MEAN TESTS FOR HIGH-DIMENSIONAL TIME SERIES

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The supplement provides the theoretical proofs of Theorems 1–4, Corollaries 1 and 1, and Lemmas 1–5.

S1 Proofs

S1.1 Proofs of Lemmas 1–3

Below, we provide some useful lemmas and proofs.

Lemma 1. *For any two $p \times p$ random matrices $\boldsymbol{A} = (a_{j_1j_2})_{p\times p}$, $\boldsymbol{B} = (b_{j_1j_2})_{p\times p}$ and four p-dimensional random vectors $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3, \boldsymbol{\phi}_4$ with finite fourth-order moments for all elements,*

$$\mathrm{tr}\big\{E(\boldsymbol{AB})\big\} - \mathrm{tr}\big\{E(\boldsymbol{A})E(\boldsymbol{B})\big\} = \sum_{j_1,j_2=1}^{p} Cov(a_{j_1j_2}, b_{j_2j_1}),$$

$$\mathrm{tr}\big\{E(\boldsymbol{A}\boldsymbol{\phi}_1\boldsymbol{\phi}_2^{\mathrm{T}})\big\} - \mathrm{tr}\big\{E(\boldsymbol{A})E(\boldsymbol{\phi}_1)E(\boldsymbol{\phi}_2^{\mathrm{T}})\big\} = \sum_{j_1,j_2=1}^{p} \big\{Cov(a_{j_1j_2}, \phi_{1j_2}\phi_{2j_1}) + E(a_{j_1j_2})Cov(\phi_{1j_2}, \phi_{2j_1})\big\},$$

$$\mathrm{tr}\big\{E(\boldsymbol{\phi}_1\boldsymbol{\phi}_2^{\mathrm{T}}\boldsymbol{\phi}_3\boldsymbol{\phi}_4^{\mathrm{T}})\big\} - \mathrm{tr}\big\{E(\boldsymbol{\phi}_1)E(\boldsymbol{\phi}_2^{\mathrm{T}})E(\boldsymbol{\phi}_3)E(\boldsymbol{\phi}_4^{\mathrm{T}})\big\} = \sum_{j_1,j_2=1}^{p} \left\{E(\phi_{1j_1}\phi_{2j_2}\phi_{3j_2}\phi_{4j_1}) - \prod_{k=1}^{4} E(\phi_{1j_k})\right\}.$$

The proof of Lemma 1 is straightforward and hence not presented here.

Lemma 2. *Under Assumption 1 with $q > 4$, and Assumptions 3 and 5, we have as $p \to +\infty$, for any $i, i' = 1, 2$ and $\boldsymbol{\mu}_{i'} \neq \boldsymbol{0}$,*

$$\sum_{|k_1|,|k_2|>K} |\mathrm{tr}(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2})| = o\big\{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)\big\} \quad \textit{and} \quad \sum_{|k|>K} |\boldsymbol{\mu}_{i'}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}\boldsymbol{\mu}_{i'}| = o\big(\boldsymbol{\mu}_{i'}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,\infty}\boldsymbol{\mu}_{i'}\big),$$

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where K satisfies $K \geq c_1 \log p$ for a sufficiently large positive constant c_1.

Proof. Recall that $\mathbf{\Sigma}_{i,k} = (\sigma_{i,k,j_1j_2})_{p\times p}$. Under Assumption 3, by the Davydov's inequality (Davydov, 1968),

$$|\sigma_{i,k,j_1j_2}| = |\mathrm{Cov}(X_{i,t+k,j_1}, X_{i,t,j_2})| \leq \frac{2q\Delta^2}{q-2}\beta_i^x(|k|)^{1-2/q} \leq \frac{2qc^{1-2/q}\Delta^2}{q-2}\exp\{-a(1-2/q)|k|\}. \tag{S1.1}$$

Under the conditions in Lemma 2, we have as $n, p \to +\infty$,

$$\begin{aligned}
\frac{\sum_{|k_1|,|k_2|>K}|\mathrm{tr}(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2})|}{\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)} &\leq \frac{\sum_{|k_1|,|k_2|>K}|\mathrm{tr}(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2})|}{C_1^2p^{1+2\eta}} \\
&\leq \frac{\sum_{|k_1|,|k_2|>K}\sum_{j_1,j_2=1}^p|\sigma_{i,k_1,j_1j_2}\sigma_{i,k_2,j_2j_1}|}{C_1^2p^{1+2\eta}} \\
&\leq \frac{4q^2c^{2-4/q}\Delta^4p^2}{(q-2)^2C_1^2p^{1+2\eta}}\left[\sum_{|k|>K}\exp\{-a(1-2/q)|k|\}\right]^2 \\
&\leq \frac{16q^2c^{2-4/q}\Delta^4p^{1-2\eta}}{(q-2)^2C_1^2}\exp\{-2a(1-2/q)K\} \\
&= \frac{16q^2c^{2-4/q}\Delta^4}{(q-2)^2C_1^2}\exp\{-2a(1-2/q)K + (1-2\eta)\log p\} \to 0,
\end{aligned}$$

where we use the inequality $\sum_{k>K} e^{-k} \leq \int_{k>K} e^{-x}dx = e^{-K}$. On the other hand,

$$\begin{aligned}
\frac{\sum_{|k|>K}|\boldsymbol{\mu}_{i'}^{\mathrm{T}}\mathbf{\Sigma}_{i,k}\boldsymbol{\mu}_{i'}|}{\boldsymbol{\mu}_{i'}^{\mathrm{T}}\mathbf{\Sigma}_{i,\infty}\boldsymbol{\mu}_{i'}} &\leq \frac{\left|\sum_{|k|>K}\boldsymbol{\mu}_{i'}^{\mathrm{T}}\mathbf{\Sigma}_{i,k}\boldsymbol{\mu}_{i'}\right|}{C_1p^{\eta}\boldsymbol{\mu}_{i'}^{\mathrm{T}}\boldsymbol{\mu}_{i'}} \leq \frac{2qc^{1-2/q}\Delta^4p^2}{(q-2)C_1p^{\eta}\boldsymbol{\mu}_{i'}^{\mathrm{T}}\boldsymbol{\mu}_{i'}}\sum_{|k|>K}\exp\{-a(1-2/q)|k|\} \\
&\leq \frac{4qc^{1-2/q}\Delta^4p^{2-\eta}}{(q-2)C_1\boldsymbol{\mu}_{i'}^{\mathrm{T}}\boldsymbol{\mu}_{i'}}\exp\{-a(1-2/q)K\} \\
&= \frac{4qc^{1-2/q}\Delta^4}{(q-2)C_1\boldsymbol{\mu}_{i'}^{\mathrm{T}}\boldsymbol{\mu}_{i'}}\exp\{-a(1-2/q)K + (2-\eta)\log p\} \to 0.
\end{aligned}$$

The proof has now been completed. □

Lemma 3. *Let $|k_1| \vee |k_2| = \max\{|k_1|, |k_2|\}$. Suppose $K = o(n)$. Under Assumption 4, as $n, p \to +\infty$, for any $i, i_1, i_2 = 1, 2$, $m > 0$ and $\boldsymbol{\mu}_{i_1}, \boldsymbol{\mu}_{i_2}$ such that $\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\mathbf{\Sigma}_{i,\infty}\boldsymbol{\mu}_{i_2} \neq \mathbf{0}$,*

$$\sum_{|k_1|,|k_2|\leq K}\left(\frac{|k_1|+|k_2|}{n_i}\right)^m \mathrm{tr}(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2}) = o\big\{\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)\big\},$$

$$\sum_{|k_1|,|k_2|\le K}\left(\frac{|k_1|\vee|k_2|}{n_i}\right)^m \mathrm{tr}(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2}) = o\big\{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)\big\},$$

$$\sum_{|k|\le K}\left(\frac{|k|}{n_i}\right)^m \boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}\boldsymbol{\mu}_{i_2} = o\left(\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,\infty}\boldsymbol{\mu}_{i_2}\right), \qquad \textit{and}$$

$$K^{-1}\sum_{|k_1|,|k_2|\le K}\left(\frac{|k_1|\vee|k_2|}{n_i}\right)^m \boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\mu}_{i_2} = o\left(\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,\infty}\boldsymbol{\mu}_{i_2}\right).$$

Proof. For any $m>0$, since

$$\left|\frac{\sum_{|k_1|,|k_2|\le K}\left(\frac{|k_1|+|k_2|}{n_i}\right)^m \mathrm{tr}(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2})}{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)}\right| \le \left(\frac{2K}{n_i}\right)^m \cdot \frac{\sum_{|k_1|,|k_2|\le K}|\mathrm{tr}(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2})|}{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)}$$

$$= \left(\frac{2K}{n_i}\right)^m \cdot \frac{\sum_{|k_1|,|k_2|\le K}|\mathrm{tr}(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2})|}{\sum_{k_1,k_2=-\infty}^{+\infty}|\mathrm{tr}(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2})|} \cdot \frac{\sum_{k_1,k_2=-\infty}^{+\infty}|\mathrm{tr}(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2})|}{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)} \quad \text{and}$$

$$\left|\frac{\sum_{|k|\le K}\left(\frac{|k|}{n_i}\right)^m \boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}\boldsymbol{\mu}_{i_2}}{\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,\infty}\boldsymbol{\mu}_{i_2}}\right| \le \left(\frac{K}{n_i}\right)^m \cdot \frac{\sum_{|k|\le K}|\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}\boldsymbol{\mu}_{i_2}|}{|\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,\infty}\boldsymbol{\mu}_{i_2}|}$$

$$= \left(\frac{K}{n_i}\right)^m \cdot \frac{\sum_{|k|\le K}|\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}\boldsymbol{\mu}_{i_2}|}{\sum_{k=-\infty}^{+\infty}|\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}\boldsymbol{\mu}_{i_2}|} \cdot \frac{\sum_{k=-\infty}^{+\infty}|\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}\boldsymbol{\mu}_{i_2}|}{|\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,\infty}\boldsymbol{\mu}_{i_2}|}.$$

Under Assumptions 4 and $K=o(n_i)$, we have as $n,p\to+\infty$,

$$\sum_{|k_1|,|k_2|\le K}\left(\frac{|k_1|+|k_2|}{n_i}\right)^m \mathrm{tr}(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2}) = o\big\{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}\boldsymbol{\Sigma}_{i,\infty})\big\} \quad \text{and}$$

$$\sum_{|k|\le K}\left(\frac{|k|}{n_i}\right)^m \boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}\boldsymbol{\mu}_{i_2} = o\big(\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,\infty}\boldsymbol{\mu}_{i_2}\big).$$

Since $|k_1|\vee|k_2| \le |k_1|+|k_2|$, we have

$$\sum_{|k_1|,|k_2|\le K}\left(\frac{|k_1|\vee|k_2|}{n_i}\right)^m \mathrm{tr}(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2}) = o\big\{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}\boldsymbol{\Sigma}_{i,\infty})\big\}.$$

The last equation holds because

$$\left|\sum_{|k_1|,|k_2|\le K}\left(\frac{|k_1|\vee|k_2|}{n_i}\right)^m \boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\mu}_{i_2}\right| \le \sum_{|k_1|,|k_2|\le K}\frac{|k_1|^m+|k_2|^m}{n_i^m}|\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\mu}_{i_2}|$$

$$\le 2K\sum_{|k|\le K}\left(\frac{|k|}{n_i}\right)^m |\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}\boldsymbol{\mu}_{i_2}| + \frac{2K^{m+1}}{n_i^m}\sum_{|k|\le K}|\boldsymbol{\mu}_{i_1}^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}\boldsymbol{\mu}_{i_2}|.$$

The proof has been completed. ∎

S1.2 Definitions 1 – 2

In the following, we provide two definitions that will be used in proving Proposition 1.

Definition 1. For two numbers $k_1, k_0 \in \mathbb{N}_+$ such that $k_1 < k_0$ and a given vector $\boldsymbol{\tau} \in \mathbb{R}^{k_0-k_1}$, a set $\mathcal{R}(\boldsymbol{\tau};\mathcal{S}) \subset \mathbb{R}^{k_1}$ is said to be the restriction domain specified by a set $\mathcal{S} \subset \mathbb{R}^{k_0}$ if and only if for any vector $\boldsymbol{s} \in \mathcal{R}(\boldsymbol{\tau};\mathcal{S})$, $(\boldsymbol{s}^{\mathrm{T}}, \boldsymbol{\tau}^{\mathrm{T}})^{\mathrm{T}} \in \mathcal{S}$.

Definition 2. For two numbers $k_2, k_0 \in \mathbb{N}_+$ such that $k_2 < k_0$, a set $\mathcal{F}(\mathcal{S}) \subset \mathbb{R}^{k_2}$ is said to be the feasible domain specified by a set $\mathcal{S} \subset \mathbb{R}^{k_0}$ if and only if for any vector $\boldsymbol{\tau} \in \mathcal{F}(\mathcal{S})$, the induced restriction domain $\mathcal{R}(\boldsymbol{\tau};\mathcal{S})$ specified by $\mathcal{S}$ satisfies $\mathcal{R}(\boldsymbol{\tau};\mathcal{S}) \neq \emptyset$.

S1.3 Proof of Proposition 1

Proof. Proposition 1 has two claims. One is the explicit form for the expectation of $T(b)$, and the other provides the variance of $T(b)$. We will deal with the two claims respectively.

(1) Derivations of E$\{T(b)\}$.

Note that $\mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}) = \mathrm{E}\{(\boldsymbol{X}_{i,t_1} - \boldsymbol{\mu}_i + \boldsymbol{\mu}_i)^{\mathrm{T}}(\boldsymbol{X}_{i,t_2} - \boldsymbol{\mu}_i + \boldsymbol{\mu}_i)\} = \mathrm{tr}(\boldsymbol{\Sigma}_{i,t_2-t_1}) + \boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\mu}_i,\ i = 1,2$. Since $\{\boldsymbol{X}_{1,t}\}$ and $\{\boldsymbol{X}_{2,t}\}$ are mutually independent, it is straightforward to show that

$$\mathrm{E}\{T(b)\} = \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|^2 + \frac{2}{n_1(b)}\sum_{k=b}^{n_1-1}(n_1-k)\mathrm{tr}(\boldsymbol{\Sigma}_{1,k}) + \frac{2}{n_2(b)}\sum_{k=b}^{n_2-1}(n_2-k)\mathrm{tr}(\boldsymbol{\Sigma}_{2,k}).$$

According to (S1.1), for any $k = 0, \pm1, \pm2, \ldots$ and $i = 1,2$,

$$|\mathrm{tr}(\boldsymbol{\Sigma}_{i,k})| \leq 2q(q-2)^{-1}c^{1-2/q}\Delta^2 p\exp\{-a(1-2/q)|k|\}.$$

If $b \geq c_1(\log n + \log p)$ for a sufficiently large $c_1 > 0$, we have for $i = 1,2$, as $n,p \to +\infty$,

$$\begin{aligned}\frac{2}{n_i(b)}\left|\sum_{k=b}^{n_i-1}(n_i-k)\mathrm{tr}(\boldsymbol{\Sigma}_{i,k})\right| &\leq \frac{4qc^{1-2/q}\Delta^2 p}{(q-2)(n_i-b)}\sum_{k=b}^{n_i-1}\exp\{-a(1-2/q)k\} \\ &\leq \frac{4qc^{1-2/q}\Delta^2 n_i}{(q-2)(n_i-b)}\exp\{-a(1-2/q)(b-1) - \log(n_i/p)\} \to 0. \qquad \text{(S1.2)}\end{aligned}$$

Thus we have $2n_i(b)^{-1}\sum_{k=b}^{n_i-1}(n_i-k)\mathrm{tr}(\boldsymbol{\Sigma}_{i,k}) = o(1)$ for $i = 1, 2$. Then

$$\mathrm{E}\{T(b)\} = (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^{\mathrm{T}}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\{1 + o(1)\}.$$

The first claim has been proved. In the next, we derive the variance of $T(b)$.

(2) Derivations of Var$\{T(b)\}$.

By the definition in (3.2), we have the decomposition $T(b) = T^{(1)}_{n_1,n_2}(b) + T^{(2)}_{n_1,n_2}(b) - 2T^{(3)}_{n_1,n_2}(b)$, where

$$T^{(1)}_{n_1,n_2}(b) = \frac{1}{n_1(b)}\sum_{|t_1-t_2|\geq b}\boldsymbol{X}^{\mathrm{T}}_{1,t_1}\boldsymbol{X}_{1,t_2},\ T^{(2)}_{n_1,n_2}(b) = \frac{1}{n_2(b)}\sum_{|t_1-t_2|\geq b}\boldsymbol{X}^{\mathrm{T}}_{2,t_1}\boldsymbol{X}_{2,t_2}\ \text{ and}$$

$$T^{(3)}_{n_1,n_2}(b) = \frac{1}{n_1 n_2}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}\boldsymbol{X}^{\mathrm{T}}_{1,t_1}\boldsymbol{X}_{2,t_2}.$$

Thus we have

$$\begin{aligned}\mathrm{Var}\{T(b)\} =& \mathrm{Var}\{T^{(1)}_{n_1,n_2}(b)\} + \mathrm{Var}\{T^{(2)}_{n_1,n_2}(b)\} + 4\mathrm{Var}\{T^{(3)}_{n_1,n_2}(b)\} + 2\mathrm{Cov}\{T^{(1)}_{n_1,n_2}(b), T^{(2)}_{n_1,n_2}(b)\}\\ &- 4\mathrm{Cov}\{T^{(1)}_{n_1,n_2}(b), T^{(3)}_{n_1,n_2}(b)\} - 4\mathrm{Cov}\{T^{(1)}_{n_1,n_2}(b), T^{(3)}_{n_1,n_2}(b)\}.\end{aligned}$$

Since $\{\boldsymbol{X}_{1,t} : t = 1, 2, \ldots, n_1\}$ and $\{\boldsymbol{X}_{2,t} : t = 1, 2, \ldots, n_2\}$ are assumed to be independent of each other, $\mathrm{Cov}\{T^{(1)}_{n_1,n_2}(b), T^{(2)}_{n_1,n_2}(b)\} = 0$. Below, we derive $\mathrm{Var}\{T^{(1)}_{n_1,n_2}(b)\}$, $\mathrm{Var}\{T^{(2)}_{n_1,n_2}(b)\}$, $\mathrm{Var}\{T^{(3)}_{n_1,n_2}(b)\}$, $\mathrm{Cov}\{T^{(1)}_{n_1,n_2}(b), T^{(3)}_{n_1,n_2}(b)\}$ and $\mathrm{Cov}\{T^{(1)}_{n_1,n_2}(b), T^{(3)}_{n_1,n_2}(b)\}$, respectively.

We take the derivations of $\mathrm{Var}\{T^{(1)}_{n_1,n_2}(b)\}$ as example. According to the definition of $T^{(1)}_{n_1,n_2}(b)$, $\mathrm{Var}\{T^{(1)}_{n_1,n_2}(b)\}$ can be decomposed into two parts as follows,

$$\mathrm{Var}\{T^{(1)}_{n_1,n_2}(b)\} = \mathrm{Var}\left\{\frac{1}{n_1(b)}\sum_{|t_1-t_2|\geq b}\boldsymbol{X}^{\mathrm{T}}_{1,t_1}\boldsymbol{X}_{1,t_2}\right\} \tag{S1.3}$$

$$= \frac{1}{n_1(b)^2}\sum_{|t_1-t_2|\geq b}\sum_{|t_3-t_4|\geq b}\mathrm{Cov}(\boldsymbol{X}^{\mathrm{T}}_{1,t_1}\boldsymbol{X}_{1,t_2}, \boldsymbol{X}^{\mathrm{T}}_{1,t_3}\boldsymbol{X}_{1,t_4})$$

$$= \frac{4}{n_1(b)^2}\sum_{t_1-t_2\geq b}\sum_{t_3-t_4\geq b}\left\{\mathrm{E}(\boldsymbol{X}^{\mathrm{T}}_{1,t_1}\boldsymbol{X}_{1,t_2}\boldsymbol{X}^{\mathrm{T}}_{1,t_3}\boldsymbol{X}_{1,t_4}) - \mathrm{E}(\boldsymbol{X}^{\mathrm{T}}_{1,t_1}\boldsymbol{X}_{1,t_2})\mathrm{E}(\boldsymbol{X}^{\mathrm{T}}_{1,t_3}\boldsymbol{X}_{1,t_4})\right\}$$

$=: V_1 - V_2.$

We shall deal with V_1 and V_2 in the following respectively.

To deal with V_1, firstly we rewrite V_1 as

$$V_1 = \frac{4}{n_1(b)^2} \sum_{t_1-t_2\geq b} \sum_{t_3-t_4\geq b} \mathrm{tr}\big\{\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}})\big\}.$$

Notice that $\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}})$ can be approximated by different formulas according to the distances between $(t_1, t_2)^{\mathrm{T}}$ and $(t_3, t_4)^{\mathrm{T}}$. For example, if t_1 and t_3 are close to each other and so t_2 and t_4 are, $\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}})$ should be equivalent to $\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}})$; if t_1 and t_3 are close to each other while t_2 and t_4 are far away, $\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}})$ should be equivalent to $\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{1,t_2})\mathrm{E}(\boldsymbol{X}_{1,t_4}^{\mathrm{T}})$. According to the distances between $(t_1, t_2)^{\mathrm{T}}$ and $(t_3, t_4)^{\mathrm{T}}$, there are totally six scenarios as follows.

