Testing and modelling for the structural change in covariance matrix
time series with multiplicative form

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Supplementary Material

This supplement provides four appendices for the paper. Appendix A gives the proofs of Theorems 2.1–2.2. Appendix B offers the proofs of Theorems 3.1–3.3. Appendix C lists some basic derivatives results, and Appendix D provides some numerical evidences on spurious long memory phenomena caused by the structural change. In what follows, we define the pseudo data

\[ u_t = u_{-t}, \Sigma_t = \Sigma_{-t} \text{ for } -[Th] \leq t \leq -1, \text{ and } u_t = u_{2T-t}, \Sigma_t = \Sigma_{2T-t} \text{ for } T + 1 \leq t \leq T + [Th] \]

obtained by the reflection method.

A Proofs of Theorems 2.1–2.2

Define \( \nu_t = \text{vech}(y_t - \Sigma_0 - \Sigma_{1t}/c_T) \) and let

\[
\Pi_1(x) = \frac{1}{T} \sum_{s=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h(x - s/T)v_s,
\]

\[
\Pi_2(x) = \left[ \frac{1}{T} \sum_{s=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h(x - s/T) - 1 \right] \text{vech}(\Sigma_0)
\]
\[ + c_T^{-1} \left[ \frac{1}{T} \sum_{s=1-[Th]}^{T-[Th]} K_h(x - s/T) \text{vech} (\Sigma_{1s}) - \frac{1}{T} \sum_{s=1}^{T} \text{vech} (\Sigma_{1s}) \right], \]

and \( \Pi_3 = \frac{1}{T} \sum_{s=1}^{T} v_s \). Then,

\[
Th^{1/2} \hat{S} = h^{1/2} \sum_{t=1}^{T} [\Pi_1(t/T) + \Pi_2(t/T) - \Pi_3][\Pi_1(t/T) + \Pi_2(t/T) - \Pi_3]
\]

\[ \triangleq S_1 - 2S_2 + S_3 + S_4, \tag{A.1} \]

where

\[
S_1 = h^{1/2} \sum_{t=1}^{T} \Pi_1(t/T)' \Pi_1(t/T), \quad S_2 = h^{1/2} \sum_{t=1}^{T} \Pi_1(t/T)' \Pi_3,
\]

\[
S_3 = h^{1/2} \sum_{t=1}^{T} \Pi_3' \Pi_3, \quad S_4 = h^{1/2} \sum_{t=1}^{T} [2\Pi_1(t/T) + \Pi_2(t/T) - 2\Pi_3]' \Pi_2(t/T).
\]

Next, let \( \Sigma_{t*} = [(\Sigma_0 + \Sigma_{1t}/c_T)^{1/2}] \otimes 2 D_n \). Under \( H_{1T} \),

\[ v_t = \Sigma_{t*} z_t, \tag{A.2} \]

where \( z_t \) is defined as in (2.10). Particularly, \( v_t = v_{0t} \) under \( H_0 \) (i.e., \( \Sigma_{1t} \equiv 0 \)), where \( v_{0t} = \Sigma_{0*} z_t \) is a stationary process, and \( \Sigma_{0*} \) is defined as in (2.3). Since \( \Sigma_0 \) and \( \Sigma_{1t} \) are bounded deterministic matrices, Assumption 2.2 implies

\[ v_t \text{ is a strictly stationary } \beta\text{-mixing process with mixing coefficients } (A.3) \]

\[ \beta(j) \text{ satisfying } \sum_{j=1}^{\infty} j^2 \beta(j)^{\delta/(1+\delta)} < \infty \text{ for some } 0 < \delta < 1; \]

\[ \max_t E \|v_t\|^{4(1+\delta)} < \infty. \tag{A.4} \]
Moreover, since $c_T \to \infty$, we have \( \| \Sigma_0^{-1/4} \Sigma_{1t} \Sigma_0^{-1/4} / c_T \| \ll 1 \). Hence,

\[
(S_0 + \Sigma_{1t} / c_T)^{1/2} = \Sigma_0^{1/4} \left( I_n + \Sigma_0^{-1/2} \Sigma_{1t} \Sigma_0^{-1/2} / c_T \right)^{1/2} \Sigma_0^{1/4} \\
= \Sigma_0^{1/4} \left[ I_n + \frac{1}{2c_T} \Sigma_0^{-1/2} \Sigma_{1t} \Sigma_0^{-1/2} + O\left( \frac{1}{c_T^2} \right) \right] \Sigma_0^{1/4} \\
= \Sigma_0^{1/2} + \frac{1}{2c_T} \Sigma_0^{-1/4} \Sigma_{1t} \Sigma_0^{-1/4} + O\left( \frac{1}{c_T^2} \right)
\]

by Taylor’s expansion, and it entails

\[\Sigma_{t*} - \Sigma_{0*} = \epsilon_t / c_T, \quad (A.5)\]

where $\epsilon_t = [\Sigma_0^{-1/4} \Sigma_{1t} \Sigma_0^{-1/4} \otimes \Sigma_0] / 2 + [\Sigma_0 \otimes \Sigma_0^{-1/4} \Sigma_{1t} \Sigma_0^{-1/4}] / 2 + O(1/c_T^2)$.

In order to prove Theorems 2.1 and Theorem 2.2 (i)–(ii), Propositions 1–5 below are needed. These five propositions and their related lemmas are all proved under $H_{1T}$ with $T^{1/2} h^{1/4} = O(c_T)$, and Assumptions 2.2–2.4.

**Proposition 1.** $S_1 - B \to_L N(0, \mathcal{V})$, where $B = h^{-1/2} tr(M) \left[ \int K^2(x) dx \right]$, $M$ is defined in (2.5), and $\mathcal{V}$ is defined as in Theorem 2.2.

**Proposition 2.** $S_2 = o_p(1)$.

**Proposition 3.** $S_3 = o_p(1)$.

**Proposition 4.** $S_4 - \frac{T h^{1/2}}{c_T^2} B_t = o_p(1)$, where $B_t$ is defined as in Theorem 2.2.

**Proposition 5.** $\hat{M} - M = o_p(h^{1/2})$, where $M$ is defined in (2.3).
Proof of Proposition \(\text{iii}\). Note that

\[
S_1 = \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \left[ \sum_{s=1-T_{th}}^{T+T_{th}} K\left(\frac{t-s}{Th}\right) v_s \right] \left[ \sum_{s=1-T_{th}}^{T+T_{th}} K\left(\frac{t-s}{Th}\right) v_s \right]
\]

\[
= \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \left[ \sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right) v_s \right] \left[ \sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right) v_s \right]
\]

\[
+ \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \left[ \left( \sum_{s=1-T_{th}}^{T} + \sum_{s=T+1}^{T+T_{th}} \right) K\left(\frac{t-s}{Th}\right) v_s \right] \left[ \left( \sum_{s=1-T_{th}}^{T} + \sum_{s=T+1}^{T+T_{th}} \right) K\left(\frac{t-s}{Th}\right) v_s \right]
\]

\[
+ \frac{2}{T^2 h^{3/2}} \sum_{t=1}^{T} \left[ \left( \sum_{s=1-T_{th}}^{T} + \sum_{s=T+1}^{T+T_{th}} \right) K\left(\frac{t-s}{Th}\right) v_s \right] \left[ \sum_{r=1}^{T} K\left(\frac{t-r}{Th}\right) v_r \right]
\]

\[
= \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} K^2(0) v_t v_t + \frac{1}{T^2 h^{3/2}} \sum_{s \neq t} T \left[ \sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right) v_s v_s \right]
\]

\[
+ \frac{2}{T^2 h^{3/2}} \sum_{s \neq t} \left[ \sum_{s=1-T_{th}}^{T} + \sum_{s=T+1}^{T+T_{th}} \right] K\left(\frac{t-s}{Th}\right) K\left(\frac{t-r}{Th}\right) v_s v_r
\]

\[
+ \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \left[ \left( \sum_{s=1-T_{th}}^{T} + \sum_{s=T+1}^{T+T_{th}} \right) K\left(\frac{t-s}{Th}\right) v_s \right] \left[ \left( \sum_{s=1-T_{th}}^{T} + \sum_{s=T+1}^{T+T_{th}} \right) K\left(\frac{t-s}{Th}\right) v_s \right]
\]

\[
+ \frac{2}{T^2 h^{3/2}} \sum_{t=1}^{T} \left[ \left( \sum_{s=1-T_{th}}^{T} + \sum_{s=T+1}^{T+T_{th}} \right) K\left(\frac{t-s}{Th}\right) v_s \right] \left[ \sum_{r=1}^{T} K\left(\frac{t-r}{Th}\right) v_r \right]
\]

\[
\triangleq \sum_{i=1}^{6} S_{1i}.
\]

By Lemmas \(\text{iv}, \text{v}, \text{vi}\) below,

\[
S_1 - B_1 - B_2 = S_{142} + o_p(1)
\]

(A.6)

Next, we have that \(\sum_{j=-\infty}^{\infty} E(v_{0t}v_{0t+j}) = \text{tr}(M)\), which entails that \(B_1 + B_2 = B\). Therefore, the conclusion holds by (A.6) and Lemma \(\text{vii}\) below.

\[\Box\]
Lemma A.1. $S_{11} = o_p(1)$.

Lemma A.2. $S_{12} - B_1 = o_p(1)$, where $B_1 = h^{-1/2}[E(v_t'v_{0t})] \int K^2(x) dx$.

Lemma A.3. $S_{13} = o_p(1)$.

Lemma A.4. $S_{14} - B_2 = S_{142} + o_p(1)$, where $S_{142}$ is defined as in (A.14) below, and $B_2 = h^{-1/2}[\sum_{j=1}^{\infty} E(v_{0t}v_{0t+j})] \int K^2(x) dx$.

Lemma A.5. $S_{15} = o_p(1)$.

Lemma A.6. $S_{16} = o_p(1)$.

Lemma A.7. $S_{142} \rightarrow_{\mathcal{L}} N(0, \mathcal{V})$.

**Proof of Lemma A.1.** By (A.4), $E(S_{11}) = O\left(\frac{1}{T^3 h^3}\right) = o(1)$ and

$$\text{Var}(S_{11}) = \frac{K^4(0)}{T^4 h^3} \sum_{t=1}^{T} \text{Var}(v_t'v_t) + \frac{K^4(0)}{T^4 h^3} \sum_{s \neq t} \text{Cov}(v_t'v_t, v_s'v_s)$$

$$= O\left(\frac{1}{T^3 h^3}\right) + O\left(\frac{1}{T^4 h^3}\right) \sum_{s \neq t} \text{Cov}(v_t'v_t, v_s'v_s)$$

$$= O\left(\frac{1}{T^3 h^3}\right) + O\left(\frac{1}{T^4 h^3}\right) \times \left[ \sum_{t=1}^{T} \sum_{j=1}^{t-1} \text{Cov}(v_t'v_t, v_{t-j}'v_{t-j}) + \sum_{t=1}^{T} \sum_{j=1}^{T-t} \text{Cov}(v_t'v_t, v_{t+j}'v_{t+j}) \right] \quad \text{(A.7)}$$

By (A.4) and Davydov’s inequality ([Davydov, 1968]),

$$|\text{Cov}(v_t'v_t, v_{t+j}'v_{t+j})| \leq C \beta(j)^{\delta/(1+\delta)} \|v_t'v_t\|_{2(1+\delta)} \|v_{t+j}'v_{t+j}\|_{2(1+\delta)}$$
\[
C(j) = (1 + \beta(j))^{\delta/(1+\delta)}
\]  
(A.8)

for some \( \delta \in (0, 1) \) and all \( t \geq 1 \). Hence, \( \sum_{t=1}^{T} \sum_{j=1}^{t-1} |\text{Cov}(v'_tv_t, v'_{t-j}v_{t-j})| + \sum_{t=1}^{T} \sum_{j=1}^{T-t} |\text{Cov}(v'_tv_t, v'_{t+j}v_{t+j})| \leq C \sum_{t=1}^{T} \sum_{j=1}^{\infty} \beta(j)^{\delta/(1+\delta)} = O(T) \) by (A.3).

Together with (A.7), it follows that \( \text{Var}(S_{11}) = O\left( \frac{1}{T^{3/2}} \right) = o(1) \). Now, we can conclude \( S_{11} = o_p(1) \) by Chebyshev’s inequality.

