Supplementary Materials for "CLT for high-dimensional R^2 statistics under a general independent components model"

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Supplementary Material

This Supplementary Material contains proofs of Theorems 1 and 2 in Sections S1 and S2, respectively. Some lemmas are given in the Appendix.

S1 Proof of Theorem 1

S1.1 Transformation on the population z

Recall that the R^2 statistic is invariant under any invertible affine transformation on y and $\mathbf{x} = (x_1, \ldots, x_p)'$, respectively. Thus, we construct such a transformation to simplify the covariance structure of the population $\mathbf{z} = (y, x_1, \ldots, x_p)'$ and the conditions in Assumption (c).

When $\rho_p \neq 0$, let $\mathbf{q}_1 = \mathbf{\Sigma}_{xx}^{-1}\boldsymbol{\sigma}_{xy}$ and $\mathbf{Q}_2$ be a $p \times (p-1)$ full column rank matrix satisfying $\boldsymbol{\sigma}'_{xy}\mathbf{Q}_2 = \mathbf{0}$. Define

$$\mathbf{Q} = (\mathbf{q}_1, \mathbf{Q}_2)' \quad \text{and} \quad \breve{\mathbf{u}} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{pmatrix} (\mathbf{z} - \boldsymbol{\mu}).$$

Then, we observe that

$$\mathrm{Var}(\breve{\mathbf{u}}) = \begin{pmatrix} \sigma_{yy} & \rho_p^2\sigma_{yy} & \mathbf{0} \\ \rho_p^2\sigma_{yy} & \rho_p^2\sigma_{yy} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M} \end{pmatrix},$$

where $\mathbf{M} = \mathbf{Q}_2'\boldsymbol{\Sigma}_{xx}\mathbf{Q}_2$ is a positive definite matrix. Take

$$\mathbf{u} = \begin{pmatrix} \frac{1}{\sqrt{\sigma_{yy}}} & 0 & \mathbf{0} \\ 0 & \frac{1}{\rho_p\sqrt{\sigma_{yy}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{-\frac{1}{2}} \end{pmatrix} \breve{\mathbf{u}}$$

It's easy to see that $\mathbf{u}$ has the same multiple correlation coefficient as $\mathbf{z}$ but possesses a much simpler covariance structure, that is,

$$\mathbf{u} = \begin{pmatrix} \mathbf{a}'_{1\mathbf{u}} \\ \mathbf{A}_{2\mathbf{u}} \end{pmatrix} \mathbf{w} \quad \text{with} \quad \mathrm{Var}(\mathbf{u}) = \begin{pmatrix} 1 & \rho_p & \mathbf{0} \\ \rho_p & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{p-1} \end{pmatrix},$$

where

$$\mathbf{a}_{1\mathbf{u}} = \frac{\mathbf{a}_1}{\sqrt{\sigma_{yy}}} \quad \text{and} \quad \mathbf{A}_{2\mathbf{u}} = \begin{pmatrix} \frac{\boldsymbol{\sigma}'_{xy}\boldsymbol{\Sigma}_{xx}^{-1}}{\rho_p\sqrt{\sigma_{yy}}} \\ \mathbf{M}^{-\frac{1}{2}}\mathbf{Q}_2' \end{pmatrix} \mathbf{A}_2.$$

When $\rho_p = 0$, we can simply take

$$\mathbf{u} = \begin{pmatrix} \sigma_{yy}^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{xx}^{-\frac{1}{2}} \end{pmatrix} (\mathbf{z} - \boldsymbol{\mu})$$

and then obtain similar conclusions.

Therefore, we only need to give the proof for the case

$$\boldsymbol{\mu} = \mathbf{0}, \quad \mathbf{a}_1'\mathbf{a}_1 = 1, \quad \mathbf{A}_2\mathbf{A}_2' = \mathbf{I}_p, \quad \mathbf{A}_2\mathbf{a}_1 = (\rho_p, 0, \ldots, 0)' \triangleq \mathbf{r}. \tag{S1.1}$$

Moreover, the conditions in Assumption (c) reduce to

$$\rho_p^{k-1} \sum_{i=1}^{m} (\tau_i - 3) a_i^{5-k} a_{1i}^{k-1} \to \zeta_k, \quad k = 1, \ldots, 5, \tag{S1.2}$$

where $(a_1, \ldots, a_m)$ and $(a_{11}, \ldots, a_{1m})$ are the components of $\mathbf{a}_1'$ and the first row of $\mathbf{A}_2$, respectively.

In the following sections, we will denote by K some constant that can vary from place to place.

S1.2 Truncation on the variables (w_{ij})

For the sample $\{\mathbf{z}_j = \mathbf{A}\mathbf{w}_j, j = 1, \ldots, n\}$, we denote $(\mathbf{w}_1, \ldots, \mathbf{w}_n) = (w_{ij})$. By the moment conditions in Assumption (b), we can select a non-random sequence $\{\delta_n\}$ such that $\delta_n \downarrow 0$ at an arbitrarily slow convergence rate and

$$\delta_n^{-6} \max_{i,j} \mathbb{E} \left\{ |w_{ij}|^6 I_{(|w_{ij}| > \delta_n n^{1/3})} \right\} \to 0.$$

Now, we truncate the variables (w_{ij}) at $\delta_n n^{1/3}$. Set $\breve{w}_{ij} = w_{ij} I_{(|w_{ij}| \leqslant \delta_n n^{1/3})}$, $\tilde{w}_{ij} = \breve{w}_{ij} - \mathbb{E}\breve{w}_{ij}$ and $\check{w}_{ij} = \tilde{w}_{ij}/\sigma_{ij}$ with $\sigma_{ij}^2 = \mathbb{E}\tilde{w}_{ij}^2$ for $i = 1, \ldots, m$ and $j = 1, \ldots n$. Accordingly, we denote by, $\breve{R}$, $\tilde{R}$ and $\check{R}$, the sample multiple correlation coefficients from the truncated samples $\breve{\mathbf{z}}_j = \mathbf{A}\breve{\mathbf{w}}_j$, $\tilde{\mathbf{z}}_j = \mathbf{A}\tilde{\mathbf{w}}_j$ and $\check{\mathbf{z}}_j = \mathbf{A}\check{\mathbf{w}}_j$ with $\breve{\mathbf{w}}_j = (\breve{w}_{1j}, \ldots, \breve{w}_{mj})'$, $\tilde{\mathbf{w}}_j = (\tilde{w}_{1j}, \ldots, \tilde{w}_{mj})'$ and $\check{\mathbf{w}}_j = (\check{w}_{1j}, \ldots, \check{w}_{mj})'$ for $j = 1, \ldots n$, respectively. Then, we get

$$
\begin{aligned}
\mathrm{P}(R \neq \breve{R}) &\leqslant \sum_{i,j} \mathrm{P}(|w_{ij}| > \delta_n n^{1/3}) \\
&\leqslant \delta_n^{-6} n^{-1} m \max_{i,j} \mathbb{E}\left\{ |w_{ij}|^6 I_{(|w_{ij}| > \delta_n n^{1/3})} \right\} = o(1). \qquad \text{(S1.3)}
\end{aligned}
$$

In addition, we have $\breve{R} = \tilde{R}$ since the multiple correlation coefficients are invariant under location transforms.

Next, we show that $\tilde{R}$ shares the same limiting distribution as $\check{R}$. Recall that the statistic $\tilde{R}$ is from the sample

$$
\tilde{\mathbf{z}}_j = \mathbf{A}\tilde{\mathbf{w}}_j = \mathbf{A}\mathbf{\Lambda}\check{\mathbf{w}}_j, \quad j = 1, \ldots n,
$$

where $\mathbf{A}$ satisfies the conditions in (S1.1)–(S1.2) and $\mathbf{\Lambda} = \mathrm{diag}(\sigma_{11}, \ldots, \sigma_{m1})$ with $\sigma_{ij}^2 = \mathbb{E}\tilde{w}_{ij}^2$. The population squared multiple correlation coefficient for this model is

$$
\tilde{\rho}_p^2 = \frac{\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'(\mathbf{A}_2\mathbf{\Lambda}^2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{\Lambda}^2\mathbf{a}_1}{\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{a}_1}.
$$

From the truncation, we have

$$0 \leq \max_i (1-\sigma_{i1}^2) \leqslant 2 \max_i \mathbb{E}\left\{|w_{1i}|^2 I_{(|w_{ij}|>\delta_n n^{1/3})}\right\} \leqslant K\delta_n^{-4} n^{-4/3}, \quad \text{(S1.4)}$$

which implies the spectral norm of $\mathbf{\Lambda}^2 - \mathbf{I}_m$ is $o(n^{-1})$. It thus follows that

$$\sqrt{n}(\tilde{\rho}_p^2 - \rho_p^2) = o(1). \quad \text{(S1.5)}$$

To further simplify the model, we apply the affine transform in Section S1.1 to the sample $\{\tilde{\mathbf{z}}_j\}$, by which the matrix $\mathbf{A}\mathbf{\Lambda}$ can be changed into a new matrix with clearer structure, denoted as $\tilde{\mathbf{A}}$. This new matrix possesses similar properties as $\mathbf{A}$, illustrated in (S1.1) and (S1.2). That is,

$$\tilde{\mathbf{A}} = \begin{pmatrix} \tilde{\mathbf{a}}_1' \\ \tilde{\mathbf{A}}_2 \end{pmatrix}, \quad \tilde{\mathbf{a}}_1'\tilde{\mathbf{a}}_1 = 1, \quad \tilde{\mathbf{A}}_2\tilde{\mathbf{A}}_2' = \mathbf{I}_p, \quad \tilde{\mathbf{A}}_2\tilde{\mathbf{a}}_1 = (\tilde{\rho}_p, 0, \ldots, 0)', \quad \text{(S1.6)}$$

$$\begin{aligned} &\tilde{\rho}_p^{k-1}\sum_{i=1}^m (\tau_i - 3)\tilde{a}_i^{5-k}\tilde{a}_{1i}^{k-1} \\ &= \sum_{i=1}^m (\tau_i - 3)\frac{[\mathbf{a}_1'\mathbf{\Lambda}\mathbf{e}_i]^{5-k}\left[\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'(\mathbf{A}_2\mathbf{\Lambda}^2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{\Lambda}\mathbf{e}_i\right]^{k-1}}{\left(\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{a}_1\right)^2} \\ &\to \zeta_k, \quad k = 1, \ldots, 5, \end{aligned} \quad \text{(S1.7)}$$

where $(\tilde{a}_i, \tilde{a}_{1i})$ are similarly defined as (a_i, a_{1i}) in (S1.2). The five convergences in (S1.7) will be proved later. With the findings in (S1.5), (S1.6) and (S1.7), the limiting distribution of $\tilde{R}$ must be the same as that of $\check{R}$, if the latter exists (as claimed by our theorem).

