Homogeneity Tests for High-dimensional Mean Vectors and Covariance Matrices

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Supplementary Material

All Technical proofs are listed in this Supplementary Material.

S1 Proof of Lemma 2.1

Proof. First, we can obtain the following relations of equivalence.

$$
\begin{aligned}
& \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_R = \boldsymbol{\mu} \\
\Longleftrightarrow \quad & \mathrm{E}(\mathbf{X}|Y=r) = \mathrm{E}(\mathbf{X}), \ \forall r = 1, 2, \cdots, R \\
\Longleftrightarrow \quad & \mathrm{var}_Y\{\mathrm{E}(\mathbf{X}|Y)\} = \sum_{r=1}^{R} p_r \|\mathrm{E}(\mathbf{X}|Y=r) - \mathrm{E}(\mathbf{X})\|^2 = 0 \\
\Longleftrightarrow \quad & \sum_{r=1}^{R} p_r \left\{\mathrm{E}(\mathbf{X}_1|Y=r) - \mathrm{E}(\mathbf{X}_1)\right\}^{\mathrm{T}} \left\{\mathrm{E}(\mathbf{X}_2|Y_2=r) - \mathrm{E}(\mathbf{X}_2)\right\} = 0 \\
\Longleftrightarrow \quad & \sum_{r=1}^{R} p_r \mathrm{E}(\mathbf{X}_1^{\mathrm{T}}\mathbf{X}_2) \left\{\frac{I(Y_1=r)}{p_r} - 1\right\} \left\{\frac{I(Y_2=r)}{p_r} - 1\right\} = 0
\end{aligned}
$$

$$\Longleftrightarrow \quad \mathrm{E}(\mathbf{X}_1^{\mathrm{T}}\mathbf{X}_2)\sum_{r=1}^{R} p_r \left\{\frac{I(Y_1=r)}{p_r}-1\right\}\left\{\frac{I(Y_2=r)}{p_r}-1\right\}=0$$

$$\Longleftrightarrow \quad \mathrm{E}(\mathbf{X}_1^{\mathrm{T}}\mathbf{X}_2)\left\{\sum_{r=1}^{R}\frac{I(Y_1=r)I(Y_2=r)}{p_r}-1\right\}=0.$$

Hence, by Definition 1, $\mathcal{U}(\mathbf{X}|Y)=\mathrm{var}_Y\{\mathrm{E}(\mathbf{X}|Y)\}\geq 0$ is true, and the equality holds if and only if $\boldsymbol{\mu}_1=\cdots=\boldsymbol{\mu}_R$, which completes the proof. □

S2 Proof of equation (2.2)

Proof. By the definition of $M_{n,p}$, we have

$$\begin{aligned}
M_{n,p} &= \sum_{i\neq j}\mathbf{X}_i^{\mathrm{T}}\mathbf{X}_j\left\{\sum_{r=1}^{R}\frac{I(Y_i=r)I(Y_j=r)}{(N_r-1)/(n-1)}-\sum_{r=1}^{R}\sum_{s=1}^{R}I(Y_i=r)I(Y_j=s)\right\}\\
&= \sum_{r=1}^{R}\frac{(n-1)}{(N_r-1)}\sum_{i\neq j}\mathbf{X}_{ri}^{\mathrm{T}}\mathbf{X}_{rj}-\sum_{r=1}^{R}\sum_{i\neq j}\mathbf{X}_{ri}^{\mathrm{T}}\mathbf{X}_{rj}-\sum_{r\neq s}\sum_{i=1}^{N_r}\sum_{j=1}^{N_s}\mathbf{X}_{ri}^{\mathrm{T}}\mathbf{X}_{sj}\\
&= \sum_{r\neq s}\frac{N_s}{(N_r-1)}\sum_{i\neq j}\mathbf{X}_{ri}^{\mathrm{T}}\mathbf{X}_{rj}-\sum_{r\neq s}\sum_{i=1}^{N_r}\sum_{j=1}^{N_s}\mathbf{X}_{ri}^{\mathrm{T}}\mathbf{X}_{sj}\\
&= \sum_{r\neq s}N_rN_s\frac{\sum_{i\neq j}\mathbf{X}_{ri}^{\mathrm{T}}\mathbf{X}_{rj}}{N_r(N_r-1)}-\sum_{r\neq s}N_rN_s\frac{\sum_{i=1}^{N_r}\sum_{j=1}^{N_s}\mathbf{X}_{ri}^{\mathrm{T}}\mathbf{X}_{sj}}{N_rN_s}\\
&= \sum_{r>s}^{R}N_rN_s\left\{\frac{\sum_{i\neq j}^{N_r}\mathbf{X}_{ri}^{\mathrm{T}}\mathbf{X}_{rj}}{N_r(N_r-1)}+\frac{\sum_{i\neq j}^{N_s}\mathbf{X}_{si}^{\mathrm{T}}\mathbf{X}_{sj}}{N_s(N_s-1)}-2\frac{\sum_{i=1}^{N_r}\sum_{j=1}^{N_s}\mathbf{X}_{ri}^{\mathrm{T}}\mathbf{X}_{sj}}{N_rN_s}\right\},
\end{aligned}$$

which completes the proof. □

S3 Proof of Theorem 2.1

Proof. Write $g_{ij} = \sum_{r=1}^{R} I(Y_i = r)I(Y_j = r)/p_r - 1$ and $\hat{g}_{ij} = \sum_{r=1}^{R} I(Y_i = r)I(Y_j = r)/\hat{p}_r - 1$. Then reconstruct the statistic $M_{n,p}$ as follows:

$$
\begin{aligned}
M_{n,p} &= \sum_{i\neq j} \mathbf{X}_i^{\mathrm{T}}\mathbf{X}_j\hat{g}_{ij} = \sum_{i\neq j}(\boldsymbol{\mu}_i + \boldsymbol{\Gamma}_i\mathbf{Z}_i)^{\mathrm{T}}(\boldsymbol{\mu}_j + \boldsymbol{\Gamma}_j\mathbf{Z}_j)\hat{g}_{ij} \\
&= \sum_{i\neq j} \boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\mu}_j\hat{g}_{ij} + \sum_{i\neq j} \boldsymbol{\mu}_i^{\mathrm{T}}\boldsymbol{\Gamma}_j\mathbf{Z}_j\hat{g}_{ij} + \sum_{i\neq j} \boldsymbol{\mu}_j^{\mathrm{T}}\boldsymbol{\Gamma}_i\mathbf{Z}_i\hat{g}_{ij} \\
&\quad + \sum_{i\neq j} \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\Gamma}_i^{\mathrm{T}}\boldsymbol{\Gamma}_j\mathbf{Z}_j(\hat{g}_{ij} - g_{ij}) + \sum_{i\neq j} \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\Gamma}_i^{\mathrm{T}}\boldsymbol{\Gamma}_j\mathbf{Z}_j g_{ij} \\
&=: W_1 + W_2 + W_3 + W_4 + W_5.
\end{aligned}
$$