- $t_1,\ t_3$ are close and $t_2,\ t_4$ are close:

$$\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}}) \approx \mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}});$$

- $t_1,\ t_3$ are close and $t_2,\ t_4$ are far away:

$$\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}}) \approx \mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{1,t_2})\mathrm{E}(\boldsymbol{X}_{1,t_4}^{\mathrm{T}});$$

- $t_1,\ t_3$ are far away and $t_2,\ t_4$ are close:

$$\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}}) \approx \mathrm{E}(\boldsymbol{X}_{1,t_3})\mathrm{E}(\boldsymbol{X}_{1,t_1}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}});$$

- $t_2,\ t_3$ are close ($t_1,\ t_3$ and $t_2,\ t_4$ must be far away):

$$\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}}) \approx \mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_2}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{1,t_1})\mathrm{E}(\boldsymbol{X}_{1,t_4}^{\mathrm{T}});$$

- $t_1,\ t_4$ are close ($t_1,\ t_3$ and $t_2,\ t_4$ must be far away):

$$\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}}) \approx \mathrm{E}(\boldsymbol{X}_{1,t_3})\mathrm{E}(\boldsymbol{X}_{1,t_2}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{1,t_1}\boldsymbol{X}_{1,t_4}^{\mathrm{T}});$$

- $t_1,\ t_2,\ t_3,\ t_4$ are all far away to each other:

$$\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}}) \approx \mathrm{E}(\boldsymbol{X}_{1,t_3}\}\mathrm{E}(\boldsymbol{X}_{1,t_1}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{1,t_2})\mathrm{E}(\boldsymbol{X}_{1,t_4}^{\mathrm{T}}).$$

To quantify "close" and "far away", we introduce a distance parameter K satisfying $b/3 \le K < b/2$. Define the set of the feasible time points in the summation of V_1,

$$\mathcal{S}_0 = \{\boldsymbol{t} = (t_1, t_2, t_3, t_4)^{\mathrm{T}} : t_1 - t_2 \ge b, t_3 - t_4 \ge b, 1 \le t_1, t_2, t_3, t_4 \le n_1\}. \tag{S1.4}$$

We can decompose the set $\mathcal{S}_0$ into the non-overlapping union of six subsets, according to the distances between $t_1,\ t_3$ and $t_2,\ t_4$:

$$\mathcal{S}_1 = \{\boldsymbol{t} = (t_1, t_2, t_3, t_4)^{\mathrm{T}} : |t_1 - t_3| \le K, |t_2 - t_4| \le K\} \cap \mathcal{S}_0, \tag{S1.5}$$

$$\mathcal{S}_2 = \{\boldsymbol{t} = (t_1, t_2, t_3, t_4)^{\mathrm{T}} : |t_1 - t_3| \le K, |t_2 - t_4| > K\} \cap \mathcal{S}_0, \tag{S1.6}$$

$$\mathcal{S}_3 = \{\boldsymbol{t} = (t_1, t_2, t_3, t_4)^{\mathrm{T}} : |t_1 - t_3| > K, |t_2 - t_4| \le K\} \cap \mathcal{S}_0, \tag{S1.7}$$

$$\mathcal{S}_4 = \{\boldsymbol{t} = (t_1, t_2, t_3, t_4)^{\mathrm{T}} : |t_2 - t_3| \le K\} \cap \mathcal{S}_0, \tag{S1.8}$$

$$\mathcal{S}_5 = \{\boldsymbol{t} = (t_1, t_2, t_3, t_4)^{\mathrm{T}} : |t_1 - t_4| \le K\} \cap \mathcal{S}_0 \quad \text{and} \tag{S1.9}$$

$$\mathcal{S}_6 = \{\boldsymbol{t} = (t_1, t_2, t_3, t_4)^{\mathrm{T}} : |t_i - t_j| > K, i = 1, 2, j = 3, 4\} \cap \mathcal{S}_0. \tag{S1.10}$$

Then $\mathcal{S}_0 = \bigcup_{h=1}^{6} \mathcal{S}_h$. Furthermore, V_1 can be decomposed by $V_1 = \sum_{h=1}^{6} V_1^{(h)}$, where

$$V_1^{(h)} = \frac{4}{n_1(b)^2} \sum_{\boldsymbol{t}\in\mathcal{S}_h} \mathrm{tr}\big\{\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}})\big\}, \quad h = 1, 2, \ldots, 6.$$

For $h = 1, 2, \ldots, 6$, $V_1^{(h)}$ can be approximated respectively by

$$\tilde{V}_1^{(1)} = \frac{4}{n_1(b)^2} \sum_{\boldsymbol{t}\in\mathcal{S}_1} \mathrm{tr}\big\{(\boldsymbol{\Sigma}_{1,t_3-t_1} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})(\boldsymbol{\Sigma}_{1,t_2-t_4} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})\big\},$$

$$\tilde{V}_1^{(2)} = \frac{4}{n_1(b)^2} \sum_{\boldsymbol{t}\in\mathcal{S}_2} \mathrm{tr}\big\{(\boldsymbol{\Sigma}_{1,t_3-t_1} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})\boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}}\big\},$$

$$\tilde{V}_1^{(3)} = \frac{4}{n_1(b)^2} \sum_{\boldsymbol{t}\in\mathcal{S}_3} \mathrm{tr}\big\{\boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}}(\boldsymbol{\Sigma}_{1,t_2-t_4} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})\big\},$$

$$\tilde{V}_1^{(4)} = \frac{4}{n_1(b)^2} \sum_{\boldsymbol{t}\in\mathcal{S}_4} \mathrm{tr}\big\{(\boldsymbol{\Sigma}_{1,t_3-t_2} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})\boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}}\big\},$$

$$\tilde{V}_1^{(5)} = \frac{4}{n_1(b)^2}\sum_{\boldsymbol{t}\in\mathcal{S}_5}\mathrm{tr}\big\{\boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}}(\boldsymbol{\Sigma}_{1,t_1-t_4}+\boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})\big\}, \text{ and}$$

$$\tilde{V}_1^{(6)} = \frac{4}{n_1(b)^2}\sum_{\boldsymbol{t}\in\mathcal{S}_6}\mathrm{tr}\big(\boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}}\big).$$

Define $\tilde{V}_1 = \sum_{h=1}^{6}\tilde{V}_1^{(h)}$ as the candidate to approximate V_1.

The explicit form of V_1 is derived in two steps. The first step is to derive the difference between V_1 and $\tilde{V}_1$, and the second step is to derive the explicit form of $\tilde{V}_1$. By Lemma 1, we have

$$V_1^{(1)} - \tilde{V}_1^{(1)} = \frac{4}{n_1(b)^2}\sum_{\boldsymbol{t}\in\mathcal{S}_1}\sum_{j_1,j_2=1}^{p}\mathrm{Cov}(X_{1,t_3,j_1}X_{1,t_1,j_2}, X_{1,t_2,j_2}X_{1,t_4,j_1}).$$

For any two constants b_1 and b_2, let $b_1\wedge b_2 = \min\{b_1,b_2\}$. Under Assumption 1 with $q>4$ and Assumption 3, by the Davydov's inequality (Davydov, 1968),

$$\begin{aligned}|V_1^{(1)} - \tilde{V}_1^{(1)}| &\le \frac{4}{n_1(b)^2}\sum_{\boldsymbol{t}\in\mathcal{S}_1}\sum_{j_1,j_2=1}^{p}\frac{2q\Delta^4}{q-4}\beta_1^x(b\wedge|t_1-t_4|\wedge|t_2-t_3|)^{1-4/q}\\ &\le \frac{8q\Delta^4c^{1-4/q}p^2}{(q-4)n_1(b)^2}\exp\{-a(1-4/q)b/2\}\sum_{\boldsymbol{t}\in\mathcal{S}_1}1 \le \frac{2q\Delta^4c^{1-4/q}p^2}{q-4}\exp\{-a(1-4/q)b/2\}.\end{aligned}$$

Analogously, under Assumption 3 and the moment condition and by Lemma 1, we can obtain the upper bound of $V_1^{(2)}-\tilde{V}_1^{(2)},\dots,V_1^{(6)}-\tilde{V}_1^{(6)}$ by

$$\begin{aligned}
|V_1^{(2)} - \tilde{V}_1^{(2)}| \le& \frac{2q\Delta^4c^{1-4/q}p^2}{q-4}\exp\left\{-\frac{ab(q-4)}{2q}\right\} + \frac{2q(3q-2)\Delta^4c^{1-2/q}p^2}{(q-2)^2}\exp\left\{-\frac{ab(q-2)}{3q}\right\},\\
|V_1^{(3)} - \tilde{V}_1^{(3)}| \le& \frac{2q\Delta^4c^{1-4/q}p^2}{q-4}\exp\left\{-\frac{ab(q-4)}{2q}\right\} + \frac{2q(3q-2)\Delta^4c^{1-2/q}p^2}{(q-2)^2}\exp\left\{-\frac{ab(q-2)}{3q}\right\},\\
|V_1^{(4)} - \tilde{V}_1^{(4)}| \le& \frac{2q\Delta^4c^{1-4/q}p^2}{q-4}\exp\left\{-\frac{ab(q-4)}{2q}\right\} + \frac{2q(3q-2)\Delta^4c^{1-2/q}p^2}{(q-2)^2}\exp\left\{-\frac{3ab(q-2)}{2q}\right\},\\
|V_1^{(5)} - \tilde{V}_1^{(5)}| \le& \frac{2q\Delta^4c^{1-4/q}p^2}{q-4}\exp\left\{-\frac{ab(q-4)}{2q}\right\} + \frac{2q(3q-2)\Delta^4c^{1-2/q}p^2}{(q-2)^2}\exp\left\{-\frac{3ab(q-2)}{2q}\right\},\\
|V_1^{(6)} - \tilde{V}_1^{(6)}| \le& \frac{2q\Delta^4c^{1-4/q}p^2}{q-4}\exp\left\{-\frac{ab(q-4)}{2q}\right\} + \frac{4q^2\Delta^4c^{2-4/q}p^2}{(q-2)^2}\exp\left\{-\frac{ab(q-2)}{q}\right\}\\
&+\frac{4q\Delta^4c^{1-2/q}p^2}{q-2}\exp\left\{-\frac{ab(q-2)}{2q}\right\}.
\end{aligned}$$

Since $|V_1 - \tilde{V}_1| \leq \sum_{h=1}^{6} |V_1^{(h)} - \tilde{V}_1^{(h)}|$, we have

$$
\begin{aligned}
|V_1 - \tilde{V}_1| \leq & \frac{12q\Delta^4 c^{1-4/q} p^2}{q-4} \exp\{-a(1-4/q)b/2\} \\
& + \frac{4q(3q-2)\Delta^4 c^{1-2/q} p^2}{(q-2)^2} \exp\{-a(1-2/q)b/3\} \\
& + \frac{4q(3q-2)\Delta^4 c^{1-2/q} p^2}{(q-2)^2} \exp\{-3a(1-2/q)b/2\} \\
& + \frac{4q^2\Delta^4 c^{2-4/q} p^2}{(q-2)^2} \exp\{-a(2-4/q)b/2\} \\
& + \frac{4q\Delta^4 c^{1-2/q} p^2}{q-2} \exp\{-a(1-2/q)b/2\}.
\end{aligned} \tag{S1.11}
$$

We will compare the right hand side of the above inequality with the leading order term of $\tilde{V}_1$ later.

Next, we derive the explicit form of the leading order term of $\tilde{V}_1$, which can be achieved by considering each $\tilde{V}_1^{(h)}$ for $h = 1, 2, \ldots, 6$, respectively. We take the derivations for $\tilde{V}_1^{(1)}$ as an example. We re-parametrize the set $\mathcal{S}_1$ by define the following transformation

$$
\boldsymbol{t}' = \begin{pmatrix} t_1' \\ t_2' \\ t_3' \\ t_4' \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 - t_3 \\ t_3 - t_1 \\ t_4 - t_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \boldsymbol{G}_1 \boldsymbol{t},
$$

where $\boldsymbol{G}_1$ is invertible. Hence the projection induced by $\boldsymbol{G}_1$ from $\mathbb{R}^4$ to $\mathbb{R}^4$ is one-to-one such that the rows of $\boldsymbol{G}_1$ can be served as another basis for $\mathbb{R}^4$.

Recall that

$$
\mathcal{S}_1 = \{\boldsymbol{t} = (t_1, t_2, t_3, t_4)^{\mathrm{T}} : t_1 - t_2, t_3 - t_4 \geq b, |t_1 - t_3|, |t_2 - t_4| \leq K, 1 \leq t_1, t_2, t_3, t_4 \leq n_1\}.
$$

Under the new parametrization, we define

$$
\mathcal{S}_1' = \{\boldsymbol{t}' = (t_1', t_2', t_3', t_4')^{\mathrm{T}} : -t_3' - t_2' \geq b, -t_2' - t_4' \geq b, |t_3'| \leq K, |t_4'| \leq K, 1 \leq t_1' \leq n_1,
$$

$$1 \le t'_1 + t'_3 \le n_1, 1 \le t'_1 + t'_2 + t'_3 \le n_1, 1 \le t'_1 + t'_2 + t'_3 + t'_4 \le n_1\}$$

$$= \{\boldsymbol{t'} = (t'_1, t'_2, t'_3, t'_4)^{\mathrm{T}} : |t'_3| \le K, |t'_4| \le K, 1 - 0 \wedge t'_3 \le t'_1 \le n_1 - 0 \vee t'_3, t'_2 \le -b - t'_3 \vee t'_4,$$

$$1 - t'_3 - 0 \wedge t'_4 \le t'_1 + t'_2 \le n_1 - t'_3 - 0 \vee t'_4\}.$$

Then $\tilde{V}_1^{(1)}$ can be represented by

$$\begin{aligned}
\tilde{V}_1^{(1)} &= \frac{4}{n_1(b)^2} \sum_{\boldsymbol{t} \in \mathcal{S}_1} \mathrm{tr}\big\{(\boldsymbol{\Sigma}_{1,t_3-t_1} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})(\boldsymbol{\Sigma}_{1,t_2-t_4} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})\big\} \\
&= \frac{4}{n_1(b)^2} \sum_{\boldsymbol{t'} \in \mathcal{S}'_1} \mathrm{tr}\big\{(\boldsymbol{\Sigma}_{1,t'_3} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})(\boldsymbol{\Sigma}_{1,-t'_4} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})\big\} \\
&= \frac{4}{n_1(b)^2} \sum_{(t'_3,t'_4)^{\mathrm{T}} \in \mathcal{F}(\mathcal{S}'_1)} \sum_{(t'_1,t'_2)^{\mathrm{T}} \in \mathcal{R}(t'_3,t'_4;\mathcal{S}'_1)} \mathrm{tr}\big\{(\boldsymbol{\Sigma}_{1,t'_3} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})(\boldsymbol{\Sigma}_{1,-t'_4} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1)\big\} \\
&= \frac{4}{n_1(b)^2} \sum_{(t'_3,t'_4)^{\mathrm{T}} \in \mathcal{F}(\mathcal{S}'_1)} |\mathcal{R}(t'_3, t'_4; \mathcal{S}'_1)| \mathrm{tr}\big\{(\boldsymbol{\Sigma}_{1,t'_3} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})(\boldsymbol{\Sigma}_{1,-t'_4} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})\big\},
\end{aligned}$$

where we call $\mathcal{F}(\mathcal{S}'_1)$ the feasible domain of $(t'_3, t'_4)^{\mathrm{T}}$ specified by the set $\mathcal{S}'_1$ and $\mathcal{R}(t'_3, t'_4; \mathcal{S}'_1)$ the restriction domain of $(t'_1, t'_2)^{\mathrm{T}}$ given $(t'_3, t'_4)^{\mathrm{T}}$ specified by the set $\mathcal{S}'_1$, and $|\cdot|$ represents the cardinality of one set. The definitions of the restriction domain and the feasible domain are provided in Definitions 1 and 2. According to Definitions 1 and 2, for given $(t'_3, t'_4)^{\mathrm{T}}$, the restriction domain $\mathcal{R}(t'_3, t'_4; \mathcal{S}'_1)$ specified by $\mathcal{S}'_1$ is the set of all the points $(t'_1, t'_2)^{\mathrm{T}}$ such that $(t'_1, t'_2, t'_3, t'_4)^{\mathrm{T}}$ in $\mathcal{S}'_1$, while the feasible domain $\mathcal{F}(\mathcal{S}'_1)$ specified by $\mathcal{S}'_1$ is the set of all the points $(t'_3, t'_4)^{\mathrm{T}}$ such that $\mathcal{R}(t'_3, t'_4; \mathcal{S}'_1) \neq \emptyset$.

From the above discussion, we can see that to calculate $\tilde{V}_1^{(1)}$ is equivalent to search for the feasible domain $\mathcal{F}(\mathcal{S}'_1)$ for $(t'_3, t'_4)^{\mathrm{T}}$ and count the number of pairs $(t'_1, t'_2)^{\mathrm{T}}$ that fall into $\mathcal{R}(t'_3, t'_4; \mathcal{S}'_1)$ given $(t'_3, t'_4)^{\mathrm{T}} \in \mathcal{F}(\mathcal{S}'_1)$. According to the definition of $\mathcal{S}'_1$, we shall discuss the form of the feasible domain $\mathcal{F}(\mathcal{S}'_1)$ by comparing t'_3 and t'_4 with 0, respectively, leading to four subsets $\mathcal{F}_h(\mathcal{S}_1')$ for $h = 1, 2, 3, 4$ where

$$\mathcal{F}_1(\mathcal{S}_1') \subset \{(t_3', t_4')^{\mathrm{T}} : t_3' \geq 0, -t_4' \geq 0\},\ \mathcal{F}_2(\mathcal{S}_1') \subset \{(t_3', t_4')^{\mathrm{T}} : t_3' > 0, -t_4' < 0\},$$

$$\mathcal{F}_3(\mathcal{S}_1') \subset \{(t_3', t_4')^{\mathrm{T}} : t_3' < 0, -t_4' > 0\} \text{ and } \mathcal{F}_4(\mathcal{S}_1') \subset \{(t_3', t_4')^{\mathrm{T}} \neq (0,0)^{\mathrm{T}} : t_3' \leq 0, -t_4' \leq 0\}$$

such that $\mathcal{F}_{h_1}(\mathcal{S}_1') \cap \mathcal{F}_{h_2}(\mathcal{S}_1') = \emptyset$ for $h_1 \neq h_2$ and $\mathcal{F}(\mathcal{S}_1') = \cup_{h=1}^4 \mathcal{F}_h(\mathcal{S}_1')$. For $(t_3', t_4')^{\mathrm{T}}$ in different subsets of $\mathcal{F}(\mathcal{S}_1')$, the corresponding restriction domain can have different forms. It is mathematically allowable to use a uniform notation $\mathcal{R}(t_3', t_4'; \mathcal{S}_1')$ for $(t_3', t_4')^{\mathrm{T}}$ in different subsets of $\mathcal{F}(\mathcal{S}_1')$. However, to make the notation homologous, we may as well denote $\mathcal{R}(t_3', t_4'; \mathcal{S}_1')$ by $\mathcal{R}_h(t_3', t_4'; \mathcal{S}_1')$ for $(t_3', t_4')^{\mathrm{T}} \in \mathcal{F}_h(\mathcal{S}_1')$ for $h = 1, 2, 3, 4$.

(a) If $t_3' \geq 0, -t_4' \geq 0$, the restriction domain $\mathcal{R}_1(t_3', t_4'; \mathcal{S}_1')$ for given $(t_3', t_4')^{\mathrm{T}} \in \mathcal{F}_1(\mathcal{S}_1')$ is

$$\mathcal{R}_1(t_3', t_4'; \mathcal{S}_1') = \{(t_1', t_2')^{\mathrm{T}} : 1 \leq t_1' \leq n_1 - t_3', t_2' \leq -b - t_3', 1 - t_3' - t_4' \leq t_1' + t_2' \leq n_1 - t_3'\}.$$

By searching for the pairs $(t_3', t_4')^{\mathrm{T}}$ that make $\mathcal{R}_1(t_3', t_4'; \mathcal{S}_1')$ nonempty, we can obtain the corresponding feasible domain $\mathcal{F}_1(\mathcal{S}_1')$, say

$$\mathcal{F}_1(\mathcal{S}_1') = \{(t_3', t_4')^{\mathrm{T}} : 1 \leq n_1 - t_3', 1 - t_4' \leq n_1, 1 - n_1 - t_4' \leq -b - t_3', 0 \leq t_3', -t_4' \leq K\}.$$

Under the assumption $b = o(n^{1/4})$, we have as $n, p \to +\infty$,

$$\mathcal{F}_1(\mathcal{S}_1') = \{(t_3', t_4')^{\mathrm{T}} : 0 \leq t_3' \leq K, 0 \leq -t_4' \leq K\}.$$

For any $(t_3', t_4')^{\mathrm{T}} \in \mathcal{F}_1(\mathcal{S}_1')$, the number of pairs $(t_1', t_2')^{\mathrm{T}}$ that fall into $\mathcal{R}_1(t_3', t_4'; \mathcal{S}_1')$ is

$$|\mathcal{R}_1(t_3', t_4'; \mathcal{S}_1')| = \frac{1}{2}(n_1 - b - t_3' + t_4')(n_1 - b + 1 - t_3' + t_4').$$

(b) If $t_3' > 0, -t_4' < 0$, the restriction domain $\mathcal{R}_2(t_3', t_4'; \mathcal{S}_1')$ given $(t_3', t_4')^{\mathrm{T}} \in \mathcal{F}_2(\mathcal{S}_1')$ is

$$\mathcal{R}_2(t_3', t_4'; \mathcal{S}_1') = \{(t_1', t_2')^{\mathrm{T}} : 1 \leq t_1' \leq n_1 - t_3', t_2' \leq -b - t_3' \vee t_4', 1 \leq t_1' + t_2' + t_3' \leq n_1 - t_4'\}.$$

The corresponding feasible domain $\mathcal{F}_2(\mathcal{S}_1')$ is

$$\mathcal{F}_2(\mathcal{S}_1') = \{(t_3', t_4')^{\mathrm{T}} : 1 \leq n_1 - t_3', 1 \leq n_1 - t_4', 1 - n_1 \leq -b - t_3' \vee t_4', 0 < t_3', t_4' \leq K\}.$$

Under the assumption $b = o(n^{1/4})$, we have as $n, p \to +\infty$,

$$\mathcal{F}_2(\mathcal{S}_1') = \{(t_3', t_4')^{\mathrm{T}} : 0 < t_3' \leq K, -K \leq -t_4' < 0\}.$$

For any $(t_3', t_4')^{\mathrm{T}} \in \mathcal{F}_2(\mathcal{S}_1')$, the number of pairs $(t_1', t_2')^{\mathrm{T}}$ that fall into $\mathcal{R}_2(t_3', t_4'; \mathcal{S}_1')$ is

$$|\mathcal{R}_2(t_3', t_4'; \mathcal{S}_1')| = \frac{1}{2}(n_1 - b - t_3' \vee t_4')(n_1 - b + 1 - t_3' \vee t_4').$$

(c) If $t_3' < 0, -t_4' > 0$, the restriction domain $\mathcal{R}_3(t_3', t_4'; \mathcal{S}_1')$ given $(t_3', t_4')^{\mathrm{T}} \in \mathcal{F}_3(\mathcal{S}_1')$ is

$$\mathcal{R}_3(t_3', t_4'; \mathcal{S}_1') = \{(t_1', t_2')^{\mathrm{T}} : 1 - t_3' \leq t_1' \leq n_1, t_2' \leq -b - t_3' \vee t_4', 1 - t_4' \leq t_1' + t_2' + t_3' \leq n_1\}.$$

The feasible domain $\mathcal{F}_3(\mathcal{S}_1')$ correspondent with $\mathcal{R}_3(t_3', t_4'; \mathcal{S}_1')$ is

$$\mathcal{F}_3(\mathcal{S}_1') = \{(t_3', t_4')^{\mathrm{T}} : (1 - t_3') \vee (1 - t_4') \leq n_1, 1 - n_1 \leq -b + t_3' \wedge t_4', -K \leq t_3', t_4' < 0\}.$$

Under the assumption $b = o(n^{1/4})$, we have as $n, p \to +\infty$,

$$\mathcal{F}_3(\mathcal{S}_1') = \{(t_3', t_4')^{\mathrm{T}} : -K \leq t_3' < 0, 0 < -t_4' \leq K\}.$$

For any $(t_3', t_4')^{\mathrm{T}} \in \mathcal{F}_3(\mathcal{S}_1')$, the number of pairs $(t_1', t_2')^{\mathrm{T}}$ that fall into $\mathcal{R}_3(t_3', t_4'; \mathcal{S}_1')$ is

$$|\mathcal{R}_3(t_3', t_4'; \mathcal{S}_1')| = \frac{1}{2}\{n_1 - b - (-t_3') \vee (-t_4')\}\{n_1 - b + 1 - (-t_3') \vee (-t_4')\}.$$

(d) If $t_3' \leq 0, -t_4' \leq 0$ and $(t_3', t_4')^{\mathrm{T}} \neq (0, 0)^{\mathrm{T}}$, the restriction domain $\mathcal{R}_4(t_3', t_4'; \mathcal{S}_1')$ given $(t_3', t_4')^{\mathrm{T}} \in \mathcal{F}_4(\mathcal{S}_1')$ is

$$\mathcal{R}_4(t_3', t_4'; \mathcal{S}_1') = \{(t_1', t_2')^{\mathrm{T}} : 1 - t_3' \leq t_1' \leq n_1, t_2' \leq -b - t_4', 1 - t_3' \leq t_1' + t_2' \leq n_1 - t_3' - t_4'\}.$$

By searching for the pairs $(t_3', t_4')^{\mathrm{T}}$ that make $\mathcal{R}_4(t_3', t_4'; \mathcal{S}_1')$ nonempty, we can obtain the feasible domain $\mathcal{F}_4(\mathcal{S}_1')$ correspondent with $\mathcal{R}_4(t_3', t_4'; \mathcal{S}_1')$, say

$$\mathcal{F}_4(\mathcal{S}_1') = \{(t_3', t_4')^{\mathrm{T}} \neq (0, 0)^{\mathrm{T}} : 1 - t_3' \leq n_1, 1 - n_1 + t_4' \leq (t_3' - b) \wedge 0, -K \leq t_3', -t_4' \leq 0\}.$$

Under the assumption $b = o(n^{1/4})$, we have as $n, p \to +\infty$,

$$\mathcal{F}_4(\mathcal{S}_1') = \{(t_3', t_4')^{\mathrm{T}} \neq (0, 0)^{\mathrm{T}} : -K \leq t_3' \leq 0, -K \leq -t_4' \leq 0\}.$$

For any $(t_3', t_4')^{\mathrm{T}} \in \mathcal{F}_4(\mathcal{S}_1')$, the number of pairs $(t_1', t_2')^{\mathrm{T}}$ that fall into $\mathcal{R}_4(t_3', t_4'; \mathcal{S}_1')$ is

$$|\mathcal{R}_4(t_3', t_4'; \mathcal{S}_1')| = \frac{1}{2}(n_1 - b + t_3' - t_4')(n_1 - b + 1 + t_3' - t_4').$$

