilde{\boldsymbol{\theta}} \neq \check{\boldsymbol{\theta}}$ such that $F(\widetilde{\boldsymbol{\theta}};\boldsymbol{\kappa}) = \mathbf{0}$, we define $h(t) = (t\widetilde{\boldsymbol{\theta}}+(1-t)\check{\boldsymbol{\theta}})^\top F(t\widetilde{\boldsymbol{\theta}}+(1-t)\check{\boldsymbol{\theta}};\boldsymbol{\kappa})$, and

$$h'(t) = (\widetilde{\boldsymbol{\theta}} - \check{\boldsymbol{\theta}})^\top \left[\frac{\partial F(t\widetilde{\boldsymbol{\theta}} + (1-t)\check{\boldsymbol{\theta}}; \boldsymbol{\kappa})}{\partial \boldsymbol{\theta}}\right] (\widetilde{\boldsymbol{\theta}} - \check{\boldsymbol{\theta}}).$$

Since $-\partial F(\boldsymbol{\theta}; \boldsymbol{\kappa})/\partial \boldsymbol{\theta}$ is a diagonally dominant matrix with positive diagonals (Yan et al., 2016), it implies that $-\partial F(\boldsymbol{\theta}; \boldsymbol{\kappa})/\partial \boldsymbol{\theta}$ is a positive definite matrix, and thus $h'(t) < 0$ for any $t \in [0, 1]$. This contradicts with the fact that $h(0) = h(1) = 0$, and the uniqueness of $\check{\boldsymbol{\theta}}$ then follows immediately. □

Proof of Lemma 4. Let $\mathbf{h} = -F(\check{\boldsymbol{\theta}}; \boldsymbol{\kappa}) - F'(\boldsymbol{\theta}^*; \boldsymbol{\kappa})(\check{\boldsymbol{\theta}} - \boldsymbol{\theta}^*)$ and $\mathbf{g} = (g_1, ..., g_{2n-1})^\top$. Further denote

$$\mathbf{g}(\boldsymbol{\theta}; \boldsymbol{\kappa}) = (g_1(\boldsymbol{\theta}; \boldsymbol{\kappa}), ..., g_{2n-1}(\boldsymbol{\theta}; \boldsymbol{\kappa}))^\top,$$

where $g_i(\boldsymbol{\theta}; \boldsymbol{\kappa}) = \sum_{k=1, k\neq i}^n \frac{e^{\alpha_i + \beta_k - \kappa_i}}{1 + ^{\alpha_i + \beta_k}}$ and $g_{n+j}(\boldsymbol{\theta}; \boldsymbol{\kappa}) = \sum_{k=1, k\neq j}^n \frac{e^{\alpha_k + \beta_j - \kappa_k}}{1 + ^{\alpha_k + \beta_j}}$ for $i, j \in [n]$. Note that $\mathbf{g}(\boldsymbol{\theta}^*; \boldsymbol{\kappa}^*) = \mathbb{E}\mathbf{g}$. Then,

$$\check{\boldsymbol{\theta}} - \boldsymbol{\theta}^* = -[F'(\boldsymbol{\theta}^*; \boldsymbol{\kappa})]^{-1}(\mathbf{g} - \mathbb{E}\mathbf{g}) - [F'(\boldsymbol{\theta}^*; \boldsymbol{\kappa})]^{-1}(\mathbf{g}(\boldsymbol{\theta}^*; \boldsymbol{\kappa}^*) - \mathbf{g}(\boldsymbol{\theta}^*; \boldsymbol{\kappa})) - [F'(\boldsymbol{\theta}^*; \boldsymbol{\kappa})]^{-1}\mathbf{h}.$$