**Proof of Lemma A.2.** By the symmetry of \( K(\cdot) \), we can write

\[
S_{12} = \frac{1}{T^2h^{3/2}} \sum_{s=1}^{T} (\Delta_{1s} + \Delta_{2s}),
\]

where

\[
\Delta_{1s} = \left[ \sum_{j=1}^{s-1} K^2\left( \frac{j}{Th} \right) \right] v'_sv_s \quad \text{and} \quad \Delta_{2s} = \left[ \sum_{j=1}^{T-s} K^2\left( \frac{j}{Th} \right) \right] v'_sv_s.
\]

By (A.2) and (A.3), we have \( v_t = \Sigma_{st}(\Sigma_{0\cdot})^{-1}v_0t = (I_n + \epsilon_t \Sigma_{0\cdot}^{-1}/c_T)v_0t \) and

\[
v'_sv_t = v'_sv_0t + v'_s\epsilon_t\Sigma_{0\cdot}^{-1}v_0t/c_T
\]

\[
+ v'_s\Sigma_{0\cdot}^{-1}\epsilon_tv_0t/c_T + v'_s\Sigma_{0\cdot}^{-1}\epsilon_t\Sigma_{0\cdot}^{-1}v_0t/c^2_T,
\]

(A.9)

where \( v_0t \) is stationary with mean zero by Assumption A.2.

Since \( \frac{1}{Th} \sum_{j=1}^{T-1} K^2\left( \frac{j}{Th} \right) = \int_{0}^{1} K^2(x)dx + O\left( \frac{1}{Th} \right) \) and \( \frac{1}{Th} \sum_{j=1}^{T-1} \frac{j}{Th} K^2\left( \frac{j}{Th} \right) = h\left[ \int_{0}^{1} xK^2(x)dx \right] + O\left( \frac{1}{h} \right) \), by the boundedness of \( \epsilon_t \) and \( \Sigma_{0\cdot} \), (A.9) and the stationarity of \( v_0t \), it is not hard to see

\[
E(S_{12}) = \sum_{s=1}^{T} \frac{2E(v'_sv_s)}{Th^{1/2}} \left[ \sum_{j=1}^{s-1} K^2\left( \frac{j}{Th} \right) + \sum_{j=1}^{T-s} K^2\left( \frac{j}{Th} \right) \right]
\]

\[
= \sum_{s=1}^{T} \frac{E(v'_sv_s)}{Th^{1/2}} \left[ \frac{2}{Th} \sum_{j=1}^{T-1} \left( 1 - \frac{j}{T} \right) K^2\left( \frac{j}{Th} \right) \right]
\]
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\[ E(v_s'v_0s)h^{1/2} = \frac{2}{T\theta} \sum_{j=1}^{T-1} \left( 1 - \frac{j}{T} \right) K^2 \left( \frac{j}{T\theta} \right) + O \left( \frac{1}{cT}h^{1/2} \right) + O \left( \frac{2}{cT}h^{1/2} \right) \]

\[ = B_1 + O \left( \frac{1}{T\theta h^{3/2}} \right) + O(h^{1/2}) + O \left( \frac{1}{cT}h^{1/2} \right) + O \left( \frac{2}{cT}h^{1/2} \right) \rightarrow B_1. \]

Moreover, since \( \frac{1}{T\theta} \sum_{j=1}^{T-1} K^2 \left( \frac{j}{T\theta} \right) = O(1) \), we can show

\[ \text{Var} \left( \frac{1}{T^2h^{3/2}} \sum_{s=1}^{T} \Delta_{1s} \right) = \frac{1}{T^4h^3} \sum_{s=1}^{T} \text{Var}(\Delta_{1s}) + \frac{2}{T^4h^3} \sum_{s=1}^{T} \sum_{s<r} \text{Cov}(\Delta_{1s}, \Delta_{1r}) \]

\[ \leq \frac{1}{T^4h^3} \sum_{s=1}^{T} \left[ \sum_{j=1}^{T-1} K^2 \left( \frac{j}{T\theta} \right) \right]^2 \text{Var}(v_s'v_s) \]

\[ + \frac{2}{T^4h^3} \sum_{s=1}^{T-1} \sum_{r=s+1}^{T-1} \left[ \sum_{i=1}^{T-1} K^2 \left( \frac{i}{T\theta} \right) \right]^2 |\text{Cov}(v_s'v_s, v_r'v_r)| \]

\[ \leq O \left( \frac{1}{T\theta} \right) + O \left( \frac{1}{T^2h^3} \right) \sum_{s=1}^{T} \sum_{j=1}^{T-1} \text{Cov}(v_s'v_s, v_s'v_{s+j}) \]

\[ \leq O \left( \frac{1}{T\theta} \right) + O \left( \frac{1}{T^2h^3} \right) \sum_{s=1}^{T} \sum_{j=1}^{T-1} \beta(j) \delta/(1+\delta) \]

\[ = O \left( \frac{1}{T\theta} \right) = o(1), \]

where the last inequality holds by (A.8). Similarly, \( \text{Var} \left( \frac{1}{T^2h^{3/2}} \sum_{s=1}^{T} \Delta_{2s} \right) = o(1) \), which implies that \( \text{Var}(S_{12}) = o(1) \) by the Cauchy-Schwarz inequality.

Proof of Lemma A.3. Define \( \Psi_1(\psi_s, \psi_t) = K \left( \frac{s-t}{T\theta} \right) v_s'v_s \), where \( \psi_s = (v_s, \frac{s}{T\theta}) \), and \( \Psi_1(\cdot, \cdot) : \mathbb{R}^{(n+1)(n+2)/2} \times \mathbb{R}^{(n+1)(n+2)/2} \rightarrow \mathbb{R} \) is a symmetric function. Then, \( S_{13} = \frac{4K(0)}{T^2h^{3/2}} \sum_{s<t}^{T} \Psi_1(\psi_s, \psi_t). \)
By the symmetry and boundedness of $K(\cdot)$, we have
\[ |E(S_{13})| = \left| \frac{4K(0)}{T^2 h^{3/2}} \sum_{t=1}^{T} \sum_{j=1}^{T-1} K\left( \frac{j}{Th} \right) E\left( v'_t v_{t-j} \right) \right| \]
\[ \leq \frac{C}{T^2 h^{3/2}} \sum_{t=1}^{T} \sum_{j=1}^{T-1} |E(\nu'_t v_{t-j})| \]
\[ \leq \frac{C}{Th^{3/2}} \sum_{j=1}^{\infty} \beta(j)^{\delta/(1+\delta)} = O\left( \frac{1}{Th^{3/2}} \right) = o(1), \quad (A.10) \]
where the last inequality holds by a similar argument as for (A.8).

Moreover, since $E[\Psi_1(\psi_s, x)] = 0$ for any fixed $x \in \mathbb{R}^{(n+1)(n+2)/2}$, by (A.9)–(A.10) we have
\[ \text{Var}(S_{13}) = \frac{16K^2(0)}{T^4 h^3} \text{Var}\left( \sum_{s<t} \Psi_1(\psi_s, \psi_t) \right) \]
\[ \leq \frac{C}{T^2 h^{3/2}} \max_{s<t} \left[ E(\Psi_1(\psi_s, \psi_t)) \right]^{2/(1+\delta)} \sum_{j=1}^{\infty} j \beta(j)^{\delta/(1+\delta)} \]
\[ = O\left( \frac{1}{T^2 h^{3/2}} \right) = o(1), \quad (A.11) \]
where the inequality holds by Lemma A(ii) of [Hjellvik et al. (1998)], which relies on some minor modifications for the proof of Lemma 1 in [Yoshihara (1976)]. Hence, by (A.10)–(A.11) it follows that $S_{13} = o_p(1)$. \hfill \Box

**Proof of Lemma A.4.** Write
\[ S_{14} = \frac{1}{T^2 h^{3/2}} \sum_{s \neq r, s \neq t, t \neq r} K\left( \frac{t-s}{Th} \right) K\left( \frac{t-s + s-r}{Th} \right) v'_t v_r \]
\[ = \frac{1}{Th^{1/2}} \sum_{s \neq r} \left[ \int K(x) K\left( x + \frac{s-r}{Th} \right) dx \right] v'_s v_r + O\left( \frac{1}{T^2 h^{3/2}} \right) \sum_{s \neq r} v'_s v_r \]
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\[ = \frac{1}{T h^{1/2}} \sum_{s \neq r}^{T} \left[ \int K(x)K \left( x + \frac{s - r}{T h} \right) dx \right] v_s' v_r + O_p \left( \frac{1}{T h^{3/2}} \right) \]

\[ = S_{14}^* + O_p \left( \frac{1}{T h^{3/2}} \right), \quad (A.12) \]

where the second equality holds since

\[ \frac{1}{T h^{1/2}} \sum_{t \neq s, t \neq r}^{T} K \left( \frac{t - s}{T h} \right) K \left( \frac{t - s}{T h} + \frac{s - r}{T h} \right) = \int K(x)K \left( x + \frac{s - r}{T h} \right) dx + O \left( \frac{1}{T h} \right) \]

for any \( s \) and \( r \), and the third equality holds since \( \sum_{s \neq r}^{T} v_s' v_r = O_p(T) \) by a similar argument as for \((A.10)\).

Next, we introduce a truncation lag \( p_T \) such that

\[ p_T \to \infty, \quad p_T = o(Th), \quad p_T h^{3/2} \to \infty \text{ and } \sum_{j = p_T}^{\infty} j^2 \beta(j) < Cp_T^{-1}. \quad (A.13) \]

Denote \( S_1 = \{(s, r) : 1 \leq |s - r| \leq p_T, 1 \leq r \neq s \leq T \} \) and \( S_2 = \{(s, r) : p_T < |s - r| < T, 1 \leq r \neq s \leq T \} \). Then,

\[ S_{14}^* = \frac{1}{T h^{1/2}} \sum_{S_1} \Delta_{3rs} v_s' v_r + \frac{1}{T h^{1/2}} \sum_{S_2} \Delta_{3rs} v_s' v_r \overset{\triangle}{=} S_{141} + S_{142}, \quad (A.14) \]

where \( \Delta_{3rs} = \int K(x)K \left( x + \frac{s - r}{T h} \right) dx. \)

Furthermore, we re-write \( S_{141} \) as

\[ S_{141} = \left[ \int K^2(x) dx \right] \frac{1}{T h^{1/2}} \sum_{S_1} v_s' v_r + \frac{1}{T h^{1/2}} \sum_{S_1} \Delta_{4rs} v_s' v_r \overset{\triangle}{=} S_{1411} + S_{1412}, \quad (A.15) \]

where \( \Delta_{4rs} = \int K(x)[K \left( x + \frac{s - r}{T h} \right) - K(x)] dx. \)

For \( S_{1411} \), \( E(S_{1411}) = \left[ \int K^2(x) dx \right] \left[ \frac{1}{T h^{1/2}} \sum_{r=1}^{T} E \sum_{j=1}^{p_T} (v_r' v_{r+j} + v_{r-j}' v_{r-j}) \right] \)

via some simple calculations. Hence, by \((A.2), (A.3)\), the stationarity of \( z_t, \)
and a similar argument as for (A.10), we can show

\[ E(S_{1411}) = B_2 + O\left(\frac{1}{c_T h^{1/2}}\right) = B_2 + o(1). \]  

(A.16)

Moreover, by defining \( \varpi_1(\psi_s, \psi_r) = v'_s v_r - E(v'_s v_r) \), we have

\[
E(S_{1411}) = B_2 + o(1). 
\]

(A.17)

where the third equality holds by Proposition A.4 in [Hong et al., 2017], and the fourth equality holds by (A.13). Here, Proposition A.4 in [Hong et al., 2017] is valid due to some minor modifications for the proof of Lemma 1 in Yoshihara (1976). By (A.16)–(A.17), it follows that \( S_{1411} = B_2 + o_p(1) \).

For \( S_{1412} \), since \(|K(x + \frac{j}{T h}) - K(x)| \leq C \frac{j}{T h} \), a similar argument as for (A.16) entails

\[
|E(S_{1412})| \leq C \int K(x) dx \sum_{r=1}^{T} \sum_{j=1}^{p_T} \frac{j}{T h} [|E(v'_s v_{r+j})| + |E(v'_s v_{r-j})|] \\
= O\left(\frac{1}{T h^{3/2}} \sum_{j=1}^{p_T} j \beta(j) j^{\delta/(1+\delta)}\right) = O\left(\frac{1}{T h^{3/2}}\right) = o(1). \]  

(A.18)

Moreover, since \( \Delta_{4rs} \leq C \frac{|s-r|}{T h} \), we have

\[
\left| \text{Var}(S_{1412}) \right| = \left| \frac{1}{T^2 h} \sum_{s,r \in S_1} \sum_{k,l \in S_1} \Delta_{4rs} \Delta_{4kl} [E(v'_s v_r v'_k v_l) - E(v'_s v_r) E(v'_k v_l)] \right| 
\]
\[ \frac{C}{T^2 h} \sum_{s,r \in S_1} \sum_{k,l \in S_1} \frac{|s - r|}{Th} \frac{|k - l|}{Th} E|\varpi_1(\psi_s, \psi_r) \varpi_1(\psi_k, \psi_l)| \]
\[ \leq \frac{C p_r^2}{T^4 h^3} \sum_{s,r \in S_1} \sum_{k,l \in S_1} E|\varpi_1(\psi_s, \psi_r) \varpi_1(\psi_k, \psi_l)| \]
\[ = O\left( \frac{p_r^3}{T^3 h^3} \right) = o(1), \quad (A.19) \]

where (A.19) holds by a similar argument as for (A.17). Hence, by (A.18)–(A.14), it follows that \( S_{1412} = o_p(1) \). Now, the conclusion holds by (A.12) and (A.14)–(A.13).

