# Testing exogeneity in the functional linear regression model

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## Supplementary Material

In this supplement we provide additional simulation results for non-stationary data and show the complete proofs of Propositions A.5, C.1 and C.2 of the main article.

## S1 Additional Simulation Results

### S1.1 Non-stationary Data

While we showed simulation results in a model meeting all model assumptions, especially joint second order stationarity, we now would like to demonstrate that the method still works for non-stationary data. To this end we generate data according to

$$X(t) = (t + 0.5)Z_1, \quad W(t) = (t + 0,5)Z_2 + H$$
$$Y = \frac{1}{p+1} \sum_{l=0}^{p} X(l/p) \cdot \beta(l/p) + U$$

with

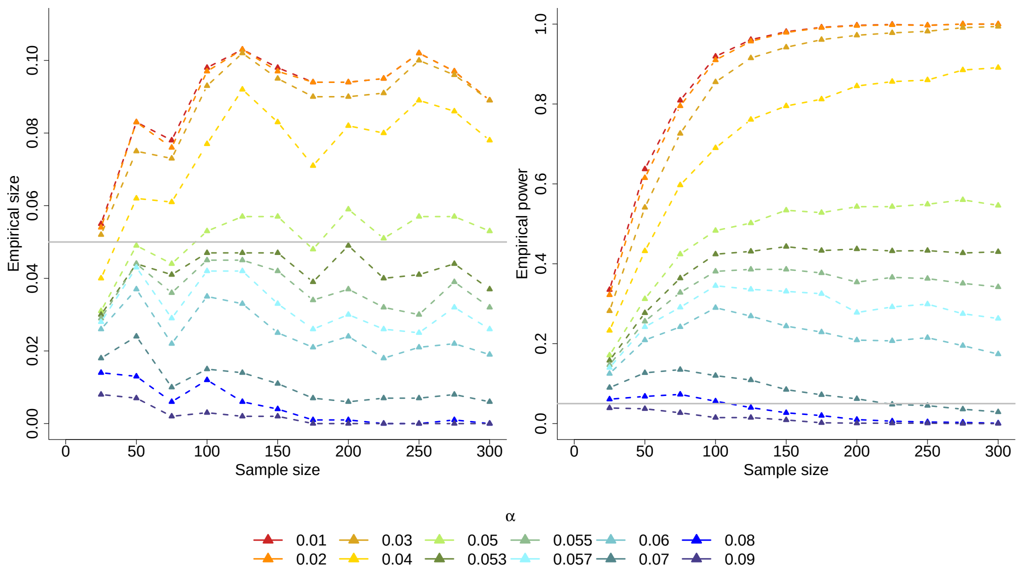


Figure 1: Empirical size and power of the asymptotic test for several choices of $\alpha$. The gray solid line shows the target level $\gamma = 0.05$. The true slope parameter function is $\beta_1$.

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \varepsilon \end{pmatrix} \sim \mathcal{N}_3 \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 & \nu\sqrt{6} & \rho\sqrt{3} \\ \nu\sqrt{6} & 2 & 0 \\ \rho\sqrt{3} & 0 & 1 \end{pmatrix} \right)$$

with $corr(Z_1, Z_2) = \nu$, $corr(Z_1, \varepsilon) = \rho$ and $U = \frac{7}{5}\varepsilon$. The random variable $H$ is uniformly distributed on $(-1/2, 1/2)$ and independent of $(Z_1, Z_2, \varepsilon)'$. The parameter $\rho$ controls the severity of endogeneity (if $\rho = 0$ we are in the exogenous case, i.e. under the null $H_0$) and $\nu$ the strength of the instrument $W$. The standard deviation is assumed to be $\sigma = 7/5$. As slope functions we use again the functions specified in (5.1). In contrast to the

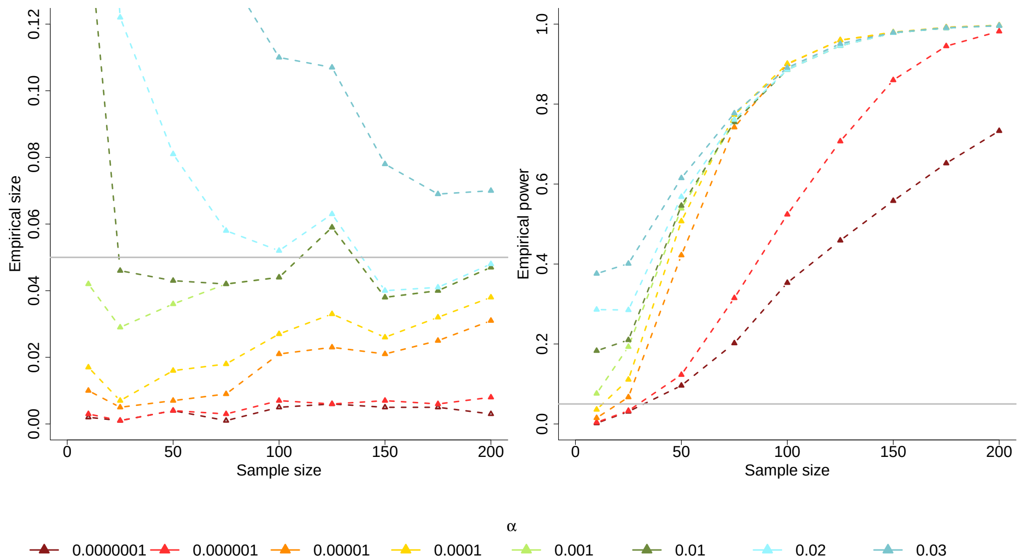


Figure 2: Empirical size and power of the bootstrap tests for several choices of $\alpha$. The gray solid line shows the target level $\gamma = 0.05$. The true slope function is $\beta_1$.

example in Section 5, $(X, W)$ is not second order stationary and we show these simulation results to demonstrate that the assumption of stationarity might not be necessary.

In a first step, we inspect the influence of the choice of $\alpha$ on the performance of the resulting tests. To this end, we fix the degree of endogeneity with $\rho = 0.4$ and the strength of the instrument with $\nu = 0.6$ and the slope function $\beta = \beta_1$. In Figure 1, the results for the asymptotic test using $\beta_1$ as slope parameter and different choices of $\alpha$ are shown. We see that the best results are obtained for $\alpha$ between 0.05 and 0.055. For smaller $\alpha$, the test does not hold the prescribed level, while for larger $\alpha$ the power is

| | $n$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 25 | 50 | 75 | 100 | 125 | 150 | 175 | 200 | 225 | 250 | 275 | 300 |
| $\beta_1$ | 0.111 | 0.507 | 0.773 | 0.901 | 0.960 | 0.980 | 0.992 | 0.997 | 0.998 | 0.998 | 1 | 1 |
| $\beta_2$ | 0.164 | 0.568 | 0.798 | 0.912 | 0.958 | 0.979 | 0.992 | 0.997 | 0.999 | 0.998 | 1 | 1 |
| $\beta_3$ | 0.255 | 0.560 | 0.733 | 0.853 | 0.904 | 0.961 | 0.978 | 0.990 | 0.993 | 0.994 | 0.997 | 0.998 |

Table 1: Empirical power of the bootstrap tests for slope functions defined in (5.1) using $\rho = 0.4$, $\nu = 0.6$ and $\alpha = 0.0001$.

comparably small up to biased tests for $\alpha$ larger than 0.07. We see that the asymptotic test has only moderate power even for larger sample sizes. This is a well known effect with asymptotic tests using plug-in estimators.