By Lemma 3, and Lemmas 8, 9 of Yan et al. (2016), we have $\|[F'(\boldsymbol{\theta}^*; \boldsymbol{\kappa})]^{-1}\mathbf{h}\|_\infty \lesssim e^{6\|\boldsymbol{\theta}^*\|_\infty}\|\check{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|_\infty^2 \lesssim_P e^{18\|\boldsymbol{\theta}^*\|_\infty} \log n/n$. Similarly, we have $\|[F'(\boldsymbol{\theta}^*; \boldsymbol{\kappa})]^{-1}(\mathbf{g}(\boldsymbol{\theta}^*; \boldsymbol{\kappa}^*) - \mathbf{g}(\boldsymbol{\theta}^*; \boldsymbol{\kappa}))\|_\infty \lesssim e^{6\|\boldsymbol{\theta}^*\|_\infty}\|\boldsymbol{\kappa} - \boldsymbol{\kappa}^*\|_\infty \lesssim e^{18\|\boldsymbol{\theta}^*\|_\infty} \log n/n$. The desired results follows immediately. □

Proof of Lemma 5. Note that for any $l \in [n-1]$,

$$u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*; \boldsymbol{\kappa})}{\partial \alpha_i \partial \beta_l}\right] \sqrt{n} v_{n+l}^{-1}(g_{n+l} - \mathbb{E}g_{n+l})$$

is a sub-Gaussian random variable with zero mean and by Lemma 1,

$$\max_{1\le l\le n-1} \mathrm{var}\left\{u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}v_{n+l}^{-1}(g_{n+l}-\mathbb{E}g_{n+l})\right\} \le \frac{c_1 e^{8\|\boldsymbol{\theta}^*\|_\infty}}{n^2},$$

where c_1 is a constant. We have

$$\begin{aligned}
&\Pr\left(\max_{1\le i\le n}\left|\sum_{l=1}^{n-1} u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}v_{n+l}^{-1}(g_{n+l}-\mathbb{E}g_{n+l})\right| \le c_2 e^{4\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}}\right)\\
\ge&1-n\max_{1\le i\le n}\Pr\left(\left|\sum_{l=1}^{n-1} u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}v_{n+l}^{-1}(g_{n+l}-\mathbb{E}g_{n+l})\right| \le c_2 e^{4\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}}\right)\\
\ge&1-2n\exp\left(-\frac{c_2^2\log n}{2c_1}\right) = 1-\frac{2}{n},
\end{aligned}$$

where the second inequality is due to Hoeffding's inequality, and the last equality holds with $c_2 = 2\sqrt{c_1}$. Similarly, we can establish the bound for

$$\max_{1\le j\le n}\left|\sum_{k=1}^{n} u_{n+j}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_j}\right]\sqrt{n}v_k^{-1}(g_k-\mathbb{E}g_k)\right|,$$

and thus we complete the proof of (S1.5).

For (S1.6), note that for any $l \in [n-1]$,

$$u_i^{-1}\mathcal{I}_{il}\sqrt{n}v_{n+l}^{-1}(g_{n+l\setminus i}-\mathbb{E}g_{n+l\setminus i})$$

is also a sub-Gaussian random variable with zero mean and by Lemma 1,

$$\max_{1\le l\le n-1}\mathrm{var}\left\{u_i^{-1}\mathcal{I}_{il}\sqrt{n}v_{n+l}^{-1}(g_{n+l\setminus i}-\mathbb{E}g_{n+l\setminus i})\right\} \le \frac{c_1 e^{8\|\boldsymbol{\theta}^*\|_\infty}}{n^2},$$

where c_1 is a constant. The rest of the proof is similar to that of (S1.5). $\square$