To demonstrate the five convergences in (S1.7), we employ the inequalities in (S1.4) and the fact $(1-\sigma_{1i}) = (1-\sigma_{1i}^2)/(1+\sigma_{1i}) \leqslant (1-\sigma_{1i}^2)$. For $k=1$, using the fact $\sum_{i=1}^m a_i^2 = 1$, we have

$$\left|\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{a}_1 - 1\right| = \left|\sum_{i=1}^m a_i^2\left(\sigma_{i1}^2 - 1\right)\right| \leqslant \max_i(1-\sigma_{i1}^2) = o(1) \tag{S1.8}$$

and

$$\begin{aligned}&\left|\sum_{i=1}^m(\tau_i-3)\left(\mathbf{a}_1'\mathbf{\Lambda}\mathbf{e}_i\right)^4 - \sum_{i=1}^m(\tau_i-3)\left(\mathbf{a}_1'\mathbf{e}_i\right)^4\right|\\ \leqslant& K\sum_{i=1}^m|a_i(\sigma_{i1}-1)| \leqslant Kn\max_i(1-\sigma_{i1}) = o(1).\end{aligned} \tag{S1.9}$$

Combining (S1.2), (S1.8) and (S1.9), we obtain

$$\frac{1}{\left(\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{a}_1\right)^2}\sum_{i=1}^m(\tau_i-3)\left[\mathbf{a}_1'\mathbf{\Lambda}\mathbf{e}_i\right]^4 \to \zeta_1.$$

For $k=5$, we have

$$\begin{aligned}&\left|\sum_{i=1}^m(\tau_i-3)\left[\left\{\mathbf{a}_1'\mathbf{A}_2'(\mathbf{A}_2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{e}_i\right\}^4 - \left\{\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'(\mathbf{A}_2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{e}_i\right\}^4\right]\right|\\ \leqslant& K\sum_{i=1}^m\left|\mathbf{a}_1'(\mathbf{I}_m-\mathbf{\Lambda}^2)\mathbf{A}_2'(\mathbf{A}_2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{e}_i\right|\\ \leqslant& Kn\max_i(1-\sigma_{i1}^2) = o(1),\end{aligned} \tag{S1.10}$$

$$\begin{aligned}&\left|\sum_{i=1}^m(\tau_i-3)\left[\left\{\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'(\mathbf{A}_2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{e}_i\right\}^4 - \left\{\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'(\mathbf{A}_2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{\Lambda}\mathbf{e}_i\right\}^4\right]\right|\\ \leqslant& K\sum_{i=1}^m\left|\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'(\mathbf{A}_2\mathbf{A}_2')^{-1}\mathbf{A}_2(\mathbf{I}_m-\mathbf{\Lambda})\mathbf{e}_i\right|\\ \leqslant& Kn\max_i(1-\sigma_{i1}) = o(1),\end{aligned} \tag{S1.11}$$

$$\left|\sum_{i=1}^{m}(\tau_i-3)\left[\left\{\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'(\mathbf{A}_2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{\Lambda}\mathbf{e}_i\right\}^4\right.\right.$$
$$\left.\left.-\left\{\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'(\mathbf{A}_2\mathbf{\Lambda}^2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{\Lambda}\mathbf{e}_i\right\}^4\right]\right|$$
$$\leqslant K\sum_{i=1}^{m}\left|\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'\left\{(\mathbf{A}_2\mathbf{A}_2')^{-1}-(\mathbf{A}_2\mathbf{\Lambda}^2\mathbf{A}_2')^{-1}\right\}\mathbf{A}_2\mathbf{\Lambda}\mathbf{e}_i\right|$$
$$=K\sum_{i=1}^{m}\left|\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'\mathbf{A}_2(\mathbf{I}_m-\mathbf{\Lambda}^2)\mathbf{A}_2'(\mathbf{A}_2\mathbf{\Lambda}^2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{\Lambda}\mathbf{e}_i\right|$$
$$\leqslant Kn\|\mathbf{\Lambda}\mathbf{A}_2'(\mathbf{A}_2\mathbf{\Lambda}^2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{\Lambda}\|\max_i\sigma_{i1}^{-1}\max_i(1-\sigma_{i1}^2)=o(1), \tag{S1.12}$$

where $\|\cdot\|$ denotes the spectral norm. From these, together with (S1.8), we have

$$\frac{1}{\left(\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{a}_1\right)^2}\sum_{i=1}^{m}(\tau_i-3)\left[\mathbf{a}_1'\mathbf{\Lambda}^2\mathbf{A}_2'(\mathbf{A}_2\mathbf{\Lambda}^2\mathbf{A}_2')^{-1}\mathbf{A}_2\mathbf{\Lambda}\mathbf{e}_i\right]^4\to\zeta_5.$$

By the same approach, one can verify the remaining three results with $k=2,3,4$. We omit the details.

Thus, we shall proceed our proof with the assumptions in (S1.1)–(S1.2) and the following conditions

$$\mathbb{E}(w_{ij})=0,\quad \mathbb{E}(w_{ij}^2)=1,\quad \mathbb{E}(w_{ij}^4)=\tau_{i,p}=\tau_i+o(1),$$
$$\sup_{i,j}\mathbb{E}(w_{ij}^6)<\infty,\quad |w_{ij}|\leqslant\delta_n n^{1/3}, \tag{S1.13}$$

where the $o(1)$ term is uniform in i and j.

S1.3 Notations and a sketch of the proof

The following notations will be frequently used throughout this section. For $i, j, k \in \{1, \ldots, n\}$, define

$$u_k = n^{-1/2}\mathbf{a}_1'\mathbf{w}_k, \quad \mathbf{r}_k = n^{-1/2}\mathbf{A}_2\mathbf{w}_k, \quad \mathbf{s} = \sum_{k=1}^{n} u_k\mathbf{r}_k, \quad \mathbf{s}_i = \mathbf{s} - u_i\mathbf{r}_i,$$

$$\mathbf{D} = \sum_{k=1}^{n} \mathbf{r}_k\mathbf{r}_k', \quad \mathbf{D}_i = \mathbf{D} - \mathbf{r}_i\mathbf{r}_i', \quad \mathbf{D}_{ij} = \mathbf{D} - \mathbf{r}_i\mathbf{r}_i' - \mathbf{r}_j\mathbf{r}_j',$$

$$\Delta_{(i)} = \mathbf{r}_i'\mathbf{D}_i^{-1}\mathbf{r}_i - \frac{1}{n}\operatorname{tr}\mathbf{D}_i^{-1}, \quad \Delta_{i(j)} = \mathbf{r}_i'\mathbf{D}_{ij}^{-1}\mathbf{r}_i - \frac{1}{n}\operatorname{tr}\mathbf{D}_{ij}^{-1}, \tag{S1.14}$$

$$\beta_{(i)} = \frac{1}{1 + \mathbf{r}_i'\mathbf{D}_i^{-1}\mathbf{r}_i}, \quad \beta_{i(j)} = \frac{1}{1 + \mathbf{r}_i'\mathbf{D}_{ij}^{-1}\mathbf{r}_i}, \quad \overline{\beta}_{(i)} = \frac{1}{1 + \frac{1}{n}\operatorname{tr}\mathbf{D}_i^{-1}}, \tag{S1.15}$$

$$\overline{\beta}_{i(j)} = \frac{1}{1 + \frac{1}{n}\operatorname{tr}\mathbf{D}_{ij}^{-1}}, \quad b_{(i)} = \frac{1}{1 + \frac{1}{n}\mathbb{E}\operatorname{tr}\mathbf{D}_i^{-1}}, \quad b_{i(j)} = \frac{1}{1 + \frac{1}{n}\mathbb{E}\operatorname{tr}\mathbf{D}_{ij}^{-1}}. \tag{S1.16}$$

Also, $\mathbf{D}_{k_1k_2k_3k_4}$, $\beta_{k_1(k_2k_3k_4)}$, $\overline{\beta}_{k_1(k_2k_3k_4)}$ and $\Delta_{k_1(k_2k_3k_4)}$ are similarly defined.

Note that the six quantities in (S1.15) and (S1.16) are all bounded in absolute value by 1. In addition, the inverses of the three matrices in (S1.14) can be treated as being bounded in spectral norm. To be specific, let $\lambda_{\min}(\mathbf{T})$ and $\lambda_{\max}(\mathbf{T})$ denote the smallest and largest eigenvalues of a real symmetric matrix $\mathbf{T}$. From the conditions in (S1.13) and (9.7.9) in Bai and Silverstein (2010), we have

$$\mathrm{P}(\lambda_{\min}(\mathbf{D}) < \eta) \leqslant \mathrm{P}(\lambda_{\min}(\mathbf{D}_i) < \eta) \leqslant \mathrm{P}(\lambda_{\min}(\mathbf{D}_{ij}) < \eta) = o(n^{-l}) \tag{S1.17}$$

for any positive l, whenever $0 < \eta < (1 - \sqrt{c'})^2$ with $c' = \limsup_{n\to\infty}(m/n)$.

Then, we get

$$R^2 = R^2 I_{(\lambda_{\max}(\mathbf{D}^{-1})\leqslant 1/\eta)} + R^2 I_{(\lambda_{\max}(\mathbf{D}^{-1})>1/\eta)} = R^2 I_{(\lambda_{\max}(\mathbf{D}^{-1})\leqslant 1/\eta)} + o_p(n^{-l}). \tag{S1.18}$$

During our proof, we will deal with some terms having the form $\mathbf{t}'_{k_1}\mathbf{T}\mathbf{t}_{k_2}$, where $\mathbf{t}_k$ can be $\mathbf{r}_k$ or $u_k\mathbf{r}_k$ and $\mathbf{T}$ can be $\mathbf{D}^{-1}$ or $\mathbf{D}_i^{-1}$. By the truncation condition in (S1.13), we can obtain

$$|\mathbf{t}'_{k_1}\mathbf{T}\mathbf{t}_{k_2}| \leqslant \lambda_{\max}(\mathbf{T})\sqrt{\mathbf{t}'_{k_1}\mathbf{t}_{k_1}\mathbf{t}'_{k_2}\mathbf{t}_{k_2}} \leqslant \lambda_{\max}(\mathbf{T})Kn^{\alpha}$$

for some positive number α. Then, similar to (S1.18), we have

$$\mathbf{t}'_{k_1}\mathbf{D}^{-1}\mathbf{t}_{k_2} = \mathbf{t}'_{k_1}\mathbf{D}^{-1}\mathbf{t}_{k_2} I_{(\lambda_{\max}(\mathbf{D}^{-1})\leqslant 1/\eta)} + o_p(n^{-l}). \tag{S1.19}$$

Therefore, by (S1.18)–(S1.19), we can assume that $\mathbf{D}^{-1}$, $\mathbf{D}_i^{-1}$ and $\mathbf{D}_{ij}^{-1}$ have bounded spectral norm and omit the indicator function in the proof for simplicity.

We next give a sketch of the proof. First, we have

$$\begin{aligned}
&\sqrt{n}\{R^2 - c_n - (1-c_n)\rho_p^2\} \\
=&\frac{1}{\sum_{j=1}^n |u_j|^2}\left[\sqrt{n}\left\{\mathbf{s}'\mathbf{D}^{-1}\mathbf{s} - c_n - (1-c_n)\rho_p^2\right\}\right] \\
&\qquad - \frac{c_n + (1-c_n)\rho_p^2}{\sum_{j=1}^n |u_j|^2}\left\{\sqrt{n}(\sum_{j=1}^n |u_j|^2 - 1)\right\} + o_p(1).
\end{aligned}$$

This approximation was obtained by Zheng et al. (2014) under finite fourth moments of the i.i.d. variables (w_{ij}). By similar steps, one can verify that this approximation still holds true under our model assumptions.

Then, we need to establish the asymptotic normality of

$$\begin{cases} T_1 \triangleq \sqrt{n}\left\{\mathbf{s}'\mathbf{D}^{-1}\mathbf{s} - \mathbb{E}(\mathbf{s}'\mathbf{D}^{-1}\mathbf{s})\right\} = \sqrt{n}\sum_{j=1}^{n}(\mathbb{E}_j - \mathbb{E}_{j-1})\mathbf{s}'\mathbf{D}^{-1}\mathbf{s}, \\ T_2 \triangleq \sqrt{n}(\sum_{j=1}^{n}|u_j|^2 - 1) = \sqrt{n}\sum_{j=1}^{n}(\mathbb{E}_j - \mathbb{E}_{j-1})|u_j|^2, \end{cases} \tag{S1.20}$$

by the martingale CLT (see Lemma 1), and calculate the limit of

$$T_3 \triangleq \sqrt{n}\left\{\mathbb{E}(\mathbf{s}'\mathbf{D}^{-1}\mathbf{s}) - c_n - (1-c_n)\rho_p^2\right\}, \tag{S1.21}$$

where $\mathbb{E}_0(\cdot)$ is the expectation and $\mathbb{E}_j(\cdot)$ denotes the conditional expectation given the σ-field $\mathcal{F}_j$ generated by $\mathbf{z}_1, \ldots, \mathbf{z}_j$, for $j = 1, \ldots, n$. To obtain the the asymptotic covariance matrix of (T_1, T_2), we need to deal with the following decompositions:

$$\begin{aligned} &n\sum_{j=1}^{n}\mathbb{E}_{j-1}\left\{(\mathbb{E}_j - \mathbb{E}_{j-1})\mathbf{s}'\mathbf{D}^{-1}\mathbf{s}\right\}^2 = Q_1 + Q_2 + o_p(1), \\ &n\sum_{j=1}^{n}\mathbb{E}_{j-1}\left\{\left((\mathbb{E}_j - \mathbb{E}_{j-1})\mathbf{s}'\mathbf{D}^{-1}\mathbf{s}\right)\left(u_j^2 - \frac{1}{n}\right)\right\} = Q_3 + Q_4 + o_p(1), \\ &n\sum_{j=1}^{n}\mathbb{E}_{j-1}\left(u_j^2 - \frac{1}{n}\right)^2 = 2 + \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m}(\tau_{i,p} - 3)(\mathbf{a}_1\mathbf{a}_1')_{ii}^2 + o_p(1) \\ &\qquad\qquad\qquad\qquad\qquad = 2 + \zeta_1 + o_p(1), \end{aligned}$$

where

$$\begin{aligned} Q_1 = \frac{1}{n}\sum_{j=1}^{n}\Bigg[&4(1-c_n)^2\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\check{\mathbf{D}}_j^{-1}\check{\mathbf{s}}_j) + c_n^2\mathbb{E}(nu_j^2 - 1)^2 \\ &+ 2(1-c_n)^2\left\{\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\check{\mathbf{D}}_j^{-1}\check{\mathbf{s}}_j)\right\}^2 - 4c_n(1-c_n)\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{r})^2 \end{aligned}$$

$$+ 4(1-c_n)^2 \left\{\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{r})\right\}^2 + 8c_n(1-c_n)\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{r})$$

$$\left. - 8(1-c_n)^2\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{r}\mathbf{s}_j'\mathbf{D}_j^{-1}\check{\mathbf{D}}_j^{-1}\check{\mathbf{s}}_j)\right],$$

$$Q_2 = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m}(\tau_{i,p}-3)\left[4c_n(1-c_n)(\mathbf{a}_1\mathbf{a}_1')_{ii}\left\{\mathbb{E}_j(\mathbf{a}_1\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}\right.$$

$$- 4(1-c_n)^2\left\{\mathbb{E}_j(\mathbf{a}_1\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}\left\{\mathbb{E}_j(\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}$$

$$- 2c_n(1-c_n)(\mathbf{a}_1\mathbf{a}_1')_{ii}\left\{\mathbb{E}_j(\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}$$

$$+ (1-c_n)^2\left\{\mathbb{E}_j(\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}^2$$

$$\left. + 4(1-c_n)^2\left\{\mathbb{E}_j(\mathbf{a}_1\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}^2 + c_n^2(\mathbf{a}_1\mathbf{a}_1')_{ii}^2\right],$$

$$Q_3 = \frac{1}{n}\sum_{j=1}^{n}\left[4(1-c_n)\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{r})\right.$$

$$\left. + c_n\mathbb{E}(nu_j^2-1)^2 - 2(1-c_n)\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{r})^2\right],$$

$$Q_4 = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m}(\tau_{i,p}-3)\left[2(1-c_n)(\mathbf{a}_1\mathbf{a}_1')_{ii}\left\{\mathbb{E}_j(\mathbf{a}_1\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}\right.$$

$$\left. + c_n(\mathbf{a}_1\mathbf{a}_1')_{ii}^2 - (1-c_n)(\mathbf{a}_1\mathbf{a}_1')_{ii}\left\{\mathbb{E}_j(\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}\right],$$

with $(\check{\mathbf{D}}_j, \check{\mathbf{s}}_j)$ in Q_1 being the same as $(\mathbf{D}_j, \mathbf{s}_j)$ except that $\mathbf{w}_{j+1}, \ldots, \mathbf{w}_n$ are substituted by $\check{\mathbf{w}}_{j+1}, \ldots, \check{\mathbf{w}}_n$, i.i.d. copies of $\mathbf{w}_{j+1}, \ldots, \mathbf{w}_n$.

Under our model assumptions, by carefully checking the proof of the main theorem in Zheng et al. (2014), we find that the non-random quantity T_3 in (S1.21) still converges to 0, the Lyapunoff condition in Lemma 1 for T_1 and T_2 in (S1.20) can been verified directly, and moreover the limits of

Q_1 and Q_3 remain unchanged, that is,

$$Q_1 = -2(1-c)(\rho^2)^2 + 4(1-c)\rho^2 + 2c(1-c) + 2c^2 + o_p(1),$$

$$Q_3 = -2(1-c)(\rho^2)^2 + 4(1-c)\rho^2 + 2c + o_p(1).$$

Therefore, our main task here is to prove the convergence of Q_2 and Q_4, which involves finding the limits of the following terms:

$$\begin{cases} \sum_{i=1}^{m}(\tau_{i,p}-3)\left\{\mathbb{E}_j(\mathbf{a}_1\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}^2 \\ \qquad\qquad = \sum_{i=1}^{m}(\tau_{i,p}-3)a_i^2\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_j^{-1}\check{\mathbf{s}}_j), \\ \sum_{i=1}^{m}(\tau_{i,p}-3)(\mathbf{a}_1\mathbf{a}_1')_{ii}^2, \quad \sum_{i=1}^{m}(\tau_{i,p}-3)\left\{\mathbb{E}_j(\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}^2, \\ \sum_{i=1}^{m}(\tau_{i,p}-3)(\mathbf{a}_1\mathbf{a}_1')_{ii}\left\{\mathbb{E}_j(\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}, \\ \sum_{i=1}^{m}(\tau_{i,p}-3)(\mathbf{a}_1\mathbf{a}_1')_{ii}\left\{\mathbb{E}_j(\mathbf{a}_1\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}, \\ \sum_{i=1}^{m}(\tau_{i,p}-3)\left\{\mathbb{E}_j(\mathbf{a}_1\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}\left\{\mathbb{E}_j(\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}. \end{cases} \tag{S1.22}$$

Our tactic to prove the convergence consists of two parts: one is to control the convergence rate of the following three conditional expectations:

$$\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_j^{-1}\check{\mathbf{s}}_j),$$

$$\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j) \quad \text{and} \quad \mathbb{E}_j(\mathbf{e}_i'\mathbf{a}_1\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i), \tag{S1.23}$$

and the other is to find a bound for $\mathbb{E}(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j)^2$. They are given successively in the following two subsections.

S1.4 The convergence rate

We control the convergence rate of the three conditional expectations in (S1.23). Our approach to handling these three terms is the same, and thus we only present the details for $\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_j^{-1}\check{\mathbf{s}}_j)$.

Denote $\check{\mathbf{s}}_j = \sum_{k\neq j}\check{u}_k\check{\mathbf{r}}_k$ and $\check{\beta}_{k(j)} = 1/(1+\check{\mathbf{r}}_k'\check{\mathbf{D}}_{kj}^{-1}\check{\mathbf{r}}_k)$, and recall $\mathbf{r} = (\rho_p, 0, \ldots, 0)'$. First, we have

$$\begin{aligned}
&\mathbb{E}\left|\sum_{k>j} u_k\mathbf{r}_k'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_j^{-1}\check{\mathbf{r}}_k\check{u}_k\right|^2\\
=&\mathbb{E}\left|\sum_{k>j} u_k\mathbf{r}_k'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk}^{-1}\check{\mathbf{r}}_k\check{u}_k\beta_{k(j)}\check{\beta}_{k(j)}\right|^2 \leqslant 2\mathbb{E}(B_{1j}^2+B_{2j}^2), \quad \text{(S1.24)}
\end{aligned}$$

where

$$\begin{aligned}
B_{1j} &= \frac{1}{n}\sum_{k>j}\mathbf{r}'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk}^{-1}\check{\mathbf{r}}_k\check{u}_k,\\
B_{2j} &= \sum_{k>j}\left(u_k\mathbf{r}_k'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk}^{-1}\check{\mathbf{r}}_k\check{u}_k - \frac{1}{n}\mathbf{r}'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk}^{-1}\check{\mathbf{r}}_k\check{u}_k\right).
\end{aligned}$$

By Lemma 3 and Lemma 4, we get

$$\begin{aligned}
&\mathbb{E}|B_{1j}|^2 \leqslant Kn^{-2}, \quad \text{(S1.25)}\\
&\mathbb{E}|B_{2j}|^2 \leqslant K\left\{\mathbb{E}(\check{u}_k^2\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{jk}^{-2}\mathbf{A}_2\mathbf{e}_i\check{\mathbf{r}}_k'\check{\mathbf{D}}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk}^{-1}\check{\mathbf{r}}_k)\right.\\
&\qquad\left.+\mathbb{E}(\mathbf{r}'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk}^{-1}\check{\mathbf{r}}_k\check{u}_k)^2\right\}
\end{aligned}$$

$$\leqslant K\left(\mathbb{E}|\check{u}_k|^4\mathbb{E}|\check{\mathbf{r}}_k'\check{\mathbf{D}}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk}^{-1}\check{\mathbf{r}}_k|^2\right)^{1/2}$$

$$\leqslant Kn^{-2}. \tag{S1.26}$$

Second, noticing that when $k<j$, $\check{u}_k\check{\mathbf{r}}_k = u_k\mathbf{r}_k$, we have

$$\mathbb{E}\left|\sum_{k<j} u_k\mathbf{r}_k'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_j^{-1}\check{\mathbf{r}}_k\check{u}_k\right|^2$$

$$=\mathbb{E}\left|\sum_{k<j} u_k^2\mathbf{r}_k'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk}^{-1}\mathbf{r}_k\beta_{k(j)}\check{\beta}_{k(j)}\right|^2$$

$$\leqslant Kn^2(\mathbb{E}|u_k|^8\mathbb{E}|\mathbf{r}_k'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk}^{-1}\mathbf{r}_k|^4)^{1/2} \leqslant Kn^2\times n^{-4}\times n^{2/3}$$

$$=Kn^{-4/3}. \tag{S1.27}$$

Third, when $k\neq w>j$, $\mathbf{D}_{jk}^{-1}$ and $\check{\mathbf{D}}_{jw}^{-1}$ are independent of $u_k\mathbf{r}_k$ and $\check{u}_w\check{\mathbf{r}}_w$, and then we have

$$\mathbb{E}\left|\sum_{k\neq w>j}\mathbb{E}_j\left(u_k\mathbf{r}_k'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_j^{-1}\check{\mathbf{r}}_w\check{u}_w\right)\right|^2$$

$$=\mathbb{E}\left|\sum_{k\neq w>j}\mathbb{E}_j\left(u_k\mathbf{r}_k'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jw}^{-1}\check{\mathbf{r}}_w\check{u}_w\beta_{k(j)}\check{\beta}_{w(j)}\right)\right|^2. \tag{S1.28}$$

Similar to (S1.24)–(S1.26), one can get

$$\mathbb{E}|u_k\mathbf{r}_k'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jw}^{-1}\check{\mathbf{r}}_w\check{u}_w|^3 \leqslant Kn^{-6}, \tag{S1.29}$$

and by Lemma 3, we have

$$\mathbb{E}|\beta_{k(j)}-\overline{\beta}_{k(j)}|^6 \leqslant \mathbb{E}\left|\mathbf{r}_k'\mathbf{D}_{jk}^{-1}\mathbf{r}_k - \frac{1}{n}\operatorname{tr}\mathbf{D}_{jk}^{-1}\right|^6 \leqslant Kn^{-3}. \tag{S1.30}$$

Also, applying Lemma 2, we get

$$\begin{aligned}\mathbb{E}|\overline{\beta}_{k(j)} - b_{k(j)}|^6 \leqslant & Kn^{-6}\mathbb{E}\left|\sum_{t\neq j,k}^{n}(\mathbb{E}_t - \mathbb{E}_{t-1})(\operatorname{tr}\mathbf{D}_{jk}^{-1} - \operatorname{tr}\mathbf{D}_{jkt}^{-1})\right|^4 \\ = & Kn^{-6}\mathbb{E}\left|\sum_{t\neq j,k}^{n}(\mathbb{E}_t - \mathbb{E}_{t-1})(\mathbf{r}_t'\mathbf{D}_{jkt}^{-2}\mathbf{r}_t\beta_{t(jk)})\right|^6 \\ \leqslant & Kn^{-3}\mathbb{E}\left|\mathbf{r}_t'\mathbf{D}_{jkt}^{-2}\mathbf{r}_t\right|^6 \leqslant Kn^{-3}. \end{aligned} \quad \text{(S1.31)}$$

