An Extreme-Value Test for Structural Breaks in Spatial Trends

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Supplementary Material

S1 Proofs

Let $T_Z = \sum_{i=1}^{4}(Z_{i+1} - Z_i)^2$, where $Z_1, ..., Z_4$ are i.i.d. standard normal sample and $Z_5 = Z_1$. First, a lemma.

Lemma 1. *For i.i.d. random samples $T_{Z_1}, ..., T_{Z_n}$ from T_Z with respective order statistics $T_{Z_{(1)}} \leq ... \leq T_{Z_{(n)}}$, we have*

$$(T_{Z_{(n)}} - d_n)/6 \to_d \Lambda,$$

where $d_n = 6[\log n + \frac{1}{3}\log(\log n) - \log\Gamma(4/3)]$, Λ is a standard Gumbel distribution with the distribution function $\Lambda(x) = exp(-e^{-x})$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ is the Gamma function.

Proof. Denote $E_1 = (Z_2 - Z_1)^2 + (Z_4 - Z_3)^2$, $E_2 = (Z_2 - Z_3)^2 + (Z_4 - Z_1)^2$, then $E_1,\ E_2 \sim \Gamma(1, 1/4)$ and $\mathrm{corr}(E_1, E_2) = 0.5$.

By Kotz and Neumann (1963), Kotz and Adams (1964), the distribution of $T_Z = \sum_{i=1}^{4}(Z_{i+1} - Z_i)^2 = E_1 + E_2$ is $\Gamma(4/3, 1/6)$.

For the maximum $T_{Z_{(n)}}$ of the random sample, it is easy to check that T_Z is in the set of von Mises distributions and thus we have $P(T_{Z_n} \leq c_n x + d_n) \rightarrow_d \Lambda$ as $n \rightarrow \infty$. Using the same method as in Example 3.3.29 of Embrechts, Klüppelberg, and Mikosch (2013), we have the following equation:

$$\frac{1}{6}d_n - \frac{1}{3}\log d_n = \log(Cn),$$

where $C = (6^{1/3}\Gamma(\frac{4}{3}))^{-1}$.

By solving for d_n, we obtain $d_n = 6[\log n + \frac{1}{3}\log(\log n) - \log\Gamma(4/3)]$, and $c_n = 6$. □

Remark 1. According to the high–dimensional strong invariance principle in El Machkouri et al. (2013), when the μ_i are the same, $k_n^2(T_n)/\sigma^2 \rightarrow_d \Gamma(4/3, 1/6)$, then $(k_n^2 \max(T_n)/\sigma^2 - d_n)/6 \rightarrow_d \Lambda$ for an i.i.d. sample from T_n. Here Λ is standard Gumbel distribution with the distribution function $\Lambda(x) = \exp(-e^{-x})$.

Now we come to the setting in which a standard normal sample is distributed in a high-dimensional space (WLOG, a two–dimensional space),

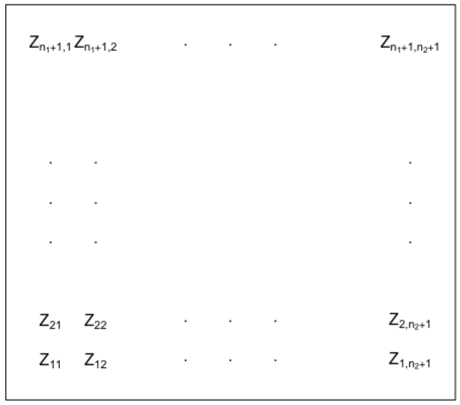

Figure 1: Setting for i.i.d. standard normal samples.

where in each row and column we have (n_1+1) and (n_2+1) i.i.d. standard normal samples Z_{ij}, as shown in Figure 1.

For each 2×2 sub-square in the space, we calculate a respective T_Z. (For example, $T_{Z_{11}}$ is the calculated result of the most left-bottom 2×2 block, that is, $T_{Z_{11}} = (Z_{11}-Z_{12})^2 + (Z_{12}-Z_{22})^2(+Z_{22}-Z_{21})^2 + (Z_{21}-Z_{11})^2$.) In such a setting, we have $n = n_1 n_2$ T_Z samples, denoted as $T_{Z_{ij}}$. The locations of the $T_{Z_{ij}}$s are shown in Figure 2. Here, we note that such $T_{Z_{ij}}$ are $(2,2)$-independent, which means that $T_{Z_{ij}}$ is only dependent on $T_{Z_{i+c,j+c}}$, where $c = -1,\ 0,\ 1$.