With elemental calculation, it is obtained that

$$
\begin{aligned}
& W_1 = \sum_{r=1}^{R} \boldsymbol{\mu}_r^{\mathrm{T}}\boldsymbol{\mu}_r N_r(n - N_r) - \sum_{r\neq s} \boldsymbol{\mu}_r^{\mathrm{T}}\boldsymbol{\mu}_s N_r N_s = \sum_{r>s} N_r N_s \|\boldsymbol{\mu}_r - \boldsymbol{\mu}_s\|^2, \\
& \mathrm{E}(W_2|Y_1, \cdots, Y_n) = \mathrm{E}(W_3|Y_1, \cdots, Y_n) = \mathrm{E}(W_4|Y_1, \cdots, Y_n) = 0, \\
& \mathrm{var}(W_2|Y_1, \cdots, Y_n) = \sum_{j=1}^{n} \left(\sum_{i\neq j}^{n} \boldsymbol{\mu}_i^{\mathrm{T}}\hat{g}_{ij}\right) \boldsymbol{\Sigma}_j \left(\sum_{k\neq j}^{n} \boldsymbol{\mu}_k\hat{g}_{kj}\right) \\
= \ & \sum_{j=1}^{n} \left\{\sum_{r\neq s} I(Y_j = r)N_s(\boldsymbol{\mu}_r - \boldsymbol{\mu}_s)^{\mathrm{T}}\right\} \boldsymbol{\Sigma}_j \left\{\sum_{r\neq s} I(Y_j = r)N_s(\boldsymbol{\mu}_r - \boldsymbol{\mu}_s)\right\} \\
= \ & \sum_{r=1}^{R} N_r \left\{\sum_{s\neq r} N_s(\boldsymbol{\mu}_r - \boldsymbol{\mu}_s)^{\mathrm{T}}\right\} \boldsymbol{\Sigma}_r \left\{\sum_{s\neq r} N_s(\boldsymbol{\mu}_r - \boldsymbol{\mu}_s)\right\} \\
\leq \ & O(n^3) \sum_{r\neq s} (\boldsymbol{\mu}_r - \boldsymbol{\mu}_s)^{\mathrm{T}}\boldsymbol{\Sigma}_r(\boldsymbol{\mu}_r - \boldsymbol{\mu}_s) = o(n^2 p), \\
& \mathrm{var}(W_3|Y_1, \cdots, Y_n) = \mathrm{var}(W_2|Y_1, \cdots, Y_n), \\
& \mathrm{var}(W_4|Y_1, \cdots, Y_n) = 2\sum_{r=1}^{R} \mathrm{tr}(\boldsymbol{\Sigma}_r^2)N_r(N_r - 1)\frac{(\hat{p}_r - p_r)^2}{\hat{p}_r^2 p_r^2}
\end{aligned}
$$

$$\leq \quad 2O(n^2)\sum_{r=1}^{R}\mathrm{tr}(\mathbf{\Sigma}_r^2)\frac{(\hat{p}_r-p_r)^2}{\hat{p}_r^2p_r^2},$$

$$\mathrm{var}(W_5) = 2n(n-1)\left\{\sum_{r=1}^{R}\mathrm{tr}(\mathbf{\Sigma}_r^2)-2\sum_{r=1}^{R}p_r\mathrm{tr}(\mathbf{\Sigma}_r^2)+\sum_{r=1}^{R}\sum_{s=1}^{R}p_rp_s\mathrm{tr}(\mathbf{\Sigma}_r\mathbf{\Sigma}_s)\right\}$$

$$= \quad 2n(n-1)\left\{\sum_{r=1}^{R}(1-p_r)^2\mathrm{tr}(\mathbf{\Sigma}_r^2)+\sum_{r\neq s}p_rp_s\mathrm{tr}(\mathbf{\Sigma}_r\mathbf{\Sigma}_s)\right\}.$$

Write $Z_{ni} = 2\sum_{j=1}^{i-1}g_{ij}\mathbf{Z}_i^{\mathrm{T}}\mathbf{\Gamma}_i^{\mathrm{T}}\mathbf{\Gamma}_j\mathbf{Z}_j$, and $\mathcal{F}_i = \sigma\{\binom{\mathbf{Z}_1}{Y_1},\cdots,\binom{\mathbf{Z}_i}{Y_i}\}$ be the $\sigma-$field generated by $\{(\mathbf{Z}_j^{\mathrm{T}},Y_j),j\leq i\}$, $d_{n,p}=\mathrm{var}(W_5)$, $v_{ni}=\mathrm{E}(Z_{ni}^2|\mathcal{F}_{i-1}), 2\leq i\leq n$, and $V_n=\sum_{i=2}^{n}v_{ni}$. Then $W_5/\sqrt{d_{n,p}}=\sum_{i=2}^{n}Z_{ni}/\sqrt{d_{n,p}}$. It is easy to see that $E(Z_{ni}|\mathcal{F}_{i-1})=0$ and that is to say $\{\sum_{i=2}^{k}Z_{ni},\mathcal{F}_k: 2\leq j\leq n\}$ is zero mean martingale. The central limit theorem in Hall and Heyde (1980) will hold if

$$d_{n,p}^{-1}V_n \xrightarrow{p} 1, \tag{S3.1}$$

and for any $\varepsilon>0$

$$\sum_{i=2}^{n}d_{n,p}^{-1}\mathrm{E}\{Z_{ni}^2I(|Z_{ni}|>\varepsilon\sqrt{d_{n,p}})|\mathcal{F}_{i-1}\}\xrightarrow{p} 0. \tag{S3.2}$$

Denote $\tilde{g}_{ij}^r = p_r^{-1}\{I(Y_i=r)-p_r\}\{I(Y_j=r)-p_r\}$. Then it can be shown that

$$v_{ni} \quad = \quad 4\sum_{j=1}^{i-1}\sum_{r=1}^{R}\tilde{g}_{jj}^r\mathbf{Z}_j^{\mathrm{T}}\mathbf{\Gamma}_j^{\mathrm{T}}\mathbf{\Sigma}_r\mathbf{\Gamma}_j\mathbf{Z}_j+8\sum_{1\leq j<k<i}\sum_{r=1}^{R}\tilde{g}_{jk}^r\mathbf{Z}_j^{\mathrm{T}}\mathbf{\Gamma}_j^{\mathrm{T}}\mathbf{\Sigma}_r\mathbf{\Gamma}_k\mathbf{Z}_k,$$

and thus

$$d_{n,p}^{-1}V_n$$

$$
\begin{aligned}
&= d_{n,p}^{-1} \sum_{r=1}^{R} \left\{ 4 \sum_{j=1}^{n-1} (n-j) \tilde{g}_{jj}^{r} \mathbf{Z}_j^{\mathrm{T}} \mathbf{\Gamma}_j^{\mathrm{T}} \mathbf{\Sigma}_r \mathbf{\Gamma}_j \mathbf{Z}_j + 8 \sum_{1 \le j < k < n} (n-k) \tilde{g}_{jk}^{r} \mathbf{Z}_j^{\mathrm{T}} \mathbf{\Gamma}_j^{\mathrm{T}} \mathbf{\Sigma}_r \mathbf{\Gamma}_k \mathbf{Z}_k \right\} \\
&=: W_6 + W_7.
\end{aligned}
$$