**Proof of Lemma A.5.** By the Cauchy-Schwarz inequality, we only need to prove
\[ S_{151} = \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \left[ \sum_{s=1-\lceil Th \rceil}^{0} K\left( \frac{t-s}{Th} \right) v_s \right]' \left[ \sum_{s=1-\lceil Th \rceil}^{0} K\left( \frac{t-s}{Th} \right) v_s \right] = o_p(1), \]
\[ S_{152} = \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \left[ \sum_{s=T+1}^{T+\lceil Th \rceil} K\left( \frac{t-s}{Th} \right) v_s \right]' \left[ \sum_{s=T+1}^{T+\lceil Th \rceil} K\left( \frac{t-s}{Th} \right) v_s \right] = o_p(1). \]

Since \( S_{151} \) deals with the left boundary while \( S_{152} \) deals with the right boundary, by symmetry, we only have to prove the result for \( S_{151} \).

By the symmetry of \( K(\cdot) \) and the fact that \( y_s = y_{-s} \) for \( \lfloor Th \rfloor \leq s \leq 0 \), we have
\[ S_{151} = \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \left[ \sum_{s=1}^{\lfloor Th \rfloor} K\left( \frac{t+s-1}{Th} \right) v_s \right]' \left[ \sum_{s=1}^{\lfloor Th \rfloor} K\left( \frac{t+s-1}{Th} \right) v_s \right] \]
\[ = \frac{1}{T^2 h^{3/2}} \sum_{t=1}^{T} \sum_{s=1}^{\lfloor Th \rfloor} K\left( \frac{t+s-1}{Th} \right)^2 v_s' v_s \]
\begin{align*}
+ \frac{1}{T^2h^{3/2}} \sum_{t=1}^{T} \sum_{s \neq r}^{T} K\left(\frac{t + s - 1}{Th}\right) K\left(\frac{t + r - 1}{Th}\right) v'_s v_r \\
&\triangleq S_{1511} + S_{1512}.
\end{align*}

Note that \( K(x) = 0 \) for \( x > 1 \). Hence,

\[ E|S_{1511}| \leq \frac{1}{T^2h^{3/2}} \sum_{t=1}^{T} \sum_{s=1}^{T} K\left(\frac{t + s - 1}{Th}\right)^2 E|v'_s v_s| = O(h^{1/2}) = o(1), \]

which implies that \( S_{1511} = o_p(1) \).

By the fact that

\[ \frac{1}{Th} \sum_{t=1}^{T} \sum_{s \neq r}^{T} K\left(\frac{t + s - 1}{Th}\right) K\left(\frac{t + r - 1}{Th}\right) = \int K\left(\frac{s}{Th}+x\right) K\left(\frac{r}{Th}+x\right) dx + o(1), \]

we can obtain

\[ S_{1512} = \frac{1}{Th^{1/2}} \sum_{s \neq r}^{T} \left[ \int K\left(\frac{s}{Th}+x\right) K\left(\frac{r}{Th}+x\right) dx + o(1) \right] v'_s v_r \]

\[ = \frac{1}{Th^{1/2}} \sum_{s \neq r}^{T} [\Delta_{5rs} + o(1)] v'_s v_r, \]

where \( \Delta_{5rs} = \int K\left(\frac{s}{Th}+x\right) K\left(\frac{r}{Th}+x\right) dx \).

By the similar arguments used in the proof of (A.11), we can prove

\[ \text{Var}(S_{1512}) \leq \frac{CT^2h^2}{T^2h} = O(h) = o(1), \]

which implies that \( S_{1512} = o_p(1) \) by Chebyshev’s inequality. Hence, it follows that \( S_{151} = o_p(1) \). \( \square \)
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PROOF OF LEMMA A.6. Note that

$$S_{16} = \frac{2}{T^2 h^{3/2}} \sum_{t=1}^{T} \sum_{s=1-[T h]}^{0} \sum_{r=1}^{T} K\left(\frac{t-s}{T h}\right) K\left(\frac{t-r}{T h}\right) v'_s v_r$$

$$+ \frac{2}{T^2 h^{3/2}} \sum_{t=1}^{T} \sum_{s=T+1}^{T+[T h]} \sum_{r=1}^{T} K\left(\frac{t-s}{T h}\right) K\left(\frac{t-r}{T h}\right) v'_s v_r$$

$$\triangleq S_{161} + S_{162}.$$

By symmetry, we only need to prove that $S_{161} = o_p(1)$.

By the symmetry of $K(\cdot)$ and the fact that $y_s = y_{-s}$ for $[T h] \leq s \leq 0$, we can further decompose $S_{161}$ as

$$S_{161} = \frac{2}{T^2 h^{3/2}} \sum_{t=1}^{[T h]} \sum_{s=1}^{[T h]} \sum_{r=1}^{[T h]} K\left(\frac{t+s-1}{T h}\right) K\left(\frac{t-r}{T h}\right) v'_s v_r$$

$$= \frac{2}{T^2 h^{3/2}} \sum_{t=1}^{[T h]} \sum_{s=1}^{[T h]} K\left(\frac{t+s-1}{T h}\right) K\left(\frac{t-s}{T h}\right) v'_s v_s$$

$$+ \frac{2}{T^2 h^{3/2}} \sum_{t=1}^{[T h]} \sum_{s \neq r}^{[T h]} K\left(\frac{t+s-1}{T h}\right) K\left(\frac{t-r}{T h}\right) v'_s v_r$$

$$\triangleq S_{1611} + S_{1612}.$$

Then, we have

$$E|S_{1611}| \leq C \frac{T^2 h^2}{T^2 h^{3/2}} = O(h^{1/2}) = o(1),$$

which implies that $S_{161} = o_p(1)$ by Markov’s inequality. Using the similar arguments used in the proof of (A.11), we can obtain that $\text{Var}(S_{1612}) = O(h) = o(1)$, and so $S_{1612} = o_p(1)$. Hence, it follows that $S_{161} = o_p(1)$. \qed
Proof of Lemma A.7. First, by the symmetry of $K(\cdot)$, we have

$$S_{142} = \frac{1}{h^{1/2}} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \{ \int K(x) \left[ K \left( x + \frac{s-r}{h} \right) + K \left( x + \frac{r-s}{h} \right) \right] dx \} v'_s v_r$$

$$= \frac{2}{h^{1/2}} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \Delta_{3rs} v'_s v_r.$$

By the boundedness of $K(\cdot)$, (A.13), and a similar argument as for (A.10), it follows that $|E(S_{142})| \leq \frac{C}{h^{1/2}} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} |E(v'_s v_r)| \leq \frac{C}{h^{1/2}} \sum_{j=p_T+1}^{T} \beta(j) = O \left( \frac{1}{p_T h^{1/2}} \right) = o(1)$.

Next, define $S_{0,142}$ in the same way as $S_{142}$ with $v_t$ replaced by $v_{0t}$. Then, since $E(S_{142}) = o(1)$, we have that $E(S_{0,142}) = o(1)$ and

$$\text{Var}(S_{0,142}) = E(S_{0,142}^2) + o(1), \quad (A.20)$$

where

$$E(S_{0,142}^2) = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \sum_{l=1}^{T-p_T-1} \sum_{k=l+p_T+1}^{T} \Delta_{3rs} \Delta_{3lk} E(v'_s v_{0r} v'_k v_{0l})$$

$$= V_1 + V_2 + V_3 + V_4 \quad (A.21)$$

with

$$V_1 = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \Delta_{3rs}^2 E(v'_s v_{0r} v'_0 v_{0r}),$$

$$V_2 = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \sum_{k=r+p_T+1}^{T} \Delta_{3rs} \Delta_{3rk} E(v'_s v_{0r} v'_k v_{0r}),$$

$$V_3 = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{l=1}^{T-p_T-1} \sum_{s=\max\{r,l\}+p_T+1}^{T} \Delta_{3rs} \Delta_{3ls} E(v'_s v_{0r} v'_0 v_{0l}).$$
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\[ V_4 = \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{l=1,l\neq r}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \sum_{k=l+p_T+1}^{T} \Delta_{3rs} \Delta_{3lk} E(v_0^s v_0^r v_0^l v_0^k). \]

Define \( \delta_j = \int K(x) K(x + \frac{j}{T h}) dx \). By (A.21) and Lemmas A.8–A.11 below, we can obtain

\[
E(S_{0,142}^2) = V_1^* + V_2^* + V_3^* + V_4^*
\]

\[
= \frac{4}{T^2 h} \sum_{r=1}^{T-p_T-1} \sum_{j=p_T+1}^{T-r} \delta_j^2 \left[ \min\{T-p_T-1-r,p_T\} \right] \sum_{m=-\min\{T-p_T-1-r,p_T\}}^{\min\{T-j-r,p_T\}} \left[ \sum_{m'=\min\{T-j-r,p_T\}}^{} E(\text{vec}(v_0^r v_0^j v_0^r v_0^j)) \right].
\]

Note that \( \frac{2}{T h} \sum_{j=p_T+1}^{T-r} \delta_j^2 = \int \left[ \int K(x) K(x + \lambda) dx \right]^2 d\lambda + o(1) \) for all \( r \), and \( v_0^l = \Sigma_0 s_t \). By (A.22), it follows that \( E(S_{0,142}^2) = V + o(1) \). Hence, by (A.20) and Lemma A.4 in Kim et al. (2011), we have that \( S_{0,142} \rightarrow_L N(0, V) \).

Third, it only suffices to show that \( S_{142} - S_{0,142} = o_p(1) \). By (A.3), we can get

\[
S_{142} - S_{0,142} = \frac{2}{Th^{1/2}} \sum_{r=1}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \Delta_{3rs}
\]

\[
\times \left[ v_0^s \epsilon_t \Sigma_0^{-1} v_0^r / c_T + v_0^s \Sigma_0^{-1} \epsilon_s v_0^r / c_T + v_0^s \Sigma_0^{-1} \epsilon_s \Sigma_0^{-1} v_0^r / c_T^2 \right].
\]

Because \( \epsilon_t \) and \( \Sigma_0 \) are bounded, by using the similar argument as for (A.22), we can prove that \( E(S_{142} - S_{0,142})^2 = O\left(\frac{1}{c_T^2}\right) = o(1) \), which implies \( S_{142} - S_{0,142} = o_p(1) \) by Chebyshev’s inequality.
Lemma A.8. $V_1 = V_1^* + o(1)$, where

$$V_1^* = \frac{4}{T^2 h} \sum_{r=1}^{T-r-1} \sum_{j=pr+1}^{T-r} \delta_j^2 E(vec(v_{0r+j}v_{0r+j}')) E(vec(v_{0r}v_{0r}')).$$

Lemma A.9. $V_2 = V_2^* + o(1)$, where

$$V_2^* = \frac{4}{T^2 h} \sum_{r=1}^{T-pr-1} \sum_{j=pr+1}^{T-r-1} \min\{T-r-j, pr\} \delta_j^2 \sum_{m'=1}^{\min\{T-r-j, pr\}} \left[ E(vec(v_{0r+j+m'}v_{0r+j}')) E(vec(v_{0r}v_{0r}')) + E(vec(v_{0r+j+m'}v_{0r+j}')) E(vec(v_{0r+m}v_{0r}'))) \right].$$

Lemma A.10. $V_3 = V_3^* + o(1)$, where

$$V_3^* = \frac{4}{T^2 h} \sum_{r=1}^{T-pr-2} \sum_{j=pr+1}^{T-r-1} \min\{T-pr-1-r, pr\} \delta_j^2 \sum_{m'=1}^{\min\{T-pr-1-r, pr\}} \left[ E(vec(v_{0r+j+m'}v_{0r+j}')) E(vec(v_{0r+m}v_{0r}'))) + E(vec(v_{0r+j+m'}v_{0r+j}')) E(vec(v_{0r+m}v_{0r}'))) \right].$$

Lemma A.11. $V_4 = V_4^* + o(1)$, where

$$V_4^* = \frac{4}{T^2 h} \sum_{r=1}^{T-pr-2} \sum_{j=pr+1}^{T-r-1} \min\{T-pr-1-r, pr\} \delta_j^2 \sum_{m'=1}^{\min\{T-r-j, pr\}} \left[ \sum_{m'=1}^{\min\{T-r-j, pr\}} E(vec(v_{0r+m+m'}v_{0r+j+k} + v_{0r+m}v_{0r+j+k}')) \times E(vec(v_{0r+j+m'}v_{0r+j+k} + v_{0r+j+m'}v_{0r+j+k}'))) \right].$$

In the sequel, we only give the proof of Lemma A.11, since the proofs of Lemmas A.8 – A.10 are similar and much easier.