From (a)(b)(c)(d), the feasible domain for $(t_3', t_4')^{\mathrm{T}}$ is

$$\mathcal{F}(\mathcal{S}_1') = \cup_{h=1}^{4}\mathcal{F}_h(\mathcal{S}_1') = \{(t_3', t_4')^{\mathrm{T}} : -K \le t_3' \le K, -K \le t_4' \le K\},$$

and we have

$$\begin{aligned}\tilde{V}_1^{(1)} &= \frac{4}{n_1(b)^2}\sum_{h=1}^{4}\sum_{(k_1,k_2)^{\mathrm{T}}\in\mathcal{F}_h(\mathcal{S}_1')} |\mathcal{R}_h(k_1, k_2; \mathcal{S}_1')|\mathrm{tr}\big\{(\boldsymbol{\Sigma}_{1,k_1} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})(\boldsymbol{\Sigma}_{1,-k_2} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})\big\} \\ &= \frac{2}{n_1(b)}\sum_{k_1=-K}^{K}\sum_{k_2=-K}^{K} C(n_1, b, k_1, k_2)\mathrm{tr}\big\{(\boldsymbol{\Sigma}_{1,k_1} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})(\boldsymbol{\Sigma}_{1,k_2} + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^{\mathrm{T}})\big\}.\end{aligned}$$

where

$$C(n_1, b, k_1, k_2) = \begin{cases} \left(1 - \frac{|k_1|+|k_2|}{n_1-b}\right)\left(1 - \frac{|k_1|+|k_2|}{n_1-b+1}\right), & \text{if } (k_1, -k_2)^{\mathrm{T}} \in \mathcal{F}_1(\mathcal{S}_1') \cup \mathcal{F}_4(\mathcal{S}_1'), \\ \left(1 - \frac{|k_1|\vee|k_2|}{n_1-b}\right)\left(1 - \frac{|k_1|\vee|k_2|}{n_1-b+1}\right), & \text{if } (k_1, -k_2)^{\mathrm{T}} \in \mathcal{F}_2(\mathcal{S}_1') \cup \mathcal{F}_3(\mathcal{S}_1'). \end{cases}$$

The leading order term of $\tilde{V}_1^{(1)}$ in the above can be simplified furthermore. By Lemmas 2 and 3, we have as $n, p \to +\infty$,

$$\tilde{V}_1^{(1)} = \left\{\frac{2}{n_1^2}\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\Sigma}_{1,\infty}) + \frac{8K}{n_1^2}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1\right\}\{1 + o(1)\} + \frac{4}{n_1(b)}|\mathcal{S}_1|(\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1)^2.$$

Analogously, by using Lemmas 1 – 3, it can be derived that

$$\begin{aligned}\tilde{V}_1^{(2)} &= \left\{\frac{4}{3n_1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1\right\}\{1 + o(1)\} + \frac{4}{n_1(b)}|\mathcal{S}_2|(\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1)^2, \\ \tilde{V}_1^{(3)} &= \left\{\frac{4}{3n_1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1\right\}\{1 + o(1)\} + \frac{4}{n_1(b)}|\mathcal{S}_3|(\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1)^2, \\ \tilde{V}_1^{(4)} &= \left\{\frac{2}{3n_1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1\right\}\{1 + o(1)\} + \frac{4}{n_1(b)}|\mathcal{S}_4|(\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1)^2, \\ \tilde{V}_1^{(5)} &= \left\{\frac{2}{3n_1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1\right\}\{1 + o(1)\} + \frac{4}{n_1(b)}|\mathcal{S}_5|(\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1)^2 \quad \text{and} \\ \tilde{V}_1^{(6)} &= \frac{4}{n_1(b)}|\mathcal{S}_6|(\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1)^2.\end{aligned}$$

To sum up the results, we have

$$\tilde{V}_1 = \sum_{h=1}^{6} \tilde{V}_1^{(h)} = \left\{ \frac{2}{n_1^2}\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}^2) + \frac{4}{n_1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1 \right\} \{1 + o(1)\} + (\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1)^2.$$

Next, we compare $\tilde{V}_1$ and $V_1 - \tilde{V}_1$. By (S1.11), if $b \geq c_1(\log p + \log n)$ for some sufficiently large $c_1 > 0$, we have for $\boldsymbol{\mu}_1 \neq \mathbf{0}$, as $n, p \to +\infty$,

$$\left| \frac{V_1 - \tilde{V}_1}{2n_1^{-2}\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}^2)} \right| \leq \frac{|V_1 - \tilde{V}_1|}{2n_1^{-2}C_1^2 p^{1+2\eta}} \to 0 \quad \text{and} \quad \left| \frac{V_1 - \tilde{V}_1}{4n_1^{-1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1} \right| \leq \frac{|V_1 - \tilde{V}_1|}{4n_1^{-1}C_1 p^{\eta}|\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1|} \to 0.$$

Thus as $n, p \to +\infty$,

$$V_1 - \tilde{V}_1 = o\big\{2n_1^{-2}\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}^2)\big\} \quad \text{and} \quad V_1 - \tilde{V}_1 = o(4n_1^{-1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1).$$

Then as $n, p \to +\infty$,

$$V_1 = \left\{ \frac{2}{n_1^2}\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\Sigma}_{1,\infty}) + \frac{4}{n_1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1 \right\} \{1 + o(1)\} + (\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1)^2.$$

According to (S1.3), we still need to consider V_2. By the definition of V_2,

$$\begin{aligned} V_2 &= \frac{4}{n_1(b)^2} \sum_{t_1 - t_2 \geq b} \sum_{t_3 - t_4 \geq b} \mathrm{E}(\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2})\mathrm{E}(\boldsymbol{X}_{1,t_3}^{\mathrm{T}}\boldsymbol{X}_{1,t_4}) \\ &= \left\{ \frac{2}{n_1(b)} \sum_{k=b}^{n_1-1} (n_1 - k)\mathrm{tr}(\boldsymbol{\Sigma}_{1,k}) + \boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1 \right\}^2 \\ &= \left\{ \frac{2}{n_1(b)} \sum_{k=b}^{n_1-1} (n_1 - k)\mathrm{tr}(\boldsymbol{\Sigma}_{1,k}) \right\}^2 + \frac{4\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1}{n_1(b)} \sum_{k=b}^{n_1-1} (n_1 - k)\mathrm{tr}(\boldsymbol{\Sigma}_{1,k}) + (\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1)^2. \end{aligned}$$

By (S1.2) and Lemma 2, we can obtain that for $\boldsymbol{\mu}_1 \neq \mathbf{0}$, as $n, p \to +\infty$,

$$\begin{aligned} \left| \frac{\sum_{k=b}^{n_1-1}(n_1 - k)\mathrm{tr}(\boldsymbol{\Sigma}_{1,k})}{\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}^2)} \right| &\leq C \cdot \frac{n_1 - b + 1}{p^{2\eta}} \exp\{-a(1 - 2/q)(b - 1)\} \to 0, \\ \left| \frac{\sum_{k=b}^{n_1-1}(n_1 - k)\mathrm{tr}(\boldsymbol{\Sigma}_{1,k})}{n_1\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1} \right| &\leq C \cdot \frac{n_1 - b + 1}{p^{\eta-1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\mu}_1} \exp\{-a(1 - 2/q)(b - 1)\} \to 0. \end{aligned}$$

Then as $n, p \to +\infty$,

$$\sum_{k=b}^{n_1-1} (n_1 - k)\mathrm{tr}(\boldsymbol{\Sigma}_{1,k}) = o\big\{\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}^2) + n_1\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1\big\}.$$

Similarly, we have as $n, p \to +\infty$,

$$\left\{\sum_{k=b}^{n_1-1}(n_1-k)\mathrm{tr}(\boldsymbol{\Sigma}_{1,k})\right\}^2 = o\left\{n_1^2\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}^2) + n_1^3\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1\right\}.$$

Hence as $n, p \to +\infty$,

$$\mathrm{Var}\{T_{n_1,n_2}^{(1)}(b)\} = \left\{\frac{2}{n_1^2}\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}^2) + \frac{4}{n_1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_1\right\}\{1+o(1)\}.$$

Analogously with the derivations of $\mathrm{Var}\{T_{n_1,n_2}^{(1)}(b)\}$, we have, as $n, p \to +\infty$,

$$\mathrm{Var}\{T_{n_1,n_2}^{(2)}(b)\} = \left\{\frac{2}{n_2^2}\mathrm{tr}(\boldsymbol{\Sigma}_{2,\infty}\boldsymbol{\Sigma}_{2,\infty}) + \frac{4}{n_2}\boldsymbol{\mu}_2^{\mathrm{T}}\boldsymbol{\Sigma}_{2,\infty}\boldsymbol{\mu}_2\right\}\{1+o(1)\},$$

$$\mathrm{Var}\{T_{n_1,n_2}^{(3)}(b)\} = \left\{\frac{1}{n_1 n_2}\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\Sigma}_{2,\infty}) + \frac{1}{n_1}\boldsymbol{\mu}_2^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_2 + \frac{1}{n_2}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{2,\infty}\boldsymbol{\mu}_1\right\}\{1+o(1)\},$$

$$\mathrm{Cov}\{T_{n_1,n_2}^{(1)}(b), T_{n_1,n_2}^{(2)}(b)\} = 0,$$

$$\mathrm{Cov}\{T_{n_1,n_2}^{(1)}(b), T_{n_1,n_2}^{(3)}(b)\} = \left(\frac{2}{n_1}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\mu}_2\right)\{1+o(1)\} \quad \text{and}$$

$$\mathrm{Cov}\{T_{n_1,n_2}^{(2)}(b), T_{n_1,n_2}^{(3)}(b)\} = \left(\frac{2}{n_2}\boldsymbol{\mu}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{2,\infty}\boldsymbol{\mu}_2\right)\{1+o(1)\}.$$

Since

$$\begin{aligned}\mathrm{Var}\{T(b)\} = {} & \mathrm{Var}\{T_{n_1,n_2}^{(1)}(b)\} + \mathrm{Var}\{T_{n_1,n_2}^{(2)}(b)\} + 4\mathrm{Var}\{T_{n_1,n_2}^{(3)}(b)\} \\ & + 2\mathrm{Cov}\{T_{n_1,n_2}^{(1)}(b), T_{n_1,n_2}^{(2)}(b)\} - 4\mathrm{Cov}\{T_{n_1,n_2}^{(1)}(b), T_{n_1,n_2}^{(3)}(b)\} \\ & - 4\mathrm{Cov}\{T_{n_1,n_2}^{(1)}(b), T_{n_1,n_2}^{(3)}(b)\},\end{aligned}$$

it can obtained that as $n, p \to +\infty$,

$$\mathrm{Var}\{T(b)\} = \left\{\frac{2}{n^2}\mathrm{tr}\left(\boldsymbol{M}_{\infty}^2\right) + \frac{4}{n}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^{\mathrm{T}}\boldsymbol{M}_{\infty}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)\right\}\{1+o(1)\},$$

which completes the proof. □

S1.4 Proof of Corollary 1

Corollary 1. *Suppose Assumption 5 hold. Under the null hypothesis, as $n, p \to \infty$, $Var\{T(b)\}$ satisfies the following inequality,*

$$Var\{T(b)\} \geq 2\kappa_0^{-2}(1-\kappa_0)^{-2}C_1 p^{2\eta+1}n^{-2},$$

where κ_0 is provided in (3.3).

Proof. According to Theorem 1, as $n, p \to \infty$, $\mathrm{Var}\{T(b)\}$ can be denoted by

$$\mathrm{Var}\{T(b)\} = \left\{2n^{-2}\mathrm{tr}(\boldsymbol{M}_\infty^2) + 4n^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^{\mathrm{T}}\boldsymbol{M}_\infty(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right\}\{1 + o(1)\}.$$

By the definition of $\boldsymbol{M}_k$ in (3.3),

$$\begin{aligned}
\mathrm{tr}\left(\boldsymbol{M}_\infty^2\right) &= \mathrm{tr}\left\{\left(\kappa_0^{-1}\boldsymbol{\Sigma}_{1,\infty} + (1-\kappa_0)^{-1}\boldsymbol{\Sigma}_{2,\infty}\right)^2\right\} \\
&= \mathrm{tr}\left\{\kappa_0^{-2}\boldsymbol{\Sigma}_{1,\infty}^2 + (1-\kappa_0)^{-2}\boldsymbol{\Sigma}_{2,\infty}^2 + 2\kappa_0^{-1}(1-\kappa_0)^{-1}\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\Sigma}_{2,\infty}\right\} \\
&\geq \kappa_0^{-2}C_1^2 p^{2\eta+1} + (1-\kappa_0)^{-2}C_1^2 p^{2\eta+1} + 2\kappa_0^{-1}(1-\kappa_0)^{-1}C_1^2 p^{2\eta+1} = \kappa_0^{-2}(1-\kappa_0)^{-2}C_1^2 p^{2\eta+1}.
\end{aligned}$$

Under the null hypothesis, plugging the above inequality into the expression for $\mathrm{Var}\{T(b)\}$ leads to the claim in Corollary 1. □

S1.5 Proof of Theorem 1

Proof. To derive the asymptotic normality of the BEU-statistic $T(b)$, we use the coupling method for time series and the martingale central limit theorem for the U-statistics. For both samples, we partition the time points $\{1, \ldots, n_i\}$ into a sequence of large segments of length a_1 followed by small segments of length a_2 where $a_2 = o(a_1)$. For the two-dimensional index array $\{(t_1, t_2) : 1 \leq t_1 \leq t_2 \leq n_i\}$, the above partition results in big square blocks of size $a_1 \times a_1$, colored in blue in Figure 1, which are separated by the smaller horizontal and vertical rectangles with sizes $a_1 \times a_2$ and small square blocks of size $a_2 \times a_2$. There

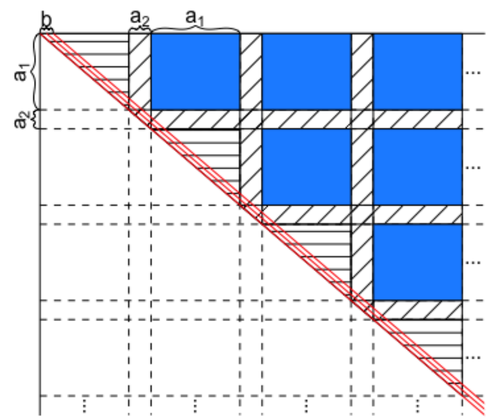

Figure 1: The band exclusion (marked by red lines) and the small-and-big blocking in the upper triangle temporal index array $\{(t_1, t_2) : 1 \leq t_1 \leq t_2 \leq n_i\}$.

are also big and small triangular blocks of length a_1 and a_2, respectively, along the main diagonal. The red lines in Figure 1 denotes the bands that are excluded in the formulation of the BEU-statistic $T(b)$.

Let $d_i = \lfloor n_i/(a_1 + a_2) \rfloor$ be the total number of large and small segments for $i = 1, 2$, where $\lfloor \cdot \rfloor$ denotes the floor function. Let $\bar{\boldsymbol{X}}_{i,m}$ be the average of $\boldsymbol{X}_{i,t}$ over the mth large segment for $m = 1, \dots, d_i$ and $i = 1, 2$. By the coupling method, $\bar{\boldsymbol{X}}_{i,m_1}$ and $\bar{\boldsymbol{X}}_{i,m_2}$ can be regarded as independent, since they are separated by at least one small block. Then, the martingale central limit theorem (Hall and Heyde, 1980) for independent observations can be applied to show the asymptotic normality of the BEU-statistic $T(b)$.

As discussed above, we group the indices $\{1, 2, \dots, n_i\}$ into a sequence of large segments of length a_1 and small segments of length a_2 which are denoted by

$$L_m = \{(m-1)(a_1 + a_2) + 1, \dots, ma_1 + (m-1)a_2\} \text{ and}$$

$$S_m = \{ma_1 + (m-1)a_2 + 1, \dots, m(a_1 + a_2)\},$$

for $m = 1, \dots, d_i$ respectively, and a remainder segment $R_i = \{d_i(a_1 + a_2) + 1, \dots, n_i\}$. Here, we assume $a_2 = o(a_1)$ and $b = o(a_2)$. Accordingly, $T(b)$ can be decomposed into the summation of five terms

$$T(b) = T_1(b) + T_2(b) + T_3(b) + T_4(b) + T_5(b), \tag{S1.12}$$

where

$$\begin{aligned}
T_1(b) &= \frac{1}{n_1(b)} \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} + \frac{1}{n_2(b)} \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} \qquad \text{(S1.13)} \\
&\quad - \frac{2}{n_1 n_2} \sum_{m_1=1}^{d_1} \sum_{m_2=1}^{d_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2}, \\
T_2(b) &= \frac{1}{n_1(b)} \sum_{m=1}^{d_1} \sum_{t_1, t_2 \in L_m}^{*} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} + \frac{1}{n_2(b)} \sum_{m=1}^{d_2} \sum_{t_1, t_2 \in L_m}^{*} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2}, \\
T_3(b) &= \frac{1}{n_1(b)} \sum_{m_1, m_2=1}^{d_1} \left(\sum_{t_1 \in L_{m_1}, t_2 \in S_{m_2}}^{*} + \sum_{t_1 \in S_{m_1}, t_2 \in L_{m_2}}^{*} \right) \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} \\
&\quad + \frac{1}{n_2(b)} \sum_{m_1, m_2=1}^{d_2} \left(\sum_{t_1 \in L_{m_1}, t_2 \in S_{m_2}}^{*} + \sum_{t_1 \in S_{m_1}, t_2 \in L_{m_2}}^{*} \right) \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} \\
&\quad - \frac{2}{n_1 n_2} \sum_{m_1=1}^{d_1} \sum_{m_2=1}^{d_2} \left(\sum_{t_1 \in L_{m_1}, t_2 \in S_{m_2}} + \sum_{t_1 \in S_{m_1}, t_2 \in L_{m_2}} \right) \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2}, \\
T_4(b) &= \frac{1}{n_1(b)} \sum_{m_1, m_2=1}^{d_1} \sum_{t_1 \in S_{m_1}, t_2 \in S_{m_2}}^{*} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} + \frac{1}{n_2(b)} \sum_{m_1, m_2=1}^{d_2} \sum_{t_1 \in S_{m_1}, t_2 \in S_{m_2}}^{*} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} \\
&\quad - \frac{2}{n_1 n_2} \sum_{m_1=1}^{d_1} \sum_{m_2=1}^{d_2} \sum_{t_1 \in S_{m_1}, t_2 \in S_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2}, \quad \text{and} \\
T_5(b) &= \frac{1}{n_1(b)} \sum_{t_1 \in R_1 \text{ or } t_2 \in R_1}^{*} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} + \frac{1}{n_2(b)} \sum_{t_1 \in R_2 \text{ or } t_2 \in R_2}^{*} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} \\
&\quad - \frac{2}{n_1 n_2} \sum_{t_1 \in R_1 \text{ or } t_2 \in R_2} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2}.
\end{aligned}$$

Here, $\sum^*$ represents the summation over $|t_1 - t_2| \geq b$. Then, we construct the coupling process for $\{(\boldsymbol{X}_{i,t},\ t \in L_m)\}_{m=1}^{d_i}$ in the following way. For the innovation process $\boldsymbol{Z}_{i,t}$ in Assumption 2, there exists a sequence of mutually independent random vectors $\{(\widetilde{\boldsymbol{Z}}_{i,t},\ t \in L_m)\}_{m=1}^{d_i}$ such that $(\widetilde{\boldsymbol{Z}}_{i,t},\ t \in L_m)$ and $(\boldsymbol{Z}_{i,t},\ t \in L_m)$ are identically distributed and $\mathrm{P}\{(\widetilde{\boldsymbol{Z}}_{i,t},\ t \in L_m) \neq (\boldsymbol{Z}_{i,t},\ t \in L_m)\} = \beta\big\{\sigma(\boldsymbol{Z}_{i,t},\ t \in L_m), \sigma(\boldsymbol{Z}_{i,t},\ t \in L_{m-1})\big\}$ for $i = 1, 2$

(Bosq, 1996). Let

$$\widetilde{\boldsymbol{X}}_{i,t} = \boldsymbol{\Gamma}_i \widetilde{\boldsymbol{Z}}_{i,t} + \boldsymbol{\mu}_i, \quad i = 1, 2,$$

where $\boldsymbol{\Gamma}_i$ is given in Assumption 2. Then, $\{(\widetilde{\boldsymbol{X}}_{i,t},\ t \in L_m)\}_{m=1}^{d_i}$ are independent, $(\widetilde{\boldsymbol{X}}_{i,t},\ t \in L_m)$ is identically distributed as $(\boldsymbol{X}_{i,t},\ t \in L_m)$, and $\mathrm{P}\{(\widetilde{\boldsymbol{X}}_{i,t},\ t \in L_m) \neq (\boldsymbol{X}_{i,t},\ t \in L_m)\} \leq \mathrm{P}\{(\widetilde{\boldsymbol{Z}}_{i,t},\ t \in L_m) \neq (\boldsymbol{Z}_{i,t},\ t \in L_m)\}$ for $i = 1, 2$. By replacing $(\boldsymbol{X}_{1,t_1}, \boldsymbol{X}_{2,t_2})$ with $(\widetilde{\boldsymbol{X}}_{1,t_1}, \widetilde{\boldsymbol{X}}_{2,t_2})$ in $T_1(b)$, define

$$\begin{aligned} T_1^*(b) =& \frac{1}{n_1(b)} \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \widetilde{\boldsymbol{X}}_{1,t_1}^{\mathrm{T}} \widetilde{\boldsymbol{X}}_{1,t_2} + \frac{1}{n_2(b)} \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \widetilde{\boldsymbol{X}}_{2,t_1}^{\mathrm{T}} \widetilde{\boldsymbol{X}}_{2,t_2} \\ & - \frac{2}{n_1 n_2} \sum_{m_1=1}^{d_1} \sum_{m_2=1}^{d_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \widetilde{\boldsymbol{X}}_{1,t_1}^{\mathrm{T}} \widetilde{\boldsymbol{X}}_{2,t_2}. \end{aligned} \tag{S1.14}$$

In the following, we will prove as $n, p \to \infty$, (i) $T_1(b) - T_1^*(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$ and (ii) $T(b) - T_1(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$.

(i) Proof of $T_1(b) - T_1^*(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$.