Obviously, we have $\mathrm{E}(W_6) = 1$ and

$$
\begin{aligned}
\mathrm{E}(W_6^2) &= \frac{16}{d_{n,p}^2} \mathrm{E} \Bigg\{ \sum_{r=1}^{R} \sum_{s=1}^{R} \sum_{i=1}^{n-1} (n-i)^2 \tilde{g}_{ii}^{r} \tilde{g}_{ii}^{s} (\mathbf{Z}_i^{\mathrm{T}} \mathbf{\Gamma}_i^{\mathrm{T}} \mathbf{\Sigma}_r \mathbf{\Gamma}_i \mathbf{Z}_i)(\mathbf{Z}_i^{\mathrm{T}} \mathbf{\Gamma}_i^{\mathrm{T}} \mathbf{\Sigma}_s \mathbf{\Gamma}_i \mathbf{Z}_i) \\
&\quad + \sum_{r=1}^{R} \sum_{s=1}^{R} \sum_{i \neq j}^{n-1} (n-i)(n-j) \tilde{g}_{jj}^{r} \tilde{g}_{ii}^{s} (\mathbf{Z}_j^{\mathrm{T}} \mathbf{\Gamma}_j^{\mathrm{T}} \mathbf{\Sigma}_r \mathbf{\Gamma}_j \mathbf{Z}_j)(\mathbf{Z}_i^{\mathrm{T}} \mathbf{\Gamma}_i^{\mathrm{T}} \mathbf{\Sigma}_s \mathbf{\Gamma}_i \mathbf{Z}_i) \Bigg\} \\
&= \frac{16}{d_{n,p}^2} \mathrm{E} \Bigg[\sum_{r=1}^{R} \sum_{i=1}^{n-1} (n-i)^2 p_r^{-2} \left\{ I(Y_i = r) - p_r \right\}^4 (\mathbf{Z}_i^{\mathrm{T}} \mathbf{\Gamma}_i^{\mathrm{T}} \mathbf{\Sigma}_r \mathbf{\Gamma}_i \mathbf{Z}_i)^2 \\
&\quad + \sum_{r \neq s} \sum_{i=1}^{n-1} (n-i)^2 \tilde{g}_{ii}^{r} \tilde{g}_{ii}^{s} (\mathbf{Z}_j^{\mathrm{T}} \mathbf{\Gamma}_j^{\mathrm{T}} \mathbf{\Sigma}_r \mathbf{\Gamma}_j \mathbf{Z}_j)(\mathbf{Z}_i^{\mathrm{T}} \mathbf{\Gamma}_i^{\mathrm{T}} \mathbf{\Sigma}_s \mathbf{\Gamma}_i \mathbf{Z}_i) \\
&\quad + \sum_{r=1}^{R} \sum_{s=1}^{R} \sum_{i \neq j}^{n-1} (n-i)(n-j) \left\{ \mathrm{tr}(\mathbf{\Sigma}_r^2) - 2 p_r \mathrm{tr}(\mathbf{\Sigma}_r^2) + p_r \mathrm{tr}(\mathbf{\Sigma}_r \mathbf{\Sigma}) \right\} \\
&\qquad \cdot \left\{ \mathrm{tr}(\mathbf{\Sigma}_s^2) - 2 p_s \mathrm{tr}(\mathbf{\Sigma}_s^2) + p_s \mathrm{tr}(\mathbf{\Sigma}_s \mathbf{\Sigma}) \right\} \Bigg] \\
&= \frac{O\{n^3 \mathrm{tr}^2(\mathbf{\Sigma}^2)\}}{d_{n,p}^2} + \frac{O\{n^3 \mathrm{tr}^2(\mathbf{\Sigma}^2)\}}{d_{n,p}^2} + \left\{ \frac{d_{n,p}^2}{d_{n,p}^2} + \frac{O\{n^3 \mathrm{tr}^2(\mathbf{\Sigma}^2)\}}{d_{n,p}^2} \right\} = 1 + o(1),
\end{aligned}
$$

which implies that $\mathrm{var}(W_6) = o(1)$. Hence, we have $W_6 \xrightarrow{p} 1$. Similarly, $E(W_7) = 0$ and

$$
\begin{aligned}
&\mathrm{var}(W_7) \\
&= \frac{64}{d_{n,p}^2} \sum_{r=1}^{R} \sum_{s=1}^{R} \sum_{i<j<n} \sum_{k<t<n} (n-j)(n-t) \mathrm{E} \left(\tilde{g}_{ij}^{r} \tilde{g}_{kt}^{s} \mathbf{Z}_i^{\mathrm{T}} \mathbf{\Gamma}_i^{\mathrm{T}} \mathbf{\Sigma}_r \mathbf{\Gamma}_j \mathbf{Z}_j \mathbf{Z}_k^{\mathrm{T}} \mathbf{\Gamma}_k^{\mathrm{T}} \mathbf{\Sigma}_s \mathbf{\Gamma}_t \mathbf{Z}_t \right) \\
&= \frac{64}{d_{n,p}^2} \sum_{r=1}^{R} \sum_{s=1}^{R} \sum_{i<j<n} (n-j)^2 \mathrm{E} \left(\tilde{g}_{ij}^{r} \tilde{g}_{ij}^{s} \mathbf{Z}_i^{\mathrm{T}} \mathbf{\Gamma}_i^{\mathrm{T}} \mathbf{\Sigma}_r \mathbf{\Gamma}_j \mathbf{Z}_j \mathbf{Z}_j^{\mathrm{T}} \mathbf{\Gamma}_j^{\mathrm{T}} \mathbf{\Sigma}_s \mathbf{\Gamma}_i \mathbf{Z}_i \right)
\end{aligned}
$$

$$= \frac{64}{d_{n,p}^2}\sum_{r=1}^{R}\sum_{s=1}^{R}\sum_{i<j<n}(n-j)^2\mathrm{E}\left\{\tilde{g}_{ij}^r\tilde{g}_{ij}^s\mathrm{tr}(\boldsymbol{\Sigma}_r\boldsymbol{\Sigma}_j\boldsymbol{\Sigma}_s\boldsymbol{\Sigma}_i)\right\} = \frac{o\{n^4\mathrm{tr}^2(\boldsymbol{\Sigma}^2)\}}{d_{n,p}^2} \longrightarrow 0,$$

which implies $W_7 \xrightarrow{p} 0$. Thus, (S3.1) follows. Next, we shall show (S3.2).