Lemma 2. *In such spatial and dependency setting of $T_{Z_{ij}}$, when $n\to\infty$,*

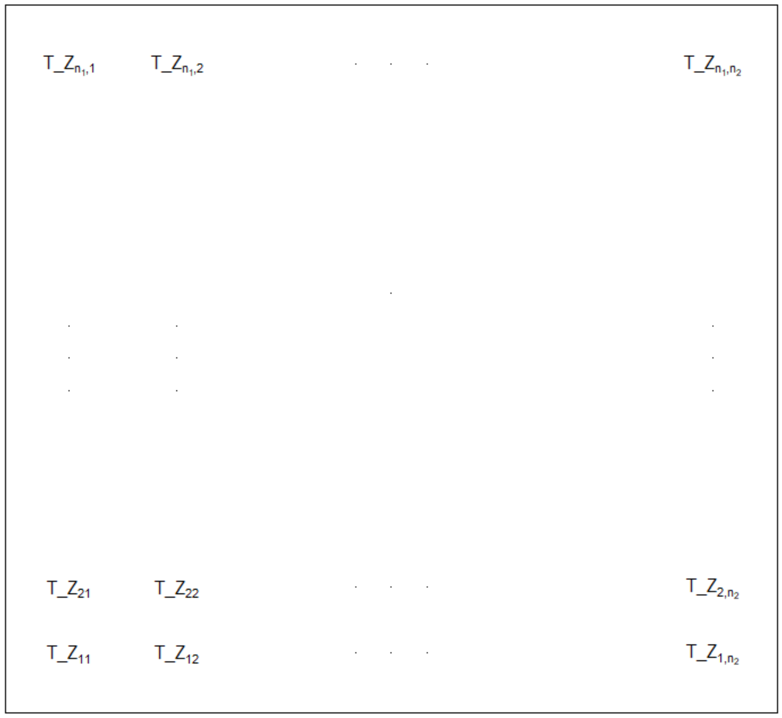

Figure 2: Setting for (2,2)-dependent $T_{Z_{ij}}$s.

we have

$$(\max T_{Z_{ij}} - d_n)/6 \to_d \Lambda$$

where Λ *and* d_n *are defined as in Lemma A.1.*

Proof. Without loss of generality, we suppose $n_1 = n_2 = \sqrt{n}$. First, we transform the index of $T_{Z_{ij}}$ to one dimension row by row: $T_{Z_r} = T_{Z_{ij}}$, where $r = (i-1)\sqrt{n} + j$. The samples are $T_{Z_1}, ..., T_{Z_n}$ and are $(\sqrt{n}+1)$-dependent. We denote a single T_{Z_n}'s distributional function as F and see that $\lim_{n\to\infty} n[1 - F(d_n + c_n x)] = e^{-x}$, and, for any given $T_{Z_i},\ i = 1, ..., n$, there are at most eight T_{Z_j}s, $j \neq i$ that are dependent on T_{Z_i}. Such T_{Z_j}s satisfy $\mid j - i \mid \leq \sqrt{n} + 1$. By Theorem 3.7.1 of Galambos (1987), select $\tau(s_n, u_n) = 1$ when $s_n < \sqrt{n} + 1$, and $\tau(s_n, u_n) = 0$ when $s_n \geq \sqrt{n} + 1$, and $s_n/n \to 0$. The τ function satisfies Equation (59) and the conditions in Theorem 3.7.1 of Galambos (1987). Finally, we need to verify that Equation (62) in Galambos (1987) is satisfied. It suffices to show that Equation (64) holds when $u \to \infty$, and X_i is T_{Z_i}. For those T_{Z_j} independent of T_{Z_1}, the limit holds as it equals $\lim_{u\to\infty}(1 - F(u)) = 0$. In addition, when $u \to \infty$, $0 < \lambda < 0.09$,