As way out we again inspect the bootstrap tests. The results for the residual-based bootstraps proposed in Section 4 and again $\beta_1$ are shown in Figure 2.

It turns out, that the regularization parameter can be chosen considerably smaller than for the asymptotic test and the procedure is much more robust in choosing $\alpha$. Nearly all tests hold the size of $\gamma = 0.05$ for larger sample sizes and the power increases with sample size for most choices of $\alpha$ up to a value close to 1 already for $n = 300$.

Comparing the performance of the bootstrap test for different slope functions, we discover that in all models the bootstrap test holds the size $\gamma = 0.05$ while we see in Table 1 that the power is similarly good for all

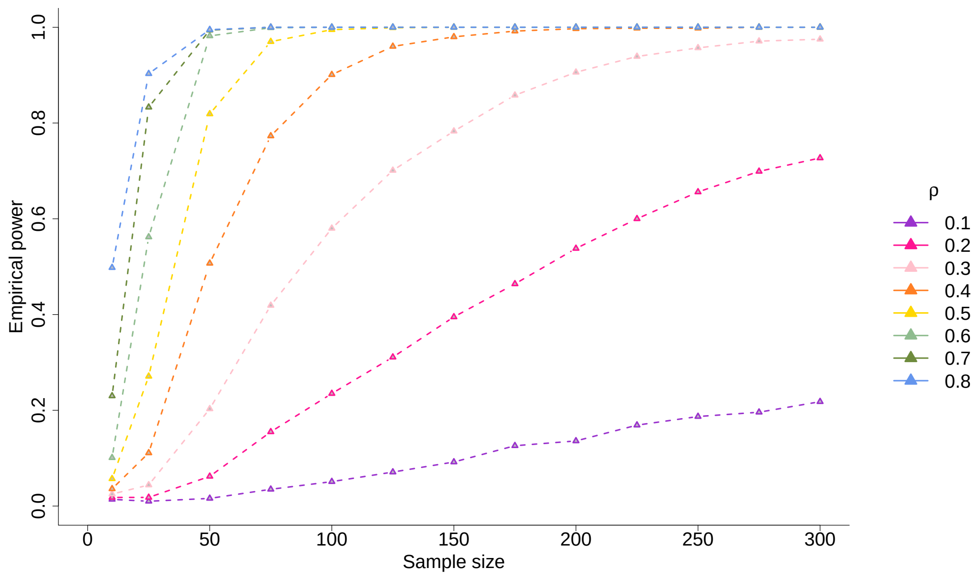


Figure 3: Power of the bootstrap test for different degrees $\rho$ of endogeneity

settings with only slight disadvantages for the smoothed indicator function $\beta_3$.

Finally, we inspect the influence of the degree of endogeneity and the strength of the instrument on the performance of the test. In Figure 3, we see that the power of the bootstrap test increases with increasing degree $\rho$ of endogeneity being already acceptable for $\rho = 0.3$. Figure 4 shows, that the performance of the test is highly dependent on the strength of the instrument. If the instrument is too weak, the power is too low and the test does not hold the size. It turns out, that for the setting with slope function $\beta_1$, $\rho = 0.4$ and $\alpha = 0.0001$, the bootstrap test performs best for

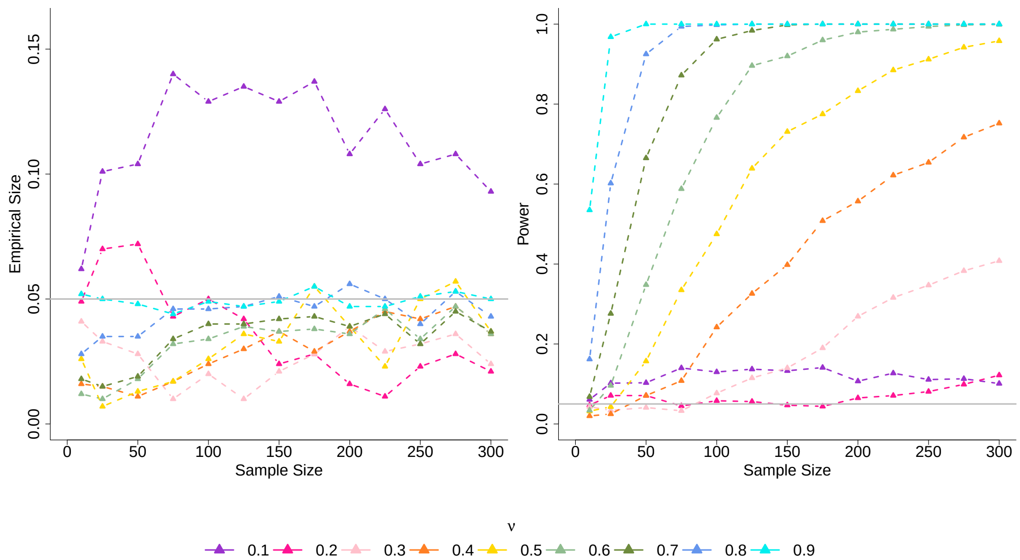


Figure 4: Power and size of the bootstrap test for different strengths $\nu$ of the instrument

a strength of the instrument around $\nu = 0.7$. In conclusion, we have seen, that all simulation results without stationarity are as good as the ones with stationarity assumed. This backs our hypothesis that the results still hold without the assumption of joint second order stationarity.

## S1.2 Runtime analysis

Since bootstrap and cross validation are time intensive methods it is interesting to get an impression of the runtime for the evaluation of one data set for different sample sizes and numbers of bootstrap repetitions. To this end we generated a sample of size $n \in \{50, 100, 200, 300\}$ from the model in Subsection S1.1 once, performed a cross validation, calculated the test

| $n \backslash B$ | 250 | 500 | 1000 | 10000 |
|---|---|---|---|---|
| 50 | 10.45 | 8.91 | 8.89 | 8.94 |
| 100 | 51.05 | 33.51 | 33.4 | 33.44 |
| 200 | 323.57 | 139.16 | 140.55 | 138.86 |
| 300 | 343.26 | 330.21 | 330.01 | 331.52 |

Table 2: Runtime in sec. for different sample sizes and bootstrap repetitions

statistic and performed $B \in \{250, 500, 1000, 10000\}$ bootstrap repetitions on which finally the decision is based. The runtime for each combination is given in Table 2. Due to parallelization in the bootstrap procedure an increase in the bootstrap replications does not necessarily have an effect on the runtime.