Proof of Lemma 6. By Lemma 4, we have

$$
\begin{aligned}
&\sum_{k=1}^{n} u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right]\sqrt{n}(\check{\alpha}_k-\alpha_k^*)+\sum_{l=1}^{n-1} u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}(\check{\beta}_l-\beta_l^*)\\
=&\sum_{k=1}^{n} u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right]\sqrt{n}\left[v_k^{-1}(g_k-\mathbb{E}g_k)+v_{2n}^{-1}(g_{2n}-\mathbb{E}g_{2n})+\epsilon_i\right]\\
&+\sum_{l=1}^{n-1} u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}\left[v_{n+l}^{-1}(g_{n+l}-\mathbb{E}g_{n+l})-v_{2n}^{-1}(b_n-\mathbb{E}b_n)+\epsilon_{n+l}\right]\\
=&\sum_{k=1}^{n} u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right]\sqrt{n}v_k^{-1}(g_k-\mathbb{E}g_k)+\sum_{l=1}^{n-1} u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}v_{n+l}^{-1}(g_{n+l}-\mathbb{E}g_{n+l})\\
&+u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_n}\right]\sqrt{n}v_{2n}^{-1}(g_{2n}-\mathbb{E}g_{2n})+\sum_{k=1}^{n} u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right]\sqrt{n}\epsilon_i\\
&+\sum_{l=1}^{n-1} u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}\epsilon_{n+l}\\
=:&\sum_{k=1}^{n} u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right]\sqrt{n}v_k^{-1}(g_k-\mathbb{E}g_k)+\sum_{l=1}^{n-1} u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}v_{n+l}^{-1}(g_{n+l}-\mathbb{E}g_{n+l})+r_i.
\end{aligned}
$$

By (S1.5) in Lemma 5, we have

$$
\begin{aligned}
\left|\sum_{k=1}^{n} u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right]\sqrt{n}v_k^{-1}(g_k-\mathbb{E}g_k)\right| &\lesssim_P e^{4\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}};\\
\max_{1\le i\le n}\left|\sum_{l=1}^{n-1} u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}v_{n+l}^{-1}(g_{n+l}-\mathbb{E}g_{n+l})\right| &\lesssim_P e^{4\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}},
\end{aligned}
$$

and it follows from Lemmas 1, 2 and 4 that for any $i\in[n]$,

$$
|r_i|\lesssim_P e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n/\sqrt{n}+e^{8\|\boldsymbol{\theta}^*\|_\infty}\sqrt{n}\|\boldsymbol{\kappa}-\boldsymbol{\kappa}^*\|_\infty\lesssim e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n/\sqrt{n}.
$$

Therefore, it holds true that

$$
\max_{1\le i\le n}\left|\sum_{k=1}^{n} u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right]\sqrt{n}(\check{\alpha}_k-\alpha_k^*)+\sum_{l=1}^{n-1} u_i^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_i\partial\beta_l}\right]\sqrt{n}(\check{\beta}_l-\beta_l^*)\right|\lesssim_P\frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n}{\sqrt{n}}.
$$

Similarly, we can also show that

$$
\max_{1\le j\le n-1}\left|\sum_{k=1}^{n}\sqrt{n}(\check{\alpha}_k-\alpha_k^*)\left\{u_{n+j}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_j}\right]-u_{2n}^{-1}\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\theta}^*;\boldsymbol{\kappa})}{\partial\alpha_k\partial\beta_n}\right]\right\}\right|\lesssim_P\frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n}{\sqrt{n}}.
$$