By the definition of $T_1(b)$ in (S1.13), we can decompose $T_1(b)$ into the following three parts,

$$T_1(b) = T_1^{(1)}(b) + T_1^{(2)}(b) - 2T_1^{(3)}(b),$$

where

$$\begin{aligned} T_1^{(1)}(b) &= \frac{1}{n_1(b)} \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2}, \\ T_1^{(2)}(b) &= \frac{1}{n_2(b)} \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} \text{ and} \\ T_1^{(3)}(b) &= \frac{1}{n_1 n_2} \sum_{m_1=1}^{d_1} \sum_{m_2=1}^{d_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2}. \end{aligned}$$

Analogously, by the definition of $T_1^*(b)$ in (S1.14), $T_1^*(b)$ has the following decomposition

$$T_1^*(b) = T_1^{*(1)}(b) + T_1^{*(2)}(b) - 2T_1^{*(3)}(b),$$

where

$$T_1^{*(1)}(b) = \frac{1}{n_1(b)} \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \widetilde{\boldsymbol{X}}_{1,t_1}^{\mathrm{T}} \widetilde{\boldsymbol{X}}_{1,t_2},$$

$$T_1^{*(2)}(b) = \frac{1}{n_2(b)} \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \widetilde{\boldsymbol{X}}_{2,t_1}^{\mathrm{T}} \widetilde{\boldsymbol{X}}_{2,t_2} \text{ and}$$

$$T_1^{*(3)}(b) = \frac{1}{n_1 n_2} \sum_{m_1=1}^{d_1} \sum_{m_2=1}^{d_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \widetilde{\boldsymbol{X}}_{1,t_1}^{\mathrm{T}} \widetilde{\boldsymbol{X}}_{2,t_2}.$$

Then

$$T_1(b) - T_1^*(b) = \sum_{l=1}^{2} \{T_1^{(l)}(b) - T_1^{*(l)}(b)\} - 2\{T_1^{(3)}(b) - T_1^{*(3)}(b)\}.$$

For the first part in the decomposition of $T_1(b) - T_1^*(b)$, define

$$\Delta_{1n}^{(1)}(b) = \frac{T_1^{(1)}(b) - T_1^{*(1)}(b)}{\sqrt{\mathrm{Var}\{T(b)\}}}.$$

For any $\epsilon > 0$, as $n, p \to \infty$,

$$\begin{aligned}
\mathrm{P}\big(|\Delta_{1n}^{(1)}| > \epsilon\big) &= \mathrm{P}\left(|T_1^{(1)}(b) - T_1^{*(1)}(b)| > \sqrt{\mathrm{Var}\{T(b)\}}\epsilon\right) \\
&\leq \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \mathrm{P}\left[|\boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} - \widetilde{\boldsymbol{X}}_{1,t_1}^{\mathrm{T}} \widetilde{\boldsymbol{X}}_{1,t_2}| > \frac{n_1(b)\sqrt{\mathrm{Var}\{T(b)\}}\epsilon}{d_1^2 a_1^2}\right] \\
&\leq \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \left(\mathrm{P}\left[|\boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} - \widetilde{\boldsymbol{X}}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2}| > \frac{n_1(b)\sqrt{\mathrm{Var}\{T(b)\}}\epsilon}{2d_1^2 a_1^2}\right]\right. \\
&\quad \left. + \mathrm{P}\left[|\widetilde{\boldsymbol{X}}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} - \widetilde{\boldsymbol{X}}_{1,t_1}^{\mathrm{T}} \widetilde{\boldsymbol{X}}_{1,t_2}| > \frac{n_1(b)\sqrt{\mathrm{Var}\{T(b)\}}\epsilon}{2d_1^2 a_1^2}\right]\right) \\
&\leq \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} \left\{\mathrm{P}\big(\boldsymbol{X}_{1,t_1} \neq \widetilde{\boldsymbol{X}}_{1,t_1}\big) + \mathrm{P}\big(\boldsymbol{X}_{1,t_2} \neq \widetilde{\boldsymbol{X}}_{1,t_2}\big)\right\}.
\end{aligned}$$

Under Assumption 3,

$$\mathrm{P}\big(|\Delta_{1n}^{(1)}| > \epsilon\big) \leq \sum_{m_1 \neq m_2} \sum_{t_1 \in L_{m_1}, t_2 \in L_{m_2}} 2c \exp(-a a_2) \leq 2c d_1^2 a_1^2 \exp(-a a_2).$$

Under the condition of $\log a_1 = o(a_2)$ and $\log d_1 = o(a_2)$, as $n, p \to \infty$, we have $\mathrm{P}\big(|\Delta_{1n}^{(1)}| > \epsilon\big) \to 0$. For the second and third parts in the decomposition of $T_1(b)$ and $T_1^*(b)$, define

$$\Delta_{1n}^{(2)}(b) = \frac{T_1^{(2)}(b) - T_1^{*(2)}(b)}{\sqrt{\mathrm{Var}\{T(b)\}}} \text{ and } \Delta_{1n}^{(3)}(b) = \frac{T_1^{(3)}(b) - T_1^{*(3)}(b)}{\sqrt{\mathrm{Var}\{T(b)\}}}.$$

Analogously, under the condition of $\log a_1 = o(a_2)$, $\log d_1 = o(a_2)$ and $\log d_2 = o(a_2)$, for any $\epsilon > 0$, as $n, p \to \infty$,

$$\mathrm{P}\big(|\Delta_{1n}^{(2)}| > \epsilon\big) \to 0 \quad \text{and} \quad \mathrm{P}\big(|\Delta_{1n}^{(3)}| > \epsilon\big) \to 0.$$

Thus for any $\epsilon > 0$, as $n, p \to \infty$,

$$\mathrm{P}\left[\left|\frac{T_1(b) - T_1^*(b)}{\sqrt{\mathrm{Var}\{T(b)\}}}\right| > \epsilon\right] \leq \mathrm{P}\left(|\Delta_{1n}^{(1)}| > \epsilon/4\right) + \mathrm{P}\left(|\Delta_{1n}^{(2)}| > \epsilon/4\right) + \mathrm{P}\left(|\Delta_{1n}^{(3)}| > \epsilon/4\right) \to 0.$$

Thus, $T_1(b) - T_1^*(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$ as $n, p \to \infty$.

(ii) Proof of $T(b) - T_1(b) = o_p\big[\sqrt{\mathbf{Var}\{T(b)\}}\big]$.

In the following, we prove $T(b) - T_1(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$ as $n, p \to \infty$, which is equivalent to $T_k(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$ for any $k = 2, 3, 4, 5$ according to (S1.12). Without loss of generality, we assume $\boldsymbol{\mu}_i = \mathrm{E}(\boldsymbol{X}_i) = \mathbf{0}$ for $i = 1, 2$ in the following derivation. It can be easily generalized to the case of $\boldsymbol{\mu}_i \neq \mathbf{0}$.

Firstly, we prove $T_2(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$, which holds if $\mathrm{Var}\{T_2(b)\} = o\big[\mathrm{Var}\{T(b)\}\big]$ as $n, p \to \infty$. Define

$$\Delta_{2n}^{(1)} = \frac{1}{n_1(b)} \sum_{m=1}^{d_1} \sum_{t_1, t_2 \in L_m}^{*} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} / \sqrt{\mathrm{Var}\{T(b)\}} \quad \text{and}$$

$$\Delta_{2n}^{(2)} = \frac{1}{n_2(b)} \sum_{m=1}^{d_2} \sum_{t_1, t_2 \in L_m}^{*} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} / \sqrt{\mathrm{Var}\{T(b)\}}.$$

Then,

$$\frac{T_2(b)}{\sqrt{\mathrm{Var}\{T(b)\}}} = \Delta_{2n}^{(1)} + \Delta_{2n}^{(2)}.$$

Note that $\mathrm{Var}(\Delta_{2n}^{(1)}) = \mathrm{E}\big\{(\Delta_{2n}^{(1)})^2\big\} - \mathrm{E}^2(\Delta_{2n}^{(1)}) \leq \mathrm{E}\big\{(\Delta_{2n}^{(1)})^2\big\}$. Moreover,

$$\mathrm{E}\big\{(\Delta_{2n}^{(1)})^2\big\} = \frac{1}{n_1^2(b)\mathrm{Var}\{T(b)\}} \sum_{m_1, m_2 = 1}^{d_1} \sum_{t_1, t_2 \in L_{m_1}}^{*} \sum_{t_3, t_4 \in L_{m_2}}^{*} \mathrm{E}(\boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} \boldsymbol{X}_{1,t_3}^{\mathrm{T}} \boldsymbol{X}_{1,t_4}).$$

Let

$$\tilde{V}_0 = \frac{1}{n_1^2(b)\mathrm{Var}\{T(b)\}} \sum_{m_1, m_2 = 1}^{d_1} \sum_{t_1, t_2 \in L_{m_1}}^{*} \sum_{t_3, t_4 \in L_{m_2}}^{*} \mathrm{tr}\{E(\boldsymbol{X}_{1,t_3} \boldsymbol{X}_{1,t_1}^{\mathrm{T}}) \mathrm{E}(\boldsymbol{X}_{1,t_2} \boldsymbol{X}_{1,t_4})^{\mathrm{T}}\}$$

and $\mathcal{D} = \cup_{m_1,m_2=1}^{d_1}\{\boldsymbol{t} = (t_1,t_2,t_3,t_4)^{\mathrm{T}} : t_1,t_2 \in L_{m_1}, t_3,t_4 \in L_{m_2}\}$. For a distance parameter K satisfying $b/3 \le K < b/2$, let

$$\mathcal{S}_1 = \{\boldsymbol{t} = (t_1,t_2,t_3,t_4)^{\mathrm{T}} : |t_1 - t_3| \le K, |t_2 - t_4| \le K\},$$

$$\mathcal{S}_2 = \{\boldsymbol{t} = (t_1,t_2,t_3,t_4)^{\mathrm{T}} : |t_1 - t_3| \le K, |t_2 - t_4| > K\},$$

$$\mathcal{S}_3 = \{\boldsymbol{t} = (t_1,t_2,t_3,t_4)^{\mathrm{T}} : |t_1 - t_3| > K, |t_2 - t_4| \le K\},$$

$$\mathcal{S}_4 = \{\boldsymbol{t} = (t_1,t_2,t_3,t_4)^{\mathrm{T}} : |t_2 - t_3| \le K\},$$

$$\mathcal{S}_5 = \{\boldsymbol{t} = (t_1,t_2,t_3,t_4)^{\mathrm{T}} : |t_1 - t_4| \le K\} \text{ and}$$

$$\mathcal{S}_6 = \{\boldsymbol{t} = (t_1,t_2,t_3,t_4)^{\mathrm{T}} : |t_1 - t_3| > K, |t_2 - t_4| > K, |t_2 - t_3| > K, |t_1 - t_4| > K\},$$

and $\mathcal{S}_{D,l} = \mathcal{S}_l \cap \mathcal{D}$. Then $\mathcal{D} = \cup_{l=1}^{6}\mathcal{S}_{D,l}$. Accordingly, for $l = 1,2,\ldots,6$, define

$$V_l = \frac{1}{n_1^2(b)\mathrm{Var}\{T(b)\}} \sum_{\boldsymbol{t}\in\mathcal{S}_{D,l}} \mathrm{tr}\big\{\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}})\big\} I(|t_1 - t_2| \ge b) I(|t_3 - t_4| \ge b).$$

Then, $\mathrm{E}\left\{\left(\Delta_{2n}^{(1)}\right)^2\right\} = \sum_{l=1}^{6} V_l$. If $\mathrm{E}(\boldsymbol{X}_{i,t}) = \boldsymbol{0}$, $V_l = o(1)$ for $l = 2,3,\ldots,6$. Define

$$\tilde{V}_1 = \frac{1}{n_1^2(b)\mathrm{Var}\{T(b)\}} \sum_{\boldsymbol{t}\in\mathcal{S}_{D,1}} \mathrm{tr}\big\{\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}})\big\} I(|t_1 - t_2| \ge b) I(|t_3 - t_4| \ge b).$$

In the following, we prove $|V_1 - \tilde{V}_1| = o(1)$ as $n, p \to \infty$. By Lemma 1,

$$\begin{aligned}
&|V_1 - \tilde{V}_1| \\
\le& \frac{1}{n_1^2(b)\mathrm{Var}\{T(b)\}} \sum_{\boldsymbol{t}\in\mathcal{S}_{D,1}} |\mathrm{tr}\big\{\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}})\big\} - \mathrm{tr}\big\{\mathrm{E}(\boldsymbol{X}_{1,t_3}\boldsymbol{X}_{1,t_1}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{1,t_2}\boldsymbol{X}_{1,t_4}^{\mathrm{T}})\big\}| \\
&\cdot I(|t_1 - t_2| \ge b) I(|t_3 - t_4| \ge b) \\
\le& \frac{1}{n_1^2(b)\mathrm{Var}\{T(b)\}} \sum_{\boldsymbol{t}\in\mathcal{S}_{D,1}} \sum_{j_1,j_2=1}^{p} |\mathrm{Cov}\left(X_{1,t_3,j_1}X_{1,t_1,j_2}, X_{1,t_2,j_2}X_{1,t_4,j_1}\right)| I(|t_1 - t_2|, |t_3 - t_4| \ge b).
\end{aligned}$$

By the Davydov's inequality and Corollary 1, if there exists a sufficiently large $c_1 > 0$ such that $b \ge c_1(\log n + \log p)$, we have, as $n, p \to \infty$,

$$|V_1 - \tilde{V}_1| \to 0. \tag{S1.15}$$

Moreover, it can be derived that

$$\begin{aligned}\tilde{V}_1 &= \frac{4}{n_1^2(b)\mathrm{Var}\{T(b)\}}\sum_{m_1,m_2=1}^{d_1}\sum_{t_1,t_2\in L_{m_1},t_3,t_4\in L_{m_2}}\mathrm{tr}\left(\boldsymbol{\Sigma}_{1,t_3-t_1}\boldsymbol{\Sigma}_{1,t_2-t_4}\right)\\
&\qquad\cdot I(t_2-t_1\geq b)I(t_4-t_3\geq b)I(|t_1-t_3|\leq K)I(|t_2-t_4|\leq K)\\
&= \frac{4}{n_1^2(b)\mathrm{Var}\{T(b)\}}\sum_{m=1}^{d_1}\sum_{t_1,t_2,t_3,t_4\in L_m}\mathrm{tr}\left(\boldsymbol{\Sigma}_{1,t_3-t_1}\boldsymbol{\Sigma}_{1,t_2-t_4}\right)I(t_2-t_1\geq b)I(t_4-t_3\geq b)\\
&\qquad\cdot I(|t_1-t_3|\leq K)I(|t_2-t_4|\leq K)\\
&\leq \frac{2d_1a_1^2}{n_1^2(b)\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}^2)}\sum_{k_1,k_2=-\infty}^{\infty}\left|\mathrm{tr}\left(\boldsymbol{\Sigma}_{1,k_1}\boldsymbol{\Sigma}_{1,-k_2}\right)\right| \to 0 \text{ as } n,p\to\infty.\end{aligned}$$

Thus, $\mathrm{Var}(\Delta_{2n}^{(1)})\to 0$. Analogously, $\mathrm{Var}(\Delta_{2n}^{(2)})\to 0$ as $n,p\to\infty$. Since $\mathrm{Var}\{T_2(b)\}\leq 2\mathrm{Var}(\Delta_{2n}^{(1)})+2\mathrm{Var}(\Delta_{2n}^{(2)})$, $\mathrm{Var}\{T_2(b)\}\to 0$ as $n,p\to\infty$. Then $T_2(b)=o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$.

Next, we prove $T_3(b)=o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$ as $n,p\to\infty$. The basic idea is to prove $\mathrm{Var}\{T_3(b)\}=o\big[\mathrm{Var}\{T(b)\}\big]$. Note that $T_3(b)=T_{3,1}(b)+T_{3,2}(b)$ where

$$\begin{aligned}T_{3,1}(b) &= \frac{1}{n_1(b)}\sum_{m_1,m_2=1}^{d_1}\sum_{t_1\in L_{m_1},t_2\in S_{m_2}}^{*}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}+\frac{1}{n_2(b)}\sum_{m_1,m_2=1}^{d_2}\sum_{t_1\in L_{m_1},t_2\in S_{m_2}}^{*}\boldsymbol{X}_{2,t_1}^{\mathrm{T}}\boldsymbol{X}_{2,t_2}\\
&\quad-\frac{2}{n_1n_2}\sum_{m_1=1}^{d_1}\sum_{m_2=1}^{d_2}\sum_{t_1\in L_{m_1},t_2\in S_{m_2}}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{2,t_2} \quad\text{and}\\
T_{3,2}(b) &= \frac{1}{n_1(b)}\sum_{m_1,m_2=1}^{d_1}\sum_{t_1\in S_{m_1},t_2\in L_{m_2}}^{*}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}+\frac{1}{n_2(b)}\sum_{m_1,m_2=1}^{d_2}\sum_{t_1\in S_{m_1},t_2\in L_{m_2}}^{*}\boldsymbol{X}_{2,t_1}^{\mathrm{T}}\boldsymbol{X}_{2,t_2}\\
&\quad-\frac{2}{n_1n_2}\sum_{m_1=1}^{d_1}\sum_{m_2=1}^{d_2}\sum_{t_1\in S_{m_1},t_2\in L_{m_2}}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{2,t_2}.\end{aligned}$$

For $T_{3,1}(b)$, define

$$\begin{aligned}\Delta_{3n,1}^{(1)} &= \frac{1}{n_1(b)}\sum_{m_1,m_2=1}^{d_1}\sum_{t_1\in L_{m_1},t_2\in S_{m_2}}^{*}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}/\sqrt{\mathrm{Var}\{T(b)\}}\\
&= \frac{1}{n_1(b)}\sum_{m_1,m_2=1}^{d_1}\sum_{t_1\in L_{m_1},t_2\in S_{m_2}}\boldsymbol{X}_{1,t_1}^{\mathrm{T}}\boldsymbol{X}_{1,t_2}I(|t_1-t_2|\geq b)/\sqrt{\mathrm{Var}\{T(b)\}},\\
\Delta_{3n,1}^{(2)} &= \frac{1}{n_2(b)}\sum_{m_1,m_2=1}^{d_2}\sum_{t_1\in L_{m_1},t_2\in S_{m_2}}^{*}\boldsymbol{X}_{2,t_1}^{\mathrm{T}}\boldsymbol{X}_{2,t_2}/\sqrt{\mathrm{Var}\{T(b)\}}\end{aligned}$$

$$= \frac{1}{n_2(b)} \sum_{m_1,m_2=1}^{d_2} \sum_{t_1 \in L_{m_1}, t_2 \in S_{m_2}} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} I(|t_1 - t_2| \geq b) / \sqrt{\mathrm{Var}\{T(b)\}} \quad \text{and}$$

$$\Delta_{3n,1}^{(12)} = \frac{2}{n_1 n_2} \sum_{m_1,m_2=1}^{d_1} \sideset{}{^*}\sum_{t_1 \in L_{m_1}, t_2 \in S_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} / \sqrt{\mathrm{Var}\{T(b)\}}$$

$$= \frac{2}{n_1 n_2} \sum_{m_1,m_2=1}^{d_1} \sum_{t_1 \in L_{m_1}, t_2 \in S_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} I(|t_1 - t_2| \geq b) / \sqrt{\mathrm{Var}\{T(b)\}}$$

such that

$$\frac{T_3(b)}{\sqrt{\mathrm{Var}\{T(b)\}}} = \Delta_{3n,1}^{(1)} + \Delta_{3n,1}^{(2)} - \Delta_{3n,1}^{(12)}.$$

Analogously, for $T_{3,2}(b)$, define

$$\Delta_{3n,2}^{(1)} = \frac{1}{n_1(b)} \sum_{m_1,m_2=1}^{d_1} \sideset{}{^*}\sum_{t_1 \in S_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} / \sqrt{\mathrm{Var}\{T(b)\}}$$

$$= \frac{1}{n_1(b)} \sum_{m_1,m_2=1}^{d_1} \sum_{t_1 \in S_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} I(|t_1 - t_2| \geq b) / \sqrt{\mathrm{Var}\{T(b)\}},$$

$$\Delta_{3n,2}^{(2)} = \frac{1}{n_2(b)} \sum_{m_1,m_2=1}^{d_2} \sideset{}{^*}\sum_{t_1 \in S_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} / \sqrt{\mathrm{Var}\{T(b)\}}$$

$$= \frac{1}{n_2(b)} \sum_{m_1,m_2=1}^{d_2} \sum_{t_1 \in S_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} I(|t_1 - t_2| \geq b) / \sqrt{\mathrm{Var}\{T(b)\}} \quad \text{and}$$

$$\Delta_{3n,2}^{(12)} = \frac{2}{n_1 n_2} \sum_{m_1,m_2=1}^{d_1} \sideset{}{^*}\sum_{t_1 \in S_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} / \sqrt{\mathrm{Var}\{T(b)\}}$$

$$= \frac{2}{n_1 n_2} \sum_{m_1,m_2=1}^{d_1} \sum_{t_1 \in S_{m_1}, t_2 \in L_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} I(|t_1 - t_2| \geq b) / \sqrt{\mathrm{Var}\{T(b)\}}$$

so that

$$\frac{T_{3,2}(b)}{\sqrt{\mathrm{Var}\{T(b)\}}} = \Delta_{3n,2}^{(1)} + \Delta_{3n,2}^{(2)} - \Delta_{3n,2}^{(12)}.$$

Since $\mathrm{Var}\{T_{3,i}(b)\} = O\big\{\mathrm{Var}(\Delta_{3n,i}^{(1)}) + \mathrm{Var}(\Delta_{3n,i}^{(2)}) + \mathrm{Var}(\Delta_{3n,i}^{(12)})\big\}$ for $i = 1, 2$, we only need to prove $\mathrm{Var}(\Delta_{3n,l}^{(1)}) = o(1)$, $\mathrm{Var}(\Delta_{3n,l}^{(2)}) = o(1)$ and $\mathrm{Var}(\Delta_{3n,l}^{(12)}) = o(1)$ for $l = 1, 2$.

First, we consider $\Delta_{3n,1}^{(1)}$ and $\Delta_{3n,2}^{(1)}$. The variance of $\Delta_{3n,1}^{(1)}$ and $\Delta_{3n,2}^{(1)}$ satisfy $\mathrm{Var}(\Delta_{3n,1}^{(1)}) \leq \mathrm{E}\big\{(\Delta_{3n,1}^{(1)})^2\big\}$ and $\mathrm{Var}(\Delta_{3n,2}^{(1)}) \leq \mathrm{E}\big\{(\Delta_{3n,2}^{(1)})^2\big\}$. Analogously, using the proof of $\mathrm{E}\big\{(\Delta_{2n}^{(1)})^2\big\}$, it can be shown that $\mathrm{E}\big\{\big(\Delta_{3n,1}^{(1)}\big)^2\big\} \to 0$ and $\mathrm{E}\big\{\big(\Delta_{3n,2}^{(1)}\big)^2\big\} \to 0$. Then, $\mathrm{Var}\big(\Delta_{3n,1}^{(1)}\big) + \mathrm{Var}\big(\Delta_{3n,2}^{(1)}\big) \to 0$ as $n, p \to \infty$. It can be derived similarly that as $n, p \to \infty$,

$$\mathrm{Var}\big(\Delta_{3n,1}^{(2)}\big) + \mathrm{Var}\big(\Delta_{3n,2}^{(2)}\big) \to 0 \quad \text{and} \quad \mathrm{Var}\big(\Delta_{3n,1}^{(12)}\big) + \mathrm{Var}\big(\Delta_{3n,2}^{(12)}\big) \to 0.$$

Since $\mathrm{Var}\{T_{3,i}(b)\} = O\big\{\mathrm{Var}(\Delta_{3n,i}^{(1)}) + \mathrm{Var}(\Delta_{3n,i}^{(2)}) + \mathrm{Var}(\Delta_{3n,i}^{(12)})\big\}$ for $i = 1, 2$, we have

$$\mathrm{Var}\{T_{3,1}(b)\} + \mathrm{Var}\{T_{3,2}(b)\} \to 0,$$

as $n, p \to \infty$. Thus, $T_3(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$ as $n, p \to \infty$.

To show $T_4(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$, similarly to the proof in (ii.2), define

$$\begin{aligned}
\Delta_{4n}^{(1)} &= \frac{1}{n_1(b)} \sum_{m_1,m_2=1}^{d_1} \sideset{}{^*}\sum_{t_1 \in S_{m_1}, t_2 \in S_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} / \sqrt{\mathrm{Var}\{T(b)\}} \\
&= \frac{1}{n_1(b)} \sum_{m_1,m_2=1}^{d_1} \sum_{t_1 \in S_{m_1}, t_2 \in S_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} I(|t_1 - t_2| \geq b) / \sqrt{\mathrm{Var}\{T(b)\}}, \\
\Delta_{4n}^{(2)} &= \frac{1}{n_2(b)} \sum_{m_1,m_2=1}^{d_2} \sideset{}{^*}\sum_{t_1 \in S_{m_1}, t_2 \in S_{m_2}} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} / \sqrt{\mathrm{Var}\{T(b)\}} \\
&= \frac{1}{n_2(b)} \sum_{m_1,m_2=1}^{d_2} \sum_{t_1 \in S_{m_1}, t_2 \in S_{m_2}} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} I(|t_1 - t_2| \geq b) / \sqrt{\mathrm{Var}\{T(b)\}} \quad \text{and} \\
\Delta_{4n}^{(12)} &= \frac{2}{n_1 n_2} \sum_{m_1,m_2=1}^{d_1} \sideset{}{^*}\sum_{t_1 \in S_{m_1}, t_2 \in S_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} / \sqrt{\mathrm{Var}\{T(b)\}} \\
&= \frac{2}{n_1 n_2} \sum_{m_1,m_2=1}^{d_1} \sum_{t_1 \in S_{m_1}, t_2 \in S_{m_2}} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} I(|t_1 - t_2| \geq b) / \sqrt{\mathrm{Var}\{T(b)\}},
\end{aligned}$$

such that

$$\frac{T_4(b)}{\sqrt{\mathrm{Var}\{T(b)\}}} = \Delta_{4n}^{(1)} + \Delta_{4n}^{(2)} - \Delta_{4n}^{(12)}.$$

Analogously, it can be proved that $\mathrm{Var}\big(\Delta_{4n}^{(1)}\big) = o(1)$, $\mathrm{Var}\big(\Delta_{4n}^{(2)}\big) = o(1)$ and $\mathrm{Var}\big(\Delta_{4n}^{(12)}\big) = o(1)$. Then $T_4(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$ holds as $n, p \to \infty$.