## S2 Proof of Proposition A.5

We give only the proof for $R_{n,2}$. We have

$$\begin{aligned}
&\frac{n^2}{t_n^2}\mathrm{E}|R_{n,2}-\mathfrak{R}_n|^2 \\
&\le \frac{1}{t_n^2 n^2}\sum_{k\in\mathbb{Z}} x_k^2 I\{\lambda_k \ge \alpha\gamma_k^\nu\}\mathrm{E}\left|\sum_{i=1}^n \left(|D_{i,k}\mathscr{U}_{i,k}|^2 - \mathfrak{V}^{1/2}\left(\frac{w_k}{|c_k|^2}-\frac{1}{x_k}\right)\right)\right|^2 \\
&\quad + \frac{1}{t_n^2 n^2}\sum_{\substack{k,l\in\mathbb{Z},\\ |k|\neq|l|}} x_k I\{\lambda_k \ge \alpha\gamma_k^\nu\} x_l I\{\lambda_l \ge \alpha\gamma_l^\nu\} \\
&\qquad\qquad \sum_{i=1}^n \mathrm{E}\Big[\Big(|D_{i,k}\mathscr{U}_{i,k}|^2 - \mathfrak{V}^{1/2}\Big(\frac{w_k}{|c_k|^2}-\frac{1}{x_k}\Big)\Big)\Big(|D_{i,l}\mathscr{U}_{i,l}|^2 - \mathfrak{V}^{1/2}\Big(\frac{w_l}{|c_l|^2}-\frac{1}{x_l}\Big)\Big)\Big] \\
&\quad + \frac{1}{t_n^2 n^2}\sum_{\substack{k,l\in\mathbb{Z},\\ |k|\neq|l|}} x_k I\{\lambda_k \ge \alpha\gamma_k^\nu\} x_l I\{\lambda_l \ge \alpha\gamma_l^\nu\} \\
&\qquad\qquad \sum_{\substack{i,p=1,\\ i\neq p}}^n \mathrm{E}\Big[|D_{i,k}\mathscr{U}_{i,k}|^2 - \mathfrak{V}^{1/2}\Big(\frac{w_k}{|c_k|^2}-\frac{1}{x_k}\Big)\Big]\mathrm{E}\Big[|D_{p,l}\mathscr{U}_{p,l}|^2 - \mathfrak{V}^{1/2}\Big(\frac{w_l}{|c_l|^2}-\frac{1}{x_l}\Big)\Big].
\end{aligned}$$

The terms quadratic in $k\in\mathbb{Z}$ can be estimated by Lemma B.1 und (S4.6),

while the other terms except the one coming from $|\langle\beta,\phi_k\rangle|^2 x_k$ vanish

$$\mathrm{E}\left|\sum_{i=1}^n \left(|D_{i,k}\mathscr{U}_{i,k}|^2 - \mathfrak{V}^{1/2}\left(\frac{w_k}{|c_k|^2}-\frac{1}{x_k}\right)\right)\right|^2 \le \frac{Cn}{\alpha^2} + Cn^2\left(\frac{x_k w_k}{|c_k|^2}-1\right)^2 |\langle\beta,\phi_k\rangle|^4\left(1+\frac{1}{n}\right).$$

Using the Cauchy-Schwarz inequality (S3.3), leads to

$$\mathrm{E}\left[\left(|D_{i,k}\mathscr{U}_{i,k}|^2 - \mathfrak{V}^{1/2}\left(\frac{w_k}{|c_k|^2}-\frac{1}{x_k}\right)\right)\left(|D_{i,l}\mathscr{U}_{i,l}|^2 - \mathfrak{V}^{1/2}\left(\frac{w_l}{|c_l|^2}-\frac{1}{x_l}\right)\right)\right] \le \frac{C}{\alpha^2}.$$

The expectations with $k, l \in \mathbb{Z}$, $|k| \neq |l|$ und $i, p \in \{1, \dots, n\}$, $i \neq p$ can be estimated by (S3.4). This finally yields

$$\begin{aligned}
&\frac{n^2}{t_n^2}\mathrm{E}|R_{n,2} - \mathfrak{R}_n|^2 \\
&\leq \frac{1}{t_n^2 n^2}\sum_{k\in\mathbb{Z}} x_k^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\}\left\{\frac{Cn}{\alpha^2} + Cn^2\left(\frac{x_k w_k}{|c_k|^2} - 1\right)^2 |\langle \beta, \phi_k\rangle|^4 \left(1 + \frac{1}{n}\right)\right\} \\
&\quad + \frac{C}{t_n^2 n^2}\sum_{\substack{k,l\in\mathbb{Z},\\ |k|\neq|l|}} x_k I\{\lambda_k \geq \alpha\gamma_k^\nu\} x_l I\{\lambda_l \geq \alpha\gamma_l^\nu\} \\
&\qquad\qquad \left\{\frac{n}{\alpha^2} + n(n-1)\left(\frac{x_k w_k}{|c_k|^2} - 1\right)|\langle\beta, \phi_k\rangle|^2 \left(\frac{x_l w_l}{|c_l|^2} - 1\right)|\langle\beta, \phi_l\rangle|^2\right\} \\
&= o\left(1 + \frac{1}{t_n^2}\right).
\end{aligned}$$

The second part can be shown by using

$$\frac{x_k w_k}{|c_k|^2} - 1 \leq \frac{1}{\alpha}(x_k - \lambda_k), \tag{S2.1}$$

for all $k \in \mathscr{K}_n$ together with Lemma B.1. □

All the other parts of Proposition A.5 as well as Lemmas A.2-A.4 follow by very similar techniques. For details we refer to [5] in the main article.

## S3 Details for the proof of Proposition C.1

Using that $\mathscr{U}_{j,k}D_{j,k}\overline{\mathscr{U}_{j,l}D_{j,l}}$ is independent of $(\mathcal{F}_{n,j-1})_{j=1,\dots,n}$, we can decompose

$$\begin{aligned}\mathfrak{V}_n &= \frac{1}{t_n^2 n^2}\sum_{j=2}^{n}\mathrm{E}\Big[\Big|\sum_{k\in\mathbb{Z}}\mathscr{U}_{j,k}D_{j,k}Z_{n,j,k}\Big|^2 \mid \mathcal{F}_{n,j-1}\Big]\\ &= \frac{1}{t_n^2 n}\sum_{k\in\mathbb{Z}} x_k\left(\frac{x_k w_k}{|c_k|^2}-1\right) I\{\lambda_k \geq \alpha\gamma_k^{\nu}\}\mathrm{E}|\mathscr{U}_{1,k}|^2\\ &\qquad\qquad \left(\sum_{i=1}^{n-1}|\mathscr{U}_{i,k}D_{i,k}|^2 + \sum_{\substack{i,p=1,\\ i\neq p}}^{n-1}\mathscr{U}_{i,k}D_{i,k}\overline{\mathscr{U}_{p,k}D_{p,k}}\right)\\ &= \mathfrak{V}_{n,1}+\mathfrak{V}_{n,2}.\end{aligned}$$