$$\begin{aligned} P(T_{Z_1} > u, (Z_1 - Z_{\sqrt{n}+2})^2 \leq \lambda u) &= P((Z_1 - Z_2)^2 + (Z_2 - Z_{\sqrt{n}+3})^2 \\ &\quad + (Z_{\sqrt{n}+3} - Z_{\sqrt{n}+2})^2 > (1-\lambda)u) \\ \leq P((Z_1 - Z_2)^2 + (Z_2 - Z_{\sqrt{n}+3})^2 + (Z_{\sqrt{n}+3} &- Z_{\sqrt{n}+2})^2 + (Z_5 - Z_6)^2 > (1-\lambda)u). \end{aligned}$$

Note that $R_1 := (Z_1 - Z_2)^2 + (Z_{\sqrt{n}+3} - Z_{\sqrt{n}+2})^2 \sim \Gamma(1, 1/4)$, $R_2 := (Z_2 - Z_{\sqrt{n}+3})^2 + (Z_5 - Z_6)^2 \sim \Gamma(1, 1/4)$, and $corr(R_1, R_2) = \sqrt{2}/4$, then $R_1 + R_2 \sim \Gamma(1.4775, 0.185)$. Thus, $\frac{P(R_1+R_2>(1-\lambda)u)}{P(T_{Z1}>u)} \sim e^{u/6-0.185u(1-\lambda)}$ as u is sufficiently large. When $0 < \lambda < 0.09$, we have $P(T_{Z_1} > u, (Z_1 - Z_{\sqrt{n}+2})^2 \leq$

$\lambda u) = o(P(T_{Z_1} < u))$. Therefore

$$P(T_{Z_1} > u, T_{Z_2} > u) \leq P(T_{Z_1} > u, (Z_1 - Z_{\sqrt{n}+2})^2 \leq \lambda u) + P(T_{Z_2} > u, (Z_1 - Z_{\sqrt{n}+2})^2 > \lambda u).$$

As our construction T_{Z_2} is independent of Z_1 and $Z_{\sqrt{n}+2}$, $P(T_{Z_1} > u, T_{Z_2} > u) = o(P(T_{Z_1} < u))$. Thus, we have verified all of the conditions in Theorem 3.7.1 of Galambos (1987). Therefore, arrangement of T_{Z_n} in Figure A.2 with a $\sqrt{n}+1$ dependence also has an extreme value convergence, as follows:

$$(\max(T_{Z_n}) - d_n)/6 \to_d \Lambda.$$

Here, T_{Z_n} is as same as our target series $T_{Z_{ij}}$. □

Remark 2. If the $(\sqrt{n}+1)^2$ Z_is are not arranged in a square, that is, we have $n = n_1 n_2$ T_Z's distributed in n_1 rows and n_2 columns, then it is easy to see that the series $\{T_{Z_n}\}$ is $(\min(n_1, n_2) + 1)$-dependent, and $\min(n_1, n_2) < \sqrt{n}$. The conditions in Galambos (1987), as well as Lemma A.2, continues to hold.

Proof of Theorem 1. (i)(a) In the two-dimensional sample space, the sample size is $n = n_1 n_2$ with n_1 rows and n_2 columns, and the blocking length is k_n. In one block we have k_n^2 samples. First, we need to ensure $k_n/n_1 \to 0$, and $k_n/n_2 \to 0$, Denote $[n/k_n^2] = m$, $[n_1/k_n] = m_1$,