This completes the proof of Lemma 6. □

Proof of Lemma 7. By Lemma 4, we have

$$
\begin{aligned}
&\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}(\check{\alpha}_k-\alpha_k^*)+\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}(\check{\beta}_l-\beta_l^*)\\
=&\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}\left[v_k^{-1}(g_k-\mathbb{E}g_k)+v_{2n}^{-1}(g_{2n}-\mathbb{E}g_{2n})+\epsilon_i\right]\\
&+\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}\left[v_{n+l}^{-1}(g_{n+l}-\mathbb{E}g_{n+l})-v_{2n}^{-1}(b_n-\mathbb{E}b_n)+\epsilon_{n+l}\right]\\
=&\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}v_k^{-1}(g_k-\mathbb{E}g_k)+\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}v_{n+l}^{-1}(g_{n+l}-\mathbb{E}g_{n+l})\\
&+u_{2n}^{-1}\sqrt{n}v_{2n}^{-1}(g_{2n}-\mathbb{E}g_{2n})\sum_{k=1}^{n}\mathcal{I}_{kn}-u_i^{-1}\sqrt{n}v_{2n}^{-1}(b_n-\mathbb{E}b_n)\sum_{l=1}^{n-1}\mathcal{I}_{il}\\
&+\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}\epsilon_i+\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}\epsilon_{n+l}\\
=&\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}v_k^{-1}(g_{k\backslash n}-\mathbb{E}g_{k\backslash n})+\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}v_k^{-1}(y_{kn}-\mathbb{E}y_{kn})\\
&+\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}v_{n+l}^{-1}(g_{n+l\backslash i}-\mathbb{E}g_{n+l\backslash i})+\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}v_{n+l}^{-1}(y_{il}-\mathbb{E}y_{il})\\
&+\sqrt{n}v_{2n}^{-1}(g_{2n}-\mathbb{E}g_{2n})\frac{u_{2n}-u_{2n}}{u_{2n}}-\sqrt{n}v_{2n}^{-1}(b_n-\mathbb{E}b_n)\frac{u_i-u_i}{u_i}+\sqrt{n}v_{2n}^{-1}(b_n-\mathbb{E}b_n)\frac{\mathcal{I}_{in}}{u_i}\\
&+\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}\epsilon_i+\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}\epsilon_{n+l}\\
=:&\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}v_k^{-1}(g_{k\backslash n}-\mathbb{E}g_{k\backslash n})+\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}v_{n+l}^{-1}(g_{n+l\backslash i}-\mathbb{E}g_{n+l\backslash i})+s_i.
\end{aligned}
$$

By (S1.6) in Lemma 5, we have

$$\left|\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}v_k^{-1}(g_{k\setminus n} - \mathbb{E}g_{k\setminus n})\right| \lesssim_P e^{4\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}};$$

$$\max_{1\le i\le n}\left|\sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}v_{n+l}^{-1}(g_{n+l\setminus i} - \mathbb{E}g_{n+l\setminus i})\right| \lesssim_P e^{4\|\boldsymbol{\theta}^*\|_\infty}\sqrt{\frac{\log n}{n}},$$

and it follows from Lemmas 1, 2 and 4 that

$$\max_{1\le i\le n}|s_i| \lesssim_P e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n/\sqrt{n} + e^{8\|\boldsymbol{\theta}^*\|_\infty}\sqrt{n}\|\boldsymbol{\kappa}-\boldsymbol{\kappa}^*\|_\infty \lesssim e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n/\sqrt{n}.$$

Therefore, it holds true that

$$\max_{1\le i\le n}\left|\sum_{k=1}^{n} u_{2n}^{-1}\mathcal{I}_{kn}\sqrt{n}(\check{\alpha}_k - \alpha_k^*) + \sum_{l=1}^{n-1} u_i^{-1}\mathcal{I}_{il}\sqrt{n}(\check{\beta}_l - \beta_l^*)\right| \lesssim_P \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n}{\sqrt{n}}.$$

Similarly, we can also show that

$$\max_{1\le j\le n-1}\left|\sum_{k=1}^{n}\sqrt{n}(\check{\alpha}_k - \alpha_k^*)\left\{u_{n+j}^{-1}\mathcal{I}_{kj} - u_{2n}^{-1}\mathcal{I}_{kn}\right\}\right| \lesssim_P \frac{e^{20\|\boldsymbol{\theta}^*\|_\infty}\log n}{\sqrt{n}}.$$