We define

$$\mathfrak{H}_n = \frac{\mathfrak{V}}{t_n^2 n}\sum_{k\in\mathbb{Z}} x_k\left(\frac{x_k w_k}{|c_k|^2}-1\right) I\{\lambda_k \geq \alpha\gamma_k^{\nu}\}\sum_{i=1}^{n-1}\mathrm{E}|D_{i,k}|^2$$

and show

$$\mathfrak{V}_{n,1} = \mathfrak{H}_n + o(1)$$

by proving the corresponding $L_2$-convergence. Afterwards we show that $\mathfrak{H}_n$ converges in probability to $\mathfrak{V}$. Writing for $i\in\{1,\dots,n\}$ and $k\in\mathbb{Z}$

$$\begin{aligned}&|\mathscr{U}_{i,k}D_{i,k}|^2\mathrm{E}|\mathscr{U}_{1,k}|^2 - \mathfrak{V}\mathrm{E}|D_{i,k}|^2\\ &= \mathfrak{V}^{1/2}\Big[|\mathscr{U}_{i,k}D_{i,k}|^2 - \mathfrak{V}^{1/2}\mathrm{E}|D_{i,k}|^2\Big] - |\mathscr{U}_{i,k}D_{i,k}|^2|\langle\beta,\phi_k\rangle|^2 x_k.\end{aligned}$$

and, observing that $\sigma^2+\sum_{m\in\mathbb{Z}}|\langle\beta,\phi_m\rangle|^2x_m \le C_1$ for some constant $C_1>0$, we get

$$\mathrm{E}\left(\mathfrak{V}_{n,1}-\mathfrak{H}_n\right)^2 \le \mathbb{V}_{n,1}+\mathbb{V}_{n,2}+\mathbb{V}_{n,3}$$

with

$$\begin{aligned}
\mathbb{V}_{n,1} = \frac{C}{t_n^4 n^2}\sum_{k\in\mathbb{Z}} x_k^2\left(\frac{x_k w_k}{|c_k|^2}-1\right)^2 I\{\lambda_k\ge\alpha\gamma_k^\nu\}& \\
\Bigg\{\sum_{i=1}^{n-1}\mathrm{E}\Big(|\mathscr{U}_{i,k}D_{i,k}|^2-\mathfrak{V}^{1/2}\mathrm{E}|D_{i,k}|^2\Big)^2& \\
+\sum_{\substack{i,p=1,\\ i\neq p}}^{n-1}\mathrm{E}\Big[|\mathscr{U}_{i,k}D_{i,k}|^2-\mathfrak{V}^{1/2}\mathrm{E}|D_{1,k}|^2\Big]\mathrm{E}\Big[(|\mathscr{U}_{p,k}D_{p,k}|^2-\mathfrak{V}^{1/2}\mathrm{E}|D_{1,k}|^2\Big]\Bigg\}&, \\
\mathbb{V}_{n,2}=\frac{C}{t_n^4 n^2}\sum_{\substack{k,l\in\mathbb{Z},\\ |k|\neq|l|}} x_k\left(\frac{x_k w_k}{|c_k|^2}-1\right)I\{\lambda_k\ge\alpha\gamma_k^\nu\}x_l\left(\frac{x_l w_l}{|c_l|^2}-1\right)I\{\lambda_l\ge\alpha\gamma_l^\nu\}& \\
\Bigg\{\sum_{i=1}^{n-1}\mathrm{E}\Big[\Big(|\mathscr{U}_{i,k}D_{i,k}|^2-\mathfrak{V}^{1/2}\mathrm{E}|D_{i,k}|^2\Big)\Big(|\mathscr{U}_{i,l}D_{i,l}|^2-\mathfrak{V}^{1/2}\mathrm{E}|D_{i,l}|^2\Big)\Big]& \\
+\sum_{\substack{i,p=1,\\ i\neq p}}^{n-1}\mathrm{E}\Big[|\mathscr{U}_{i,k}D_{i,k}|^2-\mathfrak{V}^{1/2}\mathrm{E}|D_{i,k}|^2\Big]\mathrm{E}\Big[|\mathscr{U}_{i,l}D_{i,l}|^2-\mathfrak{V}^{1/2}\mathrm{E}|D_{i,l}|^2\Big]\Bigg\}&, \\
\mathbb{V}_{n,3}=\frac{2}{t_n^4 n^2}\mathrm{E}\left(\sum_{k\in\mathbb{Z}} x_k^2\left(\frac{x_k w_k}{|c_k|^2}-1\right)|\langle\beta,\phi_k\rangle|^2 I\{\lambda_k\ge\alpha\gamma_k^\nu\}\sum_{i=1}^{n-1}|\mathscr{U}_{i,k}D_{i,k}|^2\right)^2&.
\end{aligned}$$

We have

$$\mathrm{E}|\mathscr{U}_{j,k}D_{j,k}|^2=\left(\sigma^2+\sum_{\substack{m\in\mathbb{Z},\\ |m|\neq|k|}}|\langle\beta,\phi_m\rangle|^2x_m\right)\left(\frac{w_k}{|c_k|^2}-\frac{1}{x_k}\right), \tag{S3.2}$$

because $|\mathscr{U}_{j,k}|^2$ and $|D_{j,k}|^2$ are uncorrelated for all $k \in \mathbb{Z}$ and $j \in \{1,\ldots,n\}$. With Lemma B.1 and (S4.6), for all $i \in \{1,\ldots,n\}$ and $k \in \mathcal{K}_n$, we have

$$\mathrm{E}\left(|\mathscr{U}_{i,k}D_{i,k}|^2 - \mathfrak{V}^{1/2}\mathrm{E}|D_{i,k}|^2\right)^2 \leq C\left(\mathrm{E}|D_{1,k}|^4 - \left(\mathrm{E}|D_{1,k}|^2\right)^2\right) \leq C\mathrm{E}|D_{1,k}|^4$$
$$\leq \frac{C}{\alpha^2} \tag{S3.3}$$

as well as

$$\mathrm{E}\left[|\mathscr{U}_{i,k}D_{i,k}|^2 - \mathfrak{V}^{1/2}\mathrm{E}|D_{i,k}|^2\right] = -\mathrm{E}|D_{1,k}|^2|\langle\beta,\phi_k\rangle|^2 x_k$$
$$= -\left(\frac{x_k w_k}{|c_k|^2} - 1\right)|\langle\beta,\phi_k\rangle|^2. \tag{S3.4}$$