**Proof of Lemma A.11.** By noting that

$$v_{0s}v_{0s}v_{0k}v_{0l} = v_{0s}v_{0s}v_{0l}v_{0k} = tr(v_{0s}v_{0s}v_{0l}v_{0k})$$
\[ = \text{tr}(v_0 v_0' v_0 v_0') = \text{vec}(v_0 v_0')' \text{vec}(v_0 v_0'), \]

\[ (A.23) \]

we can re-write

\[
V_4 = \frac{4}{T^2 h} \sum_{r \neq l}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \sum_{k=l+p_T+1}^{T} \delta_{s \neq k} E \text{vec}(v_0 v_0')' \text{vec}(v_0 v_0') \\
= \frac{4}{T^2 h} \sum_{r \neq l}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \sum_{k=l+p_T+1}^{T} \delta_{s \neq k} E \text{vec}(v_0 v_0')' E \text{vec}(v_0 v_0') \\
+ \frac{4}{T^2 h} \sum_{r \neq l}^{T-p_T-1} \sum_{s=r+p_T+1}^{T} \sum_{k=l+p_T+1}^{T} \delta_{s \neq k} E \text{vec}(v_0 v_0')' E \text{vec}(v_0 v_0') \\
\times [E \text{vec}(v_0 v_0')' E \text{vec}(v_0 v_0') - E \text{vec}(v_0 v_0')' E \text{vec}(v_0 v_0')] \\
\Delta V_{41} + V_{42}.
\]

First, we consider \( V_{41} \) by splitting it into four parts:

\[
V_{41} = V_{411} + V_{412} + V_{413} + V_{414}, \tag{A.24}
\]

where \( V_{41; i} \) are defined according to the following constraints on the indexes:

\[
V_{411}: 0 < |r - l| \leq p_T, 0 < |s - k| \leq p_T; \hspace{1em} V_{412}: |r - l| > p_T, 0 < |s - k| \leq p_T; \\
V_{412}: 0 < |r - l| \leq p_T, |s - k| > p_T; \hspace{1em} V_{414}: |r - l| > p_T, |s - k| > p_T.
\]

For \( V_{411} \), some calculations lead to

\[
V_{411} = \frac{4}{T^2 h} \sum_{r,l=1}^{T-p_T-1} \text{vec}(v_0 v_0')' \sum_{s=r+p_T+1}^{T} \sum_{k=l+p_T+1}^{T} \delta_{s \neq k} E \text{vec}(v_0 v_0')
\]
where the last equality holds by (A.13).
For $V_{412}$, we can show

$$|V_{412}| = \frac{4}{T^2h} \sum_{r,l=1}^{T-p_T-1} \sum_{|r-l|>p_T} E\text{vec}(v_{0r}v_{0l}')' \sum_{s=r+p_T+1}^{T} \sum_{k=l+p_T+1}^{T} \delta_{3rs} \delta_{3lk} E\text{vec}(v_{0k}v_{0s}')'$$

$$= \frac{4}{T^2h} \sum_{r=1}^{T-2p_T-2T-p_T-1} E\text{vec}(v_{0r}v_{0r+m} + v_{0r+m}v_{0r}')'$$

$$\times \sum_{s=r+p_T+1}^{T-1} \min(T-s,p_T) \sum_{m' = 1}^{m = p_T+1} E\text{vec}(v_{0s+m}v_{0s}' + v_{0s}v_{0s+m}')'$$

$$\times \int K(x)K\left(x + \frac{r-s}{Th}\right)dx \int K(x)K\left(x + \frac{s-r + \epsilon_{mm'}}{Th}\right)dx \left| E\text{vec}(v_{0r}v_{0r+m} + v_{0r+m}v_{0r}')' \right|$$

$$\leq \frac{C}{T^2h} \sum_{r=1}^{T-2p_T-2T-p_T-1} \sum_{m = p_T+1}^{T-1} \min(T-s,p_T) \sum_{m' = 1}^{m = p_T+1} \left| E\text{vec}(v_{0s+m}v_{0s}' + v_{0s}v_{0s+m}')' \right|$$

$$\leq \frac{C}{T^2h} \sum_{r=1}^{T-2p_T-2T-p_T-1} \sum_{m = p_T+1}^{T-1} \beta(m)^{\delta/(1+\delta)} \sum_{s=r+p_T+1}^{T-1} \sum_{m' = 1}^{m' = p_T+1} \beta(m')^{\delta/(1+\delta)}$$

$$\leq \frac{C}{T^2h p_T} \frac{T^2}{T^2h p_T} = O\left(\frac{h^{1/2}}{p_T h^{3/2}}\right) = o(1),$$

where the first inequality holds by the integrability of $K(\cdot)$, the second inequality holds by a similar argument as for (A.8), and the third inequality holds by Assumption A.2(i) and (A.13). Hence, $V_{412} = o(1)$. Similarly, we can prove that $V_{413} = o(1)$ and $V_{414} = o(1)$. By (A.24)–(A.25), it follows that $V_{41} = V_{4}^* + o(1)$. By (A.14) and the similar argument as for (A.10), we can show that $V_{42} = o(1)$, and hence the conclusion holds.
Proof of Proposition 2

Write

\[ S_2 = \frac{1}{T^2 h^{1/2}} \sum_{t=1}^{T} \sum_{s,r=1}^{T} K \left( \frac{s-t}{T h} \right) v'_s v_r + \frac{1}{T^2 h^{1/2}} \sum_{t=1}^{T} \left[ \sum_{s=1-\lceil T h \rceil}^{0} + \sum_{s=T+\lceil T h \rceil}^{T} \right] \sum_{r=1}^{T} K \left( \frac{s-t}{T h} \right) v'_s v_r \]

\[ = \frac{1}{T^2 h^{1/2}} \sum_{t=1}^{T} K(0) v'_t v_t + \frac{1}{T^2 h^{1/2}} \sum_{t \neq r}^{T} K(0) v'_t v_r + \frac{1}{T^2 h^{1/2}} \sum_{t \neq s}^{T} K \left( \frac{s-t}{T h} \right) v'_s v_t \]

\[ + \frac{1}{T^2 h^{1/2}} \sum_{t \neq s}^{T} \sum_{s=1-\lceil T h \rceil}^{0} \sum_{r=1}^{T} K \left( \frac{s-t}{T h} \right) v'_s v_r + \frac{1}{T^2 h^{1/2}} \sum_{t=1}^{T} \sum_{s=T+\lceil T h \rceil}^{T} \sum_{r=1}^{T} K \left( \frac{s-t}{T h} \right) v'_s v_r \]

\[ \triangleq \sum_{i=1}^{7} S_{2i}. \]

By Lemmas A.1, A.3, \( S_{21} = \frac{h}{K(0)} S_{11} = o_p(1) \) and \( S_{23} = \frac{h}{2K(0)} S_{13} = o_p(1) \).

For \( S_{22} \), we can show that \( E(S_{22}) = O \left( \frac{1}{T^{3/2}} \right) = o(1) \) by using a similar proof as for Lemma A.3. Moreover, by Lemma A(ii) of Hjellvik et al. (1998), it entails that \( \text{Var}(S_{22}) = O \left( \frac{1}{T^{1/2}} \right) = o(1) \), which implies \( S_{22} = o_p(1) \) by Chebyshev’s inequality. For \( S_{24} \), by using a similar proof as for Lemma A.2, we can show that \( |E(S_{24})| = O \left( h^{1/2} \right) + O \left( \frac{h^{1/2}}{c_T} \right) + O \left( \frac{h^{1/2}}{c_T} \right) = o(1) \) and \( \text{Var}(S_{24}) = O \left( \frac{h}{c_T} \right) = o(1) \), leading to \( S_{24} = o_p(1) \). For \( S_{25} \), we write it as \( S_{25} = h \cdot S_{25}^* \). Then, by a similar argument as for \( S_{14} \) in (A.12), we have \( S_{25}^* = O_p(1) \), and so \( S_{25} = o_p(1) \).

Note that

\[ S_{26} = \frac{1}{T^2 h^{1/2}} \sum_{t=1}^{\lceil T h \rceil} \sum_{s=1}^{\lceil T h \rceil} \sum_{r=1}^{T} K \left( \frac{t+s-1}{T h} \right) v'_s v_r \]
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\[
\frac{1}{T^2 h^{1/2}} \sum_{t=1}^{\lceil Th \rceil} \sum_{s=1}^{\lceil Th \rceil} \sum_{r=1}^{\lceil Th \rceil} K \left( \frac{t + s - 1}{Th} \right) v'_s v_r + \frac{1}{T^2 h^{1/2}} \sum_{t=1}^{\lceil Th \rceil} \sum_{s=\lceil Th \rceil+1}^{T} \sum_{r=1}^{\lceil Th \rceil} K \left( \frac{t + s - 1}{Th} \right) v'_s v_r
\]

\[\triangleq S_{261} + S_{262}.\]

Here, the similar arguments used in Lemma [A.6] indicate that \( S_{261} = o_p(1). \)

Next, since

\[\frac{1}{Th} \sum_{t=1}^{\lceil Th \rceil} K \left( \frac{t + s - 1}{Th} \right) = \int_0^1 K \left( x + \frac{s}{Th} \right) ds + o(1),\]

we have

\[S_{262} = \frac{h^{1/2}}{T} \sum_{s=1}^{\lceil Th \rceil} \sum_{r=\lceil Th \rceil+1}^{T} \left[ \int_0^1 K \left( x + \frac{s}{Th} \right) ds + o(1) \right] v'_s v_r.\]

Hence, Davydov’s inequality implies \(|ES_{262}| \leq Ch^{3/2} = o(1).\) Furthermore, by Lemma A(ii) of [Hjellvik et al. (1998)], \( \text{Var}(S_{262}) = O \left( \frac{h}{T^2} \times T^2 h \right) = O(h^2), \)

this implies that \( S_{262} = o_p(1) \) by Chebyshev’s inequality. Hence \( S_{26} = o_p(1), \)

and similarly, \( S_{27} = o_p(1). \)

Now, we can conclude that \( S_2 = o_p(1). \)

**Proof of Proposition 3.** Write \( S_3 = S_{31} + S_{32}, \) where \( S_{31} = \frac{h^{1/2}}{T^2} \sum_{s=1}^{T} v'_s v_s \) and \( S_{32} = \frac{h^{1/2}}{T^2} \sum_{s \neq t}^{T} v'_s v_t. \) Since \( S_{31} = \frac{h^2}{K^2(0)} S_{11} \) and \( S_{32} = \frac{h}{K(0)} S_{22}, \) we have \( S_{31} = o(1) \) and \( S_{32} = o(1), \) which imply \( S_3 = o_p(1). \)

**Proof of Proposition 4.** Write \( S_4 = 2S_{41} + S_{42} - 2S_{43}, \) where \( S_{41} = h^{1/2} \sum_{t=1}^{T} \Pi_1(t/T)' \Pi_2(t/T), \) \( S_{42} = h^{1/2} \sum_{t=1}^{T} \Pi_2(t/T)'\Pi_2(t/T), \) and
\[ \hat{S}_{43} = h^{1/2} \sum_{t=1}^{T} \Pi_3' \Pi_2(t/T). \]