Lastly, we prove $T_5(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$ as $n, p \to \infty$. Define

$$\Delta_{5n}^{(1)} = \frac{1}{n_1(b)} \sideset{}{^*}\sum_{t_1 \in R_1 \ \mathit{or} \ t_2 \in R_1} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} / \sqrt{\mathrm{Var}\{T(b)\}},$$

$$\Delta_{5n}^{(2)} = \frac{1}{n_2(b)} \sum_{t_1 \in R_2 \ or \ t_2 \in R_2}^{*} \boldsymbol{X}_{2,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} / \sqrt{\mathrm{Var}\{T(b)\}} \quad \text{and}$$

$$\Delta_{5n}^{(12)} = \frac{2}{n_1 n_2} \sum_{t_1 \in R_1 \ or \ t_2 \in R_2} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{2,t_2} / \sqrt{\mathrm{Var}\{T(b)\}}.$$

Then

$$\frac{T_5(b)}{\sqrt{\mathrm{Var}\{T(b)\}}} = \Delta_{5n}^{(1)} + \Delta_{5n}^{(2)} - \Delta_{5n}^{(12)}.$$

Since $\mathrm{Var}\{T_5(b)\} = O\big\{\mathrm{Var}(\Delta_{5n}^{(1)}) + \mathrm{Var}(\Delta_{5n}^{(2)}) + \mathrm{Var}(\Delta_{5n}^{(12)})\big\}$, we only need to prove $\mathrm{Var}(\Delta_{5n}^{(1)}) = o(1)$, $\mathrm{Var}(\Delta_{5n}^{(2)}) = o(1)$ and $\mathrm{Var}(\Delta_{5n}^{(12)}) = o(1)$. For $\mathrm{Var}(\Delta_{5n}^{(1)})$, we have

$$\begin{aligned}
\Delta_{5n}^{(1)} &= \frac{1}{n_1(b)} \sum_{t_1 \in R_1 \ or \ t_2 \in R_1} \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} I(|t_1 - t_2| \geq b) / \sqrt{\mathrm{Var}\{T(b)\}} \\
&= \frac{1}{n_1(b)} \left(\sum_{t_1, t_2 \in R_1} + 2 \sum_{t_1 \in R_1, t_2 \notin R_1} \right) \boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} I(|t_1 - t_2| \geq b) / \sqrt{\mathrm{Var}\{T(b)\}} =: \Delta_{5n,1}^{(1)} + \Delta_{5n,2}^{(1)}.
\end{aligned}$$

In the above equation, it can be proved in the similar way as $\mathrm{Var}\big(\Delta_{2n}^{(1)}\big)$ that $\mathrm{Var}\big(\Delta_{5n,1}^{(1)}\big) \to 0$ as $n, p \to \infty$. For $\Delta_{5n,2}^{(1)}$, notice that $\mathrm{Var}\big(\Delta_{5n,2}^{(1)}\big) \leq \mathrm{E}\big\{\big(\Delta_{5n,2}^{(1)}\big)^2\big\}$. On the other hand,

$$\mathrm{E}\big\{\big(\Delta_{5n,2}^{(2)}\big)^2\big\} = \frac{4}{n_1^2(b)\mathrm{Var}\{T(b)\}} \sum_{t_1, t_3 \in R_1, t_2, t_4 \notin R_1}^{*} \mathrm{E}(\boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2} \boldsymbol{X}_{1,t_3}^{\mathrm{T}} \boldsymbol{X}_{1,t_4}).$$

Let

$$\begin{aligned}
\tilde{V}_0 &= \frac{4}{n_1^2(b)\mathrm{Var}\{T(b)\}} \sum_{t_1, t_3 \in R_1, t_2, t_4 \notin R_1}^{*} \mathrm{E}(\boldsymbol{X}_{1,t_1}^{\mathrm{T}} \boldsymbol{X}_{1,t_2}) \mathrm{E}(\boldsymbol{X}_{1,t_3}^{\mathrm{T}} \boldsymbol{X}_{1,t_4}) \\
&= \frac{4}{n_1^2(b)\mathrm{Var}\{T(b)\}} \sum_{t_1, t_3 \in R_1, t_2, t_4 \notin R_1} \mathrm{tr}\big(\boldsymbol{\Sigma}_{1,t_3 - t_1} \boldsymbol{\Sigma}_{1,t_2 - t_4}\big) I(|t_1 - t_2| \geq b) I(|t_3 - t_4| \geq b).
\end{aligned}$$

Then, $|\mathrm{E}\big\{\big(\Delta_{5n,2}^{(2)}\big)^2\big\} - \tilde{V}_0| \to 0$ and $\tilde{V}_0 \to 0$ as $n, p \to \infty$, which imples that $\mathrm{Var}\big(\Delta_{5n}^{(1)}\big) \to 0$. Analogously, $\mathrm{Var}\big(\Delta_{5n}^{(2)}\big) \to 0$ and $\mathrm{Var}\big(\Delta_{5n}^{(12)}\big) \to 0$. Thus, we have $\mathrm{Var}\{T_5(b)\} = o\big[\mathrm{Var}\{T(b)\}\big]$, and $T_5(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$ as $n, p \to \infty$.

Summarizing the above results, we have $T(b) - T_1(b) = o_p\big[\sqrt{\mathrm{Var}\{T(b)\}}\big]$ as $n, p \to \infty$.

Let $\bar{\boldsymbol{X}}_{i,m} = a_1^{-1}\sum_{t\in L_m}\widetilde{\boldsymbol{X}}_{i,t}$ for $m = 1, \dots, d_i$. Then

$$T_1^*(b) = \left\{\frac{1}{d_1(1)}\sum_{m_1\neq m_2}\bar{\boldsymbol{X}}_{1,m_1}^{\mathrm{T}}\bar{\boldsymbol{X}}_{1,m_2} + \frac{1}{d_2(1)}\sum_{m_1\neq m_2}\bar{\boldsymbol{X}}_{2,m_1}^{\mathrm{T}}\bar{\boldsymbol{X}}_{2,m_2} - \frac{2}{d_1d_2}\sum_{m_1=1}^{d_1}\sum_{m_2=1}^{d_2}\bar{\boldsymbol{X}}_{1,m_1}^{\mathrm{T}}\bar{\boldsymbol{X}}_{2,m_2}\right\}\{1+o_p(1)\},$$

where $d_i(1) = d_i(d_i-1)$, $i = 1, 2$. For $i = 1, 2$, let $\bar{\boldsymbol{Z}}_{i,m} = a_1^{-1}\sum_{t\in L_m}\widetilde{\boldsymbol{Z}}_{i,t}$. By the coupling method, $\bar{\boldsymbol{X}}_{i,m}$ are independent and identically distributed random vectors satisfying $\bar{\boldsymbol{X}}_{i,t} = \boldsymbol{\Gamma}_i\bar{\boldsymbol{Z}}_{i,t} + \boldsymbol{\mu}_i$. Then, by Theorem 1 in Chen and Qin (2010), we can show that

$$\frac{T_1^*(b) - \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|^2}{\sqrt{\mathrm{Var}\{T(b)\}}} \xrightarrow{d} N(0,1)$$

as $n, p \to \infty$. Since

$$\frac{T(b)}{\sqrt{\mathrm{Var}\{T(b)\}}} = \frac{T_1^*(b)}{\sqrt{\mathrm{Var}\{T(b)\}}} + \frac{T_1(b) - T_1^*(b)}{\sqrt{\mathrm{Var}\{T(b)\}}} + \frac{T_2(b)+T_3(b)+T_4(b)+T_5(b)}{\sqrt{\mathrm{Var}\{T(b)\}}},$$

by the Slutsky's theorem, as $n, p \to \infty$,

$$\frac{T(b) - \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|^2}{\sqrt{\mathrm{Var}\{T(b)\}}} \xrightarrow{d} N(0,1).$$

The proof is completed. □

S1.6 Proofs of Lemmas 4 – 5

We first provide two useful lemmas and their proofs. Let $\boldsymbol{B}_i = \boldsymbol{\Gamma}_i^T\boldsymbol{\Gamma}_i = (b_{i,j_1j_2})$ and $\tilde{\boldsymbol{B}}_i = (|b_{i,j_1j_2}|)$. Let $\circ$ denote the Hadamard product of two matrix.

Lemma 4. *Under Assumption 2, for $i = 1$ and 2,*

$$\mathrm{tr}(\boldsymbol{B}_i^2 \circ \boldsymbol{B}_i^2) \le \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^4), \tag{S1.16}$$

$$\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^4 \le \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^4) \text{ and} \tag{S1.17}$$

$$\mathrm{tr}(\tilde{\boldsymbol{B}}_i^4) \le \mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2). \tag{S1.18}$$

Proof. Notice that under Assumption 2, $\mathbf{\Gamma}_i\mathbf{\Gamma}_i^T = \mathbf{\Sigma}_{i,\infty}$. Firstly, for $\mathrm{tr}(\boldsymbol{B}_i^2 \circ \boldsymbol{B}_i^2)$, we have

$$\mathrm{tr}(\boldsymbol{B}_i^2 \circ \boldsymbol{B}_i^2) = \sum_{j=1}^{r}\{(\boldsymbol{B}_i^2)_{jj}\}^2 \le \sum_{j_1,j_2=1}^{r}\{(\boldsymbol{B}_i^2)_{j_1j_2}\}^2 = \mathrm{tr}(\boldsymbol{B}_i^4) = \mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^4),$$

where $(\boldsymbol{B}_i^2)_{j_1j_2}$ represents the (j_1, j_2)-th element of the matrix $\boldsymbol{B}_i^2$. Secondly, it can be derived that

$$\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^4 \le \sum_{j_1=1}^{r}\left\{\sum_{j_2=1}^{r} b_{i,j_1j_2}^2\right\}^2 = \sum_{j_1=1}^{r}\{(\boldsymbol{B}_i^2)_{j_1j_1}\}^2 = \mathrm{tr}(\boldsymbol{B}_i^2 \circ \boldsymbol{B}_i^2) \le \mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^4).$$

Thirdly, for $\mathrm{tr}(\tilde{\boldsymbol{B}}_i^4)$, it can be seen that

$$\begin{aligned}
\mathrm{tr}(\tilde{\boldsymbol{B}}_i^4) &= \sum_{j_1,j_2,j_3,j_4=1}^{r} |b_{j_1j_2}b_{j_1j_3}b_{j_2j_4}b_{j_3j_4}| \\
&\le \sum_{j_1,j_4=1}^{r}\left(\sum_{j_2=1}^{r} b_{j_1j_2}^2\right)^{1/2}\left(\sum_{j_2=1}^{r} b_{j_2j_4}^2\right)^{1/2}\left(\sum_{j_3=1}^{r} b_{j_1j_3}^2\right)^{1/2}\left(\sum_{j_3=1}^{r} b_{j_3j_4}^2\right)^{1/2} \\
&= \sum_{j_1,j_4=1}^{r} (\boldsymbol{B}_i^2)_{j_1j_1}(\boldsymbol{B}_i^2)_{j_4j_4} = \mathrm{tr}^2(\boldsymbol{B}_i^2) = \mathrm{tr}^2(\mathbf{\Sigma}_{i,\infty}^2).
\end{aligned}$$

The proof has been completed. □

Lemma 5. *Suppose Assumptions 2, 5, 6 and 7 hold. If $\boldsymbol{\mu}_i = 0$ and there exists a positive constant $c_2 > 0$ (sufficiently large) such that $\tilde{b} \ge c_2(\log n + \log p)$,*

$$|E(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_5}^{\mathrm{T}}\boldsymbol{X}_{i,t_6}\boldsymbol{X}_{i,t_7}^{\mathrm{T}}\boldsymbol{X}_{i,t_8})| \le O\{\mathrm{tr}^2(\mathbf{\Sigma}_{i,\infty}^2)\}$$

holds for $i = 1, 2$ and any fixed $t_1, t_2, \ldots, t_8$ such that $|t_1 - t_2|, |t_3 - t_4|, |t_5 - t_6|, |t_7 - t_8| \ge \tilde{b}$.

Proof. Under Assumption 2, it can be derived that

$$\begin{aligned}
&|\mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_5}^{\mathrm{T}}\boldsymbol{X}_{i,t_6}\boldsymbol{X}_{i,t_7}^{\mathrm{T}}\boldsymbol{X}_{i,t_8})| \\
&= \left|\sum_{j_1,\ldots,j_8=1}^{r} b_{i,j_1j_2}b_{i,j_3j_4}b_{i,j_5j_6}b_{i,j_7j_8}\mathrm{E}(Z_{i,t_1,j_1}Z_{i,t_2,j_2}Z_{i,t_3,j_3}Z_{i,t_4,j_4}Z_{i,t_5,j_5}Z_{i,t_6,j_6}Z_{i,t_7,j_7}Z_{i,t_8,j_8})\right| \\
&\le \sum_{j_1,\ldots,j_8=1}^{r} |b_{i,j_1j_2}b_{i,j_3j_4}b_{i,j_5j_6}b_{i,j_7j_8}||\mathrm{E}(Z_{i,t_1,j_1}Z_{i,t_2,j_2}Z_{i,t_3,j_3}Z_{i,t_4,j_4}Z_{i,t_5,j_5}Z_{i,t_6,j_6}Z_{i,t_7,j_7}Z_{i,t_8,j_8})|.
\end{aligned}$$

In the following, we will deal with the right hand side of the above inequality by separating the summation into seven cases. To simplify the notation, let

$$R_{i,j_1j_2\dots j_8} = |\mathrm{E}(Z_{i,t_1,j_1}Z_{i,t_2,j_2}Z_{i,t_3,j_3}Z_{i,t_4,j_4}Z_{i,t_5,j_5}Z_{i,t_6,j_6}Z_{i,t_7,j_7}Z_{i,t_8,j_8})|.$$

Case 1: $j_1 = j_2 = \cdots = j_8$. According to (S1.17),

$$\sum_{j=1}^{r} b_{i,jj}^4 R_{i,j\dots j} \le \Delta_z \sum_{j_1j_2=1}^{r} b_{i,j_1j_2}^4 \le \Delta_z \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^4). \tag{S1.19}$$

Case 2: Six of $j_1, j_2, \dots, j_8$ are equal and the other two are equal. There are two scenarios that need to be considered in this case. On one hand, according to (S1.16) and (S1.19),

$$\begin{aligned}\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^2 b_{i,j_2j_2}^2 R_{i,j_1j_2j_1j_2j_2j_2j_2j_2} &\le \Delta_z \sum_{j_2=1}^{r} (\boldsymbol{B}_i^2)_{j_2j_2} b_{i,j_2j_2}^2 \\ &\le \Delta_z \left\{\sum_{j_2=1}^{r} (\boldsymbol{B}_i^2)_{j_2j_2}^2\right\}^{1/2} \left(\sum_{j_2=1}^{r} b_{i,j_2j_2}^4\right)^{1/2} \\ &= \Delta_z \mathrm{tr}^{1/2}(\boldsymbol{B}_i^2 \circ \boldsymbol{B}_i^2) \left(\sum_{j_2=1}^{r} b_{i,j_2j_2}^4\right)^{1/2} \le \Delta_z \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^4).\end{aligned}$$

On the other hand, under Assumption 2, by the Davydov's inequality,

$$\begin{aligned}R_{i,j_1j_1j_2\dots j_2} &= |\mathrm{E}(Z_{i,t_1,j_1}Z_{i,t_2,j_1})\mathrm{E}(Z_{i,t_3,j_2}Z_{i,t_4,j_2}Z_{i,t_5,j_2}Z_{i,t_6,j_2}Z_{i,t_7,j_2}Z_{i,t_8,j_2})| \\ &\le \Delta_z^{3/4}|\mathrm{Cov}(Z_{i,t_1,j_1}, Z_{i,t_2,j_1})| \le 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3.\end{aligned}$$

Then according to (S1.19),

$$\begin{aligned}\sum_{j_1,j_2=1}^{r} |b_{i,j_1j_1} b_{i,j_2j_2}^3| R_{i,j_1j_1j_2\dots j_2} &\le \frac{8}{3}\mathrm{tr}(\tilde{\boldsymbol{B}}_i) \left(\sum_{j_2=1}^{r} b_{i,j_2j_2}^4\right)^{1/2} \left(\sum_{j_2=1}^{r} b_{i,j_2j_2}^2\right)^{1/2} \Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\} \\ &\le \frac{8}{3}\mathrm{tr}(\tilde{\boldsymbol{B}}_i)\mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^4) \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^2\right)^{1/2} \Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\} \\ &= \frac{8}{3}\mathrm{tr}(\tilde{\boldsymbol{B}}_i)\mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^4)\mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2)\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}.\end{aligned}$$

Case 3: Five of $j_1, j_2, \dots, j_8$ are equal and the other three are equal. There are two scenarios in this case. Firstly,

$$\sum_{j_1,j_2=1}^{r} |b_{i,j_1j_1} b_{i,j_1j_2} b_{i,j_2j_2}^2| R_{i,j_1j_1j_1j_2\dots j_2} \le \Delta_z \sum_{j_2=1}^{r} \left(\sum_{j_1=1}^{r} b_{i,j_1j_1}^2\right)^{1/2} \left(\sum_{j_1=1}^{r} b_{i,j_1j_2}^2\right)^{1/2} b_{i,j_2j_2}^2$$

$$
\leq \Delta_z \sum_{j_2=1}^{r} \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^2 \right)^{1/2} (\boldsymbol{B}_i^2)_{j_2j_2}^{1/2} b_{i,j_2j_2}^2 = \Delta_z \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2) \sum_{j_2=1}^{r} (\boldsymbol{B}_i^2)_{j_2j_2}^{1/2} b_{i,j_2j_2}^2
$$

$$
\leq \Delta_z \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2) \left\{ \sum_{j_2=1}^{r} (\boldsymbol{B}_i^2)_{j_2j_2} \right\}^{1/2} \left(\sum_{j_2=1}^{r} b_{i,j_2j_2}^4 \right)^{1/2} \leq \Delta_z \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2) \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^4).
$$

Secondly,

$$
\begin{aligned}
\sum_{j_1,j_2=1}^{r} |b_{i,j_1j_2}^3 b_{i,j_2j_2}| R_{i,j_1j_2j_1j_2j_1j_2j_2j_2} &\leq \Delta_z \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^4 \right)^{1/2} \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^2 b_{i,j_2j_2}^2 \right)^{1/2} \\
&\leq \Delta_z \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^4) \left\{ \sum_{j_2=1}^{r} (\boldsymbol{B}_i^2)_{j_2j_2} b_{i,j_2j_2}^2 \right\}^{1/2} \\
&\leq \Delta_z \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^4) \left\{ \sum_{j_2=1}^{r} (\boldsymbol{B}_i^2)_{j_2j_2}^2 \right\}^{1/4} \left\{ \sum_{j_2=1}^{r} b_{i,j_2j_2}^4 \right\}^{1/4} \\
&\leq \Delta_z \mathrm{tr}^{3/4}(\boldsymbol{\Sigma}_{i,\infty}^4) \mathrm{tr}^{1/4}(\boldsymbol{B}_i \circ \boldsymbol{B}_i) \leq \Delta_z \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^4).
\end{aligned}
$$

Case 4: Four of $j_1, j_2, \ldots, j_8$ are equal and the other four are equal. There are three scenarios in this case. Firstly,

$$
\sum_{j_1,j_2=1}^{r} b_{i,j_1j_1}^2 b_{i,j_2j_2}^2 R_{i,j_1j_1j_1j_1j_2j_2j_2j_2} \leq \Delta_z \left(\sum_{j=1}^{r} b_{i,jj}^2 \right)^2 \leq \Delta_z \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^2 \right)^2 = \Delta_z \mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2).
$$

Secondly,

$$
\begin{aligned}
&\sum_{j_1,j_2=1}^{r} |b_{i,j_1j_1} b_{i,j_1j_2}^2 b_{i,j_2j_2}| R_{i,j_1j_1j_1j_2j_1j_2j_2j_2} \\
&\leq \Delta_z \sum_{j_2=1}^{r} \left(\sum_{j_1=1}^{r} b_{i,j_1j_1}^2 \right)^{1/2} \left(\sum_{j_1=1}^{r} b_{i,j_1j_2}^4 \right)^{1/2} |b_{i,j_2j_2}| \\
&\leq \Delta_z \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2) \sum_{j_2=1}^{r} \left(\sum_{j_1=1}^{r} b_{i,j_1j_2}^4 \right)^{1/2} |b_{i,j_2j_2}| \\
&\leq \Delta_z \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2) \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^4 \right)^{1/2} \left(\sum_{j_2=1}^{r} b_{i,j_2j_2}^2 \right)^{1/2} \\
&\leq \Delta_z \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2) \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^4).
\end{aligned}
$$

Thirdly, by (S1.17),

$$\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^4 R_{i,j_1j_2j_1j_2j_1j_2j_1j_2} \le \Delta_z \mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^4).$$

Case 5: Four of $j_1, j_2, \ldots, j_8$ are equal, and among the rest four j's, two of them are equal and the other two are equal. There are five scenarios in this case. Firstly,

$$\begin{aligned}
R_{i,j_1j_1j_1j_1j_2j_2j_3j_3} &= |\mathrm{E}(Z_{i,t_1,j_1}Z_{i,t_2,j_1}Z_{i,t_3,j_1}Z_{i,t_4,j_1})\mathrm{E}(Z_{i,t_5,j_2}Z_{i,t_6,j_2})\mathrm{E}(Z_{i,t_7,j_3}Z_{i,t_8,j_3})| \\
&\le \Delta_z^{1/2}|\mathrm{Cov}(Z_{i,t_5,j_2}, Z_{i,t_6,j_2})\mathrm{Cov}(Z_{i,t_7,j_3}, Z_{i,t_8,j_3})| \\
&\le \Delta_z(8c^{3/4}\exp\{-3a\tilde{b}/4\}/3)^2 = 64\Delta_z c^{3/2}\exp\{-3a\tilde{b}/2\}/9
\end{aligned}$$

by the Davydov's inequality. Thus we have

$$\sum_{j_1,j_2,j_3=1}^{r} b_{i,j_1j_1}^2|b_{i,j_2j_2}b_{i,j_3j_3}|R_{i,j_1j_1j_1j_1j_2j_2j_3j_3} \le \mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)\mathrm{tr}^2(\tilde{\boldsymbol{B}}_i)\cdot 64\Delta_z c^{3/2}\exp\{-3a\tilde{b}/2\}/9.$$

Secondly,

$$\sum_{j_1,j_2,j_3=1}^{r} b_{i,j_1j_1}^2 b_{i,j_2j_3}^2 R_{i,j_1j_1j_1j_1j_2j_3j_2j_3} \le \Delta_z \mathrm{tr}^2(\mathbf{\Sigma}_{i,\infty}^2).$$