Moreover, it's not difficult to verify that

$$|b_{k(j)} - b_0|^4 \leqslant Kn^{-6}, \quad \text{(S1.32)}$$

where $b_0 = (1 + n^{-1}\mathbb{E}\operatorname{tr}\mathbf{D}^{-1})^{-1}$. Then combining (S1.28)–(S1.32) and Hölder's inequality, we can get

$$\begin{aligned}\mathbb{E}\Bigg|\sum_{k\neq w>j}\mathbb{E}_j\Big(&u_k\mathbf{r}_k'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_j^{-1}\check{\mathbf{r}}_w\check{u}_w \\ & -b_0^2u_k\mathbf{r}_k'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jw}^{-1}\check{\mathbf{r}}_w\check{u}_w\Big)\Bigg|^2 \leqslant Kn^{-1}. \end{aligned} \quad \text{(S1.33)}$$

Notice that

$$\mathbb{E}_j(u_k\mathbf{r}_k'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jw}^{-1}\check{\mathbf{r}}_w\check{u}_w) = \frac{1}{n^2}\mathbb{E}_j(\mathbf{r}'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jw}^{-1}\mathbf{r})$$

and, by similar arguments as in the derivation of (S1.31),

$$\begin{aligned}\mathbb{E}\Bigg|\frac{1}{n^2}\sum_{k\neq w>j}&\mathbb{E}_j(\mathbf{r}'\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jw}^{-1}\mathbf{r}) \\ & -\frac{1}{n^2}\sum_{k\neq w>j}\left(\mathbf{r}'\mathbb{E}\mathbf{D}_{jk}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbb{E}\check{\mathbf{D}}_{jw}^{-1}\mathbf{r}\right)\Bigg|^2 \leqslant Kn^{-1}, \end{aligned} \quad \text{(S1.34)}$$

where $\mathbb{E}\mathbf{D}_{jk}^{-1}$ and $\mathbb{E}\check{\mathbf{D}}_{jk}^{-1}$ can be replaced by $\mathbb{E}\mathbf{D}^{-1}$ without changing the final order in (S1.34). In addition, following the proofs of (9.9.20) in Bai and Silverstein (2010) and Lemma 7 in Bai et al. (2011), we find that

$$|b_0 - (1-c_n)|^2 \leqslant Kn^{-1}, \quad \left|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i - \frac{\rho_p a_{1i}}{1-c_n}\right|^2 \leqslant Kn^{-1}. \tag{S1.35}$$

Then, summing up (S1.33)–(S1.35) gives

$$\mathbb{E}\left|\sum_{k\neq w>j}\mathbb{E}_j\left(u_k\mathbf{r}_k'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_j^{-1}\check{\mathbf{r}}_w\check{u}_w\right) - \frac{(n-j)(n-j-1)}{n^2}\rho_p^2 a_{1i}^2\right|^2 \leqslant Kn^{-1}. \tag{S1.36}$$

Fourth, when $k \neq w < j$, neither $\mathbf{D}_{jk}^{-1}$ nor $\check{\mathbf{D}}_{jw}^{-1}$ is independent of $u_k\mathbf{r}_k$ and $\check{u}_w\check{\mathbf{r}}_w = u_w\mathbf{r}_w$, thus we should decompose $\mathbf{D}_{jk}^{-1}$ and $\check{\mathbf{D}}_{jw}^{-1}$ into $\mathbf{D}_{jkw}^{-1} - \mathbf{D}_{jkw}^{-1}\mathbf{r}_w\mathbf{r}_w'\mathbf{D}_{jkw}^{-1}\beta_{w(jk)}$ and $\check{\mathbf{D}}_{jkw}^{-1} - \check{\mathbf{D}}_{jkw}^{-1}\check{\mathbf{r}}_k\check{\mathbf{r}}_k'\check{\mathbf{D}}_{jkw}^{-1}\check{\beta}_{k(jw)}$, respectively, where $\check{\mathbf{D}}_{jkw}$ and $\check{\beta}_{k(jw)}$ are similarly defined as $\check{\mathbf{D}}_{kw}$ and $\check{\beta}_{k(w)}$. Write

$$\sum_{k\neq w<j}\mathbb{E}_j\left(u_k\mathbf{r}_k'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_j^{-1}\check{\mathbf{r}}_w\check{u}_w\right) = C_{1j} + C_{2j} + C_{3j} + C_{4j}, \tag{S1.37}$$

where

$$C_{1j} = \sum_{k\neq w<j}\mathbb{E}_j\left(u_k\mathbf{r}_k'\mathbf{D}_{jkw}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jkw}^{-1}\mathbf{r}_w u_w\beta_{k(j)}\check{\beta}_{w(j)}\right),$$

$$C_{2j} = -\sum_{k\neq w<j}\mathbb{E}_j\left(u_k\mathbf{r}_k'\mathbf{D}_{jkw}^{-1}\mathbf{r}_w\mathbf{r}_w'\mathbf{D}_{jkw}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jkw}^{-1}\mathbf{r}_w u_w\beta_{k(j)}\check{\beta}_{w(j)}\beta_{w(jk)}\right),$$

$$C_{3j} = -\sum_{k\neq w<j}\mathbb{E}_j\left(u_k\mathbf{r}_k'\mathbf{D}_{jkw}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jkw}^{-1}\mathbf{r}_k\mathbf{r}_k'\check{\mathbf{D}}_{jkw}^{-1}\mathbf{r}_w u_w\beta_{k(j)}\check{\beta}_{w(j)}\check{\beta}_{k(jw)}\right),$$

$$C_{4j} = \sum_{k\neq w<j} \mathbb{E}_j \left(u_k \mathbf{r}_k' \mathbf{D}_{jkw}^{-1} \mathbf{r}_w \mathbf{r}_w' \mathbf{D}_{jkw}^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_k \right. \\ \left. \times \mathbf{r}_k' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_w u_w \beta_{k(j)} \check{\beta}_{w(j)} \beta_{w(jk)} \check{\beta}_{k(jw)} \right).$$

Then, applying Lemma 3,

$$\begin{aligned} \mathbb{E}|C_{2j}|^2 \leqslant & K n^4 \mathbb{E}|u_k \mathbf{r}_k' \mathbf{D}_{jkw}^{-1} \mathbf{r}_w \mathbf{r}_w' \mathbf{D}_{jkw}^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_w u_w|^2 \\ \leqslant & K n^2 \mathbb{E}|\mathbf{r}_w' \mathbf{D}_{jkw}^{-2} \mathbf{r}_w||\mathbf{r}_w' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_w|^2 u_w^2 \\ \leqslant & K n^2 \left(\mathbb{E}|\mathbf{r}_w' \mathbf{D}_{jkw}^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_w|^4 \right)^{1/2} \left(\mathbb{E}|\mathbf{r}_w' \mathbf{D}_{jkw}^{-2} \mathbf{r}_w|^4 \mathbb{E} u_w^8 \right)^{1/4} \\ \leqslant & K n^2 (n^{-4} \times \delta_n^2 n^{2/3})^{1/2} \times (n^{-4} \times \delta_n^2 n^{2/3})^{1/4} = K \delta_n^{3/2} n^{-1/2}, \end{aligned} \tag{S1.38}$$

$$\mathbb{E}|C_{3j}|^2 \leqslant K \delta_n^{3/2} n^{-1/2}, \tag{S1.39}$$

$$\begin{aligned} \mathbb{E}|C_{4j}|^2 \leqslant & K n^4 \left(\mathbb{E}|u_k \mathbf{r}_k' \mathbf{D}_{jkw}^{-1} \mathbf{r}_w u_w|^4 \right. \\ & \left. \times \mathbb{E}|\mathbf{r}_w' \mathbf{D}_{jkw}^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_k \mathbf{r}_k' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_w|^4 \right)^{1/2} \\ \leqslant & K n^{3/2} \left(\mathbb{E}|\mathbf{r}_w' \mathbf{D}_{jkw}^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_k|^8 \mathbb{E}|\mathbf{r}_k' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_w|^8 \right)^{1/4} \\ \leqslant & K n^{3/2} (n^{-2} \times n^{1/3} \times n^{-1} \times n^{1/6}) = K n^{-1}. \end{aligned} \tag{S1.40}$$

Next, we consider C_{1j}. Similar to (S1.33), one can get

$$\mathbb{E}\left| \sum_{k\neq w<j} \mathbb{E}_j \left(u_k \mathbf{r}_k' \mathbf{D}_{jkw}^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_w u_w \beta_{k(j)} \check{\beta}_{w(j)} \right. \right. \\ \left. \left. - b_0^2 u_k \mathbf{r}_k' \mathbf{D}_{jkw}^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_{jkw}^{-1} \mathbf{r}_w u_w \right) \right|^2 \leqslant K n^{-1}. \tag{S1.41}$$

Additionally, for a fixed $w < j$, we have

$$\begin{aligned}
&\mathbb{E}\left|\sum_{k\neq w<j}\mathbb{E}_j\left(u_k\mathbf{r}_k'\mathbf{D}_{jkw}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jkw}^{-1}\mathbf{r}_w u_w\right.\right.\\
&\qquad\qquad\left.\left.-\frac{1}{n}\mathbf{r}'\mathbf{D}_{jkw}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jkw}^{-1}\mathbf{r}_w u_w\right)\right|^2\\
\leqslant &Kn^2\mathbb{E}\left|\sum_{k<j}\mathbb{E}_j\left(u_k\mathbf{r}_k'\mathbf{D}_{jkw}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jkw}^{-1}\mathbf{r}_w u_w\right.\right.\\
&\qquad\qquad\left.\left.-\frac{1}{n}\mathbf{r}'\mathbf{D}_{jkw}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jkw}^{-1}\mathbf{r}_w u_w\right)\right|^2\\
= &Kn^2\mathbb{E}(C_{11j}+C_{12j}),
\end{aligned}\tag{S1.42}$$

where

$$\begin{aligned}
C_{11j} = &\sum_{k<j}\left\{\mathbb{E}_j\left(u_k\mathbf{r}_k'\mathbf{D}_{jkw}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jkw}^{-1}\mathbf{r}_w u_w\right.\right.\\
&\qquad\qquad\left.\left.-\frac{1}{n}\mathbf{r}'\mathbf{D}_{jkw}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jkw}^{-1}\mathbf{r}_w u_w\right)\right\}^2,\\
C_{12j} = &\sum_{k_1\neq k_2<j}\mathbb{E}_j\left(u_{k_1}\mathbf{r}_{k_1}'\mathbf{D}_{jk_1w}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk_1w}^{-1}\mathbf{r}_w u_w\right.\\
&\qquad\qquad\left.-\frac{1}{n}\mathbf{r}'\mathbf{D}_{jk_1w}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk_1w}^{-1}\mathbf{r}_w u_w\right)\\
&\times\mathbb{E}_j\left(u_{k_2}\mathbf{r}_{k_2}'\mathbf{D}_{jk_2w}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk_2w}^{-1}\mathbf{r}_w u_w\right.\\
&\qquad\qquad\left.-\frac{1}{n}\mathbf{r}'\mathbf{D}_{jk_2w}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk_2w}^{-1}\mathbf{r}_w u_w\right).
\end{aligned}$$

Similar to (S1.26), we have

$$\mathbb{E}(C_{11j}) \leqslant Kn^{-3}.\tag{S1.43}$$

For a pair of fixed $k_1 \neq k_2 \neq w < j$, write

$$\mathbb{E}_j\left(u_{k_1}\mathbf{r}'_{k_1}\mathbf{D}^{-1}_{jk_1w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1w}\mathbf{r}_w u_w - \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_1w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1w}\mathbf{r}_w u_w\right)$$
$$= C_{121j} + C_{122j} + C_{123j} + C_{124j}, \tag{S1.44}$$

where

$$\begin{aligned}
C_{121j} =& \mathbb{E}_j\Bigg(u_{k_1}\mathbf{r}'_{k_1}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w \\
&- \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w\Bigg), \\
C_{122j} =& -\mathbb{E}_j\Bigg(u_{k_1}\mathbf{r}'_{k_1}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{r}_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w \\
&- \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{r}_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w\Bigg)\beta_{k_2(jk_1w)}, \\
C_{123j} =& -\mathbb{E}_j\Bigg(u_{k_1}\mathbf{r}'_{k_1}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_{k_2}\mathbf{r}'_{k_2}\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w \\
&- \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_{k_2}\mathbf{r}'_{k_2}\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w\Bigg)\check{\beta}_{k_2(jk_1w)}, \\
C_{124j} =& \mathbb{E}_j\Bigg(u_{k_1}\mathbf{r}'_{k_1}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{r}_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_{k_2}\mathbf{r}'_{k_2}\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w \\
&- \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{r}_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_{k_2}\mathbf{r}'_{k_2}\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w\Bigg) \\
&\times \beta_{k_2(jk_1w)}\check{\beta}_{k_2(jk_1w)}.
\end{aligned}$$