$[n_2/k_n] = m_2$, when $m, m_1, m_2 \to \infty$ we also have

$$(\max(T_{Z_m}) - d_m)/6 \to_d \Lambda.$$

Denote a vector $\mathbf{l} = (l_1, l_2)$, where $l_1 = 1, \cdots, m_1$, $l_2 = 1, \cdots, m_2$. $D_{\mathbf{l}} = D_{(l_1,l_2)} = [(l_1-1)k_n, l_1 k_n]) \times [(l_2-1)k_n, l_2 k_n)$, and $S_{\mathbf{l}} = S_{(l_1,l_2)} = \sum X_{\mathbf{i}} I_{(\mathbf{i} \in D_{\mathbf{l}})}$, Denote $A_{\mathbf{l}} = A_{(l_1,l_2)} = S_{(l_1,l_2)}/k_n^2$. To study the convergence of $\hat{G}_n$, we need the error bound of 1-dimensional strong invariance principle from Wu (2007), so we write the index $\mathbf{i}$ of error term $\epsilon_{\mathbf{i}} = \epsilon_{(i_1,i_2)}$ to be one dimension, which is expressed as τ. Without loss of generality, to avoid tedious discussion of the cases when the samples locate on the boundaries of blocks, we assume the left-bottom point of first block is $(0.5, 0.5)$ and the block length k_n to be an integer, a bijection $b(\mathbf{i})$ from $\{\mathbf{i} = (i_1, i_2), i_1 = 1, ..., n_1,\ i_2 = 1, ..., n_2\}$ to $\{\tau : \tau = 1, ..., n = n_1 n_2\}$ is constructed as

$$\tau = b(i_1, i_2) = \phi(i_2, k_n)n_1 + \phi(i_1, k_n)k_n + (i_2 - \phi(i_2, k_n) - 1)k_n + (i_1 - \phi(i_1, k_n)),$$

where $\phi(a, b) = [\frac{a-0.5}{b}]b$, $a, b \in \mathbb{N}$. The bijection b ensures that a set of all locations $\mathbf{i}$ in a same block is transformed to a set of consecutive one-dimensional indexes.

By the same way, the index $\mathbf{i}$ of i.i.d. random variables $\eta_{\mathbf{i}}$ can be converted to one-dimensional τ. Then, as $\epsilon_{\mathbf{i}} = g(\eta_{\mathbf{i}-\mathbf{j}}, j \in Z^2)$ for a measurable function g and the transformation $b(\mathbf{i})$ from (i_1, i_2) to τ is piecewise linear, there exists a measurable function g_1 s.t. $\epsilon_\tau = g_1(\eta_{\tau-\zeta}, \zeta \in Z)$. Denote an i.i.d. copy of η_τ by η'_τ, and define

$$\eta^*_\tau = \begin{cases} \eta_\tau, & \text{if} \quad \tau \neq 0\,, \\ \eta'_\tau, & \text{if} \quad \tau = 0\,, \end{cases}$$

with $\epsilon^*_\tau = g_1(\eta^*_{\tau-\zeta}, \zeta \in Z)$. We transform the index $\mathbf{i}$, $\mathbf{l}$ into one dimension by $\tau = b(\mathbf{i})$ and $D_{l_2(m_2-1)+l_1} := D_{(l_1,l_2)}$ respectively, and let $U_t = \cup_{l=1}^t D_l$.

Following Wu (2007) and Wu and Zhao (2007), we describe the Brownian motion in terms of in the strong invariance principle as B, and as $Z_{\mathbf{l},n} = Z_{(l_1,l_2),n} = k_n^{-1}(B(|U_{l_2(m_2-1)+l_1}|) - B(|U_{l_2(m_2-1)+l_1-1}|))$, $l_1 = 1, ..., m_1$, $l_2 = 1, .., m_2$, as the error process $\{\epsilon_\mathbf{i}\}$ satisfies $\Delta_4 < \infty$, $|| \epsilon_\mathbf{i} - \epsilon^*_\mathbf{i} ||_4 \leq n^{-1}(i_1 i_2)^{-2}$, then for some constant C, $\{\epsilon_\tau\} = \{\epsilon_{b(\mathbf{i})}\}$ satisfies $\sum_{\tau=1}^n \tau \,||\, \epsilon_\tau - \epsilon^*_\tau \,||_4 < n \sum_{\tau=1}^n ||\, \epsilon_\tau - \epsilon^*_\tau \,||_4 = n \sum_{\mathbf{i} \in Z^2 \cap [1,n]^2} ||\, \epsilon_\mathbf{i} - \epsilon^*_\mathbf{i} \,||_4 \leq C \sum_{i_1=1,...,n_1,\ i_2=1,...,n_2} (i_1 i_2)^{-2} < \infty$ when $n, n_1, n_2 \to \infty$. it is determined that