Since \( \sup_{x \in [0,1]} \left| \frac{1}{T} \sum_{s=1-\lfloor T h \rfloor}^{T+\lfloor T h \rfloor} K_h(x-s/T) - 1 \right| = O\left(\frac{1}{T h}\right) \), and \( \sup_{x \in [0,1]} \frac{1}{T} \sum_{s=1-\lfloor T h \rfloor}^{T+\lfloor T h \rfloor} K_h(x-s/T) = O(1) \), we have

\[
\sup_t \| \Pi_2(t/T) \| \leq \sup_t \left| \frac{1}{T} \sum_{s=1-\lfloor T h \rfloor}^{T+\lfloor T h \rfloor} K_h\left(\frac{t-s}{T}\right) - 1 \right| \|\text{vech}(\Sigma_0)\|
\]

\[
+ c_T^{-1} \sup_t \left| \frac{1}{T} \sum_{s=1-\lfloor T h \rfloor}^{T+\lfloor T h \rfloor} K_h\left(\frac{t-s}{T}\right) \text{vech}(\Sigma_1 s) \right| + O(c_T^{-1})
\]

\[
\leq O\left((Th)^{-1} + c_T^{-1}\right)
\]

by the boundedness of \( \Sigma_0 \) and \( \Sigma_1(\cdot) \). Hence, \( \sup_s \frac{1}{T h} \sum_{t=1}^{T} K\left(\frac{t-s}{T h}\right) \|\Pi_2(t/T)\| = O((Th)^{-1} + c_T^{-1}) \). Furthermore, since

\[
S_{41} = h^{1/2} \sum_{s=1-\lfloor T h \rfloor}^{T+\lfloor T h \rfloor} v_s' \frac{1}{T h} \sum_{t=1}^{T} K\left(\frac{t-s}{T h}\right) \Pi_2(t/T)
\]

\[
= h^{1/2} \left[ \sum_{s=1}^{T} + \sum_{s=1-\lfloor T h \rfloor}^{0} + \sum_{s=T+1}^{T+\lfloor T h \rfloor} \right] v_s' \frac{1}{T h} \sum_{t=1}^{T} K\left(\frac{t-s}{T h}\right) \Pi_2(t/T)
\]

\[
\Delta S_{411} + S_{412} + S_{413}.
\]

By (A.23), Assumption 2.2, and Davydov’s inequality, it follows that

\[
ES_{411}^2 = h \sum_{s=1}^{T} E \left[ v_s' \frac{1}{T h} \sum_{t=1}^{T} K\left(\frac{t-s}{T h}\right) \Pi_2(t/T) \right]^2
\]

\[
+ 2h \sum_{s=1}^{T} \sum_{j=1}^{T-s} E \left\{ v_s' \left[ \frac{1}{T h} \sum_{t=1}^{T} K\left(\frac{t-s}{T h}\right) \Pi_2(t/T) \right] \right\}
\]

\[
\times v_{s+j}' \left[ \frac{1}{T h} \sum_{t=1}^{T} K\left(\frac{t-s-j}{T h}\right) \Pi_2(t/T) \right] \}.
\]
A. PROOFS OF THEOREMS 2.1–2.2

\[
\leq O\left(\frac{Th}{c_T^2} + \frac{1}{Th}\right) + O\left(\frac{h}{c_T^2} + \frac{1}{T^{2}h}\right) \sum_{s=1}^{T} \sum_{j=1}^{T-s} \beta(j)^{5/(1+\delta)}
\]

\[
= O\left(\frac{Th}{c_T^2}\right) + o(1) = o(1).
\]

By Chebyshev’s inequality, we have that \( S_{411} = o_p(1) \). Similarly, \( S_{412} = o_p(1) \) and \( S_{413} = o_p(1) \), implying that \( S_{41} = o_p(1) \). Using the similar arguments, we also have that \( S_{43} = o_p(1) \).

Next, it suffices to show that \( S_{42} = \frac{Th^{1/2}}{c_T^2} \mathcal{B}_t + o(1) \). Let

\[
\Pi_{21}(x) = \left[ \frac{1}{T} \sum_{s=1}^{T+[Th]} K_h\left(x - \frac{s}{T}\right) - 1 \right] \text{vech}(\Sigma_0),
\]

\[
\Pi_{22}(x) = \left[ \frac{1}{T} \sum_{s=1}^{T+[Th]} K_h\left(x - \frac{s}{T}\right) \text{vech}(\Sigma_{1s}) - \frac{1}{T} \sum_{s=1}^{T} \text{vech}[\Sigma_{1s}] \right].
\]

Then, \( S_{42} = S_{421} + S_{422} + S_{423} \), where

\[
S_{421} = \frac{h^{1/2}}{c_T^2} \sum_{t=1}^{T} \Pi_{22}(t/T)' \Pi_{22}(t/T),
\]

\[
S_{422} = \frac{h^{1/2}}{c_T^2} \sum_{t=1}^{T} \Pi_{21}(t/T)' \Pi_{21}(t/T),
\]

\[
S_{423} = \frac{2h^{1/2}}{c_T} \sum_{t=1}^{T} \Pi_{21}(t/T)' \Pi_{22}(t/T).
\]

Note that \( \sup_x \|\Pi_{21}(x)\| = O\left(\frac{1}{T^h}\right) \) and \( \sup_x \|\Pi_{22}(x)\| = O(1) \). Hence, \( |S_{422}| \leq O\left(\frac{1}{T^{h^{3/2}}h}\right) = o(1) \) and \( |S_{423}| \leq O\left(\frac{1}{h^{1/2}c_T^2}\right) = o(1) \).

Moreover, by letting \( \Sigma_{1} = \frac{1}{T} \sum_{t=1}^{T} \Sigma_{1t} \) and \( \Sigma_{1t} = \frac{1}{T_h} \sum_{s=1}^{T+[Th]} K\left(\frac{t-s}{T_h}\right) \Sigma_{1s} \),
we have
\[
S_{421} = \frac{h^{1/2}}{c_T^2} \sum_{t=1}^{T} \text{vech}(\Sigma_{1t} - \Sigma_1)\text{vech}(\Sigma_{1t} - \Sigma_1)'
\]
\[
= \frac{h^{1/2}}{c_T^2} \sum_{t=1}^{T} \text{vech}(\Sigma_{1t} - \Sigma_1 + \Sigma_{1t} - \Sigma_{1t})'\text{vech}(\Sigma_{1t} - \Sigma_1 + \Sigma_{1t} - \Sigma_{1t})
\]
\[
= S_{4211} + S_{4212} + S_{4213},
\]

where
\[
S_{4211} = \frac{h^{1/2}}{c_T^2} \sum_{t=1}^{T} \text{vech}(\Sigma_{1t} - \Sigma_1)\text{vech}(\Sigma_{1t} - \Sigma_1),
\]
\[
S_{4212} = \frac{2h^{1/2}}{c_T^2} \sum_{t=1}^{T} \text{vech}(\Sigma_{1t} - \Sigma_1)\text{vech}(\Sigma_{1t} - \Sigma_{1t}),
\]
\[
S_{4213} = \frac{h^{1/2}}{c_T^2} \sum_{t=1}^{T} \text{vech}(\Sigma_{1t} - \Sigma_{1t})'\text{vech}(\Sigma_{1t} - \Sigma_{1t}).
\]

It follows easily that \(S_{4211} = \frac{Th^{1/2}}{c_T} B_l + o(1)\). Since \(\Sigma_1\), \(\Sigma_{1t}\), and \(\Sigma_1\) all are bounded and \(\|\Sigma_{1t} - \Sigma_{1t}\| = o(1)\) except for at most \([2Th]\) points, we can show that \(S_{4212} = O(\frac{Th^{3/2}}{c_T^2}) = o(1)\) and \(S_{4213} = O(\frac{Th^{3/2}}{c_T^2}) = o(1)\). Therefore, it follows that \(S_{421} = \frac{Th^{1/2}}{c_T} B_l + o(1)\). This completes the proof.

**Proof of Proposition 5.** Since \(\tilde{M} - M = (\tilde{M} - M) + (\tilde{M} - \tilde{M})\), it suffices to show that \(\tilde{M} - M = o_p(h^{1/2})\) and \(\tilde{M} - \tilde{M} = o_p(h^{1/2})\), where \(\tilde{M}\) is defined in the same way as \(\hat{M}\) in (4.14) with \(\hat{v}_t\) replaced by \(v_t\). By Lemmas A.12, A.13 below, the conclusion follows.

**Lemma A.12.** \(\tilde{M} - M = o_p(h^{1/2})\).
Lemma A.13. $\widetilde{M} - \widetilde{M} = o_p(h^{1/2})$.

PROOF OF LEMMA A.12. Denote $\Gamma_{v,j} = E v_j' v_t$. Then, we have

$$|E(\widetilde{M} - M)| = |E\widetilde{M} - \sum_{j=-\infty}^{\infty} \Gamma_{v,j} + \sum_{j=-\infty}^{\infty} \Gamma_{v,j} - M|$$

$$\leq |E\widetilde{M} - \sum_{j=-\infty}^{\infty} \Gamma_{v,j}| + \left| \sum_{j=-\infty}^{\infty} \Gamma_{v,j} - M \right|.$$

First, we can show

$$\left| E\widetilde{M} - \sum_{j=-\infty}^{\infty} \Gamma_{v,j} \right| \leq \sum_{j=-b_T}^{b_T} \left| k\left( \frac{j}{b_T} \right) \frac{T-j}{T} - 1 \right| |\Gamma_{v,j}| + \sum_{|j|>b_T} |\Gamma_{v,j}|$$

$$\leq \frac{C}{b_T} \sum_{j=-b_T}^{b_T} |j\Gamma_{v,j}| + O\left( \frac{1}{b_T} \right) \leq O\left( \frac{1}{b_T} \right),$$

where the second inequality holds by Lipschitz condition and the fact that

$$\sum_{|j|>b_T} |\Gamma_{v,j}| \leq C/b_T \text{ for large } b_T, \text{ and the last inequality holds by Davydov’s inequality and Assumption (A.5).}$$

Second, we have

$$\left| \sum_{j=-\infty}^{\infty} \Gamma_{v,j} - M \right| = \left| \sum_{j=-\infty}^{\infty} \Sigma_{t+j} E z_{t+j} z_t' \Sigma_t' - \Sigma_{0*} E z_{t+j} z_t' \Sigma_{0*}' \right|$$

$$\leq \sum_{j=-\infty}^{\infty} \left| \Sigma_{t+j} E z_{t+j} z_t' \Sigma_t' - \Sigma_{0*} E z_{t+j} z_t' \Sigma_{0*}' \right|$$

$$\leq \frac{C}{c_T} \sum_{j=-\infty}^{\infty} \left| E z_{t+j} z_t' \Sigma_{0*}' \right| = O\left( \frac{1}{c_T} \right),$$

where we have used the fact that $\sup_{t} |\Sigma_t - \Sigma_{0*}| = O(1/c_T)$ by (A.28).

Hence, it follows that $E(\widetilde{M} - M) = o(h^{1/2})$. 

Third, by a similar argument as for (A.17), we can show that \( \text{Var}(\tilde{M}) = O\left(\frac{b_T}{T}\right) \), and then the result follows by Chebyshev’s inequality and Assumption 2.4.

**Proof of Lemma A.13.** Write \( \tilde{M} = \tilde{M}_1 + \tilde{M}_2 \), where \( \tilde{M}_1 = \sum_{j=0}^{T-1} k\left(\frac{j}{b_T}\right) \frac{1}{T} \sum_{t=j+1}^{T} (\hat{v}_t \hat{v}_{t-j}' - v_t v_{t-j}') \) and \( \tilde{M}_2 = \sum_{j=-(T-1)}^{-1} k\left(\frac{j}{b_T}\right) \frac{1}{T} \sum_{t=1-j}^{T} (\hat{v}_{t+j} \hat{v}_t' - v_{t+j} v_t') \).

It suffices to prove that \( \tilde{M}_1 = o_p\left(h^{1/2}\right) \), since the proof of \( \tilde{M}_2 \) is similar.