For the mixed terms with $k,l \in \mathbb{Z}, |k| \neq |l|$ and $i \in \{1,\ldots,n\}$ and $\frac{w_k}{|c_k|^2} - \frac{1}{x_k} \geq 0$ for all $k \in \mathbb{Z}$, we get

$$\begin{aligned}
&\mathrm{E}\Big[\Big(|\mathscr{U}_{i,k}D_{i,k}|^2 - \mathfrak{V}^{1/2}\mathrm{E}|D_{i,k}|^2\Big)\Big(|\mathscr{U}_{i,l}D_{i,l}|^2 - \mathfrak{V}^{1/2}\mathrm{E}|D_{i,l}|^2\Big)\Big] \\
&\leq \mathrm{E}\Big[|\mathscr{U}_{1,k}D_{1,k}\mathscr{U}_{1,l}D_{1,l}|^2\Big] + \Big(\frac{w_k}{|c_k|^2} - \frac{1}{x_k}\Big)\Big(\frac{w_l}{|c_l|^2} - \frac{1}{x_l}\Big) \\
&\leq C\Bigg\{\frac{1}{\alpha^2}|\langle\beta,\phi_k\rangle|^2 x_k|\langle\beta,\phi_l\rangle|^2 x_l + \frac{x_l}{\alpha}|\langle\beta,\phi_l\rangle|^2\Big(\frac{w_k}{|c_k|^2} - \frac{1}{x_k}\Big) \\
&\qquad + \frac{x_k}{\alpha}|\langle\beta,\phi_k\rangle|^2\Big(\frac{w_l}{|c_l|^2} - \frac{1}{x_l}\Big) + \Big(\frac{w_k}{|c_k|^2} - \frac{1}{x_k}\Big)\Big(\frac{w_l}{|c_l|^2} - \frac{1}{x_l}\Big)\Bigg\}.
\end{aligned} \tag{S3.5}$$

Using this, we have

$$
\begin{aligned}
\mathbb{V}_{n,1} &\leq \frac{C}{t_n^4 n^2} \sum_{k\in\mathbb{Z}} x_k^2 \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\} \left\{ \frac{n}{\alpha^2} + n^2 \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^2 |\langle \beta, \phi_k\rangle|^4 \right\} \\
&\leq \frac{C}{t_n^4 n\alpha^2} \sum_{k\in\mathbb{Z}} x_k^2 \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\} \\
&\quad + \frac{C}{t_n^4} \sum_{k\in\mathbb{Z}} x_k^2 \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^4 |\langle \beta, \phi_k\rangle|^4 I\{\lambda_k \geq \alpha\gamma_k^\nu\} \\
&= o\left(1 + \frac{1}{t_n^2}\right),
\end{aligned}
$$

with some constant $C > 0$. With similar arguments, we obtain

$$
\begin{aligned}
\mathbb{V}_{n,2} \leq \frac{C}{t_n^4} \Bigg\{ &\frac{1}{n\alpha^2} \left( \sum_{k\in\mathbb{Z}} x_k^2 \left(\frac{x_k w_k}{|c_k|^2} - 1\right) |\langle \beta, \phi_k\rangle|^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\} \right)^2 \\
&+ \frac{t_n^2}{n\alpha} \left( \sum_{l\in\mathbb{Z}} x_l^2 \left(\frac{x_l w_l}{|c_l|^2} - 1\right) |\langle \beta, \phi_l\rangle|^2 I\{\lambda_l \geq \alpha\gamma_l^\nu\} \right) + \frac{(t_n^2)^2}{n} \Bigg\} \\
+ \frac{C}{t_n^4} &\left( \sum_{k\in\mathbb{Z}} x_k \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^2 |\langle \beta, \phi_k\rangle|^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\} \right)^2,
\end{aligned}
$$

which can be further bounded using the Cauchy-Schwarz inequality to get

$$
\mathbb{V}_{n,2} = o\left(1 + \frac{1}{t_n^2} + \frac{1}{\sqrt{n}t_n}\right) + \mathcal{O}\left(\frac{1}{n}\right).
$$

Using similar arguments as for the first two terms, $V_{n,3}$ can also be bounded to get

$$\begin{aligned}
\mathbb{V}_{n,3} \leq & \frac{C}{t_n^4 n\alpha^2} \sum_{k\in\mathbb{Z}} x_k^4 \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^2 |\langle \beta, \phi_k\rangle|^4 I\{\lambda_k \geq \alpha\gamma_k^\nu\} \\
& + \frac{C}{t_n^4} \sum_{k\in\mathbb{Z}} x_k^2 \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^4 |\langle \beta, \phi_k\rangle|^4 I\{\lambda_k \geq \alpha\gamma_k^\nu\} \\
& + \frac{C}{t_n^4 n\alpha^2} \left(\sum_{k\in\mathbb{Z}} x_k^3 \left(\frac{x_k w_k}{|c_k|^2} - 1\right) |\langle \beta, \phi_k\rangle|^4 I\{\lambda_k \geq \alpha\gamma_k^\nu\}\right)^2 \\
& + \frac{C}{t_n^4 n} \left(\sum_{k\in\mathbb{Z}} x_k \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^2 |\langle \beta, \phi_k\rangle|^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\}\right)^2 \\
& + \frac{C}{t_n^4 n\alpha} \sum_{k\in\mathbb{Z}} x_k \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^2 |\langle \beta, \phi_k\rangle|^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\} \\
& \qquad \sum_{k\in\mathbb{Z}} x_l^3 \left(\frac{x_l w_l}{|c_l|^2} - 1\right) |\langle \beta, \phi_l\rangle|^4 I\{\lambda_l \geq \alpha\gamma_l^\nu\} \\
& + \frac{C}{t_n^4} \left(\sum_{k\in\mathbb{Z}} x_k \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^2 |\langle \beta, \phi_k\rangle|^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\}\right)^2 \\
= & \, o\left(1 + \frac{1}{t_n^2} + \frac{1}{n} + \frac{1}{\sqrt{n} t_n}\right).
\end{aligned}$$

Altogether, we have

$$\mathfrak{V}_{n,1} = \mathfrak{H}_n + o_P(1).$$

The stochastic convergence of $\mathfrak{H}_n$ follows by

$$\mathfrak{H}_n = \mathfrak{V}\frac{n-1}{t_n^2 n} \sum_{k\in\mathbb{Z}} \left(\frac{x_k w_k}{|c_k|^2} - 1\right)^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\} \xrightarrow{P} \mathfrak{V}$$

for $n \to \infty$. For proving, that $\mathfrak{V}_{n,2}$ converges stochastically to 0, we show again the corresponding $L_2$-convergence. To this end we bound for all