Then,

$$\mathbb{E}\Bigg\{C_{121j}\mathbb{E}_j(u_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w - \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w)\Bigg\} = 0 \tag{S1.45}$$

since C_{121j} is independent of $\mathbf{z}_{k_2}$. Also, decomposing

$$\mathbb{E}_j\left(u_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w - \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w\right)$$

into

$$\begin{aligned}
C_{125j} &= \mathbb{E}_j\Bigg(u_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w \\
&\qquad - \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w\Bigg), \\
C_{126j} &= \mathbb{E}_j\Bigg(u_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{r}_{k_1}\mathbf{r}'_{k_1}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w \\
&\qquad - \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{r}_{k_1}\mathbf{r}'_{k_1}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w\Bigg)\beta_{k_1(hk_2w)}, \\
C_{127j} &= \mathbb{E}_j\Bigg(u_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_{k_1}\mathbf{r}'_{k_1}\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w \\
&\qquad - \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_{k_1}\mathbf{r}'_{k_1}\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w\Bigg)\check{\beta}_{k_1(hk_2w)},
\end{aligned}$$

and using

$$\begin{aligned}
\mathbb{E}|C_{122j}|^2 &\leqslant Kn^{-2}\mathbb{E}|\mathbf{r}'_{k_2}\mathbf{D}^{-2}_{jk_1k_2w}\mathbf{r}_{k_2}||\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_1k_2w}\mathbf{r}_w u_w|^2 \\
&\leqslant Kn^{-2}\left(Kn^{-4}\times n^{2/3}\times\mathbb{E}|\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_1k_2w}\mathbf{A}_2\mathbf{e}_i|^4\right)^{1/2} \\
&\leqslant Kn^{-14/3}, \qquad \text{(S1.46)}
\end{aligned}$$

$$\mathbb{E}|C_{126j}|^2 \leqslant Kn^{-14/3}, \quad \mathbb{E}|C_{127j}|^2 \leqslant Kn^{-14/3}, \qquad \text{(S1.47)}$$

$$\mathbb{E}(C_{122j}C_{125j}) = 0, \qquad \text{(S1.48)}$$

we have

$$\mathbb{E}\Bigg\{C_{122j}\mathbb{E}_j(u_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w$$

$$- \frac{1}{n}\mathbf{r}'\mathbf{D}_{jk_2w}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk_2w}^{-1}\mathbf{r}_w u_w)\Big\} \leqslant Kn^{-14/3}. \tag{S1.49}$$

By symmetry, we also get

$$\mathbb{E}\Big\{C_{123j}\mathbb{E}_j(u_{k_2}\mathbf{r}_{k_2}'\mathbf{D}_{jk_2w}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk_2w}^{-1}\mathbf{r}_w u_w - \frac{1}{n}\mathbf{r}'\mathbf{D}_{jk_2w}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk_2w}^{-1}\mathbf{r}_w u_w)\Big\} \leqslant Kn^{-14/3}. \tag{S1.50}$$

For C_{124j}, we have

$$\begin{aligned}
\mathbb{E}|C_{124j}|^2 &\leqslant Kn^{-2}\mathbb{E}|\mathbf{r}_{k_2}'\mathbf{D}_{jk_1k_2w}^{-2}\mathbf{r}_{k_2}| \\
&\qquad \times |\mathbf{r}_{k_2}'\mathbf{D}_{jk_1k_2w}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk_1k_2w}^{-1}\mathbf{r}_{k_2}\mathbf{r}_{k_2}'\check{\mathbf{D}}_{jk_1k_2w}^{-1}\mathbf{r}_w u_w|^2 \\
&\leqslant Kn^{-2}\left(\mathbb{E}|\mathbf{r}_{k_2}'\mathbf{D}_{jk_1k_2w}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk_1k_2w}^{-1}\mathbf{r}_{k_2}|^4 \right.\\
&\qquad \left. \times \mathbb{E}|\mathbf{r}_{k_2}'\check{\mathbf{D}}_{jk_1k_2w}^{-1}\mathbf{r}_w u_w|^4\right)^{1/2} \\
&\quad + Kn^{-2}\left(\mathbb{E}|\mathbf{r}_{k_2}'\mathbf{D}_{jk_1k_2w}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\check{\mathbf{D}}_{jk_1k_2w}^{-1}\mathbf{r}_{k_2}|^4\right)^{1/2} \\
&\quad \times \left(\mathbb{E}\left|\mathbf{r}_{k_2}'\mathbf{D}_{jk_1k_2w}^{-2}\mathbf{r}_{k_2} - \frac{1}{n}\operatorname{tr}\mathbf{D}_{jk_1k_2w}^{-2}\right|^4 \mathbb{E}|\mathbf{r}_{k_2}'\check{\mathbf{D}}_{jk_1k_2w}^{-1}\mathbf{r}_w u_w|^8\right)^{1/4} \\
&\leqslant Kn^{-2}\Big\{n^{-10/3} + n^{-5/3} \times n^{-1/2} \\
&\qquad \times \left(n^{-10/3}\mathbb{E}|\mathbf{r}_w'\check{\mathbf{D}}_{jk_1k_2w}^{-2}\mathbf{r}_w|^4|u_w|^8\right)^{1/4}\Big\} \\
&\leqslant Kn^{-16/3} + Kn^{-5} \times (n^{-11/24} \times n^{-7/12}) \leqslant Kn^{-16/3}.
\end{aligned} \tag{S1.51}$$

Then combining,

$$\mathbb{E}\Big\{\mathbb{E}_j(u_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w - \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w)\Big\}^2 \leqslant Kn^{-4}, \tag{S1.52}$$

we obtain

$$\mathbb{E}\Big\{C_{124j}\mathbb{E}_j(u_{k_2}\mathbf{r}'_{k_2}\mathbf{D}^{-1}_{jk_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w - \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jk_2w}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jk_2w}\mathbf{r}_w u_w)\Big\} \leqslant Kn^{-14/3}. \tag{S1.53}$$

Summing up (S1.42)–(S1.45) and (S1.49)–(S1.53), we obtain

$$\mathbb{E}\left|\sum_{k\neq w<j}\mathbb{E}_j\left(u_k\mathbf{r}'_k\mathbf{D}^{-1}_{jkw}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jkw}\mathbf{r}_w u_w - \frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jkw}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jkw}\mathbf{r}_w u_w\right)\right|^2 \leqslant Kn^{-2/3}. \tag{S1.54}$$

Similarly, one can also prove

$$\mathbb{E}\left|\sum_{k\neq w<j}\mathbb{E}_j\left(\frac{1}{n}\mathbf{r}'\mathbf{D}^{-1}_{jkw}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jkw}\mathbf{r}_w u_w - \frac{1}{n^2}\mathbf{r}'\mathbf{D}^{-1}_{jkw}\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_{jkw}\mathbf{r}\right)\right|^2 \leqslant Kn^{-2/3}. \tag{S1.55}$$

Thus by similar approaches of proving (S1.36), combining (S1.35)–(S1.41), (S1.54) and (S1.55), we have

$$\mathbb{E}\left|\sum_{k\neq w<j}\mathbb{E}_j\left(u_k\mathbf{r}'_k\mathbf{D}^{-1}_j\mathbf{A}_2\mathbf{e}_i\mathbf{e}'_i\mathbf{A}'_2\check{\mathbf{D}}^{-1}_j\check{\mathbf{r}}_w\check{u}_w\right)\right.$$

$$- \frac{(j-1)(j-2)}{n^2}\rho_p^2 a_{1i}^2 \Bigg|^2 \leqslant K\delta_n^{3/2} n^{-1/2}. \quad \text{(S1.56)}$$

Last, when $k > j > w$ and $k < j < w$, following the proof for the case $k \neq w > j$ and $k \neq w < j$, one can easily get

$$\mathbb{E}\Bigg| \sum_{k>j>w} \mathbb{E}_j \left(u_k \mathbf{r}_k' \mathbf{D}_j^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_j^{-1} \check{\mathbf{r}}_w \check{u}_w\right) - \frac{(n-j)(j-1)}{n^2}\rho_p^2 a_{1i}^2 \Bigg|^2 \leqslant K\delta_n^{3/2} n^{-1/2}, \quad \text{(S1.57)}$$

$$\mathbb{E}\Bigg| \sum_{k<j<w} \mathbb{E}_j \left(u_k \mathbf{r}_k' \mathbf{D}_j^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_j^{-1} \check{\mathbf{r}}_w \check{u}_w\right) - \frac{(n-j)(j-1)}{n^2}\rho_p^2 a_{1i}^2 \Bigg|^2 \leqslant K\delta_n^{3/2} n^{-1/2}. \quad \text{(S1.58)}$$

Therefore, summing up (S1.24)–(S1.27), (S1.36) and (S1.56)–(S1.58), we finally obtain

$$\mathbb{E}\left|\mathbb{E}_j(\mathbf{s}_j' \mathbf{D}_j^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \check{\mathbf{D}}_j^{-1} \check{\mathbf{s}}_j) - \rho_p^2 a_{1i}^2\right|^2 \leqslant K\delta_n^{3/2} n^{-1/2}. \quad \text{(S1.59)}$$

Taking the same approach, we can also get

$$\mathbb{E}\left|\mathbb{E}_j(\mathbf{s}_j' \mathbf{D}_j^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \mathbf{D}_j^{-1} \mathbf{s}_j) - \rho_p^2 a_{1i}^2\right|^2 \leqslant K\delta_n^{3/2} n^{-1/2}. \quad \text{(S1.60)}$$

$$\mathbb{E}\left|\mathbb{E}_j(\mathbf{e}_i' \mathbf{a}_1 \mathbf{s}_j' \mathbf{D}_j^{-1} \mathbf{A}_2 \mathbf{e}_i) - \rho_p a_i a_{1i}\right|^2 \leqslant K\delta_n^{3/2} n^{-1/2}. \quad \text{(S1.61)}$$

S1.5 The bound for $\mathbb{E}(\mathbf{s}_j' \mathbf{D}_j^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \mathbf{D}_j^{-1} \mathbf{s}_j)^2$

Now, we find a bound for $\mathbb{E}(\mathbf{s}_j' \mathbf{D}_j^{-1} \mathbf{A}_2 \mathbf{e}_i \mathbf{e}_i' \mathbf{A}_2' \mathbf{D}_j^{-1} \mathbf{s}_j)^2$. WLOG, we assume $j \neq 1, 2, 3, 4$ for simplicity of notation. Recall that $\mathbf{s}_j = \sum_{k\neq j} u_k \mathbf{r}_k$, then we have

$$\mathbb{E}(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j)^2 \leqslant K\mathbb{E}(n^4M_{1j}+n^3M_{2j}+n^2M_{3j}+nM_{4j}) \tag{S1.62}$$

where

$$M_{1j} = u_1\mathbf{r}_1'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{r}_4u_4$$

$$= u_1\mathbf{r}_1'\mathbf{D}_{1j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{2j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{3j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{4j}^{-1}\mathbf{r}_4u_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)},$$

$$M_{2j} = u_1^2\mathbf{r}_1'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{r}_1u_2\mathbf{r}_2'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{r}_3u_3,$$

$$M_{3j} = u_1^3\mathbf{r}_1'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{r}_2u_2,$$

$$+ u_1^2u_2^2\mathbf{r}_1'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{r}_1\mathbf{r}_2'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{r}_2,$$

$$M_{4j} = u_1^4\mathbf{r}_1'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{r}_1.$$