Since

$$\mathrm{E}\{Z_{ni}^2 I(|Z_{ni}| > \varepsilon\sqrt{d_{n,p}})|\mathcal{F}_{i-1}\} \le \mathrm{E}(Z_{ni}^4|\mathcal{F}_{i-1})/d_{n,p},$$

it is sufficient to show

$$\sum_{i=2}^{n} E(Z_{ni}^4) = o(d_{n,p}^2)$$

by the law of large numbers. Because $|\tilde{g}_{ij}|$ is bounded, then we have

$$\begin{aligned}\sum_{i=2}^{n}\mathrm{E}(Z_{ni}^4) &= O(n^2)\left\{\mathrm{E}(\mathbf{Z}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1^{\mathrm{T}}\boldsymbol{\Gamma}_2\mathbf{Z}_2)^4\right\} + O(n^3)\mathrm{E}\left\{(\mathbf{Z}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1^{\mathrm{T}}\boldsymbol{\Gamma}_2\mathbf{Z}_2)^2(\mathbf{Z}_3^{\mathrm{T}}\boldsymbol{\Gamma}_3^{\mathrm{T}}\boldsymbol{\Gamma}_4\mathbf{Z}_4)^2\right\} \\ &\le O(n^3)E(\mathbf{Z}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1^{\mathrm{T}}\boldsymbol{\Gamma}_2\mathbf{Z}_2)^4 = o(d_{n,p}^2),\end{aligned}$$

which completes the proof. □

S4 Proof of Lemma 3.1

Proof. Similar to the proof of Lemma 2.1 in S1, we have

$$\begin{aligned} & \boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_R = \boldsymbol{\Sigma} \\ \Longleftrightarrow\ & \frac{1}{2}\mathrm{E}\left\{(\mathbf{X}_1-\mathbf{X}_2)(\mathbf{X}_1-\mathbf{X}_2)^{\mathrm{T}}|Y_1=r, Y_2=r\right\} \\ & = \frac{1}{2}\mathrm{E}\sum_{s=1}^{R}\frac{I(Y_1=s)I(Y_2=s)}{p_s}(\mathbf{X}_1-\mathbf{X}_2)(\mathbf{X}_1-\mathbf{X}_2)^{\mathrm{T}},\ \forall r=1,2,\cdots,R \\ \Longleftrightarrow\ & \frac{1}{2}\mathrm{E}(\mathbf{X}_1-\mathbf{X}_2)(\mathbf{X}_1-\mathbf{X}_2)^{\mathrm{T}}\frac{I(Y_1=r)I(Y_2=r)}{p_r^2}\end{aligned}$$

$$= \frac{1}{2}\mathrm{E}\sum_{s=1}^{R}\frac{I(Y_1=s)I(Y_2=s)}{p_s}(\mathbf{X}_1-\mathbf{X}_2)(\mathbf{X}_1-\mathbf{X}_2)^{\mathrm{T}},\ \forall r=1,2,\cdots,R$$

$$\iff \frac{1}{4}\sum_{r=1}^{R}p_r\mathrm{tr}\Bigg[\mathrm{E}(\mathbf{X}_1-\mathbf{X}_2)(\mathbf{X}_1-\mathbf{X}_2)^{\mathrm{T}}$$

$$\left\{\frac{I(Y_1=r)I(Y_2=r)}{p_r^2}-\sum_{s=1}^{R}\frac{I(Y_1=s)I(Y_2=s)}{p_s}\right\}\Bigg]^2=0$$

$$\iff \frac{1}{4}\mathrm{E}\left\{(\mathbf{X}_1-\mathbf{X}_2)^{\mathrm{T}}(\mathbf{X}_3-\mathbf{X}_4)\right\}^2\left\{\sum_{r=1}^{R}I(Y_1=r)I(Y_2=r)I(Y_3=r)I(Y_4=r)\right.$$

$$\left.\frac{(1-p_r)}{p_r^3}-\sum_{r\neq s}\frac{I(Y_1=r)I(Y_2=r)}{p_r}\frac{I(Y_3=s)I(Y_4=s)}{p_s}\right\}=0,$$

which implies that $\mathcal{V}(\mathbf{X}|Y)\geq 0$ is true, and the equality holds if and only if $\mathbf{\Sigma}_1=\cdots=\mathbf{\Sigma}_R$. □

S5 Proof of Theorem 3.1

Proof. Under condition 2, it is obtained that

$$\begin{aligned} T_{n,p} &= \frac{1}{4}\sum_{(i_1,i_2,i_3,i_4)}^{*}\left\{(\mathbf{\Gamma}_{i_1}\mathbf{Z}_{i_1}-\mathbf{\Gamma}_{i_2}\mathbf{Z}_{i_2})^{\mathrm{T}}(\mathbf{\Gamma}_{i_3}\mathbf{Z}_{i_3}-\mathbf{\Gamma}_{i_4}\mathbf{Z}_{i_4})\right\}^2\hat{f}_{i_1i_2i_3i_4} \\ &= \sum_{(i_1,i_2,i_3,i_4)}^{*}\Big\{(\mathbf{Z}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_3}\mathbf{Z}_{i_3})^2\hat{f}_{i_1i_2i_3i_4}-2(\mathbf{Z}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_3}\mathbf{Z}_{i_3}\mathbf{Z}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_4}\mathbf{Z}_{i_4})\hat{f}_{i_1i_2i_3i_4} \\ &\quad +(\mathbf{Z}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_3}\mathbf{Z}_{i_3}\mathbf{Z}_{i_2}^{\mathrm{T}}\mathbf{\Gamma}_{i_2}^{\mathrm{T}}\mathbf{\Gamma}_{i_4}\mathbf{Z}_{i_4})\hat{f}_{i_1i_2i_3i_4}\Big\}=:Q_1+Q_2+Q_3. \end{aligned}$$

First, we consider Q_2 and Q_3. It is easy to find that $\mathrm{E}(Q_2|Y_1,\cdots,Y_n)=\mathrm{E}(Q_3|Y_1,\cdots,Y_n)=0$. Write

$$\hat{h}_{i_1i_2i_3}=\sum_{r=1}^{R}\frac{(n-N_r)I(Y_{i_1}=Y_{i_2}=Y_{i_3}=r)}{(N_r-2)(N_r-1)/(n-1)^2}-\sum_{r\neq s}\frac{I(Y_{i_1}=r)I(Y_{i_2}=Y_{i_3}=s)}{(N_s-1)/(n-1)^2}.$$