$i \in \{1, \dots, n\}$ und $k \in \mathbb{Z}$ the term $\mathrm{E}|\mathscr{U}_{1,k}|^2$ by a constant $C < \infty$ by using that $U$ is centered and Lemma B.1, to obtain

$$\begin{aligned}
\mathrm{E}|\mathfrak{V}_{n,2}|^2 \leq \frac{C}{t_n^4 n^2} \Bigg\{ & \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\} \mathrm{E} \Bigg| \sum_{\substack{i,p=1, \\ i \neq p}}^{n-1} \mathscr{U}_{i,k} D_{i,k} \overline{\mathscr{U}_{p,k} D_{p,k}} \Bigg|^2 \\
& + \sum_{\substack{k,l \in \mathbb{Z}, \\ |k| \neq |l|}} x_k \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) I\{\lambda_k \geq \alpha \gamma_k^\nu\} x_l \left( \frac{x_l w_l}{|c_l|^2} - 1 \right) I\{\lambda_l \geq \alpha \gamma_l^\nu\} \\
& \qquad \mathrm{E} \Bigg[ \Bigg( \sum_{\substack{i,p=1, \\ i \neq p}}^{n-1} \mathscr{U}_{i,k} D_{i,k} \overline{\mathscr{U}_{p,k} D_{p,k}} \Bigg) \Bigg( \sum_{\substack{i,p=1, \\ i \neq p}}^{n-1} \overline{\mathscr{U}_{i,l} D_{i,l}} \mathscr{U}_{p,l} D_{p,l} \Bigg) \Bigg] \Bigg\}.
\end{aligned}$$

Since $\mathscr{U}_{i,k} D_{i,k}$ and $\mathscr{U}_{p,k} D_{p,k}$ are stochastically independent for $p \neq i$, only the quadratic terms for $k \in \mathbb{Z}$ are relevant

$$\begin{aligned}
\sum_{\substack{i,p=1, \\ i \neq p}}^{n-1} \mathrm{E} \Big| \mathscr{U}_{i,k} D_{i,k} \overline{\mathscr{U}_{p,k} D_{p,k}} \Big|^2 &= \sum_{\substack{i,p=1, \\ i \neq p}}^{n-1} \mathrm{E}|\mathscr{U}_{i,k} D_{i,k}|^2 \mathrm{E}|\mathscr{U}_{p,k} D_{p,k}|^2 \\
&= (n-1)(n-2) \left( \mathrm{E}|\mathscr{U}_{1,k}|^2 \mathrm{E}|D_{1,k}|^2 \right)^2 \\
&\leq C n^2 \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right)^2.
\end{aligned}$$

Under the assumptions of Theorem 2.1, this leads to

$$\mathrm{E}|\mathfrak{V}_{n,2}|^2 \leq \frac{C}{t_n^4} \sum_{k \in \mathbb{Z}} \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^4 I\{\lambda_k \geq \alpha \gamma_k^\nu\} = o(1),$$

and therefore

$$\mathfrak{V}_{n,2} = o_P(1).$$

## S4 Details for the proof of Proposition C.2

It is shown in [1] and [10] that the conditional Lindeberg condition follows from the unconditional Lyapunov condition. We will show in the following, that

$$\sum_{j=2}^{n} \mathrm{E}|Y_{n,j}|^4 = o(1)$$

and, for this purpose, we decompose

$$\sum_{j=2}^{n} \mathrm{E}|Y_{n,j}|^4 = L_{n,1} + L_{n,2} + L_{n,3} + L_{n,4},$$

where

$$L_{n,1} = \frac{1}{t_n^4 n^4} \sum_{j=2}^{n} \sum_{k\in\mathbb{Z}} \mathrm{E}\left|\mathscr{U}_{j,k} D_{j,k} Z_{n,j,k}\right|^4,$$

$$L_{n,2} = \frac{1}{t_n^4 n^4} \sum_{j=2}^{n} \sum_{\substack{k,l\in\mathbb{Z},\\ |k|\neq|l|}} \mathrm{E}\left|\mathscr{U}_{j,k} D_{j,k} Z_{n,j,k} \overline{\mathscr{U}_{j,l} D_{j,l} Z_{n,j,l}}\right|^2,$$

$$L_{n,3} = \frac{1}{t_n^4 n^4} \sum_{j=2}^{n} \sum_{\substack{k,l,q\in\mathbb{Z},\\ |k|,|l|\neq|q|,|k|\neq|l|}} \mathrm{E}\Big[|\mathscr{U}_{j,k} D_{j,k} Z_{n,j,k}|^2 \mathscr{U}_{j,l} D_{j,l} Z_{n,j,l} \overline{\mathscr{U}_{j,q} D_{j,q} Z_{n,j,q}}\Big],$$

$$L_{n,4} = \frac{1}{t_n^4 n^4} \sum_{j=2}^{n} \sum_{\substack{k,l,p,q\in\mathbb{Z},\\ |k|,|l|,|p|\neq|q|,\\ |k|,|l|\neq|p|,|k|\neq|l|}} \mathrm{E}\Big[\mathscr{U}_{j,k} D_{j,k} Z_{n,j,k} \overline{\mathscr{U}_{j,l} D_{j,l} Z_{n,j,l}} \mathscr{U}_{j,p} D_{j,p} Z_{n,j,p} \overline{\mathscr{U}_{j,q} D_{j,q} Z_{n,j,q}}\Big].$$

For $L_{n,1}$, we use that for all $k \in \mathbb{Z}, n \in \mathbb{N}, j \in \{1,\dots,n\}$, $Z_{n,j,k}$ are stochastically independent of $\mathscr{U}_{j,k} D_{j,k}$ and $\mathscr{U}_{j,k}$ are uncorrelated with $D_{j,k}$. Furthermore, the fourth absolute moment of $\mathscr{U}_{j,k}$ is uniformly bounded because $U$ is centered and due to Lemma B.1. The fourth absolute moment of $D_{j,k}$

can be estimated using Assumption 3 and $(X, W) \in \mathcal{F}_\eta^4$ as

$$\mathrm{E}|D_{j,k}|^4 \leq C\left(\frac{\mathrm{E}|\langle W,\phi_k\rangle|^4}{|c_k|^4} + \frac{\mathrm{E}|\langle X,\phi_k\rangle|^4}{x_k^4}\right) \leq C\eta\left(\frac{w_k^2}{|c_k|^4} + \frac{1}{x_k^2}\right) \leq \frac{C\eta}{\alpha^2}. \tag{S4.6}$$

Again using similar arguments, we obtain

$$\mathrm{E}\left|\mathscr{U}_{i_1,k}D_{i_1,k}\right|^2 = \mathrm{E}|\mathscr{U}_{i_1,k}|^2\mathrm{E}|D_{i_1,k}|^2 \leq C\left(\frac{w_k}{|c_k|^2} - \frac{1}{x_k}\right). \tag{S4.7}$$

This results in

$$\begin{aligned}
&\mathrm{E}\Big|\sum_{i=1}^{j-1}\mathscr{U}_{i,k}D_{i,k}x_kI\{\lambda_k \geq \alpha\gamma_k^\nu\}\Big|^4 \\
&= x_k^4 I\{\lambda_k \geq \alpha\gamma_k^\nu\}\left\{\sum_{i=1}^{j-1}\mathrm{E}|\mathscr{U}_{i,k}|^4\mathrm{E}|D_{i,k}|^4 + 2\sum_{1\leq i_1<i_2\leq j-1}\mathrm{E}|\mathscr{U}_{i_1,k}D_{i_1,k}|^2\mathrm{E}|\mathscr{U}_{i_2,k}D_{i_2,k}|^2\right\} \\
&\leq \frac{Cn}{\alpha^2}x_k^4 I\{\lambda_k \geq \alpha\gamma_k^\nu\} + Cn^2x_k^2\left(\frac{x_kw_k}{|c_k|^2} - 1\right)^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\}.
\end{aligned} \tag{S4.8}$$