This completes the proof of Lemma 7. □

Appendix D: estimation of $\boldsymbol{\kappa}$ and asymptotics

Define $\mathcal{I}_1^* = \{i \in [n] : \kappa_i = \kappa_{01}\}$ and $n_1^* = |\mathcal{I}_1^*|$. We develop a selection procedure to determine the pattern of each node in sending negative edges. Specifically, for each $i \in [n]$, we calculate $\zeta_i = n^{-1}\sum_{k=1,k\neq i}^{n} 1_{\{y_{ik}=-1\}}$, and define $\mathcal{I}_1 = \{i \in [n] : \zeta_i > \xi_n\}$, where ξ_n is a pre-specified threshold. Further, let $\kappa_i = \kappa$ if $i \in \mathcal{I}_1$, where κ is the unknown parameter we are going to estimate, and $\kappa_i = \log n/n$ if otherwise. Without loss of generality, we assume $\mathcal{I}_1 = [n_1]$ and define $\boldsymbol{\theta}_1 = (\alpha_1, ..., \alpha_{n_1}, \beta_1, ..., \beta_{n_1-1})^\top$. Then, the log likelihood based on the subnetwork with n_1 nodes takes the form

$$l(\boldsymbol{\theta}_1; \kappa) = \sum_{i,j=1, i\neq j}^{n_1} l_{ij}(\alpha_i + \beta_j; \kappa).$$

To estimate κ, we consider the following restricted MLE

$$(\widetilde{\boldsymbol{\theta}}_1, \widetilde{\kappa}) = \mathop{\arg\min}_{\substack{\|\boldsymbol{\theta}_1\|_\infty \leq \tau \log n_1 \\ \kappa \in [\gamma_n, 1-\gamma_n]}} l(\boldsymbol{\theta}_1; \kappa),$$

where $\tau \geq 1/40$ and $\gamma_n \in (0, 1/2)$. Given $\widetilde{\kappa}$, we let $\widetilde{\boldsymbol{\kappa}} = (\widetilde{\kappa}_1, ..., \widetilde{\kappa}_n)$, where $\widetilde{\kappa}_i = \widetilde{\kappa}$ if $i \in \mathcal{I}_1$ and $\widetilde{\kappa}_i = \log n/n$ if otherwise.

Lemma 8. *Suppose* $\|\boldsymbol{\theta}^*\|_\infty \leq c \log n$ *with* $0 < c < 1/40$, $\kappa_{00} \lesssim e^{12\|\boldsymbol{\theta}^*\|_\infty} \log n/n$ *and* $\kappa_{01} \in (\gamma_n, 1-\gamma_n)$. *Choose* ξ_n *such that* $\sqrt{\log n/n} \ll \xi_n \ll \gamma_n n^{-\frac{1}{10}}$. *Then,* $\Pr(\mathcal{I}_1 = \mathcal{I}_1^*) \to 1$ *as* n *grows to infinity.*

Proof of Lemma 8. By Hoeffding's inequality,

$$\max_{i\in[n]} |\zeta_i - \mathbb{E}\zeta_i| \lesssim_P \sqrt{\frac{\log n}{n}},$$

which implies that

$$\min_{i\in\mathcal{I}_1^*} |\zeta_i| \gtrsim_P \min_{i\in\mathcal{I}_1^*} |\mathbb{E}\zeta_i| - \sqrt{\frac{\log n}{n}} \gtrsim \gamma_n e^{-4\|\boldsymbol{\theta}^*\|_\infty} - \sqrt{\frac{\log n}{n}} \gtrsim \gamma_n n^{-\frac{1}{10}},$$

$$\max_{i\in[n]\setminus\mathcal{I}_1^*} |\zeta_i| \lesssim_P \max_{i\in[n]\setminus\mathcal{I}_1^*} |\mathbb{E}\zeta_i| + \sqrt{\frac{\log n}{n}} \lesssim \frac{e^{12\|\boldsymbol{\theta}^*\|_\infty}\log n}{n} + \sqrt{\frac{\log n}{n}} \lesssim \sqrt{\frac{\log n}{n}}.$$