Write \( \tilde{M}_1 = \tilde{M}_{11} + \tilde{M}_{12} + \tilde{M}_{13} \), where \( \tilde{M}_{11} = \frac{1}{T} \sum_{j=0}^{T-1} k\left(\frac{j}{b_T}\right) \sum_{t=j+1}^{T} v_t (\hat{v}_t - v_t)(\hat{v}_{t-j} - v_{t-j})' \), \( \tilde{M}_{12} = \frac{1}{T} \sum_{j=0}^{T-1} k\left(\frac{j}{b_T}\right) \sum_{t=j+1}^{T} v_t (\hat{v}_t - v_t)'(\hat{v}_{t-j} - v_{t-j}) \), and \( \tilde{M}_{13} = \frac{1}{T} \sum_{j=0}^{T-1} k\left(\frac{j}{b_T}\right) \sum_{t=j+1}^{T} (\hat{v}_t - v_t)v_{t-j}' \). Here, we note that under \( H_{1T} \),

\[
\| \hat{v}_t - v_t \| = \frac{1}{T} \sum_{s=1}^{T} v_s + \text{vech}(\Sigma_{1T}/c_T + \Sigma_{1T}/c_T)
\]  

for any \( t \), hence it follows that

\[
\sup_t \| \hat{v}_t - v_t \| = O_p\left(\frac{1}{\sqrt{T}}\right) + O\left(\frac{1}{c_T}\right).
\]  

(A.27)

For \( \tilde{M}_{11} \), by (A.27), Assumptions 2.3(ii) and 2.4(ii), and the fact that

\[
\frac{1}{b_T} \sum_{j=0}^{T-1} k\left(\frac{j}{b_T}\right) = O(1),
\]

we have

\[
\| \tilde{M}_{11} \| \leq \frac{1}{T} \sum_{j=0}^{T-1} k\left(\frac{j}{b_T}\right) \left\| \sup_j \sum_{t=j+1}^{T} (\hat{v}_t - v_t)(\hat{v}_{t-j} - v_{t-j})' \right\|
\]

\[
\leq O\left(\frac{b_T}{T}\right) \left\| \sup_j \sum_{t=j+1}^{T} (\hat{v}_t - v_t)(\hat{v}_{t-j} - v_{t-j})' \right\|
\]

\[
\leq O(b_T) \times O_p\left(\frac{1}{T} + \frac{1}{c_T}\right)
\]

(A.26)
For $M_{12}$, by \( \text{(A.26)} \) and Assumptions 2.3(ii) and 2.4(ii), the boundedness of $\Sigma_1$ and $\Sigma_1(\cdot)$, and the fact that $\frac{1}{b_T} \sum_{j=0}^{T-1} k\left(\frac{j}{b_T}\right) = O(1)$, we have

$$\|M_{12}\| \leq O\left(\frac{b_T}{T}\right) \sup_j \left\| \sum_{t=j+1}^{T} v_t(\hat{\nu}_{t-j} - \nu_{t-j})' \right\|$$

$$\leq O\left(\frac{b_T}{T}\right) \left[ \sup_j \left\| \sum_{t=j+1}^{T} v_t \left(\frac{1}{T} \sum_{s=1}^{T} v_s\right) \right\| + \frac{1}{c_T} \sup_j \sum_{t=j+1}^{T} \|v_t\| \right]$$

$$\leq O\left(\frac{b_T}{T}\right) \left[ O_p(\sqrt{T}) + O_p\left(\frac{T}{c_T}\right) \right]$$

$$= O_p\left(\frac{b_T}{\sqrt{T}} + \frac{b_T}{c_T}\right) = O_p\left(\frac{b_T}{c_T}\right)$$

Similarly, $\|M_{13}\| \leq o_p(h^{1/2})$, and hence $M_1 = o_p(h^{1/2})$. \hfill \Box

**Proof of Theorem 2.1.** Under $H_0$ (i.e., $\Sigma_1(x) \equiv 0$), $B_i \equiv 0$. Hence, the conclusion holds by \( \text{(2.4)} \), \( \text{(A.1)} \), and Propositions \( 1 \)–\( 5 \). \hfill \Box

**Proof of Theorem 2.2(i)–(ii).** The conclusion holds directly by \( \text{(2.4)} \), \( \text{(A.1)} \), Propositions \( 1 \)–\( 5 \), and the facts that when $c_T = cT^{1/2}h^{1/4}$, $S_4 = c^2B_i + o_p(1)$; and when $T^{1/2}h^{1/4} = o(c_T)$, $S_4 = o_p(1)$. \hfill \Box

**Proof of Theorem 2.2(iii).** When $c_T = o(T^{1/2}h^{1/4})$, we consider three cases: (1) $c_T^{-1}(Th)^{1/2} < \infty$; (2) $c_T(Th)^{-1/2} \to 0$ but $c_T h^{1/2} \to \infty$; (3) $c_T = O(h^{-1/2})$.

**Case (1).** Since $c_T h^{1/2} \to \infty$, a detailed investigation indicates that
the proofs of Propositions 1-3 still hold. Moreover, under the assumption that $h^{-1/2}c_T^{-1}b_T \to 0$, we can show that Proposition 4 holds. Using the fact that $S_4 = \frac{Th^{1/2}}{c_T}B_t + O_p(1)$ while $\frac{Th^{1/2}}{c_T} \to \infty$, it follows that $\hat{D} \to \infty$ in probability.

Case (2). As for Case (1), Propositions 1-3 hold. Next, it is easy to see that $Th^{1/2}\hat{S} = O_p\left(\frac{Th^{1/2}}{c_T}\right)$, $\hat{B} = O_p(h^{-1/2})$, and $\hat{V} = O_p(1)$. Since $\frac{Th}{c_T} \to \infty$, we can see that $\hat{D} = O_p\left(\frac{Th^{1/2}}{c_T}\right)$, implying that $\hat{D} \to \infty$ in probability.

Case (3). It is easy to see that $S_1 - 2S_2 + S_3 = O_p(Th^{1/2})$ and $S_1 - 2S_2 + S_3 > 0$. Furthermore, by noting that

$$S_4 = h^{1/2} \sum_{t=1}^{T} 2\Pi_1(t/T)'\Pi_2(t/T) + h^{1/2} \sum_{t=1}^{T} \Pi_2(t/T)'\Pi_2(t/T)$$

$$- h^{1/2} \sum_{t=1}^{T} 2\Pi_3(t/T)'\Pi_2(t/T),$$

we can easily show

$$h^{1/2} \sum_{t=1}^{T} \Pi_1(t/T)'\Pi_2(t/T) = O_p(T^{1/2}h),$$

$$h^{1/2} \sum_{t=1}^{T} \Pi_2(t/T)'\Pi_2(t/T) = O\left(\frac{Th^{1/2}}{c_T^{2}}\right),$$

$$h^{1/2} \sum_{t=1}^{T} \Pi_3(t/T)'\Pi_2(t/T) = O_p(T^{1/2}h).$$

Hence, $Th^{1/2}\hat{S}$ is at least of order $O_p(Th^{3/2})$ by the fact that $c_T = O(h^{-1/2})$. Since $\hat{B} = O_p(h^{-1/2}) = o_p(Th^{3/2})$ and $\hat{V} = O_p(1)$, it follows that $\hat{D} \to \infty$ in probability. \qed
B Proofs of Theorems 3.1–3.3

Let $B(x) = C_2 \Sigma''(x)/2$, $V_t(x) = \Sigma(x)^{1/2}(u_t - I_u)\Sigma(x)^{1/2}$, and $\tilde{\Sigma}(x) = \sum_{t=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h(x-t/T)\Sigma_t$. The technical lemma below plays a key role in our proof.

Lemma B.1. Suppose Assumptions 2.1, 2.3(i) and 3.1 hold, and $\Sigma(u)$ is twice continuously differentiable on $[0,1]$. Then, almost surely,

(i) $\sup_{x \in [0,1]} \left| \tilde{\Sigma}(x) - \Sigma(x) - \frac{1}{T} \sum_{t=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h(x-t/T)V_t(t/T) - h^2 B(x) \right| = O\left( \frac{1}{T h} \right) + o(h^2);

(ii) if conditions in Theorem 3.2 hold, then almost surely,

$$
\sup_{x \in [0,1]} \left| \frac{1}{T} \sum_{t=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h(x-t/T)V_t(x) \right| = O\left( \sqrt{\frac{\log T}{Th}} \right),
$$

$$
\sup_{x \in [0,1]} \left| \frac{1}{T} \sum_{t=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h(x-t/T)[V_t(x) - V_t(t/T)] \right| = O\left( \sqrt{\frac{\log T}{T}} \right).
$$

Proof of Lemma B.1. (i) Recall $u_t = u_{-t}$, $\Sigma_t = \Sigma_{-t}$ for $-\lfloor Th \rfloor \leq t \leq -1$, and $u_t = u_{2T-t}$, $\Sigma_t = \Sigma_{2T-t}$ for $T+1 \leq t \leq T+\lfloor Th \rfloor$. Note that

$$
\tilde{\Sigma}(x) - \Sigma(x) = \frac{1}{T} \sum_{t=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h\left( x - \frac{t}{T} \right)V_t(t/T) + \frac{1}{T} \sum_{t=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h\left( x - \frac{t}{T} \right)[\Sigma_t - \Sigma(x)]
$$

$$
+ \Sigma(x) \left[ \frac{1}{T} \sum_{t=1-\lfloor Th \rfloor}^{T+\lfloor Th \rfloor} K_h\left( x - \frac{t}{T} \right) - 1 \right]
$$

$$
:= \mathcal{T}_1(x) + \mathcal{T}_2(x) + \mathcal{T}_3(x).
$$

The proof of (i) is standard by using Taylor’s expansion and the approxi-
(ii) Using the fact that \( \| \Sigma(x) - \Sigma_t \| \leq C_1(1/|x - t/T| \leq h) \), it suffices to show

\[
\sup_{x \in [0,1]} \| T_1(x) \| = O(\sqrt{\log T}/Th).
\]

Our proofs below follow the similar arguments as in \cite{Masry96, Hansen08} and \cite{Vogt12}. Let \( c_T = (\log T)^{1/2} \) and write

\[
T_1(x) = \frac{1}{T} \sum_{t=1}^{T + \lfloor Th \rfloor} K_h(x - t/T) V_t(t/T) 1(\|V_t(t/T)\| > T^{1/s} c_T)
+ \frac{1}{T} \sum_{t=1}^{T + \lfloor Th \rfloor} K_h(x - t/T) V_t(t/T) 1(\|V_t(t/T)\| \leq T^{1/s} c_T)
:= T_{1,1}(x) + T_{1,2}(x).
\]

First, we consider \( T_{1,1}(x) \). Let \( s > 2 \). Then,

\[
\sum_{t=1}^{\infty} P(\|V_t(t/T)\| > t^{1/s} c_t) \leq \sum_{t=1}^{\infty} t^{-1} c_t^{-s} E \|V_t(t/T)\|^{2s} \leq CE \|u_t\|^{2s} \sum_{t=1}^{\infty} t^{-1} c_t^{-s} < \infty.
\]

Hence, by Borel-Cantelli Lemma, for \( T \) sufficiently large, \( \|V_t(t/T)\| \leq T^{1/s} c_T \) for \( t \leq T \). That is, \( T_{1,1}(x) = 0 \) almost surely.

Second, we consider \( T_{1,2}(x) \). Let \( a_T = \sqrt{\log T/Th} \). Cover the interval \([0,1]\) with \( \lfloor h^{-1} a_T^{-1} \rfloor + 1 = N \) balls \( A_j = \{ x : |x - x_j| \leq a_T h \} \). Then, for \( x \in A_j \), \( h^{-1}|x - x_j| \leq a_T \). Note that Assumption \( 2.8 \) ensures that for any \( |x_1 - x_2| \leq \delta \leq 2 \),

\[
|K(x_2) - K(x_1)| \leq \delta K^*(x_1)
\]
where $K^*(x) = C1(|x| \leq 2)$. Hence,

$$\left| K\left(\frac{Tx - t}{Th}\right) - K\left(\frac{Tx_j - t}{Th}\right) \right| \leq a_T K^*\left(\frac{Tx_j - t}{Th}\right).$$

Denote

$$\tilde{\mathcal{T}}_{1,2}(x) = \frac{1}{T} \sum_{t=1}^{T+[Th]} K_h\left(x - \frac{t}{T}\right)V_t(t/T)1(\|V_t(t/T)\| \leq T^{1/s} c_T).$$

Note that there exists a constant $0 < M < \infty$ such that $\frac{1}{T} \sum_{t=1}^{T+[Th]} K_h(x - \frac{t}{T})\|\Sigma_t\| \leq M$ and $\frac{1}{T} \sum_{t=1}^{T+[Th]} K^*_h(x - \frac{t}{T})\|\Sigma_t\| \leq M$. By triangular inequality, we obtain

$$\sup_{x \in A_j} \|\mathcal{T}_{1,2}(x)\| \leq \|\mathcal{T}_{1,2}(x_j)\| + \|\tilde{\mathcal{T}}_{1,2}(x_j)\| + 2Ma_T,$$

and hence,

$$P\left(\sup_{x \in [0,1]} \|\mathcal{T}_{1,2}(x)\| > 4Ma_T\right)$$

$$\leq N \max_{1 \leq j \leq N} P\left(\|\mathcal{T}_{1,2}(x_j)\| > Ma_T\right) + N \max_{1 \leq j \leq N} P\left(\|\tilde{\mathcal{T}}_{1,2}(x_j)\| > Ma_T\right).$$

By Theorem 2.1 in >Liebscher< (1996), the following statement holds: if the triangular array $\{Z_{t,T}\}_{t=1}^T$ satisfies $|Z_{t,T}| \leq b_T$ uniformly with triangular array $\alpha$-mixing coefficient $\alpha_T(k)$, then for $T_0 \leq T$ and $\varepsilon > 4T_0b_T$, we have

$$P\left(\left|\sum_{t=1}^T Z_{t,T}\right| > \varepsilon\right) \leq 4 \exp\left(-\frac{\varepsilon^2}{64T_0^{-1}T^2S_{T_0}^2 + 3\varepsilon T_0 b_T}\right) + \frac{4T}{T_0} \alpha_T(T_0), \quad (B.1)$$

where $S_{T_0} = \sup_{0 \leq j \leq T-1} E\left(\sum_{t=j+1}^{\min\{j+T_0,T\}} Z_{t,T}\right)^2$. 