$$A_{(l_1,l_2)} = \sigma k_n^{-1} Z_{(l_1,l_2),n} + k_n^{-2} \sum_{j_1,j_2=0,...,k_n-1} \mu\left(\frac{(l_1-1)k_n + j_1}{n_1}, \frac{(l_2-1)k_n + j_2}{n_2}\right) + k_n^{-2} o_{AS}(n^{1/4} \log n). \quad \text{(S1.1)}$$

when the error term $\{\epsilon_\mathbf{i}\}$ satisfies $\Delta_2 < \infty$, and $\{\epsilon_\tau\} = \{\epsilon_{b(\mathbf{i})}\}$ satisfies $\sum_{\tau=1}^n \tau \,||\, \epsilon_\tau - \epsilon^*_\tau \,||_4 < \infty$.

As $\mu()$ is α-order Hölder continuous, that is, $|\mu(\mathbf{i}) - \mu(\mathbf{j})| \leq ||\mathbf{i} - \mathbf{j}||^\alpha$, we have

$$k_n\sigma^{-1}[A_{(l_1+1,l_2)} - A_{(l_1,l_2)}] = Z_{(l_1+1,l_2),n} - Z_{(l_1,l_2),n} + O_{AS}(k_n^{(1+\alpha)}/n_1^\alpha + k_n^{-1}n^{1/4}\log n), \quad \text{(S1.2)}$$

$$k_n\sigma^{-1}[A_{(l_1,l_2+1)} - A_{(l_1,l_2)}] = Z_{(l_1,l_2+1),n} - Z_{(l_1,l_2),n} + O_{AS}(k_n^{(1+\alpha)}/n_2^\alpha + k_n^{-1}n^{1/4}\log n), \quad \text{(S1.3)}$$

$$k_n\sigma^{-1}[A_{(l_1+1,l_2+1)} - A_{(l_1,l_2+1)}] = Z_{(l_1+1,l_2+1),n} - Z_{(l_1,l_2+1),n} + O_{AS}(k_n^{(1+\alpha)}/n_1^\alpha + k_n^{-1}n^{1/4}\log n), \quad \text{(S1.4)}$$

$$k_n\sigma^{-1}[A_{(l_1+1,l_2+1)} - A_{(l_1+1,l_2)}] = Z_{(l_1+1,l_2+1),n} - Z_{(l_1+1,l_2),n} + O_{AS}(k_n^{(1+\alpha)}/n_2^\alpha + k_n^{-1}n^{1/4}\log n). \quad \text{(S1.5)}$$

We need $k_n^{(1+\alpha)}/\min(n_1,n_2)^\alpha + k_n^{-1}n^{1/4}\log n \to 0$, as we can see that when $\min(n_1,n_2)n^{-1/4} \to \infty$ and $k_n \sim n^p$ when $p > 1/4$, such $\alpha > 1$ exist. So, we square both sides of the equations (S1.1)–(S1.4), and then sum the four squared equations, which gives

$$k_n^2\sigma^{-2}\{[A_{(l_1+1,l_2)} - A_{(l_1,l_2)}]^2 + [A_{(l_1,l_2+1)} - A_{(l_1,l_2)}]^2 + [A_{(l_1+1,l_2+1)} - A_{(l_1,l_2+1)}]^2 + [A_{(l_1+1,l_2+1)} - A_{(l_1+1,l_2)}]^2\} = [Z_{(l_1+1,l_2),n} - Z_{(l_1,l_2),n}]^2 + [Z_{(l_1,l_2+1),n} - Z_{(l_1,l_2),n}]^2 + [Z_{(l_1+1,l_2+1),n} - Z_{(l_1,l_2+1),n}]^2 + [Z_{(l_1+1,l_2+1),n} - Z_{(l_1+1,l_2),n}]^2 + O_{AS}(k_n^{(1+\alpha)}/\min(n_1,n_2)^\alpha + k_n^{-1}n^{1/4}\log n)^2. \quad \text{(S1.6)}$$