Then, we have $Q_2 = -2\sum^*_{(i_1,i_2,i_3)}(\mathbf{Z}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_2}\mathbf{Z}_{i_2}\mathbf{Z}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_3}\mathbf{Z}_{i_3})\hat{h}_{i_1i_2i_3}$, and

$$\begin{aligned}
&\mathrm{E}(Q_2^2|Y_1,\cdots,Y_n) = 4\mathrm{E}\Bigg\{\sum^*_{(i_1,i_2,i_3)}\sum^*_{(j_1,j_2,j_3)}(\mathbf{Z}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_2}\mathbf{Z}_{i_2}\mathbf{Z}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_1}^{\mathrm{T}}\mathbf{\Gamma}_{i_3}\mathbf{Z}_{i_3})\\
&\quad(\mathbf{Z}_{j_1}^{\mathrm{T}}\mathbf{\Gamma}_{j_1}^{\mathrm{T}}\mathbf{\Gamma}_{j_2}\mathbf{Z}_{j_2}\mathbf{Z}_{j_1}^{\mathrm{T}}\mathbf{\Gamma}_{j_1}^{\mathrm{T}}\mathbf{\Gamma}_{j_3}\mathbf{Z}_{j_3})\hat{h}_{i_1i_2i_3}\hat{h}_{j_1j_2j_3}|Y_1,\cdots,Y_n\Bigg\}\\
\leq\ & O(1)\sum^*_{(i_1,i_2,i_3,i_4)}\mathrm{tr}(\mathbf{\Sigma}_{i_1}\mathbf{\Sigma}_{i_3}\mathbf{\Sigma}_{i_4}\mathbf{\Sigma}_{i_2})\hat{h}_{i_1i_2i_3}\hat{h}_{i_4i_2i_3}+O(1)\sum^*_{(i_1,i_2,i_3)}\mathrm{tr}(\mathbf{\Sigma}_{i_1}\mathbf{\Sigma}_{i_2})\mathrm{tr}(\mathbf{\Sigma}_{i_1}\mathbf{\Sigma}_{i_3})\hat{h}^2_{i_1i_2i_3}\\
\leq\ & o\{\mathrm{tr}^2(\mathbf{\Sigma}^2)\}\sum^*_{(i_1,i_2,i_3,i_4)}\hat{h}^2_{i_1i_2i_3}+O\{\mathrm{tr}^2(\mathbf{\Sigma}^2)\}\sum^*_{(i_1,i_2,i_3)}\hat{h}^2_{i_1i_2i_3}=o(\delta_{n,p}).\\
&\mathrm{E}(Q_3^2|Y_1,\cdots,Y_n)\leq O(1)\sum^*_{(i_1,i_2,i_3,i_4)}\mathrm{tr}(\mathbf{\Sigma}_{i_1}\mathbf{\Sigma}_{i_3})\mathrm{tr}(\mathbf{\Sigma}_{i_2}\mathbf{\Sigma}_{i_4})\hat{f}^2_{i_1i_2i_3i_4}\\
&+O(1)\sum^*_{(i_1,i_2,i_3,i_4)}\mathrm{tr}(\mathbf{\Sigma}_{i_1}\mathbf{\Sigma}_{i_3}\mathbf{\Sigma}_{i_4}\mathbf{\Sigma}_{i_2})\hat{f}_{i_1i_2i_3i_4}\hat{f}_{i_1i_3i_2i_4}\\
\leq\ & O\{\mathrm{tr}^2(\mathbf{\Sigma}^2)\}\sum^*_{(i_1,i_2,i_3,i_4)}\hat{f}^2_{i_1i_2i_3i_4}=O\{n^4\mathrm{tr}^2(\mathbf{\Sigma}^2)\}.
\end{aligned}$$

Given $Y_1,\cdots,Y_n$,

$$\begin{aligned}
&\mathrm{E}(T_{n,p}|Y_1,\cdots,Y_n)=\mathrm{E}(Q_1|Y_1,\cdots,Y_n)\\
=\ &\sum_{r=1}^R\mathrm{tr}(\mathbf{\Sigma}_r^2)(n-1)^2N_r(n-N_r)-\sum_{r\neq s}\mathrm{tr}(\mathbf{\Sigma}_r\mathbf{\Sigma}_s)(n-1)^2N_rN_s\\
=\ &(n-1)^2\left\{\sum_{r=1}^R N_r(n-N_r)\mathrm{tr}(\mathbf{\Sigma}_r^2)-\sum_{r\neq s}\mathrm{tr}(\mathbf{\Sigma}_r\mathbf{\Sigma}_s)N_rN_s\right\}\\
=\ &(n-1)^2\sum_{r>s}N_rN_s\mathrm{tr}\left\{(\mathbf{\Sigma}_r-\mathbf{\Sigma}_s)^2\right\}.
\end{aligned}$$

By elemental calculation, it is obtained that

$$Q_1-\mathrm{E}(Q_1|Y_1,\cdots,Y_n) = (n-1)^2\sum_{i\neq j}\left\{(\mathbf{Z}_i^{\mathrm{T}}\mathbf{\Gamma}_i^{\mathrm{T}}\mathbf{\Gamma}_j\mathbf{Z}_j)^2-\mathrm{tr}(\mathbf{\Sigma}_i\mathbf{\Sigma}_j)\right\}\hat{g}_{ij}.$$

Write $\xi_{ij} = (\mathbf{Z}_i^{\mathrm{T}}\mathbf{\Gamma}_i^{\mathrm{T}}\mathbf{\Gamma}_j\mathbf{Z}_j)^2$, $\xi_i^j = \mathrm{E}(\xi_{ij}|\mathbf{Z}_i, Y_j)$ and $\xi^{ij} = \mathrm{E}(\xi_{ij}|Y_i, Y_j)$. Then

$$
\begin{aligned}
& Q_1 - \mathrm{E}(Q_1|Y_1,\cdots,Y_n) = (n-1)^2\sum_{i\neq j}\left\{(\xi_{ij}-\xi^{ij})(\hat{g}_{ij}-g_{ij}) + (\xi_{ij}-\xi^{ij})g_{ij}\right\} \\
= \;& (n-1)^2\sum_{i\neq j}\left\{(\xi_{ij}-\xi_i^j-\xi_j^i+\xi^{ij})(\hat{g}_{ij}-g_{ij}) + (\xi_i^j+\xi_j^i-2\xi^{ij})(\hat{g}_{ij}-g_{ij})\right. \\
& \left. +(\xi_{ij}-\xi^{ij})g_{ij}\right\} =: Q_{11}+Q_{12}+Q_{13}.
\end{aligned}
$$