Thirdly, it can be seen that

$$\begin{aligned}
R_{i,j_1j_1j_1j_2j_1j_2j_3j_3} &= |\mathrm{E}(Z_{i,t_1,j_1}Z_{i,t_2,j_1}Z_{i,t_3,j_1}Z_{i,t_5,j_1})\mathrm{E}(Z_{i,t_4,j_2}Z_{i,t_6,j_2})\mathrm{E}(Z_{i,t_7,j_3}Z_{i,t_8,j_3})| \\
&\le \Delta_z^{3/4}|\mathrm{Cov}(Z_{i,t_7,j_3}, Z_{i,t_8,j_3})| \le 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{j_1,j_2,j_3=1}^{r} |b_{i,j_1j_1}b_{i,j_1j_2}^2 b_{i,j_3j_3}|R_{i,j_1j_1j_1j_2j_1j_2j_3j_3} \\
&\le \sum_{j_1=1}^{r} |b_{i,j_1j_1}|(\boldsymbol{B}_i^2)_{j_1j_1}\mathrm{tr}(\tilde{\boldsymbol{B}}_i)\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3 \\
&\le \left(\sum_{j_1=1}^{r} b_{i,j_1j_1}^2\right)^{1/2}\left\{\sum_{j_1=1}^{r}(\boldsymbol{B}_i^2)_{j_1j_1}^2\right\}^{1/2}\mathrm{tr}(\tilde{\boldsymbol{B}}_i)\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3 \\
&\le \mathrm{tr}^{1/2}(\mathbf{\Sigma}_{i,\infty}^2)\mathrm{tr}^{1/2}(\boldsymbol{B}_i^2\circ\boldsymbol{B}_i^2)\mathrm{tr}(\tilde{\boldsymbol{B}}_i)\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3 \\
&\le \mathrm{tr}^{1/2}(\mathbf{\Sigma}_{i,\infty}^2)\mathrm{tr}^{1/2}(\mathbf{\Sigma}_{i,\infty}^4)\mathrm{tr}(\tilde{\boldsymbol{B}}_i)\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3.
\end{aligned}$$

Fourthly,

$$
\begin{aligned}
&\sum_{j_1,j_2,j_3=1}^{r} |b_{i,j_1j_1}b_{i,j_1j_2}b_{i,j_1j_3}b_{i,j_2j_3}|R_{i,j_1j_1j_1j_2j_1j_3j_2j_3}\\
&\le \Delta_z \sum_{j_1,j_2=1}^{r} |b_{i,j_1j_1}b_{i,j_1j_2}|(\tilde{\boldsymbol{B}}_i^2)_{j_1j_2}\\
&\le \Delta_z \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_1}^2 b_{i,j_2j_2}^2\right)^{1/2} \left\{\sum_{j_1,j_2=1}^{r} (\tilde{\boldsymbol{B}}_i^2)_{j_1j_2}^2\right\}^{1/2}\\
&\le \Delta_z \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)\mathrm{tr}^{1/2}(\tilde{\boldsymbol{B}}_i^4) \le \Delta_z \mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2).
\end{aligned}
$$

Fifthly,

$$
\sum_{j_1,j_2,j_3=1}^{r} b_{i,j_1j_2}^2 b_{i,j_1j_3}^2 R_{i,j_1j_2j_1j_2j_1j_3j_1j_3} \le \Delta_z \sum_{j_1=1}^{r} (\boldsymbol{B}_i^2)_{j_1j_1}^2 = \Delta_z \mathrm{tr}(\boldsymbol{B}_i^2 \circ \boldsymbol{B}_i^2) \le \Delta_z \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^4).
$$

Case 6: Three of $j_1, j_2, \ldots, j_8$ are equal, and among the rest five j's, three of them are equal and the other two are equal. There are five scenarios in this case. Firstly, by the Davydov's inequality,

$$
\begin{aligned}
R_{i,j_1j_1j_1j_2j_2j_2j_3j_3} &= |\mathrm{E}(Z_{i,t_1,j_1}Z_{i,t_2,j_1}Z_{i,t_3,j_1})\mathrm{E}(Z_{i,t_4,j_2}Z_{i,t_5,j_2}Z_{i,t_6,j_2})\mathrm{E}(Z_{i,t_7,j_3}Z_{i,t_8,j_3})|\\
&\le \Delta_z^{3/4}|\mathrm{Cov}(Z_{i,t_7,j_3}, Z_{i,t_8,j_3})| \le 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\sum_{j_1,j_2,j_3=1}^{r} |b_{i,j_1j_1}b_{i,j_1j_2}b_{i,j_2j_2}b_{i,j_3j_3}|R_{i,j_1j_1j_1j_2j_2j_2j_3j_3}\\
&\le \mathrm{tr}(\tilde{\boldsymbol{B}}_i)\sum_{j_1=1}^{r}|b_{i,j_1j_1}|\left(\sum_{j_2=1}^{r} b_{i,j_1j_2}^2\right)^{1/2}\left(\sum_{j_2=1}^{r} b_{i,j_2j_2}^2\right)^{1/2}\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3\\
&\le \mathrm{tr}(\tilde{\boldsymbol{B}}_i)\mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2)\sum_{j_1=1}^{r}|b_{i,j_1j_1}|\left(\sum_{j_2=1}^{r} b_{i,j_1j_2}^2\right)^{1/2}\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3\\
&\le \mathrm{tr}(\tilde{\boldsymbol{B}}_i)\mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2)\left(\sum_{j_1=1}^{r} b_{i,j_1j_1}^2\right)^{1/2}\left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^2\right)^{1/2}\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3\\
&\le \mathrm{tr}(\tilde{\boldsymbol{B}}_i)\mathrm{tr}^{3/2}(\boldsymbol{\Sigma}_{i,\infty}^2)\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3.
\end{aligned}
$$

Secondly,

$$\sum_{j_1,j_2,j_3=1}^{r} |b_{i,j_1j_1} b_{i,j_1j_2} b_{i,j_2j_3}^2| R_{i,j_1j_1j_1j_2j_2j_3j_2j_3} \le \Delta_z \sum_{j_1,j_2=1}^{r} |b_{i,j_1j_1} b_{i,j_1j_2}| (\boldsymbol{B}_i^2)_{j_2j_2}$$

$$\le \Delta_z \sum_{j_2=1}^{r} \left(\sum_{j_1=1}^{r} b_{i,j_1j_1}^2 \right)^{1/2} \left(\sum_{j_1=1}^{r} b_{i,j_1j_2}^2 \right)^{1/2} (\boldsymbol{B}_i^2)_{j_2j_2}$$

$$\le \Delta_z \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2) \sum_{j_2=1}^{r} \left(\sum_{j_1=1}^{r} b_{i,j_1j_2}^2 \right)^{1/2} (\boldsymbol{B}_i^2)_{j_2j_2}$$

$$\le \Delta_z \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2) \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^2 \right)^{1/2} \left\{ \sum_{j_2=1}^{r} (\boldsymbol{B}_i^2)_{j_2j_2}^2 \right\}^{1/2}$$

$$= \Delta_z \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2) \mathrm{tr}^{1/2}(\boldsymbol{B}_i^2 \circ \boldsymbol{B}_i^2)$$

$$\le \Delta_z \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2) \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^4).$$

Thirdly,

$$\sum_{j_1,j_2,j_3=1}^{r} |b_{i,j_1j_1} b_{i,j_2j_2} b_{i,j_1j_3} b_{i,j_2j_3}| R_{i,j_1j_1j_2j_2j_1j_3j_2j_3}$$

$$\le \Delta_z \sum_{j_1,j_2=1}^{r} |b_{i,j_1j_1} b_{i,j_2j_2}| (\tilde{\boldsymbol{B}}_i^2)_{j_1j_2}$$

$$\le \Delta_z \sum_{j_1=1}^{r} |b_{i,j_1j_1}| \left(\sum_{j_2=1}^{r} b_{i,j_2j_2}^2 \right)^{1/2} \left\{ \sum_{j_2=1}^{r} (\tilde{\boldsymbol{B}}_i^2)_{j_1j_2}^2 \right\}^{1/2}$$

$$\le \Delta_z \mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2) \left(\sum_{j_1=1}^{r} b_{i,j_1j_1}^2 \right)^{1/2} \left\{ \sum_{j_1,j_2=1}^{r} (\tilde{\boldsymbol{B}}_i^2)_{j_1j_2}^2 \right\}^{1/2}$$

$$\le \Delta_z \mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2) \mathrm{tr}^{1/2}(\tilde{\boldsymbol{B}}_i^4) \le \Delta_z \mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2),$$

where the last inequality holds due to (S1.18). Fourthly, by the Davydov's inequality,

$$R_{i,j_1j_2j_1j_2j_1j_2j_3j_3} = |\mathrm{E}(Z_{i,t_1,j_1} Z_{i,t_3,j_1} Z_{i,t_5,j_1}) \mathrm{E}(Z_{i,t_2,j_2} Z_{i,t_4,j_2} Z_{i,t_6,j_2}) \mathrm{E}(Z_{i,t_7,j_3} Z_{i,t_8,j_3})|$$

$$\le \Delta_z^{3/4} |\mathrm{Cov}(Z_{i,t_7,j_3}, Z_{i,t_8,j_3})| \le 8\Delta_z c^{3/4} \exp\{-3a\tilde{b}/4\}/3.$$

Then we have

$$
\begin{aligned}
&\sum_{j_1,j_2,j_3=1}^{r} |b_{i,j_1j_2}^3 b_{i,j_3j_3}| R_{i,j_1j_2j_1j_2j_1j_2j_3j_3} \\
&\le \operatorname{tr}(\tilde{\boldsymbol{B}}_i) \sum_{j_1,j_2=1}^{r} |b_{i,j_1j_2}^3| \cdot 8\Delta_z c^{3/4} \exp\{-3a\tilde{b}/4\}/3 \\
&\le \operatorname{tr}(\tilde{\boldsymbol{B}}_i) \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^4\right)^{1/2} \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^2\right)^{1/2} \cdot 8\Delta_z c^{3/4} \exp\{-3a\tilde{b}/4\}/3 \\
&\le \operatorname{tr}(\tilde{\boldsymbol{B}}_i) \operatorname{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^4) \operatorname{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2) \cdot 8\Delta_z c^{3/4} \exp\{-3a\tilde{b}/4\}/3.
\end{aligned}
$$

Fifthly,

$$
\begin{aligned}
&\sum_{j_1,j_2,j_3=1}^{r} b_{i,j_1j_2}^2 |b_{i,j_1j_3} b_{i,j_2j_3}| R_{i,j_1j_2j_1j_2j_1j_3j_2j_3} \le \Delta_z \sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^2 (\tilde{\boldsymbol{B}}_i^2)_{j_1j_2} \\
&\le \Delta_z \left(\sum_{j_1,j_2=1}^{r} b_{i,j_1j_2}^4\right)^{1/2} \left\{\sum_{j_1,j_2=1}^{r} (\tilde{\boldsymbol{B}}_i^2)_{j_1j_2}^2\right\}^{1/2} \le \Delta_z \operatorname{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^4) \operatorname{tr}^{1/2}(\tilde{\boldsymbol{B}}_i^4) \\
&\le \Delta_z \operatorname{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^4) \operatorname{tr}(\boldsymbol{\Sigma}_{i,\infty}^2).
\end{aligned}
$$

Case 7: The index $j_1, j_2, \ldots, j_8$ can be divided into four group where in each group there are two elements equal to each other. There are five scenarios in this case. Firstly, by the Davydov's inequality,

$$
\begin{aligned}
&R_{i,j_1j_1j_2j_2j_3j_3j_4j_4} \\
&= |\mathrm{E}(Z_{i,t_1,j_1} Z_{i,t_2,j_1}) \mathrm{E}(Z_{i,t_3,j_2} Z_{i,t_4,j_2}) \mathrm{E}(Z_{i,t_5,j_3} Z_{i,t_6,j_3}) \mathrm{E}(Z_{i,t_7,j_4} Z_{i,t_8,j_4})| \\
&= |\mathrm{Cov}(Z_{i,t_1,j_1}, Z_{i,t_2,j_1}) \mathrm{Cov}(Z_{i,t_3,j_2}, Z_{i,t_4,j_2}) \mathrm{Cov}(Z_{i,t_5,j_3}, Z_{i,t_6,j_3}) \mathrm{Cov}(Z_{i,t_7,j_4}, Z_{i,t_8,j_4})| \\
&\le \Delta_z (8c^{3/4} \exp\{-3a\tilde{b}/4\}/3)^4 = 4096\Delta_z c^3 \exp\{-3a\tilde{b}\}/81.
\end{aligned}
$$

Then we have

$$
\sum_{j_1,j_2,j_3,j_4=1}^{r} |b_{i,j_1j_1} b_{i,j_2j_2} b_{i,j_3j_3} b_{i,j_4j_4}| R_{i,j_1j_1j_2j_2j_3j_3j_4j_4} \le \operatorname{tr}^4(\tilde{\boldsymbol{B}}_i) \cdot 4096\Delta_z c^3 \exp\{-3a\tilde{b}\}/81.
$$

Secondly, since

$$\begin{aligned} R_{i,j_1j_1j_2j_2j_3j_4j_3j_4} &= |\mathrm{E}(Z_{i,t_1,j_1}Z_{i,t_2,j_1})\mathrm{E}(Z_{i,t_3,j_2}Z_{i,t_4,j_2})\mathrm{E}(Z_{i,t_5,j_3}Z_{i,t_7,j_3})\mathrm{E}(Z_{i,t_6,j_4}Z_{i,t_8,j_4})| \\ &\le \Delta_z^{1/2}|\mathrm{Cov}(Z_{i,t_1,j_1}, Z_{i,t_2,j_1})\mathrm{Cov}(Z_{i,t_3,j_2}, Z_{i,t_4,j_2})| \\ &\le \Delta_z(8c^{3/4}\exp\{-3a\tilde{b}/4\}/3)^2 = 64\Delta_z c^{3/2}\exp\{-3a\tilde{b}/2\}/9, \end{aligned}$$

it can be derived that

$$\sum_{j_1,j_2,j_3,j_4=1}^{r} |b_{i,j_1j_1}b_{i,j_2j_2}|b_{i,j_3j_4}^2 R_{i,j_1j_1j_2j_2j_3j_4j_3j_4} \le \mathrm{tr}^2(\tilde{\boldsymbol{B}}_i)\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)\cdot 64\Delta_z c^2\exp\{-3a\tilde{b}/2\}/9.$$

Thirdly, notice that

$$\begin{aligned} R_{i,j_1j_1j_2j_3j_3j_4j_2j_4} &= |\mathrm{E}(Z_{i,t_1,j_1}Z_{i,t_2,j_1})\mathrm{E}(Z_{i,t_3,j_2}Z_{i,t_7,j_2})\mathrm{E}(Z_{i,t_4,j_3}Z_{i,t_5,j_3})\mathrm{E}(Z_{i,t_6,j_4}Z_{i,t_8,j_4})| \\ &\le \Delta_z^{3/4}|\mathrm{Cov}(Z_{i,t_1,j_1}, Z_{i,t_2,j_1})| \le 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3. \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{j_1,j_2,j_3,j_4=1}^{r} |b_{i,j_1j_1}b_{i,j_2j_3}b_{i,j_3j_4}b_{i,j_2j_4}|R_{i,j_1j_1j_2j_3j_3j_4j_2j_4} \\ &\le \mathrm{tr}(\tilde{\boldsymbol{B}}_i)\sum_{j_2,j_3=1}^{r}|b_{i,j_2j_3}|(\tilde{\boldsymbol{B}}_i^2)_{j_2j_3}\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3 \\ &\le \mathrm{tr}(\tilde{\boldsymbol{B}}_i)\left(\sum_{j_2,j_3=1}^{r} b_{i,j_2j_3}^2\right)^{1/2}\left\{\sum_{j_2,j_3=1}^{r}(\tilde{\boldsymbol{B}}_i^2)_{j_2j_3}^2\right\}^{1/2}\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3 \\ &\le \mathrm{tr}(\tilde{\boldsymbol{B}}_i)\mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2)\mathrm{tr}^{1/2}(\tilde{\boldsymbol{B}}_i^4)\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3 \\ &\le \mathrm{tr}(\tilde{\boldsymbol{B}}_i)\mathrm{tr}^{3/2}(\boldsymbol{\Sigma}_{i,\infty}^2)\cdot 8\Delta_z c^{3/4}\exp\{-3a\tilde{b}/4\}/3. \end{aligned}$$

Fourthly,

$$\begin{aligned} &\sum_{j_1,j_2,j_3,j_4=1}^{r} |b_{i,j_1j_2}b_{i,j_2j_3}b_{i,j_3j_4}b_{i,j_4j_1}|R_{i,j_1j_2j_2j_3j_3j_4j_4j_1} \\ &\le \Delta_z\sum_{j_1,j_3=1}^{r}(\tilde{\boldsymbol{B}}_i^2)_{j_1j_3}^2 = \Delta_z\mathrm{tr}(\tilde{\boldsymbol{B}}_i^4) \le \Delta_z\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^4). \end{aligned}$$

Fifthly,

$$\sum_{j_1,j_2,j_3,j_4=1}^{r} b_{i,j_1j_2}^2 b_{i,j_3j_4}^2 R_{i,j_1j_2j_1j_2j_3j_4j_3j_4} \le \Delta_z \mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2).$$

To sum up Cases 1 to 7, according to Assumption 6, we have

$$\begin{aligned}
&|\mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_5}^{\mathrm{T}}\boldsymbol{X}_{i,t_6}\boldsymbol{X}_{i,t_7}^{\mathrm{T}}\boldsymbol{X}_{i,t_8})| \\
&\le O\{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)\} + O\{\mathrm{tr}^{3/2}(\boldsymbol{\Sigma}_{i,\infty}^2)\mathrm{tr}(\tilde{\boldsymbol{B}}_i)\exp(-3a\tilde{b}/4)\} \\
&\quad + O\{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)\mathrm{tr}^2(\tilde{\boldsymbol{B}}_i)\exp(-3a\tilde{b}/2)\} + O\{\mathrm{tr}^4(\tilde{\boldsymbol{B}}_i)\exp(-3a\tilde{b})\}.
\end{aligned}$$

Under Assumptions 5 and 7, it can be derived that

$$\begin{aligned}
\frac{\mathrm{tr}(\tilde{\boldsymbol{B}}_i)}{\mathrm{tr}^{1/2}(\boldsymbol{\Sigma}_{i,\infty}^2)}\exp(-3a\tilde{b}/4) &\le \frac{p\lambda_{\max}(\tilde{\boldsymbol{B}}_i)}{p^{1/2}\lambda_{i,\min}}\exp(-3a\tilde{b}/4) \le \frac{C_2p^{\psi+1}}{C_1p^{1/2+\eta}}\exp(-3a\tilde{b}/4) \\
&= C_2C_1^{-1}\exp(-3a\tilde{b}/4 + (\psi-\eta+1/2)\log p) \to 0,
\end{aligned}$$

where the last limit holds since $\tilde{b} \ge c_2 \log p$ for sufficiently large $c_2 > 0$. It follows that

$$|\mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_5}^{\mathrm{T}}\boldsymbol{X}_{i,t_6}\boldsymbol{X}_{i,t_7}^{\mathrm{T}}\boldsymbol{X}_{i,t_8})| \le O\{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)\}.$$

The proof has been completed. □

S1.7 Proof of Theorem 2

Proof. We provide the variance of $T(b)$ under the null hypothesis in Corollary 1 by

$$\mathrm{Var}\{T(b)\} = \left\{2n_1^{-2}\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}^2) + 2n_2^{-2}\mathrm{tr}(\boldsymbol{\Sigma}_{2,\infty}^2) + 4(n_1n_2)^{-1}\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\Sigma}_{2,\infty})\right\}\{1+o(1)\}$$

and the variance estimation in (4.10) by

$$V_n\big(\tilde{b},s_0\big) = 2n_1^{-2}\widehat{\mathrm{tr}}\big(\boldsymbol{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big) + 2n_2^{-2}\widehat{\mathrm{tr}}\big(\boldsymbol{\Sigma}_{2,\infty}^2;\tilde{b},s_0\big) + 4(n_1n_2)^{-1}\widehat{\mathrm{tr}}\big(\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\Sigma}_{2,\infty};\tilde{b},s_0\big).$$

If $\widehat{\mathrm{tr}}\big(\boldsymbol{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)/\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2) \xrightarrow{P} 1$ for $i=1,2$ and $\widehat{\mathrm{tr}}\big(\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\Sigma}_{2,\infty};\tilde{b},s_0\big)/\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\Sigma}_{2,\infty}) \xrightarrow{P} 1$ as $n,p\to\infty$, it can be derived straightforwardly that $V_n\big(\tilde{b},s_0\big)/\mathrm{Var}\{T(b)\} \xrightarrow{P} 1$. In the following, we derive $\widehat{\mathrm{tr}}\big(\boldsymbol{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)/\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2) \xrightarrow{P} 1$ as $n,p\to\infty$ and the ratio consistency of

$\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{1,\infty}\mathbf{\Sigma}_{2,\infty};\tilde{b},s_0\big)$ can be obtained analogously. It can be obtained that

$$\mathrm{E}\left[\left\{\frac{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)}{\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)}-1\right\}^2\right]=\left[\mathrm{E}\left\{\frac{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)}{\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)}\right\}-1\right]^2+\mathrm{Var}\left\{\frac{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)}{\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)}\right\}. \tag{S1.20}$$

From the above equation, we only need to prove the two terms on the right hand side converge to zero respectively as $n,p\to\infty$.

Firstly, we prove $\mathrm{E}\{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)/\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)\}\to 1$ as $n,p\to\infty$. By the definition of $\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)$ in (4.9), we have

$$\mathrm{E}\left\{\frac{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)}{\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)}\right\}=\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}\sum_{k_1=-n_i+1}^{n_i-1}\sum_{k_2=-n_i+1}^{n_i-1}\mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{E}\{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2};\tilde{b}\big)\}.$$

From (4.7), $\mathrm{E}\{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2};\tilde{b}\big)\}$ can be expressed as

$$\mathrm{E}\{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2};\tilde{b}\big)\}=\mathrm{E}\{G_{i,1}\big(k_1,k_2;\tilde{b}\big)\}-\mathrm{E}\{G_{i,2}\big(k_1;\tilde{b}\big)\}-\mathrm{E}\{G_{i,2}\big(k_2;\tilde{b}\big)\}+\mathrm{E}\{G_{i,3}\big(\tilde{b}\big)\}.$$

The first term $\mathrm{E}\{G_{i,1}\big(k_1,k_2;\tilde{b}\big)\}$ on the right hand side of the above equation is equal to

$$\mathrm{E}\{G_{i,1}\big(k_1,k_2;\tilde{b}\big)\}=\Big|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\Big|^{-1}\sum_{(t_1,t_2)\in\mathcal{N}_{i,1}\left(k_1,k_2;\tilde{b}\right)}\mathrm{tr}\big\{\mathrm{E}(\boldsymbol{X}_{i_1,t_1}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_2}^{\mathrm{T}})\big\}.$$

Let $\mathbf{\Sigma}_{i,k}^0=\mathrm{E}(\boldsymbol{X}_{i,t+k}\boldsymbol{X}_{i,t}^{\mathrm{T}})$. Notice that

$$\begin{aligned}&\Big|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\Big|^{-1}\sum_{(t_1,t_2)\in\mathcal{N}_{i,1}\left(k_1,k_2;\tilde{b}\right)}\mathrm{tr}\big\{\mathrm{E}(\boldsymbol{X}_{i,t_1}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_2}^{\mathrm{T}})\big\}\\ &=\Big|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\Big|^{-1}\sum_{(t_1,t_2)\in\mathcal{N}_{i,1}\left(k_1,k_2;\tilde{b}\right)}\mathrm{tr}\big(\mathbf{\Sigma}_{i,k_1}^0\mathbf{\Sigma}_{i,k_2}^0\big)=\mathrm{tr}\big(\mathbf{\Sigma}_{i,k_1}^0\mathbf{\Sigma}_{i,k_2}^0\big).\end{aligned}$$

From Lemma 1, under Assumption 3, by the Davydov's inequality,

$$|\mathrm{E}\{G_{i,1}\big(k_1,k_2;\tilde{b}\big)\}-\mathrm{tr}\big(\mathbf{\Sigma}_{i,k_1}^0\mathbf{\Sigma}_{i,k_2}^0\big)|\le\frac{2q\Delta^4c^{1-4/q}p^2}{q-4}\exp\{-a(1-4/q)\tilde{b}\}. \tag{S1.21}$$

For $\mathrm{E}\{G_{i,2}\big(k;\tilde{b}\big)\}$, it can be obtained straightforwardly by (4.6) that

$$\mathrm{E}\{G_{i,2}\big(k;\tilde{b}\big)\}=\Big|\mathcal{N}_{i,2}\big(k;\tilde{b}\big)\Big|^{-1}\sum_{(t_1,t_2,t_3)\in\mathcal{N}_{i,2}\left(k;\tilde{b}\right)}\mathrm{E}(\boldsymbol{X}_{i,t_2}^{T}\boldsymbol{X}_{i,t_1}\boldsymbol{X}_{i,t_1-k}^{\mathrm{T}}\boldsymbol{X}_{i,t_3}).$$