Recall that when $t \in [-[Th], -1]$, $V_t(t/T) = V_{-t}(-t/T)$; and when $t \in [T + 1, T + [Th]]$, $V_t(t/T) = V_{2T-t}(2 - t/T)$. Hence,

$$
T_{1,2}(x) = \frac{1}{Th} \sum_{t=1}^{[Th]} [K(x - t/T) + K(x + t/T)]V_t(t/T) \mathbf{1}(\|V_t(t/T)\| \leq T^{1/s}c_T)
+ \frac{1}{Th} \sum_{t=[Th]+1}^{T-[Th]} K(x - t/T)V_t(t/T) \mathbf{1}(\|V_t(t/T)\| \leq T^{1/s}c_T)
+ \frac{1}{Th} \sum_{t=T-[Th]+1}^{T} [K(x - t/T) + K(x + t/T - 2)]V_t(t/T) \mathbf{1}(\|V_t(t/T)\| \leq T^{1/s}c_T)
\begin{equation}
\nonumber := \frac{1}{Th} \sum_{t=1}^{T} Z_{t,T}(x).
\end{equation}

Note that $P\left(\|T_{i,2}(x_j)\| > Ma_T\right) = P\left(\|\sum_{t=1}^{T} Z_{t,T}(x_j)\| > Ma_TTh\right)$, and a straightforward extension of Theorem 1 in [Hansen (2008)] implies that $S_{T_0} \leq C_0 T_0 h$ for some constant $0 < C_0 < \infty$. Hence, by letting $\varepsilon = Ma_TTh$, $b_T = T^{1/s}c_T$ and $T_0 = b_T^{-1}a_T^{-1}$ in (B.3), using the inequalities between mixing coefficients that $\alpha_T(k) \leq \alpha(k) \leq \beta(k)$, we have that for each $x_j$,

$$
P\left(\|T_{i,2}(x_j)\| > Ma_T\right) \leq 4 \exp\left(-\frac{M^2 \log T}{64C_0 + 3M}\right) + C \rho^{-T_0}T_0^{-1}T,
$$

by the fact that $\beta(k) \leq C\rho^k$ for some $\rho \in (0, 1)$ and $C > 0$. Similarly, we have that for each $x_j$,

$$
P\left(\|\tilde{T}_{i,2}(x_j)\| > Ma_T\right) \leq 4 \exp\left(-\frac{M^2 \log T}{64C_0 + 3M}\right) + C \rho^{-T_0}T_0^{-1}T.
$$

Hence, recall $N = \lfloor h^{-1}a_T^{-1} \rfloor + 1$,

$$
P\left(\sup_{x \in [0,1]} \|T_{i,2}(x)\| > 4M\sqrt{\frac{\log T}{Th}}\right) \leq C \left[T^{-\frac{M^2}{64C_0 + 3M}} \sqrt{Th^{-1} + \rho^{-T_0}T^2}\right] \triangleq \kappa^{(1)}(T) + \kappa^{(2)}(T).$$
Note that \( \kappa^{(1)}(T) \leq T^{1-\frac{M^2}{64C_0+3M}} \) and \( \frac{M^2}{64C_0+3CM} > 2 \) with \( M \) sufficiently large. Since \( \rho \in (0, 1) \), it follows that

\[
\sum_{t=1}^{\infty} \kappa^{(1)}(t) + \kappa^{(2)}(t) < \infty.
\]

By Borel-Cantelli Lemma, the proof is completed.

**Proof of Theorem 3.1.** (i) Note that \( h \to 0 \) when \( T \) is sufficiently large. Hence, for any \( x \in (0, 1) \), we have that \( h < x < 1 - h \), and \( \hat{\Sigma}(x) = \frac{1}{T} \sum_{t=1}^{T} K_h(x - \frac{t}{T}) y_t \). By the similar arguments as for Theorem 2 in [Xu and Phillips (2008)] and Assumption 2.3(i), we have that \( \hat{\Sigma}(x) \to \Sigma(x-) \int_{-1}^{0} K(x) dx + \Sigma(x+) \int_{0}^{1} K(x) dx = \frac{1}{2} [\Sigma(x-) + \Sigma(x+)] \). Next, it suffices to show that for \( x \in (0, 1) \),

\[
\left\| \hat{\Sigma}(x) - \tilde{\Sigma}(x) \right\| = o_p(1). \tag{B.2}
\]

Since \( \text{vech}(y_t - \Sigma_t) = L_n \Sigma_t^{1/2} \otimes 2 D_n z_t \), we have

\[
E \left[ \frac{1}{T} \sum_{t=1}^{T} K_h(x - \frac{t}{T}) \text{vech}(y_t - \Sigma_t) \right] \left[ \frac{1}{T} \sum_{t=1}^{T} K_h(x - \frac{t}{T}) \text{vech}(y_t - \Sigma_t) \right]'
\]

\[
= \frac{1}{T^2} \sum_{t=1}^{T} K_h^2 \left( x - \frac{t}{T} \right) E \text{vech}(y_t - \Sigma_t)' \text{vech}(y_t - \Sigma_t)
\]

\[
+ \frac{2}{T^2} \sum_{t=1}^{T} \sum_{j=1}^{T-t} K_h \left( x - \frac{t}{T} \right) K_h \left( x - \frac{t+j}{T} \right) E \text{vech}(y_t - \Sigma_t)' \text{vech}(y_{t+j} - \Sigma_{t+j})
\]

\[
\leq \frac{C}{Th} |E z_t' z_t| \left[ \frac{1}{Th} \sum_{t=1}^{T} K \left( x - \frac{t}{T} \right) \right]
\]
\[
+ \frac{C}{T^2h^2} \sum_{t=1}^{T} K \left( x - \frac{t}{T} \right) \sum_{j=1}^{T-t} |Ez_t'z_{t+j}| = O \left( \frac{1}{Th} \right)
\]

by Davydov’s inequality, the stationarity of \( z_t \), and the fact that \( \sup_t \| L_n(\Sigma_t^{1/2}) \| \leq 2 \) and \( \frac{1}{Th} \sum_{t=1}^{T} K \left( x - \frac{t}{T} \right) < \infty \). Hence, it follows that (B.2) holds by Chebyshev’s inequality.

(ii) Recall \( b(x) = \text{vech}(B(x)) \). Let \( v_t(x) \triangleq \text{vech}(V_t(x)) = L_n(\Sigma(x)^{1/2}) \Sigma^2 D_n z_t \).

By Lemma B.1(i), it follows that

\[
\sqrt{Th}(\hat{\sigma}(x) - \sigma(x) - h^2 b(x)) = \sqrt{Th} \frac{1}{T} \sum_{t=1}^{T} K_h(x - t/T) \left[ v_t(x) + v_t(t/T) - v_t(x) \right] .
\]

By the CLT for mixing process (see Hall and Heyde (2014)),

\[
\sqrt{Th} \frac{1}{T} \sum_{t=1}^{T} K_h(x - t/T) v_t(x) \sim N(0, V_\sigma(x)),
\]

and by noting that \( \| \Sigma(x) - \Sigma(t/T) \|_1 \| x - t/T \| \leq h = o(1) \), we have

\[
\sqrt{Th} \frac{1}{T} \sum_{t=1}^{T} K_h(x - t/T) \left[ v_t(t/T) - v_t(x) \right] = o_p(1).
\]

Hence, Slutsky’s Theorem implies the result once we give the expression of \( V_\sigma(x) \). Take \( p_T \) as in (A.13). Then,\[
\text{Var} \left( \sqrt{Th} \frac{1}{T} \sum_{t=1}^{T} K_h \left( x - \frac{t}{T} \right) z_t \right) = \frac{1}{Th} \sum_{r=1}^{T} K^2 \left( \frac{Tx - r}{Th} \right) Ez_r z_r' + \frac{1}{Th} \sum_{s_1} K \left( \frac{Tx - s}{Th} \right) K \left( \frac{Tx - r}{Th} \right) Ez_r z_s' + \frac{1}{Th} \sum_{s_2} K \left( \frac{Tx - s}{Th} \right) K \left( \frac{Tx - r}{Th} \right) Ez_r z_s',
\]
\[ V_{z,1} + V_{z,2} + V_{z,3}, \]

where \( S_1 \) and \( S_2 \) are defined as in (A.10). Note that \( \left| K\left(\frac{T x-r}{T h}\right) - K\left(\frac{T x-s}{T h}\right) \right| \leq C\frac{|r-s|}{T h} \) by Assumption 2.3. Since \( E(z_t z'_{t-j}) = O(\rho^j) \) for some \( 0 < \rho < 1 \) by Proposition 3.2(i), we can show that \( V_{z,3} = o(1) \) and

\[
V_{z,2} = \frac{1}{Th} \sum_{r=1}^{T-1} \sum_{j=1}^{\min(r-1,pr)} K^2\left(\frac{T x-r}{T h}\right) (Ez_r z'_{r+j} + Ez_{r+j} z'_r) + o(1)
\]

\[
= \frac{1}{Th} \sum_{r=1}^{T-1} \sum_{j=1}^{pr} K^2\left(\frac{T x-r}{T h}\right) (Ez_r z'_{r+j} + Ez_{r+j} z'_r)
\]

\[
- \frac{1}{Th} \sum_{r=2}^{pr} \sum_{j=r-1}^{pr} K^2\left(\frac{T x-r}{T h}\right) (Ez_r z'_{r+j} + Ez_{r+j} z'_r)
\]

\[
\rightarrow \left[ \int K^2(x)dx \right] \sum_{j=-\infty, j \neq 0}^{\infty} E(z_r z'_{r-j}),
\]

where we have used the fact that

\[
\left| \frac{1}{Th} \sum_{r=2}^{pr} \sum_{j=r-1}^{pr} K^2\left(\frac{T x-r}{T h}\right) (Ez_r z'_{r+j} + Ez_{r+j} z'_r) \right|
\]

\[
\leq C \frac{pt}{Th} \sum_{j=1}^{\infty} |Ez_r z'_{r+j} + Ez_{r+j} z'_r| = o(1).
\]

Since \( V_{z,1} = \left[ \int K^2(x)dx \right] E(z_r z'_r) + o(1) \), it implies that \( V_{z,1} + V_{z,2} + V_{z,3} = \left[ \int K^2(x)dx \right] Z_\infty + o(1) \), and hence the expression of \( V_\phi(x) \) follows.

To facilitate the proof of Theorem 3.2, we will introduce some notations.

Denote

\[
\ell_t(\phi) \triangleq \ell(y_t, \Omega_t(\phi)) = \text{tr}(G_t^{-1}(\phi)u_t) + \log \det(\Sigma_t^{1/2} G_t(\phi) \Sigma_t^{-1/2}).
\]
Recall the definitions of $G_t(\phi)$, $\tilde{G}_t(\phi)$, $\mathcal{L}_T(\phi)$ and $\tilde{\mathcal{L}}(\phi)$ in (3.2)–(3.3). We similarly define

$$\tilde{\mathcal{L}}(\phi) = \sum_{t=1}^{T} \tilde{\ell}_t(\phi)$$

with

$$\tilde{\ell}_t(\phi) = \text{tr}(\tilde{G}_t(\phi)^{-1}u_t) + \log \det(\Sigma_t^{1/2}\tilde{G}_t(\phi)\Sigma_t^{1/2}),$$

(B.3)

where $\tilde{G}_t(\phi)$ is defined in the same way as $\tilde{G}_t(\phi)$ in (3.3) with $\{u_t\}_{t=1}^{T}$ replaced by $\{\tilde{u}_t\}_{t=1}^{T}$.

In addition, we need Lemma B.2 herein, which is useful throughout the proof of Theorem 3.2. Specifically, Lemma B.2(i)-(ii) provide some useful results for $\Sigma(x)$ allowing for finite discontinuous points, and Lemma B.2(iii)-(iv) give some useful results for everywhere continuous $\Sigma(x)$. Let

$$\kappa_T \triangleq \sqrt{\frac{\log T}{Th}} + \sup_t (\tilde{\Sigma}_t - \Sigma_t) \quad \text{and} \quad \Delta_t \triangleq \tilde{u}_t - u_t,$$

(B.4)

where $\tilde{\Sigma}_t = \tilde{\Sigma}(t/T)$.