Then, $\mathrm{E}(Q_{1k}|Y_1,\cdots,Y_n) = 0$ for $k = 1,2,3$. And

$$
\begin{aligned}
& \mathrm{var}(Q_{11}|Y_1,\cdots,Y_n) = 2(n-1)^4\sum_{i\neq j}\hat{g}_{ij}^2\mathrm{E}\left\{(\xi_{ij}-\xi_i^j-\xi_j^i+\xi^{ij})^2|Y_1,\cdots,Y_n\right\} \\
= \;& 2(n-1)^4\sum_{r=1}^R N_r(N_r-1)\frac{(\hat{p}_r-p_r)^2}{\hat{p}_r^2p_r^2}\mathrm{E}\left\{(\xi_{12}-\xi_1^r-\xi_2^r+\xi^{rr})^2|Y_1=Y_2=r\right\} \\
\leq \;& O\{n^4\mathrm{tr}^2(\mathbf{\Sigma}^2)\}\sum_{r=1}^R N_r(N_r-1)\frac{(\hat{p}_r-p_r)^2}{\hat{p}_r^2p_r^2}.
\end{aligned}
$$

It is not hard to obtain that

$$
\begin{aligned}
& Q_{12} = 2(n-1)^2\sum_{i\neq j}(\xi_i^j-\xi^{ij})(\hat{g}_{ij}-g_{ij}) \\
= \;& 2(n-1)^2\sum_{i\neq j}(\xi_i^j-\xi^{ij})\left\{\sum_{r=1}^R I(Y_i=Y_j=r)(\hat{p}_r^{-1}-p_r^{-1})\right\} \\
= \;& 2(n-1)^2\left\{\sum_{r=1}^R\sum_{i=1}^n(\xi_i^r-\xi^{rr})I(Y_i=r)(N_r-1)(\hat{p}_r^{-1}-p_r^{-1})\right\}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& \mathrm{var}(Q_{12}|Y_1,\cdots,Y_n) \\
= \;& 4(n-1)^4\sum_{r=1}^R\sum_{i=1}^R I(Y_i=r)(N_r-1)^2(\hat{p}_r^{-1}-p_r^{-1})^2\mathrm{E}\left\{(\xi_i^r-\xi^{rr})^2|Y_1,\cdots,Y_n\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= 4(n-1)^4 \sum_{r=1}^{R} N_r(N_r-1)^2(\hat{p}_r^{-1}-p_r^{-1})^2 \mathrm{E}\left\{(\xi_1^r-\xi^{rr})^2|Y_1=r\right\} \\
&= 4(n-1)^4 \sum_{r=1}^{R} N_r(N_r-1)^2 \frac{(\hat{p}_r-p_r)^2}{\hat{p}_r^2 p_r^2}\left\{2\mathrm{tr}(\mathbf{\Sigma}_r^4)+\triangle \mathrm{tr}(\mathbf{\Gamma}_r^{\mathrm{T}}\mathbf{\Gamma}_r \mathrm{diag}(\mathbf{\Gamma}_{\mathrm{r}}^{\mathrm{T}}\mathbf{\Sigma}_{\mathrm{r}}\mathbf{\Gamma}_{\mathrm{r}})\mathbf{\Gamma}_{\mathrm{r}}^{\mathrm{T}}\mathbf{\Gamma}_{\mathrm{r}})\right\}.
\end{aligned}
$$

Similar to the analysis in proof of Theorem 2.1, it is obtained that $Q_{1k} = o\{n^3\mathrm{tr}(\mathbf{\Sigma}^2)\}$ for $k = 1, 2$. Up to now, the proof completes if we have $Q_{13}/\sqrt{\delta_{n,p}} \xrightarrow{d} \mathcal{N}(0,1)$ as $n, p \to \infty$.

Let $\mathrm{E}_i(\cdot)$ denote the conditional expectation given $\mathcal{F}_i$. Define $D_{ni} = (\mathrm{E}_i - \mathrm{E}_{i-1})Q_{13}$ and $\delta_{n,p} = \mathrm{var}(Q_{13})$. Then $Q_{13}/\sqrt{\delta_{n,p}} = \sum_{i=2}^{n} D_{ni}/\sqrt{\delta_{n,p}}$, and $\{D_{nk}, 1 \le k \le n\}$ is a martingale difference sequence with respect to the $\sigma-$fields $\{\mathcal{F}_k, 1 \le k \le n\}$. By elemental calculation, it is obtained that

$$
\begin{aligned}
D_{ni} &= 2(n-1)^2 \sum_{j=1}^{i-1} \left\{(\xi_{ij}-\xi^{ij})g_{ij} - \sum_{r=1}^{R}(I(Y_j=r)-p_r)(\xi_j^r-\xi^{jr})\right\} \\
&\quad +(n-i)\sum_{r=1}^{R}\left\{I(Y_i=r)-p_r\right\}(\xi_i^r-\xi^{ri}).
\end{aligned}
$$

With elemental calculation, we have

$$
\begin{aligned}
&\mathrm{var}\left\{(\mathbf{Z}_1^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{Z}_2)^2\right\} \\
&= 3\mathrm{E}\left\{(\mathbf{Z}_1^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}\mathbf{Z}_1)^2\right\} + \triangle \mathrm{tr}\left[\mathrm{E}\left\{(\mathbf{M}\mathbf{Z}_1\mathbf{Z}_1^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}) \circ (\mathbf{M}\mathbf{Z}_1\mathbf{Z}_1^{\mathrm{T}}\mathbf{M}^{\mathrm{T}})\right\}\right] - \mathrm{tr}^2(\mathbf{M}^{\mathrm{T}}\mathbf{M}) \\
&= 6\mathrm{tr}(\mathbf{M}^{\mathrm{T}}\mathbf{M}\mathbf{M}^{\mathrm{T}}\mathbf{M}) + 3\triangle\,\mathrm{tr}(\mathbf{M}^{\mathrm{T}}\mathbf{M}\circ\mathbf{M}^{\mathrm{T}}\mathbf{M}) + 2\mathrm{tr}^2(\mathbf{M}^{\mathrm{T}}\mathbf{M}) \\
&\quad + \triangle^2 \sum_i \sum_j m_{ij}^4 + 3\triangle\,\mathrm{tr}(\mathbf{M}\mathbf{M}^{\mathrm{T}}\circ\mathbf{M}\mathbf{M}^{\mathrm{T}}),
\end{aligned}
$$

where $\mathbf{M} = (m_{ij})$ is a given matrix. Notice the fact that $\sum_i \sum_j m_{ij}^4 \le$