Notice that

$$\begin{aligned}&\left|\mathcal{N}_{i,2}\big(k;\tilde{b}\big)\right|^{-1}\sum_{(t_1,t_2,t_3)\in\mathcal{N}_{i,2}\left(k;\tilde{b}\right)}\boldsymbol{\mu}_i^{\mathrm{T}}\mathrm{E}(\boldsymbol{X}_{i,t_1}\boldsymbol{X}_{i,t_1-k}^{\mathrm{T}})\boldsymbol{\mu}_i\\&=\left|\mathcal{N}_{i,2}\big(k;\tilde{b}\big)\right|^{-1}\sum_{(t_1,t_2,t_3)\in\mathcal{N}_{i,2}\left(k;\tilde{b}\right)}\boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}^{0}\boldsymbol{\mu}_i=\boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}^{0}\boldsymbol{\mu}_i.\end{aligned}$$

Then according to Lemma 1,

$$\begin{aligned}|\mathrm{E}\{G_{i,2}\big(k;\tilde{b}\big)\}-\boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}^{0}\boldsymbol{\mu}_i|\le{}&\frac{2qc^{1-4/q}\Delta^4}{q-4}\exp\{-a(1-4/q)\tilde{b}+2\log p\}\\&+\frac{2qc^{1-2/q}\Delta^4}{q-2}\exp\{-a(1-2/q)\tilde{b}+2\log p\}.\end{aligned}\tag{S1.22}$$

For $\mathrm{E}\{G_{i,3}\big(\tilde{b}\big)\}$, we can obtain by (4.6) that

$$\mathrm{E}\{G_{i,3}\big(\tilde{b}\big)\}=\left|\mathcal{N}_{i,3}\big(\tilde{b}\big)\right|^{-1}\sum_{(t_1,t_2,t_3,t_4)\in\mathcal{N}_{i,3}\left(\tilde{b}\right)}\mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}).$$

Notice that

$$\left|\mathcal{N}_{i,3}\big(\tilde{b}\big)\right|^{-1}\sum_{(t_1,t_2,t_3,t_4)\in\mathcal{N}_{i,3}\left(\tilde{b}\right)}\mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{i,t_2})\mathrm{E}(\boldsymbol{X}_{i,t_3}^{\mathrm{T}})\mathrm{E}(\boldsymbol{X}_{i,t_4})=\big(\boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\mu}_i\big)^2.$$

Then by Lemma 1, we have

$$\begin{aligned}|\mathrm{E}\{G_{i,3}\big(\tilde{b}\big)\}-\big(\boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\mu}_i\big)^2|\le{}&\frac{2qc^{1-4/q}\Delta^4p^2}{q-4}\exp\{-a(1-4/q)\tilde{b}\}\\&+\frac{4qc^{1-2/q}\Delta^4p^2}{q-2}\exp\{-a(1-2/q)\tilde{b}\}.\end{aligned}\tag{S1.23}$$

Let $\Delta(k_1,k_2;\tilde{b})=G_{i,1}\big(k_1,k_2;\tilde{b}\big)-\mathrm{tr}\big(\boldsymbol{\Sigma}_{i,k_1}^{0}\boldsymbol{\Sigma}_{i,k_2}^{0}\big)$, $\Delta(k;\tilde{b})=G_{i,2}\big(k;\tilde{b}\big)-\boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k}^{0}\boldsymbol{\mu}_i$, and $\Delta_3(\tilde{b})=G_{i,3}\big(\tilde{b}\big)-\big(\boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\mu}_i\big)^2$. Then

$$\begin{aligned}&\mathrm{E}\left\{\frac{\widehat{\mathrm{tr}}\big(\boldsymbol{\Sigma}_{i,\infty}^{2};\tilde{b},s_0\big)}{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^{2})}\right\}=\sum_{k_1=-n_i+1}^{n_i-1}\sum_{k_2=-n_i+1}^{n_i-1}\mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^{2})^{-1}\mathrm{E}\{\widehat{\mathrm{tr}}\big(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2};\tilde{b}\big)\}\\&=\sum_{k_1=-n_i+1}^{n_i-1}\sum_{k_2=-n_i+1}^{n_i-1}\mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^{2})^{-1}\big[\big\{\Delta(k_1,k_2;\tilde{b})-\Delta(k_1;\tilde{b})-\Delta(k_2;\tilde{b})\\&\quad+\Delta_3(\tilde{b})\big\}+\big\{\mathrm{tr}\big(\boldsymbol{\Sigma}_{i,k_1}^{0}\boldsymbol{\Sigma}_{i,k_2}^{0}\big)-\boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k_1}^{0}\boldsymbol{\mu}_i-\boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\Sigma}_{i,k_2}^{0}\boldsymbol{\mu}_i+\big(\boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\mu}_i\big)^2\big\}\big]\end{aligned}$$

$$= \sum_{l=1}^{4} \tilde{\Delta}_l + \sum_{k_1=-n_i+1}^{n_i-1} \sum_{k_2=-n_i+1}^{n_i-1} \mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}\mathrm{tr}\big(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2}\big),$$

where

$$\tilde{\Delta}_1 = \sum_{k_1=-n_i+1}^{n_i-1} \sum_{k_2=-n_i+1}^{n_i-1} \mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}\Delta(k_1, k_2; \tilde{b}),$$

$$\tilde{\Delta}_2 = \sum_{k_1=-n_i+1}^{n_i-1} \sum_{k_2=-n_i+1}^{n_i-1} \mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}\Delta(k_1; \tilde{b}),$$

$$\tilde{\Delta}_3 = \sum_{k_1=-n_i+1}^{n_i-1} \sum_{k_2=-n_i+1}^{n_i-1} \mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}\Delta(k_2; \tilde{b}) \quad \text{and}$$

$$\tilde{\Delta}_4 = \sum_{k_1=-n_i+1}^{n_i-1} \sum_{k_2=-n_i+1}^{n_i-1} \mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}\Delta_3(\tilde{b}).$$

By Assumption 5 , there exist $\eta, C_1 > 0$ such that $\lambda_{1,\min} \wedge \lambda_{2,\min} \geq C_1 p^{\eta}$. Moreover, if $\int |\mathcal{K}(u)| du < +\infty$, then $\sum_{k=-n_i+1}^{n_i-1} \mathcal{K}(k/s_0) = O(s_0)$. By (S1.21), (S1.22) and (S1.23),

$$\Bigg| \sum_{k_1=-n_i+1}^{n_i-1} \sum_{k_2=-n_i+1}^{n_i-1} \mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}$$

$$\cdot\big\{\Delta(k_1, k_2; \tilde{b}) - \Delta(k_1; \tilde{b}) - \Delta(k_2; \tilde{b}) + \Delta_3(\tilde{b})\big\}\Bigg|$$

$$\leq \sum_{k_1=-n_i+1}^{n_i-1} \sum_{k_2=-n_i+1}^{n_i-1} |\mathcal{K}(k_1/s_0)||\mathcal{K}(k_2/s_0)| C_1^{-2} p^{-(2\eta+1)}$$

$$\cdot\, O[\exp\{-a(1-4/q)\tilde{b} + 2\log p\} + \exp\{-a(1-4/q)\tilde{b}\} + \exp\{-a(1-2/q)\tilde{b}\}]$$

$$= \left\{ \sum_{k=-n_i+1}^{n_i-1} |\mathcal{K}(k/s_0)| \right\}^2 O[\exp\{-a(1-4/q)\tilde{b} - (2\eta-1)\log p\}$$

$$+ \exp\{-a(1-4/q)\tilde{b} - (2\eta+1)\log p\} + \exp\{-a(1-2/q)\tilde{b} - (2\eta+1)\log p\}]$$

$$= O[\exp\{-a(1-4/q)\tilde{b} - (2\eta-1)\log p + 2s_0\} + \exp\{-a(1-4/q)\tilde{b} - (2\eta+1)\log p + 2s_0\}$$

$$+ \exp\{-a(1-2/q)\tilde{b} - (2\eta+1)\log p + 2s_0\}].$$

If $\tilde{b} \geq c_2(\log p + \log n + s_0)$ for a sufficiently large $c_2 > 0$, $\sum_{l=1}^{4} \tilde{\Delta}_l \to 0$ as $n, p \to \infty$. On the other hand, we have

$$\Bigg|\sum_{k_1=-n_i+1}^{n_i-1}\sum_{k_2=-n_i+1}^{n_i-1}\mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}\mathrm{tr}\big(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2}\big) - \sum_{k_1=-n_i+1}^{n_i-1}\sum_{k_2=-n_i+1}^{n_i-1}\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}\mathrm{tr}\big(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2}\big)\Bigg|$$
$$\le \sum_{k_1=-n_i+1}^{n_i-1}\sum_{k_2=-n_i+1}^{n_i-1}|\mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)-1||\mathrm{tr}\big(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2}\big)|\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}.$$

Since $|\mathcal{K}(u)| \le 1$ and $\mathcal{K}(0) = 1$, from Assumption 4 and by the dominated convergence theorem, the right hand side in the above inequality converges to zero as $n, p \to \infty$. Then

$$\sum_{k_1=-n_i+1}^{n_i-1}\sum_{k_2=-n_i+1}^{n_i-1}\mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)^{-1}\mathrm{tr}\big(\mathbf{\Sigma}_{i,k_1}\mathbf{\Sigma}_{i,k_2}\big) \to 1.$$

Thus we have as $n, p \to \infty$,

$$\mathrm{E}\{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)/\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)\} \to 1. \tag{S1.24}$$

Secondly, we consider $\mathrm{Var}\left\{\frac{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)}{\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)}\right\}$. From (4.7) and (4.9),

$$\begin{aligned}
&\mathrm{Var}\left\{\frac{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)}{\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)}\right\}\\
&= \frac{1}{\mathrm{tr}^2(\mathbf{\Sigma}_{i,\infty}^2)}\sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1}\mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathcal{K}(k_3/s_0)\mathcal{K}(k_4/s_0)\\
&\cdot\big[\mathrm{E}\{G_{i,1}(k_1,k_2;\tilde{b})G_{i,1}(k_3,k_4;\tilde{b})\} - \mathrm{E}\{G_{i,1}(k_1,k_2;\tilde{b})G_{i,2}(k_3;\tilde{b})\}\\
&- \mathrm{E}\{G_{i,1}(k_1,k_2;\tilde{b})G_{i,2}(k_4;\tilde{b})\} + \mathrm{E}\{G_{i,1}(k_1,k_2;\tilde{b})G_{i,3}(\tilde{b})\}\\
&- \mathrm{E}\{G_{i,2}(k_1;\tilde{b})G_{i,1}(k_3,k_4;\tilde{b})\} + \mathrm{E}\{G_{i,2}(k_1;\tilde{b})G_{i,2}(k_3;\tilde{b})\} + \mathrm{E}\{G_{i,2}(k_1;\tilde{b})G_{i,2}(k_4;\tilde{b})\}\\
&- \mathrm{E}\{G_{i,2}(k_1;\tilde{b})G_{i,3}(\tilde{b})\} - \mathrm{E}\{G_{i,2}(k_2;\tilde{b})G_{i,1}(k_3,k_4;\tilde{b})\} + \mathrm{E}\{G_{i,2}(k_2;\tilde{b})G_{i,2}(k_3;\tilde{b})\}\\
&+ \mathrm{E}\{G_{i,2}(k_2;\tilde{b})G_{i,2}(k_4;\tilde{b})\}\big] - \mathrm{E}^2\left\{\frac{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)}{\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)}\right\}\\
&=: \sum_{l=1}^{16}V_l - \mathrm{E}^2\left\{\frac{\widehat{\mathrm{tr}}\big(\mathbf{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)}{\mathrm{tr}(\mathbf{\Sigma}_{i,\infty}^2)}\right\}.
\end{aligned}$$

According to (S1.24), the second term on the right hand side of the above equation satisfies

$\mathrm{E}^2\left\{\frac{\widehat{\mathrm{tr}}\left(\boldsymbol{\Sigma}_{i,\infty}^2;\tilde{b},s_0\right)}{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)}\right\} \to 1$ as $n, p \to \infty$. In the following, we will deal with the first term by considering $V_1, V_2, \ldots, V_{16}$, respectively.

We take V_1 which corresponds to $\mathrm{E}\big\{G_{i,1}(k_1,k_2)G_{i,1}(k_3,k_4)\big\}$ as an example to illustrate the main idea of the proof and provide the derived results for other terms directly. Let $K = \tilde{b}/2$ and introduce the parameter $\tilde{K}$ such that $\tilde{b} \geq c_3\tilde{K}$ and $\tilde{K} \geq c_4 s_0$ for sufficiently large $c_3, c_4 > 0$. Then we can rewrite V_1 as

$$\begin{aligned}
V_1 &= \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1} \\
&\cdot \sum_{(t_1,t_2)\in\mathcal{N}_{i,1}\left(k_1,k_2;\tilde{b}\right),(t_3,t_4)\in\mathcal{N}_{i,1}\left(k_3,k_4;\tilde{b}\right)} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}) \\
&= \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \left(\sum_{|k_1|,|k_2|,|k_3|,|k_4|\leq\tilde{K}}^{*} + \sum_{\exists l=1,2,3,4,|k_l|>\tilde{K}}\right) \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1} \\
&\cdot \Bigg(\sum_{|t_1-t_3|\leq K,|t_2-t_4|\leq K}^{*} + \sum_{|t_1-t_4|\leq K,|t_2-t_3|\leq K}^{*} + \sum_{|t_1-t_3|\leq K,|t_2-t_4|>K}^{*} + \sum_{|t_1-t_4|\leq K,|t_2-t_3|>K}^{*} \\
&+ \sum_{|t_2-t_3|\leq K,|t_1-t_4|>K}^{*} + \sum_{|t_2-t_4|\leq K,|t_1-t_3|>K}^{*} + \sum_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|>K}^{*}\Bigg) \\
&\cdot \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}) \\
&=: \sum_{l=1}^{7} W_{1,l} + \sum_{l=1}^{7} W_{2,l},
\end{aligned}$$

where

$$\begin{aligned}
W_{1,1} &= \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|,|k_3|,|k_4|\leq\tilde{K}}^{*} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1} \\
&\quad \cdot \sum_{|t_1-t_3|\leq K,|t_2-t_4|\leq K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}), \\
W_{1,2} &= \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|,|k_3|,|k_4|\leq\tilde{K}}^{*} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}
\end{aligned}$$

$$\cdot \sum_{|t_1-t_4|\le K,|t_2-t_3|\le K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{1,3} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|,|k_3|,|k_4|\le\tilde{K}}^{*} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}$$

$$\cdot \sum_{|t_1-t_3|\le K,|t_2-t_4|> K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{1,4} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|,|k_3|,|k_4|\le\tilde{K}}^{*} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}$$

$$\cdot \sum_{|t_1-t_4|\le K,|t_2-t_3|> K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{1,5} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|,|k_3|,|k_4|\le\tilde{K}}^{*} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}$$

$$\cdot \sum_{|t_2-t_3|\le K,|t_1-t_4|> K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{1,6} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|,|k_3|,|k_4|\le\tilde{K}}^{*} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}$$

$$\cdot \sum_{|t_2-t_4|\le K,|t_1-t_3|> K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{1,7} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|,|k_3|,|k_4|\le\tilde{K}}^{*} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}$$

$$\cdot \sum_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|> K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{2,1} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{\exists l=1,2,3,4,|k_l|>\tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}$$

$$\cdot \sum_{|t_1-t_3|\le K,|t_2-t_4|\le K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{2,2} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{\exists l=1,2,3,4,|k_l|>\tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}$$

$$\cdot \sum_{|t_1-t_4|\le K,|t_2-t_3|\le K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2}\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{2,3} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{\exists l=1,2,3,4, |k_l|>\tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}\big(k_1, k_2; \tilde{b}\big) \mathcal{N}_{i,1}\big(k_3, k_4; \tilde{b}\big) \right|^{-1}$$
$$\cdot \sum_{|t_1-t_3|\le K, |t_2-t_4|>K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2} \boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2+k_2} \boldsymbol{X}_{i,t_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4} \boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{2,4} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{\exists l=1,2,3,4, |k_l|>\tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}\big(k_1, k_2; \tilde{b}\big) \mathcal{N}_{i,1}\big(k_3, k_4; \tilde{b}\big) \right|^{-1}$$
$$\cdot \sum_{|t_1-t_4|\le K, |t_2-t_3|>K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2} \boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2+k_2} \boldsymbol{X}_{i,t_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4} \boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{2,5} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{\exists l=1,2,3,4, |k_l|>\tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}\big(k_1, k_2; \tilde{b}\big) \mathcal{N}_{i,1}\big(k_3, k_4; \tilde{b}\big) \right|^{-1}$$
$$\cdot \sum_{|t_2-t_3|\le K, |t_1-t_4|>K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2} \boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2+k_2} \boldsymbol{X}_{i,t_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4} \boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4+k_4}),$$

$$W_{2,6} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{\exists l=1,2,3,4, |k_l|>\tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}\big(k_1, k_2; \tilde{b}\big) \mathcal{N}_{i,1}\big(k_3, k_4; \tilde{b}\big) \right|^{-1}$$
$$\cdot \sum_{|t_2-t_4|\le K, |t_1-t_3|>K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2} \boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2+k_2} \boldsymbol{X}_{i,t_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4} \boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4+k_4}) \quad \text{and}$$

$$W_{2,7} = \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{\exists l=1,2,3,4, |k_l|>\tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}\big(k_1, k_2; \tilde{b}\big) \mathcal{N}_{i,1}\big(k_3, k_4; \tilde{b}\big) \right|^{-1}$$
$$\cdot \sum_{|t_1-t_3|, |t_1-t_4|, |t_2-t_3|, |t_2-t_4|>K}^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2} \boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2+k_2} \boldsymbol{X}_{i,t_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4} \boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4+k_4}).$$

In the above equations, $\sum^*$ represents the summation restricted in the set $\{(t_1, t_2, t_3, t_4) : (t_1, t_2) \in \mathcal{N}_{i,1}\big(k_1, k_2; \tilde{b}\big), (t_3, t_4) \in \mathcal{N}_{i,1}\big(k_3, k_4; \tilde{b}\big)\}$.

Next we consider $W_{1,1}, \ldots, W_{1,7}$. Similar with Lemma 5, we have for any $\boldsymbol{\mu}_i$,

$$|\mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2} \boldsymbol{X}_{i,t_1-k}^{\mathrm{T}} \boldsymbol{X}_{i,t_2+k_2} \boldsymbol{X}_{i,t_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4} \boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4+k_4})| \le O\{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)\}.$$

Moreover, for $(l, j) = (3, 4)$ and (4,3),

$$\left| \mathcal{N}_{i,1}\big(k_1, k_2; \tilde{b}\big) \mathcal{N}_{i,1}\big(k_3, k_4; \tilde{b}\big) \right|^{-1} \sum^{*} \mathbb{I}\{|t_1 - t_l| \le K, |t_2 - t_j| \le K\} = O\left(\frac{K^2}{n_i^2}\right),$$
$$\left| \mathcal{N}_{i,1}\big(k_1, k_2; \tilde{b}\big) \mathcal{N}_{i,1}\big(k_3, k_4; \tilde{b}\big) \right|^{-1} \sum^{*} \mathbb{I}\{|t_1 - t_l| \le K, |t_2 - t_j| > K\} = O\left(\frac{K}{n_i}\right), \quad \text{(S1.25)}$$

$$\left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}\sum^{*}\mathbb{I}\{|t_2-t_l|\le K,|t_1-t_j|>K\}=O\left(\frac{K}{n_i}\right).$$

Plugging these results into the expressions of $\{W_{1,j}\}_{j=1}^{6}$, we have for $j=1$ and 2, and $l=3,\cdots,6$,

$$W_{1,j}\le O\left(\frac{K^2s_0^4}{n_i^2}\right)\quad\text{and}\quad W_{1,l}\le O\left(\frac{Ks_0^4}{n_i}\right).$$

Under the conditions $\tilde{b}s_0^4=o(n)$ and $\tilde{b}=o(n)$, $W_{1,1},W_{1,2},\ldots,W_{1,6}\to 0$. In the following, we consider $W_{1,7}$. Define

$$\begin{aligned}\tilde{W}_{1,7}=&\frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}\sum_{|k_1|,|k_2|,|k_3|,|k_4|\le\tilde{K}}\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}\\&\cdot\sum^{*}_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|>K}\mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2})\mathrm{E}(\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4}).\end{aligned}$$

Then

$$\begin{aligned}|W_{1,7}-\tilde{W}_{1,7}|\le&\frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}\sum_{|k_1|,|k_2|,|k_3|,|k_4|\le\tilde{K}}\left|\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\right|\left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}\\&\cdot\sum^{*}_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|>K}|\mathrm{Cov}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}}\boldsymbol{X}_{i,t_2+k_2},\boldsymbol{X}_{i,t_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}}\boldsymbol{X}_{i,t_4+k_4})|\\\le&\frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}\sum_{|k_1|,|k_2|,|k_3|,|k_4|\le\tilde{K}}\left|\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\right|\left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}\sum_{j_1,j_2,j_3,j_4=1}^{p}\\&\cdot\sum^{*}_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|>K}\frac{2q}{q-8}\beta_i^x(K-2\tilde{K})^{1-8/q}\\&\cdot\|X_{i,t_1,j_1}X_{i,t_2,j_1}X_{i,t_1-k_1,j_2}X_{i,t_2+k_2,j_2}\|_{q/4}\|X_{i,t_3,j_3}X_{i,t_4,j_3}X_{i,t_3-k_3,j_4}X_{i,t_4+k_4,j_4}\|_{q/4}\\\le&\frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}\sum_{|k_1|,|k_2|,|k_3|,|k_4|\le\tilde{K}}\left|\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\right|\left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}\sum_{j_1,j_2,j_3,j_4=1}^{p}\\&\cdot\sum^{*}_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|>K}\frac{2qc^{1-8/q}}{q-8}\exp\{-a(1-8/q)(K-2\tilde{K})\}\|X_{i,t_1,j_1}\|_q\|X_{i,t_2,j_1}\|_q\\&\cdot\|X_{i,t_1-k_1,j_2}\|_q\|X_{i,t_2+k_2,j_2}\|_q\|X_{i,t_3,j_3}\|_q\|X_{i,t_4,j_3}\|_q\|X_{i,t_3-k_3,j_4}\|_q\|X_{i,t_4+k_4,j_4}\|_q\\\le&\frac{1}{p^2\lambda_{i,\min}^4}\sum_{|k_1|,|k_2|,|k_3|,|k_4|\le\tilde{K}}\left|\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\right|\left|\mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big)\mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big)\right|^{-1}\sum_{j_1,j_2,j_3,j_4=1}^{p}\end{aligned}$$

$$\cdot \sum_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|>K}^{*} \frac{2qc^{1-8/q}\Delta^8}{q-8} \exp\{-a(1-8/q)(K-2\tilde{K})\}$$

$$\leq O\left[s_0^4 p^{-4\eta+2} \exp\{-a(1-8/q)(K-2\tilde{K})\}\right]$$

$$= O\left[\exp\{-a(1-8/q)(K-2\tilde{K}) + 4\log s_0 + (-4\eta+2)\log p\}\right].$$

Since $\tilde{b} \geq c_2(\log n + \log p + s_0)$ for a sufficiently large c_2 and $\tilde{K} = o(\tilde{b})$, $|W_{1,7} - \tilde{W}_{1,7}| \to 0$ as $n, p \to \infty$. In the similar way as the proof of $W_{1,1}, \ldots, W_{1,6}$, we can prove the terms corresponding to $|t_1 - t_3| \leq K$, $|t_1 - t_4| \leq K$, $|t_2 - t_3| \leq K$ and $|t_2 - t_4| \leq K$ are all smaller order terms. Thus if we define

$$\tilde{W}_1 = \frac{1}{\text{tr}^2(\mathbf{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|,|k_3|,|k_4| \leq \tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}(k_1,k_2;\tilde{b})\mathcal{N}_{i,1}(k_3,k_4;\tilde{b})\right|^{-1}$$

$$\cdot \sum^{*} \text{E}(\boldsymbol{X}_{i,t_1}^{\text{T}}\boldsymbol{X}_{i,t_2}\boldsymbol{X}_{i,t_1-k_1}^{\text{T}}\boldsymbol{X}_{i,t_2+k_2})\text{E}(\boldsymbol{X}_{i,t_3}^{\text{T}}\boldsymbol{X}_{i,t_4}\boldsymbol{X}_{i,t_3-k_3}^{\text{T}}\boldsymbol{X}_{i,t_4+k_4})$$

$$= \frac{1}{\text{tr}^2(\mathbf{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|,|k_3|,|k_4| \leq \tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0)\text{E}\{G_{i,1}(k_1,k_2;\tilde{b})\}\text{E}\{G_{i,1}(k_3,k_4;\tilde{b})\},$$

we have $W_{1,7} - \tilde{W}_1 = o(1)$ as $n, p \to \infty$.