This completes the proof. □

Let $p_{ij}(\alpha_i, \beta_j, \kappa)$ represent the distribution of y_{ij} under parameters $(\alpha_i, \beta_j, \kappa)$ and $p_{ij} = p_{ij}(\alpha_i^*, \beta_j^*, \kappa_{01})$. Define the KL-divergence of p_{ij} from $p_{ij}(\alpha_i, \beta_j, \kappa)$ as

$$D_{KL}(p_{ij}||p_{ij}(\alpha_i, \beta_j, \kappa)) = \sum_{y\in\{-1,0,1\}} p(y \mid \alpha_i^* + \beta_j^*, \kappa_{01}) \log \frac{p(y \mid \alpha_i^* + \beta_j^*, \kappa_{01})}{p(y \mid \alpha_i + \beta_j, \kappa)}.$$

For any $\rho > 0$, define $B_n(\rho) = \{\kappa : |\kappa - \kappa_{01}| < \rho\}$ and $B_n^c(\rho) = [\gamma_n, 1 - \gamma_n] \backslash B_n(\rho)$.

Lemma 9. *Under the same conditions of Lemma 8, suppose*

$$\min_{\substack{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1 \\ \kappa \in B_n^c(\rho)}} \frac{1}{n_1(n_1-1)} \sum_{i,j=1, i\neq j}^{n_1} D_{KL}(p_{ij}||p_{ij}(\alpha_i, \beta_j, \kappa)) \gtrsim \rho e^{-12\|\boldsymbol{\theta}_1^*\|_\infty}, \tag{S3.14}$$

then we have $|\widetilde{\kappa} - \kappa_{01}| \lesssim_P e^{12\|\boldsymbol{\theta}_1^*\|_\infty} \log n_1 / n_1$.

Proof of Lemma 9. The proof is similar to the proof of Theorem 2 in Yan et al. (2019), and thus we only show the main steps and different parts. We have

$$l(\boldsymbol{\theta}_1; \kappa) - \mathbb{E} l(\boldsymbol{\theta}_1; \kappa) = \sum_{i\neq j}^{n_1} \{1_{\{y_{ij}=1\}} - \mathbb{E}1_{\{y_{ij}=1\}}\} \log p(1 \mid \alpha_i + \beta_j, \kappa)$$

$$+ \{1_{\{y_{ij}=0\}} - \mathbb{E}1_{\{y_{ij}=0\}}\} \log p(0 \mid \alpha_i + \beta_j, \kappa) + \{1_{\{y_{ij}=-1\}} - \mathbb{E}1_{\{y_{ij}=-1\}}\} \log p(-1 \mid \alpha_i + \beta_j, \kappa).$$

Note that there exists a constant c such that

$$\max_{\substack{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1 \\ \kappa \in [\gamma_n, 1-\gamma_n]}} \max_{\substack{y \in \{-1,0,1\} \\ i \neq j}} |\log p(y \mid \alpha_i + \beta_j, \kappa)| \le c \log n_1.$$

Then, by Hoeffding's equality, there exists a possibly different constant c_1 such that

$$\max_{\substack{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1 \\ \kappa \in [\gamma_n, 1-\gamma_n]}} \frac{1}{n_1(n_1-1)} |l(\boldsymbol{\theta}_1;\kappa) - \mathbb{E}l(\boldsymbol{\theta}_1;\kappa)| <_P c_1 \log n_1 \sqrt{\frac{\log n_1(n_1-1)}{n_1(n_1-1)}} < c_1 \frac{\log n_1}{n_1}, \tag{S3.15}$$

where c_1 may take different values. For any $\rho > 0$, define

$$\epsilon_n(\rho) = \frac{1}{n_1(n_1-1)} \left\{ \max_{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1} \mathbb{E}[l(\boldsymbol{\theta}_1;\kappa_{01})] - \max_{\substack{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1 \\ \kappa \in B_n^c(\rho)}} \mathbb{E}[l(\boldsymbol{\theta}_1;\kappa)] \right\},$$