$\text{tr}(\mathbf{M}\mathbf{M}^{\mathrm{T}} \circ \mathbf{M}\mathbf{M}^{\mathrm{T}})$, then we have

$$\begin{aligned}
\delta_{n,p} &= (n-1)^4 \left\{ 4 \sum_{(i,j,k)}^{*} (\xi_{ij} - \xi^{ij})(\xi_{ik} - \xi^{ik}) g_{ij} g_{ik} + 2 \sum_{i \neq j} (\xi_{ij} - \xi^{ij})^2 g_{ij}^2 \right\} \\
&= (n-1)^4 \Bigg[8n(n-1)(n-2) \sum_{r=1}^{R} p_r \text{tr} \left\{ (\mathbf{\Sigma}_r^2 - \mathbf{\Sigma}_r \mathbf{\Sigma})^2 \right\} \\
&\quad + 4\triangle n(n-1)(n-2) \sum_{r=1}^{R} p_r \text{tr} \left\{ \mathbf{\Gamma}_r^{\mathrm{T}} (\mathbf{\Sigma}_r - \mathbf{\Sigma}) \mathbf{\Gamma}_r \circ \mathbf{\Gamma}_r^{\mathrm{T}} (\mathbf{\Sigma}_r - \mathbf{\Sigma}) \mathbf{\Gamma}_r \right\} \\
&\quad + 2n(n-1) \sum_{r=1}^{R} (1-p_r)^2 \left\{ 2\text{tr}^2(\mathbf{\Sigma}_r^2) + O(p) \right\} \\
&\quad + 2n(n-1) \sum_{r \neq s} p_r p_s \left\{ 2\text{tr}^2(\mathbf{\Sigma}_r \mathbf{\Sigma}_s) + O(p) \right\} \Bigg] \\
&= 4n^6 \left\{ \sum_{r=1}^{R} (1-p_r)^2 \text{tr}^2(\mathbf{\Sigma}_r^2) + \sum_{r \neq s} p_r p_s \text{tr}^2(\mathbf{\Sigma}_r \mathbf{\Sigma}_s) \right\} (1 + o(1)) \\
&\quad + 8n^7 \sum_{r=1}^{R} p_r \text{tr} \left\{ (\mathbf{\Sigma}_r^2 - \mathbf{\Sigma}_r \mathbf{\Sigma})^2 \right\} (1 + o(1)) \\
&\quad + 4\triangle n^7 \sum_{r=1}^{R} p_r \text{tr} \left\{ \mathbf{\Gamma}_r^{\mathrm{T}} (\mathbf{\Sigma}_r - \mathbf{\Sigma}) \mathbf{\Gamma}_r \circ \mathbf{\Gamma}_r^{\mathrm{T}} (\mathbf{\Sigma}_r - \mathbf{\Sigma}) \mathbf{\Gamma}_r \right\} (1 + o(1)).
\end{aligned}$$

Similar to the analysis in the proof of Theorem 2.1, $Q_{1k}/\sqrt{\delta_{n,p}} = o_P(1)$ for $k = 1, 2$. Under H_0, $\delta_{n,p} = 4(R-1)n^6 \text{tr}^2(\mathbf{\Sigma}^2) \{1 + o(1)\}$. Write $u_{ni} = \mathrm{E}(D_{ni}^2 | \mathcal{F}_{i-1})$, $2 \le i \le n$, and $U_n = \sum_{i=2}^{n} u_{ni}$. On the basis of the proof of Theorem 2.1, it is sufficient to show $\delta_{n,p}^{-1} U_n \xrightarrow{p} 1$, and for any $\varepsilon > 0$ $\sum_{i=2}^{n} \delta_{n,p}^{-1} \mathrm{E} \left\{ D_{ni}^2 I(|D_{ni}| > \varepsilon \sqrt{\delta_{n,p}}) | \mathcal{F}_{i-1} \right\} \xrightarrow{p} 0$. $u_{ni} = \mathrm{E}_{i-1}(D_{ni}^2)$.

$$U_n = \sum_{i=2}^{n} \Bigg\{ 4(n-1)^4 \sum_{j=1}^{i-1} \sum_{k=1}^{i-1} \mathrm{E}_{i-1} (\xi_{ij} - \xi^{ij})(\xi_{ik} - \xi^{ik}) g_{ij} g_{ik}$$