Putting these results together, for $L_{n,1}$, we get

$$\begin{aligned}
L_{n,1} &= \frac{1}{t_n^4n^4}\sum_{j=2}^{n}\sum_{k\in\mathbb{Z}}\mathrm{E}|\mathscr{U}_{j,k}|^4\mathrm{E}|D_{j,k}|^4\mathrm{E}|Z_{n,j,k}|^4 \\
&\leq \frac{C}{t_n^4n^4\alpha^2}\sum_{j=2}^{n}\sum_{k\in\mathbb{Z}}\mathrm{E}\Big|\sum_{i=1}^{j-1}\mathscr{U}_{i,k}D_{i,k}x_kI\{\lambda_k \geq \alpha\gamma_k^\nu\}\Big|^4 \\
&\leq \frac{C}{t_n^4n\alpha^2}\sum_{k\in\mathbb{Z}}x_k^2I\{\lambda_k \geq \alpha\gamma_k^\nu\}\left(\frac{1}{n\alpha^2}x_k^2 + \left(\frac{x_kw_k}{|c_k|^2} - 1\right)^2\right) \\
&= o(1)\frac{1}{t_n^4}\left(\sum_{k\in\mathbb{Z}}x_k^4I\{\lambda_k \geq \alpha\gamma_k^\nu\} + \sum_{k\in\mathbb{Z}}x_k^2\left(\frac{x_kw_k}{|c_k|^2} - 1\right)^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\}\right),
\end{aligned}$$

where the first series converges due to Lemma B.1 and the second series either also converges or, if not, can be bounded by $Ct_n^2$.

Considering $L_{n,4}$, we use the stochastic independence of $Z_{n,j,k}$ and $\mathscr{U}_{j,l}D_{j,l}$ for all $k,l \in \mathbb{Z}$, which results in

$$\begin{aligned}&\mathrm{E}\big[\mathscr{U}_{j,k}D_{j,k}Z_{n,j,k}\overline{\mathscr{U}_{j,l}D_{j,l}Z_{n,j,l}}\mathscr{U}_{j,p}D_{j,p}Z_{n,j,p}\overline{\mathscr{U}_{j,q}D_{j,q}Z_{n,j,q}}\big]\\&=\mathrm{E}\big[\mathscr{U}_{j,k}D_{j,k}\overline{\mathscr{U}_{j,l}D_{j,l}}\mathscr{U}_{j,p}D_{j,p}\overline{\mathscr{U}_{j,q}D_{j,q}}\big]\mathrm{E}\big[Z_{n,j,k}\overline{Z_{n,j,l}}Z_{n,j,p}\overline{Z_{n,j,q}}\big].\end{aligned}$$

The rest of the argumentation is just calculating the expectations using that for all $j \in \{1,\dots,n\}$, $D_{j,k}, D_{j,l}, D_{j,p}$ and $D_{j,q}$ are uncorrelated with $S_{j,m}$ for all $m \in \mathbb{Z}\backslash\{m \in \mathbb{Z} : |m| = |k|, |l|, |p|, |q|\}$ and stochastically independent of $U_j$. Finally,

$$\mathrm{E}[S_{j,k}D_{j,k}] = \langle\beta,\phi_k\rangle\mathrm{E}\left[\langle\phi_k, X_j\rangle\left(\frac{\langle W_j,\phi_k\rangle}{c_k} - \frac{\langle X_j,\phi_k\rangle}{x_k}\right)\right] = \langle\beta,\phi_k\rangle\left(\frac{c_k}{c_k} - \frac{x_k}{x_k}\right) = 0 \tag{S4.9}$$

and, in the same way, $\mathrm{E}[\overline{S_{j,k}}D_{j,k}] = \mathrm{E}[S_{j,k}\overline{D_{j,k}}] = 0$, which gives $L_{n,4} = 0$.

With similar arguments as above, we get

$$L_{n,2} = \frac{1}{t_n^4 n^4}\sum_{j=2}^{n}\sum_{\substack{k,l\in\mathbb{Z},\\ k\neq l}}\mathrm{E}|\mathscr{U}_{j,k}D_{j,k}\overline{\mathscr{U}_{j,l}D_{j,l}}|^2\mathrm{E}|Z_{n,j,k}\overline{Z_{n,j,l}}|^2,$$

which can be further bounded by using

$$\begin{aligned}\mathrm{E}\big|\overline{S_{j,k}}D_{j,k}\big|^2 &\le |\langle\beta,\phi_k\rangle|^2\sqrt{\mathrm{E}\big|\langle X,\phi_k\rangle|^4\mathrm{E}\big|D_{j,k}\big|^4}\\&\le \sqrt{\eta}|\langle\beta,\phi_k\rangle|^2 x_k\left(\frac{\mathrm{E}|\langle W,\phi_k\rangle|^4}{|c_k|^4} + \frac{\mathrm{E}|\langle X,\phi_k\rangle|^4}{x_k^4}\right)^{1/2}\\&\le C|\langle\beta,\phi_k\rangle|^2 x_k\left(\frac{w_k^2}{|c_k|^4} + \frac{1}{x_k^2}\right)^{1/2} \le \frac{C|\langle\beta,\phi_k\rangle|^2 x_k}{\alpha}\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{E}|Z_{n,j,k}\overline{Z_{n,j,l}}|^2 \\
&\leq Cx_k^2x_l^2 I\{\lambda_k \geq \alpha\gamma_k^\nu\} I\{\lambda_l \geq \alpha\gamma_l^\nu\}(n-1) \\
&\quad \Bigg\{ \Bigg[ \frac{C}{\alpha^2}|\langle\beta,\phi_k\rangle|^2 x_k |\langle\beta,\phi_l\rangle|^2 x_l + \frac{C|\langle\beta,\phi_l\rangle|^2 x_l}{\alpha}\left(\frac{w_k}{|c_k|^2} - \frac{1}{x_k}\right) \\
&\qquad + \frac{C|\langle\beta,\phi_k\rangle|^2 x_k}{\alpha}\left(\frac{w_l}{|c_l|^2} - \frac{1}{x_l}\right) + \left(\frac{w_k}{|c_k|^2} - \frac{1}{x_k}\right)\left(\frac{w_l}{|c_l|^2} - \frac{1}{x_l}\right) \Bigg] \\
&\qquad + (n-2)\left(\frac{w_k}{|c_k|^2} - \frac{1}{x_k}\right)\left(\frac{w_l}{|c_l|^2} - \frac{1}{x_l}\right) \Bigg\}.
\end{aligned}$$