and $\rho_n = \inf \{\rho > 0 : \epsilon_n(\rho) > 2c_1 \log n_1/n_1\}$. Similar as proof of Theorem 2 in Yan et al. (2019), (S3.15) implies

$$\max_{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1} \frac{1}{n_1(n_1-1)} \mathbb{E}[l(\boldsymbol{\theta}_1;\widetilde{\kappa})] > \max_{\substack{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1 \\ \kappa \in B_n^c(\rho_n)}} \frac{1}{n_1(n_1-1)} \mathbb{E}[l(\boldsymbol{\theta}_1;\kappa)],$$

which further leads that $\widetilde{\kappa} \in B_n(\rho_n)$.

Note that

$$\mathbb{E}[l(\boldsymbol{\theta}_1,\kappa)] = -\sum_{i,j=1, i\neq j}^{n_1} D_{KL}(p_{ij}||p_{ij}(\alpha_i,\beta_j,\kappa)) + \sum_{i,j=1,i\neq j}^{n_1} S(p_{ij}),$$

where $D_{KL}(p_{ij}||p_{ij}(\alpha_i,\beta_j,\kappa))$ is the KL-divergence of p_{ij} from $p_{ij}(\alpha_i,\beta_j,\kappa)$ as defined earlier, and

$$S(p_{ij}) = \sum_{y\in\{-1,0,1\}} p(y \mid \alpha_i^* + \beta_j^*, \kappa_{01}) \log p(y \mid \alpha_i^* + \beta_j^*, \kappa_{01}).$$

Then by (S3.14), we have

$$\max_{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1} \mathbb{E}[l(\boldsymbol{\theta}_1; \kappa_{01})] - \max_{\substack{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1 \\ \kappa \in B_n^c(\rho)}} \mathbb{E}[l(\boldsymbol{\theta}_1; \kappa)]$$

$$= - \min_{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1} \sum_{i,j=1, i\neq j}^{n_1} D_{KL}(p_{ij}||p_{ij}(\alpha_i, \beta_j, \kappa_{01})) + \min_{\substack{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1 \\ \kappa \in B_n^c(\rho)}} \sum_{i,j=1, i\neq j}^{n_1} D_{KL}(p_{ij}||p_{ij}(\alpha_i, \beta_j, \kappa))$$

$$= \min_{\substack{\|\boldsymbol{\theta}_1\|_\infty \le \tau \log n_1 \\ \kappa \in B_n^c(\rho)}} \sum_{i,j=1, i\neq j}^{n_1} D_{KL}(p_{ij}||p_{ij}(\alpha_i, \beta_j, \kappa)) \gtrsim \rho n_1(n_1-1) e^{-12\|\boldsymbol{\theta}_1^*\|_\infty} > 0,$$

which implies that $\epsilon_n(\rho) \gtrsim e^{-12\|\boldsymbol{\theta}_1^*\|_\infty}$. Since $\epsilon_n(\rho)$ is continuous, we get $2c_1 \log n_1/n_1 = \epsilon_n(\rho_n) \gtrsim \rho e^{-12\|\boldsymbol{\theta}_1^*\|_\infty}$, which leads that $\rho_n \lesssim e^{12\|\boldsymbol{\theta}_1^*\|_\infty} \log n_1/n_1$. Therefore, $|\widetilde{\kappa} - \kappa_{01}| \lesssim_P e^{12\|\boldsymbol{\theta}_1^*\|_\infty} \log n_1/n_1$. □

Based on Lemmas 8 and 9, we have the following proposition specifying the convergence rate of $\widetilde{\boldsymbol{\kappa}}$, which satisfy the requirement in Theorem **??**.

Proposition 1. *Under the same conditions of Lemma 9, suppose $n_1^* \gtrsim n$. Then, we have $\|\widetilde{\boldsymbol{\kappa}} - \boldsymbol{\kappa}^*\|_\infty \lesssim_P e^{12\|\boldsymbol{\theta}^*\|_\infty} \log n/n$.*

Bibliography

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