We first consider M_{1j}. Using $\mathbf{D}_{kj}^{-1} = \mathbf{D}_{1kj}^{-1} - \mathbf{D}_{1kj}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1kj}^{-1}\beta_{1(kj)}$ for $k = 2,3,4$, we have $\mathbb{E}(M_{1j}) \leqslant \mathbb{E}M_{11j} + K\mathbb{E}(M_{12j}+M_{13j}+M_{14j})$, where

$$M_{11j} = u_1\mathbf{r}_1'\mathbf{D}_{1j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{12j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i$$

$$\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{14j}^{-1}\mathbf{r}_4u_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)},$$

$$M_{12j} = u_1\mathbf{r}_1'\mathbf{D}_{1j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{12j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{12j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i$$

$$\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{14j}^{-1}\mathbf{r}_4u_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)}\beta_{1(2j)},$$

$$M_{13j} = u_1\mathbf{r}_1'\mathbf{D}_{1j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{12j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{12j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i$$

$$\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{14j}^{-1}\mathbf{r}_4u_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)}\beta_{1(2j)}\beta_{1(3j)},$$

$$M_{14j} = u_1\mathbf{r}_1'\mathbf{D}_{1j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{12j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{12j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i$$
$$\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{14j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{14j}^{-1}\mathbf{r}_4u_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)}\beta_{1(2j)}\beta_{1(3j)}\beta_{1(4j)}.$$

For M_{12j}, we use the identity $\mathbf{D}_{1kj}^{-1} = \mathbf{D}_{13kj}^{-1} - \mathbf{D}_{13kj}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{13kj}^{-1}\beta_{3(1kj)}$ for $k = 0, 2, 4$ ($\mathbf{D}_{130j} = \mathbf{D}_{13j}$, $\mathbf{D}_{10j} = \mathbf{D}_{1j}$, $\beta_{3(10j)} = \beta_{3(1j)}$) to decompose M_{12j} into 16 terms, and then bound their expectations separately. For instance, we have

$$\begin{aligned}
&|\mathbb{E}u_1\mathbf{r}_1'\mathbf{D}_{13j}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{123j}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{123j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{123j}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{123j}^{-1}\mathbf{r}_2u_2\\
&\qquad \times u_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{134j}^{-1}\mathbf{r}_4u_4\\
&\qquad \times \beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)}\beta_{1(2j)}\beta_{3(12j)}^2\beta_{3(1j)}^2\beta_{3(14j)}^2|\\
\leqslant&\mathbb{E}^{1/4}|u_1\mathbf{r}_1'\mathbf{D}_{13j}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{123j}^{-1}\mathbf{r}_3|^4\mathbb{E}^{1/4}|u_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i|^4\\
&\qquad \times \mathbb{E}^{1/4}|\mathbf{r}_3'\mathbf{D}_{123j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{123j}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{123j}^{-1}\mathbf{r}_2u_2|^4\\
&\qquad \times \mathbb{E}^{1/4}|\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{134j}^{-1}\mathbf{r}_4u_4|^4\\
\leqslant&K\left(\delta_n^2n^{-10/3}\mathbb{E}|\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{123j}^{-1}\mathbf{r}_3|^4|\mathbf{r}_3'\mathbf{D}_{13j}^{-2}\mathbf{r}_3|^2\right)^{1/4} \times \delta_n^{1/2}n^{-5/6}\\
&\qquad \times \left(\delta_n^2n^{-10/3}\mathbb{E}|\mathbf{r}_3'\mathbf{D}_{123j}^{-1}\mathbf{r}_2u_2|^4|\mathbf{r}_3'\mathbf{D}_{123j}^{-2}\mathbf{r}_3|^4\right)^{1/4}\\
&\qquad \times \left(\delta_n^2n^{-10/3}\mathbb{E}|u_4|^4|\mathbf{r}_4'\mathbf{D}_{134j}^{-2}\mathbf{r}_4|^2\right)^{1/4}\\
\leqslant&K\left((\delta_n^2n^{-10/3})^2 \times +\delta_n^2n^{-10/3} \times \sqrt{n^{-2} \times n^{-8} \times (\delta_nn^{1/3})^{10}}\right)^{1/4} \times \delta_n^{1/2}n^{-5/6}\\
&\qquad \times \delta_nn^{-5/3} \times \left(\delta_n^2n^{-10/3} \times n^{-2} + \delta_n^2n^{-10/3} \times \sqrt{n^{-2} \times \delta_n^2n^{-10/3}}\right)^{1/4}\\
=&\delta_n^3n^{-11/2} = o(n^{-11/2}),
\end{aligned} \tag{S1.63}$$

where the last inequality is obtained by using the decomposition

$$\mathbf{r}_3'\mathbf{D}_{13j}^{-2}\mathbf{r}_3 = \mathbf{r}_3'\mathbf{D}_{13j}^{-2}\mathbf{r}_3 - \frac{1}{n}\operatorname{tr}\left(\mathbf{D}_{13j}^{-2}\right) + \frac{1}{n}\operatorname{tr}\left(\mathbf{D}_{13j}^{-2}\right),$$
$$\mathbf{r}_4'\mathbf{D}_{134j}^{-2}\mathbf{r}_4 = \mathbf{r}_4'\mathbf{D}_{134j}^{-4}\mathbf{r}_4 - \frac{1}{n}\operatorname{tr}\left(\mathbf{D}_{134j}^{-2}\right) + \frac{1}{n}\operatorname{tr}\left(\mathbf{D}_{134j}^{-2}\right).$$

Hence, for terms which have two or more $\mathbf{D}_{13kj}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{13kj}^{-1}$, similar to (S1.63) and by the finite sixth moment condition, one can verify that their expectations are not larger than $Kn^{-11/2}$. For the terms which have only one $\mathbf{D}_{13kj}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{13kj}^{-1}$, for example,

$$u_1\mathbf{r}_1'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{123j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{123j}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{123j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i$$
$$\times\,\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r}_4u_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)}\beta_{1(2j)}\beta_{3(12j)}, \quad \text{(S1.64)}$$

using the identities $\beta_{k(j)} = \beta_{k(4j)} + \beta_{k(j)}\beta_{k(4j)}\beta_{4(kj)}(\mathbf{r}_k'\mathbf{D}_{4kj}^{-1}\mathbf{r}_4)^2$ for $k = 1, 2, 3$ (similarly for $\beta_{1(2j)}$ and $\beta_{3(12j)}$), $\beta_{4(j)} = \overline{\beta}_{4(j)} - \overline{\beta}_{4(j)}\beta_{4(j)}\Delta_{4(j)}$ and $\mathbf{D}_{13j}^{-1} = \mathbf{D}_{134j}^{-1} - \mathbf{D}_{134j}^{-1}\mathbf{r}_4\mathbf{r}_4'\mathbf{D}_{134j}^{-1}\beta_{4(13j)}$ (similarly for $\mathbf{D}_{123j}^{-1}$), and applying the Hölder's inequality as done in (S1.63), we finally get

$$|\mathbb{E}(\text{S1.64})| \leqslant |\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2$$
$$\times\,u_2u_3\mathbf{r}_3'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r}_4u_4\beta_{1(4j)}\beta_{2(4j)}$$
$$\times\,\overline{\beta}_{3(4j)}\overline{\beta}_{4(j)}\beta_{1(24j)}\overline{\beta}_{3(124j)})| + Kn^{-11/2}$$
$$\leqslant Kn^{-1}|\mathbf{r}'\mathbb{E}\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i| \times \mathbb{E}^{1/2}|u_1\mathbf{r}_1'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_iu_3\mathbf{r}_3'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i|^2$$
$$\times\,\mathbb{E}^{1/2}|\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2|^2 + Kn^{-11/2}$$

$$\leqslant Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}$$

$$\leqslant Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}, \tag{S1.65}$$

where the second inequality is obtained by taking expectation on $\mathbf{z}_4$ first and then using the martingale decomposition of $\mathbf{D}_{134j}^{-1} - \mathbb{E}\mathbf{D}_{134j}^{-1}$. Therefore, summing up the above results, we have

$$\begin{aligned}|\mathbb{E}(M_{12j})| \leqslant& |\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{123j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{123j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i\\ &\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r}_4u_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)}\beta_{1(2j)})|\\ &+ Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}.\end{aligned}$$

By the same approach, one can also prove that

$$|\mathbb{E}(M_{12j})| \leqslant |\mathbb{E}(M_{12(34)j})| + Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}, \tag{S1.66}$$

where

$$\begin{aligned}M_{12(34)j} =& u_1\mathbf{r}_1'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\\ &\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r}_4u_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)}\beta_{1(2j)}.\end{aligned}$$

From $\beta_{k(j)} = \beta_{k(3j)} + \beta_{k(j)}\beta_{k(3j)}\beta_{3(kj)}(\mathbf{r}_k'\mathbf{D}_{3kj}^{-1}\mathbf{r}_3)^2$ for $k = 1, 2, 4$ and using the Cauchy-Schwarz inequality, one can find

$$\begin{aligned}|\mathbb{E}(M_{12(34)j})| \leqslant& |\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\\ &\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r}_4u_4\beta_{1(34j)}\beta_{2(34j)}\beta_{3(4j)}\beta_{4(3j)}\beta_{1(234j)})| + Kn^{-11/2}.\end{aligned} \tag{S1.67}$$

Plugging $\beta_{3(4j)} = \overline{\beta}_{3(4j)} - \overline{\beta}^2_{3(4j)}\Delta_{3(4j)} - \overline{\beta}^2_{3(4j)}\beta_{3(4j)}\Delta^2_{3(4j)}$ and $\beta_{4(3j)} = \overline{\beta}_{4(3j)} - \overline{\beta}^2_{4(3j)}\Delta_{4(3j)} - \overline{\beta}^2_{4(3j)}\beta_{4(3j)}\Delta^2_{4(3j)}$ into (S1.67), we get

$$|\mathbb{E}(M_{12(34)j})| \leqslant \mathbb{E}(M_{12(34)1j} + M_{12(34)2j} + M_{12(34)3j}) + Kn^{-11/2}, \quad \text{(S1.68)}$$

where

$$\begin{aligned}
M_{12(34)1j} =&\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i \\
&\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r}_4u_4\beta_{1(34j)}\beta_{2(34j)}\overline{\beta}_{3(4j)}\overline{\beta}_{4(3j)}\beta_{1(234j)}), \\
M_{12(34)2j} =&\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i \\
&\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r}_4u_4\Delta_{3(4j)}\beta_{1(34j)}\beta_{2(34j)}\overline{\beta}^2_{3(4j)}\overline{\beta}_{4(3j)}\beta_{1(234j)}), \\
M_{12(34)3j} =&\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i \\
&\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r}_4u_4\Delta_{4(3j)}\beta_{1(34j)}\beta_{2(34j)}\overline{\beta}_{3(4j)}\overline{\beta}^2_{4(3j)}\beta_{1(234j)}).
\end{aligned}$$

Similar to (S1.65), we can prove that

$$\begin{aligned}
|\mathbb{E}(M_{12(34)1j})| =&n^{-2}|\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2 \\
&\times (\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{134j}^{-1}\mathbf{r})^2\beta_{1(34j)}\beta_{2(34j)}\overline{\beta}_{3(4j)}\overline{\beta}_{4(3j)}\beta_{1(234j)})| \\
\leqslant&n^{-2}|\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2 \\
&\times (\mathbf{e}_i'\mathbf{A}_2'\mathbb{E}\mathbf{D}_{134j}^{-1}\mathbf{r})^2\beta_{1(34j)}\beta_{2(34j)}\overline{\beta}_{3(4j)}\overline{\beta}_{4(3j)}\beta_{1(234j)})| \\
&+ Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2} \\
\leqslant&Kn^{-9/2}(\mathbf{r}'\mathbb{E}\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i)^2 + Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}_{134j}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}
\end{aligned}$$

$$\leqslant Kn^{-9/2}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2 + Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}, \tag{S1.69}$$

$$|\mathbb{E}(M_{12(34)2j})| \leqslant Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}, \tag{S1.70}$$

$$|\mathbb{E}(M_{12(34)3j})| \leqslant Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}, \tag{S1.71}$$

where the first inequality in (S1.69) is obtained by Hölder's inequality and

$$\mathbb{E}\left|\mathbf{r}'(\mathbf{D}_{134j}^{-1} - \mathbb{E}\mathbf{D}_{134j}^{-1})\mathbf{A}_2\mathbf{e}_i\right|^3 \leqslant Kn^{3/2}\mathbb{E}\left|\mathbf{r}'\mathbf{D}_{134jt}^{-1}\mathbf{r}_t\mathbf{r}_t'\mathbf{D}_{134jt}^{-1}\mathbf{A}_2\mathbf{e}_i\right|^2 \leqslant Kn^{-3/2}.$$