This results in

$$\begin{aligned}
L_{n,2} &\leq \frac{C}{t_n^4(n\alpha^2)^2}\left(\sum_{k\in\mathbb{Z}}|\langle\beta,\phi_k\rangle|^4 x_k^4\right)^2 + \frac{C}{t_n^2 n^2\alpha^2}\sum_{l\in\mathbb{Z}}|\langle\beta,\phi_l\rangle|^4 x_l^4 + \frac{C}{n^2} \\
&\quad + \frac{C}{t_n^4 n\alpha^2}\left(\sum_{k\in\mathbb{Z}}|\langle\beta,\phi_k\rangle|^2 x_k^2\left(\frac{w_k x_k}{|c_k|^2} - 1\right) I\{\lambda_k \geq \alpha\gamma_k^\nu\}\right)^2 \\
&\quad + \frac{C}{t_n^2 n\alpha}\sum_{l\in\mathbb{Z}}|\langle\beta,\phi_l\rangle|^2 x_l^2\left(\frac{w_l x_l}{|c_l|^2} - 1\right) I\{\lambda_l \geq \alpha\gamma_l^\nu\} + \frac{C}{n} \\
&\leq o\left(\frac{1}{t_n^4} + \frac{1}{t_n^2 n}\right) + \mathcal{O}\left(\frac{1}{n} + \frac{1}{n^2}\right) + \frac{C}{t_n^2 n\alpha^2}\sum_{k\in\mathbb{Z}}|\langle\beta,\phi_k\rangle|^4 x_k^4 + \frac{C}{t_n n\alpha}\sqrt{\sum_{k\in\mathbb{Z}}|\langle\beta,\phi_k\rangle|^4 x_k^4} \\
&= o\left(\frac{1}{t_n^4} + \frac{1}{t_n^2 n} + \frac{1}{t_n^2} + \frac{1}{t_n\sqrt{n}}\right) + \mathcal{O}\left(\frac{1}{n} + \frac{1}{n^2}\right) \\
&= o(1),
\end{aligned}$$

using the Hölder inequality and Lemma B.1.

For the summands in $L_{n,3}$, we get

$$\mathrm{E}\big[|\mathscr{U}_{j,k}D_{j,k}Z_{n,j,k}|^2\mathscr{U}_{j,l}D_{j,l}Z_{n,j,l}\overline{\mathscr{U}_{j,q}D_{j,q}Z_{n,j,q}}\big]$$

$$= \mathrm{E}\big[|\mathscr{U}_{j,k}D_{j,k}|^2\mathscr{U}_{j,l}D_{j,l}\overline{\mathscr{U}_{j,q}D_{j,q}}\big]\mathrm{E}\big[|Z_{n,j,k}|^2Z_{n,j,l}\overline{Z_{n,j,q}}\big].$$

The first expectation is

$$\begin{aligned}
&\mathrm{E}\big[|\mathscr{U}_{j,k}D_{j,k}|^2\mathscr{U}_{j,l}D_{j,l}\overline{\mathscr{U}_{j,q}D_{j,q}}\big]\\
&= \left(\frac{w_k}{|c_k|^2}-\frac{1}{x_k}\right)|\langle\beta,\phi_l\rangle|^2|\langle\beta,\phi_q\rangle|^2\\
&\quad \mathrm{E}\left[|\langle X_j,\phi_l\rangle|^2\left(\frac{\langle W_j,\phi_l\rangle}{c_l}-\frac{\langle X_j,\phi_l\rangle}{x_l}\right)\right]\mathrm{E}\left[|\langle X_j,\phi_q\rangle|^2\left(\frac{\langle \phi_q,W_j\rangle}{\overline{c_q}}-\frac{\langle \phi_q,X_j\rangle}{x_q}\right)\right],
\end{aligned}$$

while

$$\begin{aligned}
&\mathrm{E}\big[|Z_{n,j,k}|^2Z_{n,j,l}\overline{Z_{n,j,q}}\big]\\
&= x_k^2x_lx_qI\{\lambda_k\geq\alpha\gamma_k^\nu\}I\{\lambda_l\geq\alpha\gamma_l^\nu\}I\{\lambda_q\geq\alpha\gamma_q^\nu\}\sum_{i=1}^{j-1}\mathrm{E}\big[|\mathscr{U}_{i,k}D_{i,k}|^2\mathscr{U}_{i,l}D_{i,l}\overline{\mathscr{U}_{i,q}D_{i,q}}\big].
\end{aligned}$$

Altogether, we have

$$\begin{aligned}
L_{n,3} \leq \frac{1}{t_n^4n^2}&\sum_{\substack{k,l,q\in\mathbb{Z},\\|k|,|l|\neq|q|,|k|\neq|l|}} x_k^2x_lx_qI\{\lambda_k\geq\alpha\gamma_k^\nu\}I\{\lambda_l\geq\alpha\gamma_l^\nu\}I\{\lambda_q\geq\alpha\gamma_q^\nu\}\\
&\left(\frac{w_k}{|c_k|^2}-\frac{1}{x_k}\right)^2|\langle\beta,\phi_l\rangle|^4|\langle\beta,\phi_q\rangle|^4\\
&\left(\mathrm{E}\left[|\langle X,\phi_l\rangle|^2\left(\frac{\langle W,\phi_l\rangle}{c_l}-\frac{\langle X,\phi_l\rangle}{x_l}\right)\right]\mathrm{E}\left[|\langle X,\phi_q\rangle|^2\left(\frac{\langle \phi_q,W\rangle}{\overline{c_q}}-\frac{\langle \phi_q,X\rangle}{x_q}\right)\right]\right)^2.
\end{aligned}$$

The series can be bounded by $t_n^2$. Using the Hölder inequality for $l\in\mathcal{K}_n$, we have

$$\left(\mathrm{E}\Big[|\langle X,\phi_l\rangle|^2\Big(\frac{\langle\phi_l,W\rangle}{\overline{c_l}}-\frac{\langle\phi_l,X\rangle}{x_l}\Big)\Big]\right)^2\leq \mathrm{E}|\langle X,\phi_l\rangle|^4\mathrm{E}\Big|D_{1,l}|^2\leq \eta x_l^2\left(\frac{w_l}{|c_l|^2}-\frac{1}{x_l}\right)\leq\frac{C}{\alpha^2}x_l^2.$$

Finally, relying again on Assumption 3 and Lemma B.1, also $L_{n,3}$ converges to 0 due to

$$L_{n,3}\leq\frac{C}{t_n^2n^2}\left(\sum_{k\in\mathbb{Z}}|\langle\beta,\phi_k\rangle|^4x_k\frac{x_k-\lambda_k}{\lambda_k}\right)^{1/2}\leq\frac{C}{t_n^2n^2\alpha^2}\sum_{k\in\mathbb{Z}}|\langle\beta,\phi_k\rangle|^4x_k(x_k-\lambda_k)=o\left(\frac{1}{t_n^2n}\right).$$