The discrepancy measurement $k_n^2 T_n(x_{l_1}, y_{l_2})/\sigma^2$ is on the left-hand side of Equation (S1.5). It converges almost surely to the right-hand side of Equation (S1.5), which is the $T_{Z_{l_1 l_2}}$ in the Figure 2. By taking the maximum of both sides, when $n \to \infty$, $m_1 = n_1/k_n \to \infty$, $m_2 = n_2/k_n \to \infty$, μ is α-order Hölder continuous, $k_n^{(1+\alpha)}/\min(n_1, n_2)^\alpha + k_n^{-1} n^{1/4} \log n \to 0$, and the extreme-valued test statistic, $\max T_n$, satisfies:

$$(k_n^2 \max(T_n)/\sigma^2 - d_m)/6 \to_d \Lambda.$$

(i)(b) As $b(\mathbf{i})$ is piecewise linear, the process ϵ_τ is a linear process and we can write it as $\epsilon_\tau = \sum_{\zeta=0}^{\infty} \alpha_\zeta \eta_{\tau-\zeta}$. And for a constant C_2, $\sum_{\zeta=1}^{n} \sqrt{\sum_{\rho=\zeta}^{n} \alpha_\rho^2} \leq n\sqrt{\sum_{\mathbf{j} \in Z^2 \cap [1,n]^2} \alpha(\mathbf{i}-\mathbf{j})^2} < C_2 \sqrt{\sum_{\mathbf{j} \in Z^2 \cap [1,n]^2} e^{-2\max(|j_1|,|j_2|)}} < \infty$ when $n, n_1, n_2 \to \infty$ as $\alpha(\mathbf{i}-\mathbf{j}) = O(n^{-1} e^{-\max(|j_1|,|j_2|)})$. From Proposition 2 in Wu (2007), the last item in (S1.1) could be further shrink to $o_{AS}(n^{1/q})$. Then the last item in (S1.6) is $O_{AS}(k_n^{(1+\alpha)}/\min(n_1, n_2)^\alpha + k_n^{-1} n^{1/q})^2$. We can derive (2.5) if (2.6) holds.

(ii) For a given level $\lambda \in (0,1)$, let $g_\lambda = -\log(-\log(1-\lambda))$ be the $(1-\lambda)$-th quantile of a standard Gumbel distribution. From Theorem 1(i) we reject the null hypothesis $\mu(\cdot) \in H^\alpha(I)$ if

$$\hat{G}_n > \frac{\sigma^2(6g_\lambda + d_m)}{k_n^2}. \tag{S1.7}$$

Consider the local alternative, that is, there exists a subset B of $I = [0,1]^2$, such that $\min_{\mathbf{i}\in B,\mathbf{j}\in B^c}|\mu(\mathbf{i})-\mu(\mathbf{j})|\geq C$ where the constant $C>0$ is the change-size. The location indexes for the observations in B is denoted as $B_n = B\cap\{(\frac{i}{n_1},\frac{j}{n_2}) \mid i=1,...,n_1, j=1,...,n_2\}$ with a cardinality $|B_n|$. As $\mu(\cdot)\in PH^\alpha(0,1)$, similar arguments in the proof of Theorem 1(i) yield

$$\max E(T_n) > \frac{1}{4}C^2|B_n|/k_n^2 + O(k_n/\min(n_1,n_2))^\alpha\,.$$

If the area of B is positive, $\log n = o(n)$, and $k_n^{(1+\alpha)}/\min(n_1,n_2)^\alpha + k_n^{-1}n^{1/4}\log n \to 0$ holds, then $n = O(|B_n|)$ (as the area of B is positive, $|B_n| = A_1 n$ with a constant A_1), and it follows that (S1.7) holds with probability approaching 1. □

Proof of Theorem 2. Denote the median of T_Z by M_0. From Lemma A.1, we know that M_0 is the median of $\Gamma(4/3,1/6)$, so that $M_0\approx 6.1$.