Summing up (S1.66)–(S1.71), we have

$$|\mathbb{E}(M_{12j})| \leqslant Kn^{-9/2}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2 + Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}. \tag{S1.72}$$

For M_{13j} and M_{14j}, by similar arguments, one can verify that

$$|\mathbb{E}(M_{13j})| \leqslant Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}, \tag{S1.73}$$

$$\begin{aligned}|\mathbb{E}(M_{14j})| &\leqslant |\mathbb{E}u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\\ &\quad\times \mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4\\ &\quad\times \beta_{1(234j)}^4\overline{\beta}_{2(134j)}\overline{\beta}_{3(124j)}\overline{\beta}_{4(123j)}| + o(n^{-11/2})\\ &\leqslant n^{-3}\mathbb{E}|u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i||\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{r}'\mathbf{D}_{1234j}^{-2}\mathbf{r}_1|^3 + o(n^{-11/2})\\ &= o(n^{-11/2}).\end{aligned} \tag{S1.74}$$

Then from (S1.72)–(S1.74), we obtain

$$|\mathbb{E}(M_{1j})| \leqslant |\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{1j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{12j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{13j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{14j}^{-1}\mathbf{r}_4u_4$$

$$\times \beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)})| + Kn^{-9/2}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2$$
$$+ Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}.$$

Similarly, by repeating the above process, one can prove

$$\begin{aligned}|\mathbb{E}(M_{1j})| \leqslant &|\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i \\ &\times u_4\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)})| + Kn^{-9/2}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2 \\ &+ Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}. \qquad \text{(S1.75)}\end{aligned}$$

Next, we consider the term

$$\begin{aligned}&u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4 \\ &\qquad \times (\beta_{1(j)} - \beta_{1(2j)})\beta_{2(j)}\beta_{3(j)}\beta_{4(j)} \\ =&u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4 \\ &\qquad \times (\mathbf{r}_1'\mathbf{D}_{12j}^{-1}\mathbf{r}_2)^2\beta_{1(j)}\beta_{1(2j)}\beta_{2(1j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)}. \qquad \text{(S1.76)}\end{aligned}$$

Similar to (S1.65), we have

$$\begin{aligned}|\mathbb{E}(\text{S1.76})| \leqslant &|\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i \\ &\times u_4\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4(\mathbf{r}_1'\mathbf{D}_{124j}^{-1}\mathbf{r}_2)^2 \\ &\times \beta_{1(4j)}\beta_{1(24j)}\beta_{2(14j)}\beta_{2(4j)}\beta_{3(4j)}\overline{\beta}_{4(j)})| + Kn^{-11/2} \\ =&n^{-1}|\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i \\ &\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}(\mathbf{r}_1'\mathbf{D}_{124j}^{-1}\mathbf{r}_2)^2\end{aligned}$$

$$\times \beta_{1(4j)}\beta_{1(24j)}\beta_{2(14j)}\beta_{2(4j)}\beta_{3(4j)}\overline{\beta}_{4(j)})| + Kn^{-11/2}$$

$$\leqslant Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2},$$

which implies that

$$|\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i$$
$$\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)})|$$
$$\leqslant|\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4$$
$$\times \beta_{1(2j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)})| + Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}.$$

Repeating the above steps, we can prove

$$|\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i$$
$$\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4\beta_{1(j)}\beta_{2(j)}\beta_{3(j)}\beta_{4(j)})|$$
$$\leqslant|\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4$$
$$\times \beta_{1(234j)}\beta_{2(134j)}\beta_{3(124j)}\beta_{4(123j)})| + Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}. \tag{S1.77}$$

On the other hand, we get $\beta_{k_1(k_2k_3k_4j)} = \overline{\beta}_{k_1(k_2k_3k_4j)} - \overline{\beta}^2_{k_1(k_2k_3k_4j)}\Delta_{k_1(k_2k_3k_4j)} + \overline{\beta}^3_{k_1(k_2k_3k_4j)}\Delta^2_{k_1(k_2k_3k_4j)} - \overline{\beta}_{k_1(k_2k_3k_4j)}\beta^3_{k_1(k_2k_3k_4j)}\Delta^3_{k_1(k_2k_3k_4j)}$ for $k_1 \neq k_2 \neq k_3 \neq k_4 \in \{1,2,3,4\}$. Plugging this identity into the first term on the right of (S1.77), one can get

$$|\mathbb{E}(u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4$$

$$\times \beta_{1(234j)}\beta_{2(134j)}\beta_{3(124j)}\beta_{4(123j)})|$$

$$\leqslant|\mathbb{E}(J_{1j})| + K|\mathbb{E}(J_{2j} + J_{3j} + J_{4j})| + Kn^{-11/2},$$

where

$$\begin{aligned}
J_{1j} =& u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i \\
&\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4\overline{\beta}_{1(234j)}\overline{\beta}_{2(134j)}\overline{\beta}_{3(124j)}\overline{\beta}_{4(123j)}, \\
J_{2j} =& u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i \\
&\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4\Delta_{1(234j)}\overline{\beta}_{1(234j)}^2\overline{\beta}_{2(134j)}\overline{\beta}_{3(124j)}\overline{\beta}_{4(123j)}, \\
J_{3j} =& u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i \\
&\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4\Delta_{1(234j)}\Delta_{2(134j)}\overline{\beta}_{1(234j)}^2\overline{\beta}_{2(134j)}^2\overline{\beta}_{3(124j)}\overline{\beta}_{4(123j)}, \\
J_{4j} =& u_1\mathbf{r}_1'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_2u_2u_3\mathbf{r}_3'\mathbf{D}_{1234j}^{-1}\mathbf{A}_2\mathbf{e}_i \\
&\times \mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_{1234j}^{-1}\mathbf{r}_4u_4\Delta_{1(234j)}^2\overline{\beta}_{1(234j)}^3\overline{\beta}_{2(134j)}\overline{\beta}_{3(124j)}\overline{\beta}_{4(123j)}.
\end{aligned}$$

For J_{1j},

$$\begin{aligned}
|\mathbb{E}(J_{1j})| \leqslant& Kn^{-4}\mathbb{E}(\mathbf{r}'\mathbf{D}_{1234j}^{-1}\mathbf{A}\mathbf{e}_i)^4 \leqslant Kn^{-4}\mathbb{E}(\mathbf{r}'\mathbf{D}^{-1}\mathbf{A}\mathbf{e}_i)^4 + o(n^{-11/2}) \\
\leqslant& Kn^{-4}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}\mathbf{e}_i)^4 + Kn^{-11/2}, \qquad \text{(S1.78)}
\end{aligned}$$

where the last inequality follows from

$$\begin{aligned}
\mathbb{E}\left(\mathbf{r}'(\mathbf{D}^{-1} - \mathbb{E}\mathbf{D}^{-1})\mathbf{A}\mathbf{e}_i\right)^4 \leqslant& K\mathbb{E}\left|\mathbf{r}'\sum_{t=1}^{n}(\mathbb{E}_t - \mathbb{E}_{t-1})\mathbf{D}_t^{-1}\mathbf{r}_t\mathbf{r}_t'\mathbf{D}_t^{-1}\mathbf{A}\mathbf{e}_i\beta_{(t)}\right|^3 \\
\leqslant& Kn^{3/2}\mathbb{E}|\mathbf{r}'\mathbf{D}_t^{-1}\mathbf{r}_t\mathbf{r}_t'\mathbf{D}_t^{-1}\mathbf{A}\mathbf{e}_i\beta_{(t)}|^3 \leqslant Kn^{-3/2}
\end{aligned}$$

by Lemma 2. For J_{2j}, J_{3j} and J_{4j}, by the same arguments as in the derivation of (S1.69), one can prove

$$|\mathbb{E}(J_{2j})| \leqslant Kn^{-9/2}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2 + Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}, \tag{S1.79}$$

$$|\mathbb{E}(J_{3j})| \leqslant Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}, \tag{S1.80}$$

$$|\mathbb{E}(J_{4j})| \leqslant Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}, \tag{S1.81}$$

Summing up (S1.75)–(S1.81), we obtain

$$|\mathbb{E}(M_{1j})| \leqslant Kn^{-4}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}\mathbf{e}_i)^4 + Kn^{-9/2}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2 + Kn^{-5}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-11/2}. \tag{S1.82}$$

Similarly, one can verify

$$\begin{aligned} &|\mathbb{E}(M_{2j})| \leqslant Kn^{-4}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-9/2}, \\ &|\mathbb{E}(M_{3j})| = o(n^{-7/2}) \quad \text{and} \quad |\mathbb{E}(M_{4j})| = o(n^{-5/2}). \end{aligned} \tag{S1.83}$$

From these bounds and (S1.62), we finally get

$$\mathbb{E}(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j)^2 \leqslant K(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}\mathbf{e}_i)^4 + Kn^{-1/2}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2 + Kn^{-1}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i| + Kn^{-3/2}. \tag{S1.84}$$

S1.6 Completion of the proof of Theorem 1

First, we figure out the limit of each term in (S1.22). Write

$$\begin{aligned}
&\sum_{i=1}^{m}(\tau_{i,p}-3)\left\{\mathbb{E}_j(\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2)\right\}_{ii}^2\\
=&\sum_{i=1}^{m}\Big[(\tau_{i,p}-3)\left\{\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j)-\rho_p^2a_{1i}^2\right\}\\
&\qquad\qquad\times\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j)\Big]\\
&+\rho_p^2\sum_{i=1}^{m}(\tau_{i,p}-3)a_{1i}^2\left\{\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j)-\rho_p^2a_{1i}^2\right\}\\
&\qquad\qquad+\rho_p^4\sum_{i=1}^{m}(\tau_{i,p}-3)a_{1i}^4. \qquad \text{(S1.85)}
\end{aligned}$$

Then using (S1.60) and (S1.84), we have

$$\begin{aligned}
&\sum_{i=1}^{m}\mathbb{E}^{1/2}\left|\mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j)-\rho_p^2a_{1i}^2\right|^2\\
&\qquad\qquad\times\mathbb{E}^{1/2}(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j)^2\\
\leqslant&K\Big\{\delta_n^{3/4}n^{-1/4}\sum_{i=1}^{m}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2+\delta_n^{3/4}n^{-1/2}\sum_{i=1}^{m}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i|\\
&\qquad\qquad+\delta_n^{3/4}n^{-3/4}\sum_{i=1}^{m}|\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i|^{1/2}+\delta_n^{3/4}\Big\}\\
\leqslant&K\Big[\delta_n^{3/4}n^{-1/4}\sum_{i=1}^{m}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2+\delta_n^{3/4}\left\{\sum_{i=1}^{m}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2\right\}^{1/2}\\
&\qquad\qquad+\delta_n^{3/4}\left\{\sum_{i=1}^{m}(\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2\right\}^{1/4}+\delta_n^{3/4}\Big]\\
=&o(1), \qquad \text{(S1.86)}
\end{aligned}$$

where the second inequality is obtained from Jensen's inequality

$$\sum_{i=1}^{m} |\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i|^{2/h} \leqslant \left\{ m^{(h-1)} \sum_{i=1}^{m} (\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2 \right\}^{1/h}$$

for $h \geqslant 1$, and the final conclusion follows from

$$\sum_{i=1}^{m} (\mathbf{r}'\mathbb{E}\mathbf{D}^{-1}\mathbf{A}_2\mathbf{e}_i)^2 = \mathbf{r}'(\mathbb{E}\mathbf{D}^{-1})^2\mathbf{r} \leqslant K.$$