$$
\begin{aligned}
&-4(n-1)^4\sum_{j=1}^{i-1}\sum_{k=1}^{i-1}\sum_{r=1}^{R}\sum_{s=1}^{R}(I(Y_j=r)-p_r)(I(Y_k=s)-p_s)(\xi_j^r-\xi^{rj})(\xi_k^s-\xi^{sk})\\
&+4(n-1)^4(n-i)^2\sum_{r=1}^{R}\sum_{s=1}^{R}\mathrm{E}_{i-1}(I(Y_i=r)-p_r)(I(Y_i=s)-p_s)(\xi_i^r-\xi^{ri})(\xi_i^s-\xi^{si})\\
&+8(n-1)^4(n-i)\sum_{j=1}^{i-1}\mathrm{E}_{i-1}(\xi_{ij}-\xi^{ij})g_{ij}\sum_{r=1}^{R}(I(Y_i=r)-p_r)(\xi_i^r-\xi^{ri})\Bigg\}\\
=\;& 8(n-1)^4\sum_{j=1}^{n-1}(n-j)\sum_{r=1}^{R}\tilde{g}_{jj}^r(\mathbf{Z}_j^{\mathrm{T}}\mathbf{\Gamma}_j^{\mathrm{T}}\mathbf{\Sigma}_r\mathbf{\Gamma}_j\mathbf{Z}_j)^2\\
&+4\triangle(n-1)^4\sum_{j=1}^{n-1}(n-j)\sum_{r=1}^{R}\tilde{g}_{jj}^r\mathrm{tr}(\mathbf{\Gamma}_r^{\mathrm{T}}\mathbf{\Gamma}_j\mathbf{Z}_j\mathbf{Z}_j^{\mathrm{T}}\mathbf{\Gamma}_j^{\mathrm{T}}\mathbf{\Gamma}_r\circ\mathbf{\Gamma}_r^{\mathrm{T}}\mathbf{\Gamma}_j\mathbf{Z}_j\mathbf{Z}_j^{\mathrm{T}}\mathbf{\Gamma}_j^{\mathrm{T}}\mathbf{\Gamma}_r)\\
&+4(n-1)^4\sum_{j=1}^{n-1}(n-j)\sum_{r=1}^{R}\tilde{g}_{jj}^r(1-p_r)(\xi_j^r-\xi^{jr})^2\\
&+16(n-1)^4\sum_{1\le j<k<n}(n-k)\sum_{r=1}^{R}\tilde{g}_{jk}^r(\mathbf{Z}_j^{\mathrm{T}}\mathbf{\Gamma}_j^{\mathrm{T}}\mathbf{\Sigma}_r\mathbf{\Gamma}_k\mathbf{Z}_k)^2\\
&+8\triangle(n-1)^4\sum_{1\le j<k<n}(n-k)\sum_{r=1}^{R}\tilde{g}_{jk}^r\mathrm{tr}\left\{\mathbf{Z}_j^{\mathrm{T}}\mathbf{\Gamma}_j^{\mathrm{T}}\mathbf{\Gamma}_r\mathrm{diag}(\mathbf{\Gamma}_{\mathrm{r}}^{\mathrm{T}}\mathbf{\Gamma}_{\mathrm{k}}\mathbf{Z}_{\mathrm{k}}\mathbf{Z}_{\mathrm{k}}^{\mathrm{T}}\mathbf{\Gamma}_{\mathrm{k}}^{\mathrm{T}}\mathbf{\Gamma}_{\mathrm{r}})\mathbf{\Gamma}_{\mathrm{r}}^{\mathrm{T}}\mathbf{\Gamma}_{\mathrm{j}}\mathbf{Z}_{\mathrm{j}}\right\}\\
&+8(n-1)^4\sum_{1\le j<k<n}(n-k)\sum_{r=1}^{R}\tilde{g}_{jk}^r(1-p_r)(\xi_j^r-\xi^{jr})(\xi_k^r-\xi^{kr})\\
&+8(n-1)^4\sum_{i=2}^{n}(n-i)^2\sum_{r=1}^{R}p_r\mathrm{tr}\left\{(\mathbf{\Sigma}_r^2-\mathbf{\Sigma}_r\mathbf{\Sigma})^2\right\}\\
&+4\triangle(n-1)^4\sum_{i=2}^{n}(n-i)^2\sum_{r=1}^{R}p_r\mathrm{tr}\left\{\mathbf{\Gamma}_r^{\mathrm{T}}(\mathbf{\Sigma}_r-\mathbf{\Sigma})\mathbf{\Gamma}_r\circ\mathbf{\Gamma}_r^{\mathrm{T}}(\mathbf{\Sigma}_r-\mathbf{\Sigma})\mathbf{\Gamma}_r\right\}\\
&+16(n-1)^4\sum_{i=2}^{n}(n-i)\sum_{j=1}^{i-1}\sum_{r=1}^{R}\left\{I(Y_j=r)-p_r\right\}\mathbf{Z}_j^{\mathrm{T}}\mathbf{\Gamma}_j^{\mathrm{T}}\mathbf{\Sigma}_r(\mathbf{\Sigma}_r-\mathbf{\Sigma})\mathbf{\Sigma}_r\mathbf{\Gamma}_j\mathbf{Z}_j\\
&+8\triangle(n-1)^4\sum_{i=2}^{n}(n-i)\sum_{j=1}^{i-1}\sum_{r=1}^{R}\left\{I(Y_j=r)-p_r\right\}\mathbf{Z}_j^{\mathrm{T}}\mathbf{\Gamma}_j^{\mathrm{T}}\mathbf{\Gamma}_r\mathrm{diag}\left\{\mathbf{\Gamma}_{\mathrm{r}}^{\mathrm{T}}(\mathbf{\Sigma}_{\mathrm{r}}-\mathbf{\Sigma})\mathbf{\Gamma}_{\mathrm{r}}\right\}\mathbf{\Gamma}_{\mathrm{r}}^{\mathrm{T}}\mathbf{\Gamma}_{\mathrm{j}}\mathbf{Z}_{\mathrm{j}}
\end{aligned}
$$

$$=: \sum_{k=1}^{10} Q_{13}^{(k)}$$

It is elemental to check that $\mathrm{E}(U_n) = \delta_{n,p}(1 + o(1))$ and $\mathrm{E}(|Q_{13}^{(k)}|) = O(n^6 p), k = 2, 3$. To prove $\delta_{n,p}^{-1} U_n \xrightarrow{p} 1$, it is sufficient to obtain that $\mathrm{var}(Q_{13}^{(k)}) = o(\delta_{n,p}^2)$ for $k = 1, 4, 5, 6, 9, 10$. For $Q_{13}^{(1)}$, we have

$$\begin{aligned}
\mathrm{var}(Q_{13}^{(1)}) &\leq O(n^8)O(n^3)\sum_{r=1}^{R}\sum_{s=1}^{R}(\tilde{g}_{11}^r)^2 \mathrm{E}(\mathbf{Z}_j^{\mathrm{T}}\mathbf{\Gamma}_j^{\mathrm{T}}\mathbf{\Sigma}_r\mathbf{\Gamma}_j\mathbf{Z}_j)^2(\mathbf{Z}_k^{\mathrm{T}}\mathbf{\Gamma}_k^{\mathrm{T}}\mathbf{\Sigma}_r\mathbf{\Gamma}_k\mathbf{Z}_k)^2 \\
&= O\{n^{11}\mathrm{tr}^4(\mathbf{\Sigma}^2)\}, \\
\mathrm{var}(Q_{13}^{(4)}) &\leq O(n^{13})\sum_{r \neq s}\mathrm{tr}\left\{(\mathbf{\Sigma}_r^2 - \mathbf{\Sigma}_s\mathbf{\Sigma})^2\right\} + O(n^{12})o\{\mathrm{tr}^2(\mathbf{\Sigma}^2)\} = o(\delta_{n,p}^2).
\end{aligned}$$

Similar to the analysis of $Q_{13}^{(4)}$, $\mathrm{var}(Q_{13}^{(k)}) = o(\delta_{n,p}^2)$ follows for $k = 5, 6, 9, 10$. Next we want to prove that $\sum_{i=2}^n \delta_{n,p}^{-1}\mathrm{E}\left\{D_{ni}^2 I(|D_{ni}| > \varepsilon\sqrt{\delta_{n,p}})|\mathcal{F}_{i-1}\right\} \xrightarrow{p} 0$, and it is sufficient to show $\delta_{n,p}^{-2}\sum_{i=2}^n \mathrm{E}(D_{ni}^4) \to 0$. By the definition of D_{ni}, it is obtained that

$$\begin{aligned}
\sum_{i=2}^n \mathrm{E}(D_{ni}^4) \leq\ & O(1)\Bigg[\sum_{i=2}^n \mathrm{E}\left\{2(n-1)^2\sum_{j=1}^{i-1}(\xi_{ij} - \xi_j^i)g_{ij}\right\}^4 \\
&+ \sum_{i=2}^n \mathrm{E}\left\{2(n-1)^2\sum_{j=1}^{i-1}(\xi_j^i - \xi^{ij})g_{ij}\right\}^4 \\
&+ \sum_{i=2}^n \mathrm{E}\left\{2(n-1)^2\sum_{j=1}^{i-1}(\xi_j^r - \xi^{jr})(I(Y_j = r) - p_r)\right\}^4 \\
&+ \sum_{i=2}^n (n-i)^4\mathrm{E}\left\{\sum_{r=1}^{R}(I(Y_i = r) - p_r)(\xi_i^r - \xi^{ir})\right\}^4\Bigg] \\
\leq\ & O\{n^{11}\mathrm{tr}^4(\mathbf{\Sigma}^2)\} = o(\delta_{n,p}^2),
\end{aligned}$$

which completes the proof. □