Denote $m_1=[n_1/k_n], m_2=[n_2/k_n]$, and construct $(m_1+1)(m_2+1)$ i.i.d. standard normal random variables, which are $Z_1,...,Z_{(m_1+1)(m_2+1)}$, and arrange them in a $(m_1+1)(m_2+1)$ matrix, $m_1\leq m_2$. Construct T_i' to be the sum of the squared differences of any 2×2 submatrix of Z_i. (e.g: $T_1' = (Z_2-Z_1)^2+(Z_{m_1+3}-Z_2)^2+(Z_{m_1+2}-Z_{m_1+3})^2+(Z_1-Z_{m_1+2})^2$). Then, there will be $m_1\times m_2$ T_i', and T_i' is (m_1+1) dependent. Denote $M_n = \text{median}(T_i')$, by Ruymgaart (2002), then we know that $M_n - M_0 \sim O_p(k_n/\max(n_1,n_2))$, so that $M_n - M_0 \sim O_p(k_n/\sqrt{n})$. Here, we need $k_n = o(\sqrt{n})$ to ensure convergence.

Next, square the differences of $A_{\mathbf{i}} = A_{(i_1,i_2)}$ as in the proof of Theorem 1, and sum the squared (S1.1)-(S1.4). Thus, for the median, we have:

$$\text{median}(T_n) = M_n \frac{\sigma^2}{k_n^2} + O_{AS}(k_n^{2\alpha}/\min(n_1^2, n_2^2)^{2\alpha} + k_n^{-4} n^{1/2} \log^2 n).$$

When $k_n^{2\alpha+2}/\min(n_1^2, n_2^2)^{\alpha} + k_n^{-2} n^{1/2} \log^2 n \to 0$, the median estimator $\hat{\sigma}_2^2$ of long-run variance converges almost surely, so it converges in probability. For $\alpha > 1$, we can find the k_n that satisfies the convergence.

(For example, when $n_1 \sim n_2 \sim \sqrt{n}$, and $\alpha = 2$, if $k_n \sim n^{0.3}$, the convergence is $\mid \hat{\sigma}_2^2 - \sigma^2 \mid = o_{AS}(n^{-1/10} \log^2 n)$).

□

S2 Simulation Results for Center-Jump Cases

From the results, we see that the extreme-valued test and ISE test are both effective when the breaking region is large and located at the centre of the sample region (see Figures 3–6 for an illustration).

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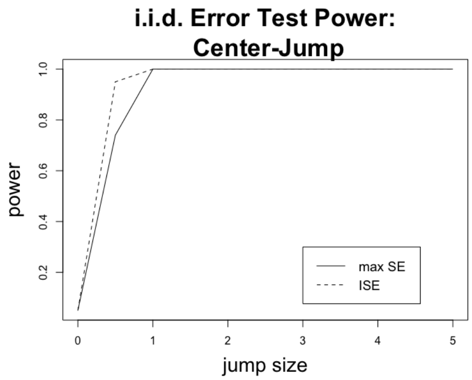
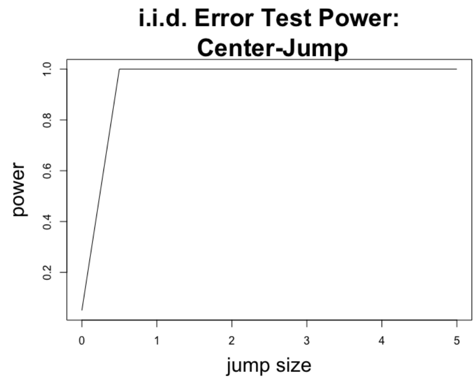


(a) Sample size $(n_1, n_2) = (125, 125)$.



(b) Sample size $(n_1, n_2) = (250, 250)$.

Figure 3: Test power-jump size plot in the i.i.d. error term, center-jump case.