Additionally, notice that $\sum_{i=1}^{m} a_{1i}^2 = 1$, we have

$$\rho_p^2 \sum_{i=1}^{m} a_{1i}^2 \mathbb{E}^{1/2} \left| \mathbb{E}_j(\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2\mathbf{e}_i\mathbf{e}_i'\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j) - \rho_p^2 a_{1i}^2 \right|^2 \leqslant K\delta_n^{3/4} n^{-1/4} = o(1). \tag{S1.87}$$

Then, (S1.85)–(S1.87) indicate that

$$\sum_{i=1}^{m} (\tau_{i,p} - 3) \left\{ \mathbb{E}_j(\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2) \right\}_{ii}^2 = \rho_p^4 \sum_{i=1}^{m} (\tau_i - 3) a_{1i}^4 + o_p(1). \tag{S1.88}$$

By the same approach, it can be shown that

$$\sum_{i=1}^{m} (\tau_{i,p} - 3)(\mathbf{a}_1\mathbf{a}_1')_{ii}^2 = \sum_{i=1}^{m} (\tau_i - 3) a_i^4 + o_p(1), \tag{S1.89}$$

$$\sum_{i=1}^{m} (\tau_{i,p} - 3) \left\{ \mathbb{E}_j(\mathbf{a}_1\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2) \right\}_{ii}^2 = \rho_p^2 \sum_{i=1}^{m} (\tau_i - 3) a_i^2 a_{1i}^2 + o_p(1), \tag{S1.90}$$

$$\sum_{i=1}^{m} (\tau_{i,p} - 3)(\mathbf{a}_1\mathbf{a}_1')_{ii} \left\{ \mathbb{E}_j(\mathbf{a}_1\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2) \right\}_{ii} = \rho_p \sum_{i=1}^{m} (\tau_i - 3) a_i^3 a_{1i} + o_p(1), \tag{S1.91}$$

$$\sum_{i=1}^{m} (\tau_{i,p} - 3)(\mathbf{a}_1\mathbf{a}_1')_{ii} \left\{ \mathbb{E}_j(\mathbf{A}_2'\mathbf{D}_j^{-1}\mathbf{s}_j\mathbf{s}_j'\mathbf{D}_j^{-1}\mathbf{A}_2) \right\}_{ii}$$

$$= \rho_p^2 \sum_{i=1}^{m} (\tau_i - 3) a_i^2 a_{1i}^2 + o_p(1), \quad \text{(S1.92)}$$

$$\sum_{i=1}^{m} (\tau_{i,p} - 3) \left\{\mathbb{E}_j(\mathbf{a}_1 \mathbf{s}_j' \mathbf{D}_j^{-1} \mathbf{A}_2)\right\}_{ii} \left\{\mathbb{E}_j(\mathbf{A}_2' \mathbf{D}_j^{-1} \mathbf{s}_j \mathbf{s}_j' \mathbf{D}_j^{-1} \mathbf{A}_2)\right\}_{ii}$$

$$= \rho_p^3 \sum_{i=1}^{m} (\tau_i - 3) a_i a_{1i}^3 + o_p(1). \quad \text{(S1.93)}$$

Summing up (S1.88)–(S1.93), we obtain

$$Q_2 \xrightarrow{i.p.} c^2\zeta_1 + 4c(1-c)\zeta_2 + 2(2-3c)(1-c)\zeta_3 - 4(1-c)^2\zeta_4 + (1-c)^2\zeta_5,$$

$$Q_4 \xrightarrow{i.p.} c\zeta_1 + 2(1-c)\zeta_2 - (1-c)\zeta_3,$$

which, together with the convergence of Q_1 and Q_3, gives the joint limiting distribution of T_1 and T_2. Finally, Theorem 1 is obtained by applying the delta method.

S2 Proof of Theorem 2

We first establish the inequality in (3.1). Let $\tilde{\beta}_i$ denote the i-th element of $\boldsymbol{\beta}'\mathbf{A}_\mathbf{x}$, $i = 1, \ldots, m_x$. Then, the multiple correlation coefficient ρ_p^2 can be represented as

$$\rho_p^2 = \frac{\boldsymbol{\beta}'\mathbf{A}_\mathbf{x}\mathbf{A}_\mathbf{x}'\boldsymbol{\beta}}{\boldsymbol{\beta}'\mathbf{A}_\mathbf{x}\mathbf{A}_\mathbf{x}'\boldsymbol{\beta} + \sigma_{\epsilon,p}^2} = \frac{\sum_{i=1}^{m_x} \tilde{\beta}_i^2}{\sum_{i=1}^{m_x} \tilde{\beta}_i^2 + \sigma_{\epsilon,p}^2},$$

which is followed by

$$\tau_y - 3 + (2\rho_p^2 - 1)(\tau_{\epsilon,p} - 3)$$

$$
\begin{aligned}
=&\left\{\sum_{i=1}^{m_x}(\tau_{(x)i,p}-3)\tilde{\beta}_i^4+3\sigma_{\epsilon,p}^4\tau_{\epsilon,p}+3\left(\sum_{i=1}^{m_x}\tilde{\beta}_i^2\right)^2+6\sigma_{\epsilon,p}^2\sum_{i=1}^{m_x}\tilde{\beta}_i^2\right\}\\
&\times\frac{1}{\sum_{i=1}^{m_x}\tilde{\beta}_i^2+\sigma_{\epsilon,p}^2}-3+(2\rho_p^2-1)(\tau_{\epsilon,p}-3)\\
=&\frac{(1-\rho_p^2)^2}{\sigma_\epsilon^4}\sum_{i=1}^{m_x}(\tau_{(x)i,p}-3)\tilde{\beta}_i^4+(\tau_{\epsilon,p}-3)\rho_p^4,
\end{aligned}
$$

where $\tau_{(x)i,p}$ denotes the kurtosis of ξ_i, the i-th component of $\boldsymbol{\xi}$. By the facts $\tau_{(x)i,p}\geqslant 1$ and $\tau_{\epsilon,p}\geqslant 1$, we have

$$
\begin{aligned}
\tau_y-3+(2\rho_p^2-1)(\tau_{\epsilon,p}-3)\geqslant&-2\frac{(1-\rho_p^2)^2}{\sigma_\epsilon^4}\sum_{i=1}^{m_x}\tilde{\beta}_i^4-2\rho_p^4\\
\geqslant&-2\frac{(1-\rho_p^2)^2}{\sigma_\epsilon^4}(\boldsymbol{\beta}'\mathbf{A_x}\mathbf{A_x}'\boldsymbol{\beta})^2-2\rho_p^4\\
=&-4\rho_p^4,
\end{aligned}
$$

which gives the bound in (3.1).

Next, we prove the convergence of $\hat{\sigma}_t^2$. For simplicity, let

$$
\begin{aligned}
&H_{n1}=\hat{\tau}_y-3+(2R_t^{*2}-1)(\hat{\tau}_\epsilon-3),\quad H_{n2}=-4R_t^{*4},\\
&H_1=\tau_y-3+(2\rho^2-1)(\tau_\epsilon-3),\quad H_2=-4\rho^4,
\end{aligned}
$$

By the consistency of R_t^{*2}, we have

$$
H_{n1}\xrightarrow{i.p.}H_1,\quad H_{n2}\xrightarrow{i.p.}H_2,\quad H_1\geqslant H_2
$$

and thus, it is sufficient to show

$$
\max\{H_{n1},H_{n2}\}\xrightarrow{i.p.}H_1, \tag{S2.1}
$$

or equivalently,

$$\max\{H_{n1} - H_{n2}, 0\} \xrightarrow{i.p.} H_1 - H_2. \tag{S2.2}$$

For any $\kappa > 0$,

$$\begin{aligned} &\mathrm{P}\left(|\max\{H_{n1} - H_{n2}, 0\} - (H_1 - H_2)| \geqslant \kappa\right) \\ =&\mathrm{P}(|(H_{n1} - H_{n2}) - (H_1 - H_2)| \geqslant \kappa, H_{n1} - H_{n2} \geqslant 0) \\ &+ \mathrm{P}(|H_1 - H_2| \geqslant \kappa, H_{n1} - H_{n2} < 0). \end{aligned} \tag{S2.3}$$

When $H_1 = H_2$,

$$\begin{aligned} \text{(S2.3)} =&\mathrm{P}(|(H_{n1} - H_{n2}) - (H_1 - H_2)| \geqslant \kappa, H_{n1} - H_{n2} \geqslant 0) \\ \leqslant&\mathrm{P}(|(H_{n1} - H_{n2}) - (H_1 - H_2)| \geqslant \kappa) \to 0, \end{aligned}$$

and when $H_1 > H_2$,

$$\text{(S2.3)} \leqslant \mathrm{P}(|(H_{n1} - H_{n2}) - (H_1 - H_2)| \geqslant \kappa) + \mathrm{P}(H_{n1} - H_{n2} < 0) \to 0,$$

which verifies the convergence in (S2.2). The proof of the theorem is complete.

S3 Appendix

Lemma 1. (Billingsley (1995), Theorem 35.12) *Suppose that for each n, $Y_{n1}, Y_{n2}, \ldots, Y_{nr_n}$ is a real martingale difference sequence with respect to the* *increasing σ-field $\mathcal{F}_{n1}, \mathcal{F}_{n2}, \ldots, \mathcal{F}_{nr_n}$ having second moments. If as $n \to \infty$,*

$$\sum_{k=1}^{r_n} \mathbb{E}\{Y_{nk}^2 | \mathcal{F}_{n,k-1}\} \xrightarrow{i.p.} \sigma^2,$$

where σ is a positive constant, and for each ϵ

$$\sum_{k=1}^{r_n} \mathbb{E}\{Y_{nk}^2 I_{(|Y_{nk}| \geqslant \epsilon)}\} \to 0.$$

Then

$$\sum_{k=1}^{r_n} Y_{nk} \xrightarrow{D} N(0, \sigma^2).$$

Lemma 2. (Burkholder (1973)) *Let $\{X_k\}$ be a martingale difference sequence with respect to the increasing σ-field $\{\mathcal{F}_k\}$. Then, for $p > 1$,*

$$\mathbb{E}\left|\sum X_k\right|^p \leqslant K_p \mathbb{E}\left(\sum |X_k|^2\right)^{p/2}.$$

Lemma 3. (Bai and Silverstein (2010), Lemma B.26) *Let $\mathbf{x} = (x_1, \ldots, x_n)^*$ be a random vector of independent entries, $\mathbf{T}$ be an $n \times n$ nonrandom matrix. Assume that $\mathbb{E}x_i = 0$, $\mathbb{E}|x_i|^2 = 1$ and $\mathbb{E}|x_i|^l \leqslant v_l$ for $i = 1, \ldots, n$. Then we have, for any $h \geqslant 1$,*

$$\mathbb{E}|\mathbf{x}^*\mathbf{T}\mathbf{x} - \operatorname{tr}(\mathbf{T})|^h \leqslant K_h \left\{(v_4 \operatorname{tr}(\mathbf{T}\mathbf{T}^*))^{h/2} + v_{2h} \operatorname{tr}(\mathbf{T}\mathbf{T}^*)^{h/2}\right\},$$

where K_h is a constant depending on h only.

Lemma 4. (Bai and Silverstein (2010), (9.8.6)) *Let $\mathbf{x} = (x_1, \ldots, x_p)^*$ be a complex random vector with independent components, $\mathbf{B} = (b_{ij})$ and $\mathbf{C} =$*

(c_{ij}) *be* $p \times p$ *complex nonrandom matrix. Assume that* $\mathbb{E}x_i = 0$, $\mathbb{E}|x_i|^2 = 1$, *then we have*

$$\begin{aligned}\mathbb{E}(\mathbf{x}^*\mathbf{B}\mathbf{x} - \operatorname{tr}\mathbf{B})(\mathbf{x}^*\mathbf{C}\mathbf{x} - \operatorname{tr}\mathbf{C}) = &\sum_{i=1}^{p}(\mathbb{E}|x_i|^4 - |\mathbb{E}x_i^2|^2 - 2)b_{ii}c_{ii} \\ &+ \operatorname{tr}(\mathbf{B}_x\mathbf{C}_x') + \operatorname{tr}(\mathbf{B}\mathbf{C}),\end{aligned}$$

where $\mathbf{B}_x = (\mathbb{E}x_i^2 b_{ij})$ *and* $\mathbf{C}_x = (\mathbb{E}x_i^2 c_{ij})$.

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