Next we consider $\{W_{2,j}\}_{j=1}^6$. According to (S1.25), we have for $j = 1$ and 2, and $l = 3, \cdots, 6$,

$$W_{2,j} \leq \sum_{\exists l=1,2,3,4, |k_l|>\tilde{K}} \mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathcal{K}(k_3/s_0)\mathcal{K}(k_4/s_0) \cdot O\left(\frac{K^2}{n_i^2}\right) \quad \text{and}$$

$$W_{2,l} \leq \sum_{\exists l=1,2,3,4, |k_l|>\tilde{K}} \mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathcal{K}(k_3/s_0)\mathcal{K}(k_4/s_0) \cdot O\left(\frac{K}{n_i}\right).$$

On the other hand, we have

$$\sum_{\exists l=1,2,3,4, |k_l|>\tilde{K}} \mathcal{K}(k_1/s_0)\mathcal{K}(k_2/s_0)\mathcal{K}(k_3/s_0)\mathcal{K}(k_4/s_0) \leq O(s_0^4) \cdot \frac{1}{s_0} \sum_{|k|>\tilde{K}} \mathcal{K}(k/s_0).$$

Thus, for $j = 1$ and 2, and $l = 3, \cdots, 6$,

$$W_{2,j} \leq O\left(\frac{K^2 s_0^4}{n_i^2}\right) \cdot \frac{1}{s_0} \sum_{|k|>\tilde{K}} \mathcal{K}(k/s_0) \quad \text{and} \quad W_{2,l} \leq O\left(\frac{K s_0^4}{n_i}\right) \cdot \frac{1}{s_0} \sum_{|k|>\tilde{K}} \mathcal{K}(k/s_0).$$

Since $\tilde{K} \to \infty$ as $n \to \infty$ and $\int |\mathcal{K}(u)| du < \infty$, $s_0^{-1} \sum_{|k|>\tilde{K}} \mathcal{K}(k/s_0) \to 0$ as $n \to \infty$, under the conditions $\tilde{b}s_0^4 = o(n)$ and $\tilde{b} = o(n)$, $W_{2,1}, \ldots, W_{2,6} \to 0$ as $n, p \to \infty$.

In the last, we deal with $W_{2,7}$. We can decompose $W_{2,7}$ into the summation of four terms where each term corresponds to one, two, three and four index of $k_1, \ldots, k_4$ with absolute values larger than $\tilde{K}$, respectively. We denote the four terms by $W_{2,7}^{(1)}, \ldots, W_{2,7}^{(4)}$ where the superscript represents the number of index with absolute values larger than $\tilde{K}$. Then $W_{2,7} = \sum_{l=1}^4 W_{2,7}^{(l)}$. Consider the four summation terms one by one. When there is only index with absolute value larger than $\tilde{K}$ for example $|k_1| > \tilde{K}$ and $|k_2|, |k_3|, |k_4| \le \tilde{K}$, by the Davydov's inequality, $W_{2,7}^{(1)} - O(\tilde{W}_{2,7}^{(1)}) = o(1)$ where

$$\begin{aligned}
\tilde{W}_{2,7}^{(1)} = & \frac{1}{\operatorname{tr}^2(\mathbf{\Sigma}_{i,\infty}^2)} \sum_{|k_1|>\tilde{K}, |k_2|,|k_3|,|k_4| \le \tilde{K}} \prod_{l=1}^4 \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}(k_1, k_2; \tilde{b}) \mathcal{N}_{i,1}(k_3, k_4; \tilde{b}) \right|^{-1} \\
& \cdot \sum^{*}_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|>K} \sum_{j_1,\ldots,j_8=1}^{r} b_{i,j_1j_2} b_{i,j_3j_4} b_{i,j_5j_6} b_{i,j_7j_8} \\
& \cdot \mathrm{E}(Z_{i,t_1,j_1}) \mathrm{E}(Z_{i,t_2,j_2} Z_{i,t_2+k_2,j_4}) \mathrm{E}(Z_{i,t_1-k_1,j_3} Z_{i,t_3,j_5} Z_{i,t_4,j_6} Z_{i,t_3-k_3,j_7} Z_{i,t_4+k_4,j_8}) = 0.
\end{aligned}$$

When there are two index with absolute values larger than $\tilde{K}$ for example $|k_1|, |k_2| > \tilde{K}$ and $|k_3|, |k_4| \le \tilde{K}$ or $|k_1|, |k_3| > \tilde{K}$ and $|k_2|, |k_4| \le \tilde{K}$, by the Davydov's inequality, $W_{2,7}^{(2)} - O(\tilde{W}_{2,7}^{(2,1)}) - O(\tilde{W}_{2,7}^{(2,2)}) = o(1)$ where

$$\begin{aligned}
\tilde{W}_{2,7}^{(2,1)} = & \frac{1}{\operatorname{tr}^2(\mathbf{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|>\tilde{K}, |k_3|,|k_4| \le \tilde{K}} \prod_{l=1}^4 \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}(k_1, k_2; \tilde{b}) \mathcal{N}_{i,1}(k_3, k_4; \tilde{b}) \right|^{-1} \\
& \cdot \sum^{*}_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|>K} \sum_{j_1,\ldots,j_8=1}^{r} b_{i,j_1j_2} b_{i,j_3j_4} b_{i,j_5j_6} b_{i,j_7j_8} \\
& \cdot \mathrm{E}(Z_{i,t_1,j_1}) \mathrm{E}(Z_{i,t_2,j_2}) \mathrm{E}(Z_{i,t_1-k_1,j_3} Z_{i,t_2+k_2,j_4} Z_{i,t_3,j_5} Z_{i,t_4,j_6} Z_{i,t_3-k_3,j_7} Z_{i,t_4+k_4,j_8}) = 0, \\
\tilde{W}_{2,7}^{(2,2)} = & \frac{1}{\operatorname{tr}^2(\mathbf{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_3|>\tilde{K}, |k_2|,|k_4| \le \tilde{K}} \prod_{l=1}^4 \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}(k_1, k_2; \tilde{b}) \mathcal{N}_{i,1}(k_3, k_4; \tilde{b}) \right|^{-1} \\
& \cdot \sum^{**}_{|t_1-t_3+k_3|,|t_3-t_1+k_1| \le K_1} \sum_{j_1,\ldots,j_8=1}^{r} b_{i,j_1j_2} b_{i,j_3j_4} b_{i,j_5j_6} b_{i,j_7j_8} \\
& \cdot \mathrm{E}(Z_{i,t_1,j_1} Z_{i,t_2,j_2} Z_{i,t_2+k_2,j_4} Z_{i,t_3-k_3,j_7}) \mathrm{E}(Z_{i,t_1-k_1,j_3} Z_{i,t_3,j_5} Z_{i,t_4,j_6} Z_{i,t_4+k_4,j_8}),
\end{aligned}$$

where $K_1 = o(n)$, $K_1 s_0^4 = o(n)$ and $\sum^{**}$ represents the summation over $t_1, \ldots, t_4$ such that $(t_1, t_2) \in \mathcal{N}_{i,1}(k_1, k_2; \tilde{b})$, $(t_3, t_4) \in \mathcal{N}_{i,1}(k_3, k_4; \tilde{b})$ and $|t_1 - t_3|, |t_1 - t_4|, |t_2 - t_3|, |t_2 - t_4| > K$. Note that $\sum^{**}_{|t_1-t_3+k_3|,|t_3-t_1+k_1|\le K_1} = O(n_i^2 K_1^2)$. By using the similar method of dealing with $W_{2,1}, \ldots, W_{2,6}$, we can prove $\tilde{W}_{2,7}^{(2,2)} = o(1)$. When there are three index with absolute values larger than $\tilde{K}$ for example $|k_1|, |k_2|, |k_3| > \tilde{K}$ and $|k_4| \le \tilde{K}$, it can be derived that $W_{2,7}^{(3)} - O(\tilde{W}_{2,7}^{(3)}) = o(1)$ where

$$
\begin{aligned}
\tilde{W}_{2,7}^{(3)} = \frac{1}{\operatorname{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} & \sum_{|k_1|,|k_2|,|k_3|>\tilde{K},|k_4|\le\tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}(k_1, k_2; \tilde{b})\mathcal{N}_{i,1}(k_3, k_4; \tilde{b})\right|^{-1} \\
& \cdot \sum^{*}_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|>K} \sum_{j_1,\ldots,j_8=1}^{r} b_{i,j_1j_2} b_{i,j_3j_4} b_{i,j_5j_6} b_{i,j_7j_8} \\
& \cdot \mathrm{E}(Z_{i,t_1,j_1} Z_{i,t_2,j_2} Z_{i,t_3-k_3,j_7}) \mathrm{E}(Z_{i,t_1-k_1,j_3} Z_{i,t_2+k_2,j_4} Z_{i,t_3,j_5} Z_{i,t_4,j_6} Z_{i,t_4+k_4,j_8}) \\
& \qquad \cdot I(|t_3 - k_3 - t_1 + k_1|, |t_3 - k_3 - t_2 - k_2| > \tilde{K}) \\
& + \frac{1}{\operatorname{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum^{*}_{|k_1|,|k_2|,|k_3|>\tilde{K},|k_4|\le\tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}(k_1, k_2; \tilde{b})\mathcal{N}_{i,1}(k_3, k_4; \tilde{b})\right|^{-1} \\
& \cdot \sum^{*}_{|t_1-t_3|,|t_1-t_4|,|t_2-t_3|,|t_2-t_4|>K} \sum_{j_1,\ldots,j_8=1}^{r} b_{i,j_1j_2} b_{i,j_3j_4} b_{i,j_5j_6} b_{i,j_7j_8} \\
& \mathrm{E}(Z_{i,t_1,j_1}) \mathrm{E}(Z_{i,t_2,j_2}) \mathrm{E}(Z_{i,t_1-k_1,j_3} Z_{i,t_2+k_2,j_4} Z_{i,t_3,j_5} Z_{i,t_4,j_6} Z_{i,t_3-k_3,j_7} Z_{i,t_4+k_4,j_8}) \\
& \qquad \cdot I\{\min(|t_3 - k_3 - t_1 + k_1|, |t_3 - k_3 - t_2 - k_2|) \le \tilde{K}\} \\
= o(1), &
\end{aligned}
$$

where the last equality holds since $|t_1 - t_2| \ge \tilde{b}$ in the term $\mathrm{E}(Z_{i,t_1,j_1} Z_{i,t_2,j_2} Z_{i,t_3-k_3,j_7})$ such that $t_3 - k_3$ can not be close to t_1 and t_2 simultaneously and hence $\mathrm{E}(Z_{i,t_1,j_1} Z_{i,t_2,j_2} Z_{i,t_3-k_3,j_7})$ can be approximate by zero. When $|k_1|, |k_2|, |k_3|, |k_4| > \tilde{K}$, we have $W_{2,7}^{(4)} - \tilde{W}_{2,7}^{(4)} = o(1)$ where

$$
\tilde{W}_{2,7}^{(4)} = \frac{1}{\operatorname{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{|k_1|,|k_2|,|k_3|,|k_4|>\tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left|\mathcal{N}_{i,1}(k_1, k_2; \tilde{b})\mathcal{N}_{i,1}(k_3, k_4; \tilde{b})\right|^{-1}
$$

$$
\begin{aligned}
&\cdot \sum^{**} \{I(|t_1 - t_3 + k_3|, |t_2 - t_4 - k_4|, |t_3 - t_1 + k_1|, |t_4 - t_2 - k_2| \leq K_2) \\
&\quad + I(|t_1 - t_3 + k_3|, |t_2 - t_4 - k_4|, |t_3 - t_2 - k_2|, |t_4 - t_1 + k_1| \leq K_2) \\
&\quad + I(|t_1 - t_4 - k_4|, |t_2 - t_3 + k_3|, |t_3 - t_1 + k_1|, |t_4 - t_2 - k_2| \leq K_2) \\
&\quad + I(|t_1 - t_4 - k_4|, |t_2 - t_3 + k_3|, |t_3 - t_2 - k_2|, |t_4 - t_1 + k_1| \leq K_2)\} \\
&\cdot \sum_{j_1,\ldots,j_8=1}^{r} b_{i,j_1j_2} b_{i,j_3j_4} b_{i,j_5j_6} b_{i,j_7j_8} \mathrm{E}(Z_{i,t_1,j_1} Z_{i,t_2,j_2} Z_{i,t_3-k_3,j_7} Z_{i,t_4+k_4,j_8}) \\
&\qquad\qquad \cdot \mathrm{E}(Z_{i,t_1-k_1,j_3} Z_{i,t_2+k_2,j_4} Z_{i,t_3,j_5} Z_{i,t_4,j_6}) \\
= {}& o(1),
\end{aligned}
$$

where $K_2 = o(n)$, $K_2 s_0^4 = o(n)$ and the last equality holds since

$$
\begin{aligned}
&\sum^{**} \{I(|t_1 - t_3 + k_3|, |t_2 - t_4 - k_4|, |t_3 - t_1 + k_1|, |t_4 - t_2 - k_2| \leq K_2) \\
&\quad + I(|t_1 - t_3 + k_3|, |t_2 - t_4 - k_4|, |t_3 - t_2 - k_2|, |t_4 - t_1 + k_1| \leq K_2) \\
&\quad + I(|t_1 - t_4 - k_4|, |t_2 - t_3 + k_3|, |t_3 - t_1 + k_1|, |t_4 - t_2 - k_2| \leq K_2) \\
&\quad + I(|t_1 - t_4 - k_4|, |t_2 - t_3 + k_3|, |t_3 - t_2 - k_2|, |t_4 - t_1 + k_1| \leq K_2)\} = o(n_i^4 s_0^{-4}).
\end{aligned}
$$

By using the similar method of dealing with $W_{2,1}, \ldots, W_{2,6}$, we have $\tilde{W}_{2,7}^{(4)} = o(1)$. Since $W_{2,7}^{(1)}, \ldots, W_{2,7}^{(4)} \to 0$ and $W_{2,7} = \sum_{l=1}^{4} W_{2,7}^{(l)}$, we have $W_{2,7} \to 0$ as $n, p \to \infty$. Analogously, it can be derived that

$$
\begin{aligned}
&\frac{1}{\mathrm{tr}^2(\mathbf{\Sigma}_{i,\infty}^2)} \sum_{\exists l=1,2,3,4, |k_l| > \tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}(k_1, k_2; \tilde{b}) \mathcal{N}_{i,1}(k_3, k_4; \tilde{b}) \right|^{-1} \\
&\cdot \sum^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2} \boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2+k_2}) \mathrm{E}(\boldsymbol{X}_{i,t_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4} \boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4+k_4}) = o(1).
\end{aligned}
$$

Now besides the above equation, we have $W_{1,1}, \ldots, W_{1,6} \to 0$, $W_{2,1}, \ldots, W_{2,7} \to 0$ and $W_{1,7} - \tilde{W}_1 \to 0$ as $n, p \to \infty$. Recall that $V_1 = \sum_{l=1}^{7} (W_{1,l} + W_{2,l})$. Define

$$
\tilde{V}_1 = \tilde{W}_1 + \frac{1}{\mathrm{tr}^2(\mathbf{\Sigma}_{i,\infty}^2)} \sum_{\exists l=1,2,3,4, |k_l| > \tilde{K}} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}(k_1, k_2; \tilde{b}) \mathcal{N}_{i,1}(k_3, k_4; \tilde{b}) \right|^{-1}
$$

$$\cdot \sum^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2} \boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2+k_2}) \mathrm{E}(\boldsymbol{X}_{i,t_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4} \boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4+k_4})$$

$$= \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \left| \mathcal{N}_{i,1}\big(k_1,k_2;\tilde{b}\big) \mathcal{N}_{i,1}\big(k_3,k_4;\tilde{b}\big) \right|^{-1}$$

$$\cdot \sum^{*} \mathrm{E}(\boldsymbol{X}_{i,t_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2} \boldsymbol{X}_{i,t_1-k_1}^{\mathrm{T}} \boldsymbol{X}_{i,t_2+k_2}) \mathrm{E}(\boldsymbol{X}_{i,t_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4} \boldsymbol{X}_{i,t_3-k_3}^{\mathrm{T}} \boldsymbol{X}_{i,t_4+k_4})$$

$$= \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,1}(k_1,k_2;\tilde{b})\} \mathrm{E}\{G_{i,1}(k_3,k_4;\tilde{b})\}.$$

Then $V_1 - \tilde{V}_1 = o(1)$ as $n, p \to \infty$.

Similarly, it can be derived that as $n, p \to \infty$,

$$-V_2 - \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,1}(k_1,k_2;\tilde{b})\} \mathrm{E}\{G_{i,2}(k_3;\tilde{b})\} = o(1),$$

$$-V_3 - \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,1}(k_1,k_2;\tilde{b})\} \mathrm{E}\{G_{i,2}(k_4;\tilde{b})\} = o(1),$$

$$V_4 - \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,1}(k_1,k_2;\tilde{b})\} \mathrm{E}\{G_{i,3}(\tilde{b})\} = o(1),$$

$$-V_5 - \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,2}(k_1;\tilde{b})\} \mathrm{E}\{G_{i,1}(k_3,k_4;\tilde{b})\} = o(1),$$

$$V_6 - \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,2}(k_1;\tilde{b})\} \mathrm{E}\{G_{i,2}(k_3;\tilde{b})\} = o(1),$$

$$V_7 - \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,2}(k_1;\tilde{b})\} \mathrm{E}\{G_{i,2}(k_4;\tilde{b})\} = o(1),$$

$$-V_8 - \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,2}(k_1;\tilde{b})\} \mathrm{E}\{G_{i,3}(\tilde{b})\} = o(1),$$

$$-V_9 - \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,2}(k_2;\tilde{b})\} \mathrm{E}\{G_{i,1}(k_3,k_4;\tilde{b})\} = o(1),$$

$$V_{10} - \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,2}(k_2;\tilde{b})\} \mathrm{E}\{G_{i,2}(k_3;\tilde{b})\} = o(1),$$

$$V_{11} - \frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)} \sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1} \prod_{l=1}^{4} \mathcal{K}(k_l/s_0) \mathrm{E}\{G_{i,2}(k_2;\tilde{b})\} \mathrm{E}\{G_{i,2}(k_4;\tilde{b})\} = o(1),$$

$$-V_{12}-\frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}\sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1}\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\mathrm{E}\{G_{i,2}(k_2;\tilde{b})\}\mathrm{E}\{G_{i,3}(\tilde{b})\}=o(1),$$

$$V_{13}-\frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}\sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1}\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\mathrm{E}\{G_{i,3}(\tilde{b})\}\mathrm{E}\{G_{i,1}(k_3,k_4;\tilde{b})\}=o(1),$$

$$-V_{14}-\frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}\sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1}\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\mathrm{E}\{G_{i,3}(\tilde{b})\}\mathrm{E}\{G_{i,2}(k_3;\tilde{b})\}=o(1),$$

$$-V_{15}-\frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}\sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1}\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\mathrm{E}\{G_{i,3}(\tilde{b})\}\mathrm{E}\{G_{i,2}(k_4;\tilde{b})\}=o(1),$$

$$V_{16}-\frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}\sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1}\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\mathrm{E}\{G_{i,3}(\tilde{b})\}\mathrm{E}\{G_{i,3}(\tilde{b})\}=o(1).$$

Thus by (4.7) and (4.9), we have

$$\sum_{l=1}^{16}V_l-\frac{1}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}\sum_{k_1,k_2,k_3,k_4=-(n_i-1)}^{n_i-1}\prod_{l=1}^{4}\mathcal{K}(k_l/s_0)\mathrm{E}\{\widehat{\mathrm{tr}}(\boldsymbol{\Sigma}_{i,k_1}\boldsymbol{\Sigma}_{i,k_2};\tilde{b})\}\mathrm{E}\{\widehat{\mathrm{tr}}(\boldsymbol{\Sigma}_{i,k_3}\boldsymbol{\Sigma}_{i,k_4};\tilde{b})\}=o(1),$$

which is equivalent to

$$\sum_{l=1}^{16}V_l-\frac{\mathrm{E}^2\{\widehat{\mathrm{tr}}(\boldsymbol{\Sigma}_{i,\infty}^2;\tilde{b},s_0)\}}{\mathrm{tr}^2(\boldsymbol{\Sigma}_{i,\infty}^2)}=o(1).$$

Since $\mathrm{E}\{\widehat{\mathrm{tr}}(\boldsymbol{\Sigma}_{i,\infty}^2;\tilde{b},s_0)\}/\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)\to 1$ as $n,p\to\infty$, $\sum_{l=1}^{16}V_l\to 1$ as $n,p\to\infty$. Thus we have $\mathrm{Var}\left\{\frac{\widehat{\mathrm{tr}}\left(\boldsymbol{\Sigma}_{i,\infty}^2;\tilde{b},s_0\right)}{\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)}\right\}\to 0$ as $n,p\to\infty$. According to (S1.20), $\widehat{\mathrm{tr}}\big(\boldsymbol{\Sigma}_{i,\infty}^2;\tilde{b},s_0\big)/\mathrm{tr}(\boldsymbol{\Sigma}_{i,\infty}^2)\to 1$ as $n,p\to\infty$. Analogously, it can be proved that $\widehat{\mathrm{tr}}\big(\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\Sigma}_{2,\infty};\tilde{b},s_0\big)/\mathrm{tr}(\boldsymbol{\Sigma}_{1,\infty}\boldsymbol{\Sigma}_{2,\infty})\to 1$ as $n,p\to\infty$. Now we are able to obtain the ratio consistency of $V_n(\tilde{b},s_0)$ such that $V_n(\tilde{b},s_0)/\mathrm{Var}\{T(b)\}\to 1$ as $n,p\to\infty$. The claim has been proved. □

S1.8 Proof of Theorem 3

Proof. Under the conditions of Theorem 1 and the local alternative (6.1), we have $\mathrm{Var}\big\{T(b)\big\}=2n^{-2}\mathrm{tr}(\boldsymbol{M}_\infty^2)\{1+o(1)\}$. According to the asymptotic normality in Theorem 1, the power function is $\beta(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)=\Phi\left\{-z_\alpha+\frac{\|\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\|_2^2}{\sqrt{2n^{-2}\mathrm{tr}(\boldsymbol{M}_\infty^2)}}\right\}\{1+o(1)\}$, where $\Phi(\cdot)$ is the standard normal distribution function and z_α is the upper α quantile of $N(0,1)$. If $n\|\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\|_2^2\mathrm{tr}^{-1/2}(\boldsymbol{M}_\infty^2)\to$

$d^2 \in [0, +\infty)$, we have $\beta(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \to \Phi(-z_\alpha + d^2/\sqrt{2})$. □

S1.9 Proof of Theorem 4

Proof. Under the conditions of Theorem 1 and the fixed alternative (6.2), we have $\mathrm{Var}\{T(b)\} = 4n^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^{\mathrm{T}}\boldsymbol{M}_\infty(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\{1 + o(1)\}$. According to the asymptotic normality in Theorem 1, the power function is $\beta(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = \Phi\left\{\frac{\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_2^2}{\sqrt{4n^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^{\mathrm{T}}\boldsymbol{M}_\infty(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)}}\right\}\{1 + o(1)\}$. Since $(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^{\mathrm{T}}\boldsymbol{M}_\infty(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \le \lambda_{\max}\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|^2$, $\beta(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \ge \Phi(1/\sqrt{4n^{-1}\lambda_{\max}})$. If $n/\lambda_{\max} \to +\infty$, we have $\beta(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \to 1$. □

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