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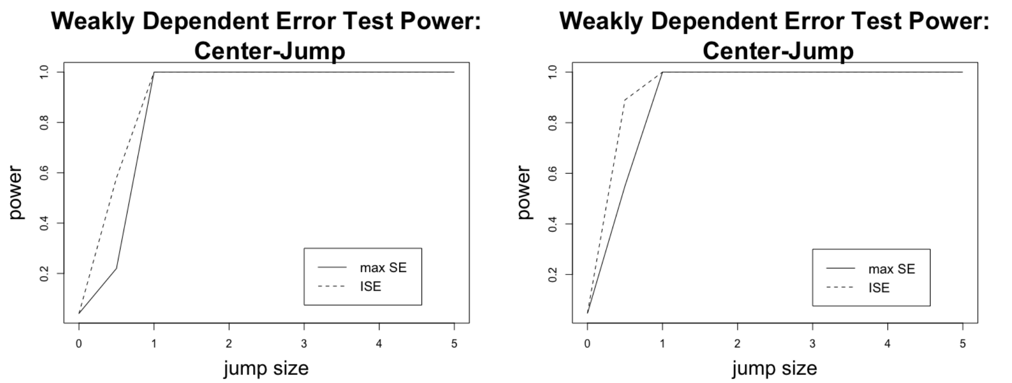
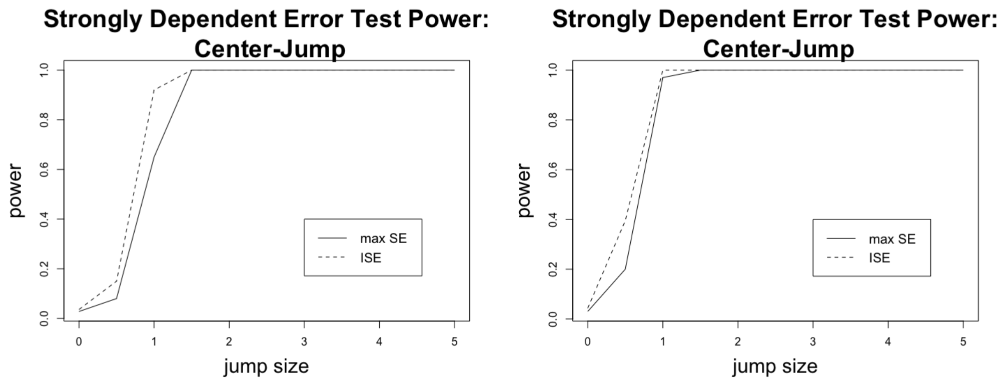


(a) Sample size $(n_1, n_2) = (125, 125)$.

(b) Sample size $(n_1, n_2) = (250, 250)$.

Figure 4: Test power-jump size plot in the weakly dependent error term, center-jump case.

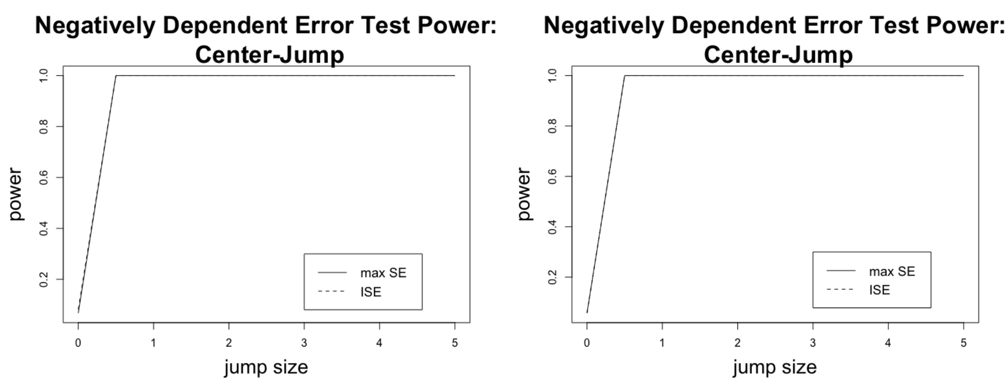
Wu, W. B. and Z. Zhao (2007). Inference of trends in time series. *Journal of the Royal Statistical Society: Series B (Statistical Methodology) 69*(3), 391–410.



(a) Sample size $(n_1, n_2) = (125, 125)$.

(b) Sample size $(n_1, n_2) = (250, 250)$.

Figure 5: Test power-jump size plot in the strongly dependent error term, center-jump case.



(a) Sample size $(n_1, n_2) = (125, 125)$.

(b) Sample size $(n_1, n_2) = (250, 250)$.

Figure 6: Test power-jump size plot in the negatively dependent error term, center-jump case.