**Lemma B.2.** Suppose Assumptions 2.1, 2.3(i) and 3.1–3.4 hold. If $\Sigma(x)$ is twice continuously differentiable at continuous points on $(0, 1)$, then almost surely,

(i) $\tilde{\Sigma}(x)^{-1/2} = \Sigma(x)^{-1/2} - \frac{1}{2} \Sigma_t^{-3/4}(\tilde{\Sigma}(x) - \Sigma_t(x))\Sigma(x)^{-3/4} + O(\kappa_T^2)$ holds uniformly for all $x$, and consequently,

(ii) $\Delta_t = \tilde{\Sigma}_t^{-1/2}y_t\Sigma_t^{-1/2} - \Sigma_t^{-1/2}y_t\Sigma_t^{-1/2} + O(\kappa_T)y_t$.

If $\Sigma(x)$ is twice continuously differentiable everywhere on $x \in (0, 1)$,
B. PROOFS OF THEOREMS 3.1–3.3

(iii) \( \hat{\Sigma}(x)^{-1/2} = \Sigma(x)^{-1/2} - \frac{1}{2} \Sigma(x)^{-3/4}(\hat{\Sigma}(x) - \Sigma(x))\Sigma(x)^{-3/4} + O(\kappa_T^2) \)

holds uniformly for all \( x \), and consequently,

(iv) \( \Delta_t = \frac{1}{2} \Sigma_t^{-3/4}(\hat{\Sigma}_t - \Sigma_t)\Sigma_t^{-3/4}y_t\Sigma_t^{-1/2} - \frac{1}{2} \Sigma_t^{-1/2}y_t\Sigma_t^{-3/4}(\hat{\Sigma}_t - \Sigma_t)\Sigma_t^{-3/4} + O(\kappa_T^2)y_t \).

**Proof of Lemma B.2.** We only prove (iii)-(vi), since the proofs of (i)-(ii) are similar.

(iii) Since \( \Sigma(x) \) is continuous everywhere on \((0, 1)\), then by Taylor’s expansion, it follows easily that \( \sup_t (\hat{\Sigma}_t - \Sigma_t) = O(1/T) \), and hence

\[
\kappa_T = O\left(\sqrt{\log T Th}\right). \tag{B.5}
\]

Moreover, since \( \hat{\Sigma}(x) = \Sigma(x)^{1/2}[I_n + \Sigma(x)^{-1/2}(\hat{\Sigma}(x) - \Sigma(x))\Sigma(x)^{-1/2}]\Sigma(x)^{1/2} \), we have

\[
\hat{\Sigma}(x)^{-1/2} = \Sigma(x)^{-1/4}[I_n + \Sigma(x)^{-1/2}(\hat{\Sigma}(x) - \Sigma(x))\Sigma(x)^{-1/2}]^{-1/2} \Sigma(x)^{-1/4} = \Sigma(x)^{-1/4}[I_n - \frac{1}{2} \Sigma(x)^{-1/2}(\hat{\Sigma}(x) - \Sigma(x))\Sigma(x)^{-1/2}] \Sigma(x)^{-1/4} + O(\kappa_T^2) = \Sigma(x)^{-1/2} - \frac{1}{2} \Sigma(x)^{-3/4}(\hat{\Sigma}(x) - \Sigma(x))\Sigma(x)^{-3/4} + O(\kappa_T^2),
\]

where the second equality holds by Taylor’s expansion of \((I_n + \epsilon)^{-1/2}\) for a \( n \times n \) matrix \( \epsilon \), and the fact that

\[
\sup_{x \in [0, 1]} \left\| \hat{\Sigma}(x) - \Sigma(x) \right\|^2 = O\left(\frac{\log T}{Th}\right) + O(h^4) = O(\kappa_T^2), \quad a.s.
\]

by Lemma B.1.
(iv) By using the result in (iii), it is straightforward to see

\[ \Delta_t = (\widehat{\Sigma}_t^{-1/2} - \Sigma_t^{-1/2})\mathbf{y}_t \mathbf{\Sigma}_t^{-1/2} + \mathbf{\Sigma}_t^{-1/2} \mathbf{y}_t (\widehat{\Sigma}_t^{-1/2} - \Sigma_t^{-1/2}) \]

\[ + (\widehat{\Sigma}_t^{-1/2} - \Sigma_t^{-1/2}) \mathbf{y}_t (\widehat{\Sigma}_t^{-1/2} - \Sigma_t^{-1/2}) \]

\[ = -\frac{1}{2} \mathbf{\Sigma}_t^{-3/4} (\widehat{\Sigma}_t - \Sigma_t) \Sigma_t^{-3/4} \mathbf{y}_t \mathbf{\Sigma}_t^{-1/2} \]

\[ - \frac{1}{2} \mathbf{\Sigma}_t^{-3/2} \mathbf{y}_t \Sigma_t^{-3/4} (\widehat{\Sigma}_t - \Sigma_t) \Sigma_t^{-3/4} + O(\kappa_T^3) \mathbf{y}_t. \]

This completes all of the proofs. \(\Box\)

**Proof of Theorem 3.2(i).** The conclusion holds by Theorem 4.1.1 in [Amemiya (1985)] and the Propositions 6–9 below. \(\Box\)

**Proposition 6.** \(\phi_0\) is the unique minimizer of \(E\ell_t(\phi)\) for \(\phi \in \Phi\).

**Proposition 7.** \(\sup_{\phi \in \Phi} |T^{-1} \mathcal{L}_T(\phi) - E\ell_t(\phi)| = o_p(1)\).

**Proposition 8.** \(\sup_{\phi \in \Phi} |T^{-1} \mathcal{\tilde{L}}_T(\phi) - T^{-1} \mathcal{\tilde{L}}_T(\phi)| = o_p(1)\).

**Proposition 9.** \(\sup_{\phi \in \Phi} |T^{-1} \mathcal{\tilde{L}}_T(\phi) - T^{-1} \mathcal{L}_T(\phi)| = o_p(1)\).

**Proof of Proposition 6.** First, we can show that \(G_t(\phi)\) is a symmetric positive definite matrix for \(\phi \in \Phi\) by Assumption 3.3(ii). Second, we have

\[ E\ell_t(\phi) - E\ell_t(\phi_0) \]

\[ = E \text{tr}(\Omega_t^{-1}(\phi) \mathbf{y}_t) + E \log \det[\Omega_t(\phi)] - E \log \det[\Omega_t(\phi_0)] - n \]
\[ E \log \det [G_t^{-1}(\phi_0) G_t(\phi)] + E \text{tr}(G_t^{-1}(\phi) G_t^{1/2}(\phi_0) E(e_t| \mathcal{F}_{t-1}) G_t^{1/2}(\phi_0)) - n \]

\[ = E \log \det [G_t^{-1}(\phi_0) G_t(\phi)] + E \text{tr}(G_t^{-1}(\phi) G_t(\phi_0)) - n \]

\[ = E \sum_{i=1}^{n} - \log \lambda_i + \lambda_i - 1, \]

where \( \lambda_i, i = 1, \cdots, n \) are the \( n \) eigenvalues of \( G_t^{-1}(\phi) G_t(\phi_0) \). Using the inequality \( x - 1 - \log(x) \geq 0 \) for \( x > 0 \), we can obtain that \( E(\ell_t(\phi)) - E(\ell_t(\phi_0)) \geq 0 \), and the equality holds if and only if \( \lambda_i = 1 \), i.e., \( G_t(\phi) = G_t(\phi_0) \) a.s., which implies \( \phi = \phi_0 \) by Assumption 3.3(iii). Hence, we have shown that \( \phi_0 \) is the unique minimizer of \( E(\ell_t(\phi)) \).

Proof of Proposition 7. By Theorem 3.1 in Ling and McAleer (2003), it suffices to show that \( E \sup_{\phi \in \Phi} \| \ell_t(\phi) \| < \infty \).

Under Assumption 3.1, \( E \log \det(G_t(\phi)) = E \sum_{i=1}^{n} \log[\lambda_i(G_t(\phi))] < E \sum_{i=1}^{n} \lambda_i(G_t(\phi)) = E \text{tr}(G_t(\phi)) \) for all \( \phi \in \Phi \), where \( \lambda_i(G_t(\phi)) > 0, i = 1, \cdots, n \) are the \( n \) eigenvalues of \( G_t(\phi) \). Hence, \( E[\log \det(G_t(\phi))]^+ < \infty \).

Obviously, \( E[\log \det(G_t(\phi))]^- < \sup_{\phi \in \Phi} (- \log \det(I_n - AA' - BB'), 0) < \infty \), which follows that \( E \sup_{\phi \in \Phi} |\log \det(G_t(\phi))| < \infty \). Since \( \Sigma_t \) is bounded, it implies that \( E \sup_{\phi \in \Phi} |\log \det(\Omega_t(\phi))| < \infty \).

It remains to show that \( E \sup_{\phi \in \Phi} \text{tr}(\Omega_t(\phi)^{-1} y_t) < \infty \). Since \( G_t(\phi) \) is positive definite by Assumption 3.3(ii), its eigenvalues are positive, and then by using the compactness of the parameter space and the Wielandt-Hoffman
we have

\[
\min_{1 \leq t \leq T} \inf_{\phi \in \Phi} \lambda_{\min}(G_t(\phi)) \geq \inf_{\phi \in \Phi} \lambda_{\min}(I_n - AA' - BB') \geq \lambda_{\min 0} > 0 \quad (B.6)
\]

for some constant \(\lambda_{\min 0} > 0\). Hence, \(\sup_{\phi \in \Phi} \|G_t(\phi)^{-1}\| < \infty\), which implies \(\sup_{\phi \in \Phi} \|\Omega_t(\phi)^{-1}\| < \infty\) by the boundedness of \(\Sigma_t\). By Hölder’s inequality and Assumptions 2.1 and 3.1, it follows that

\[
E \sup_{\phi \in \Phi} \|\Omega_t(\phi)^{-1}\| \|y_t\| \leq C \left( E \sup_{\phi \in \Phi} \|\Omega_t(\phi)^{-1}\|^2 E \|y_t\|^2 \right)^{1/2} < \infty.
\]

This completes the proof. \(\square\)

**Proof of Proposition 8.** It suffices to prove

(i) \(\frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\text{tr}(\hat{G}_t(\phi)^{-1} \tilde{u}_t - \tilde{G}_t(\phi)^{-1} u_t)| = o_p(1)\);

(ii) \(\frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\log \det \tilde{G}_t(\phi) \tilde{G}_t(\phi)^{-1}| = o_p(1)\);

(iii) \(\frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\log \det \Sigma_t \hat{\Delta}_t^{-1}| = o_p(1)\).

We first show (i)-(iii) above without discontinuous points, and then we modify the proof to allow finite discontinuous points.

[Continuous case] For (i), by letting

\[
S_t(\phi) \triangleq \hat{G}_t^{-1}(\phi) - \tilde{G}_t(\phi)^{-1} = -\hat{G}_t(\phi)^{-1}(\hat{G}_t(\phi) - \tilde{G}_t(\phi)) \tilde{G}_t(\phi)^{-1}, \quad (B.7)
\]

it is straightforward to show that \(\text{tr}(\hat{G}_t^{-1} u_t - \tilde{G}_t^{-1} u_t) = \text{tr}(S_t u_t) + \text{tr}(\hat{G}_t^{-1} \Delta_t) + \text{tr}(S_t \Delta_t)\).
Note that
\[ \|\Delta_t\| \leq O(\kappa_T)\|u_t\| \quad (B.8) \]
by Lemma B.2(iv), the boundedness of \(\Sigma_t\) and the fact that \(\sup_t \|\hat{\Sigma}_t - \Sigma_t\| = O(\kappa_T)\);
\[
\hat{G}_t - \tilde{G}_t = A\Delta_{t-1}A' + B[\hat{G}_{t-1} - \tilde{G}_{t-1}]B' = A\Delta_{t-1}A' + B[A\Delta_{t-2}A']B' + B^2[\hat{G}_{t-2} - \tilde{G}_{t-2}](B')^2 = \sum_{j=1}^{t-1} B^{j-1}[A\Delta_{t-j}A'](B')^{j-1}; \quad (B.9)
\]
and
\[ \rho_B \triangleq \rho(B) < 1 \quad (B.10) \]
by Proposition B.1. Then, we can show
\[
\|S_t\| \leq n\lambda^{-2}_{\min_0}\|\hat{G}_t - \tilde{G}_t\| \leq Cn\lambda^{-2}_{\min_0} \sum_{j=1}^{t-1} \rho_B^{j-1}\|\Delta_{t-j}\|
\leq Cn\lambda^{-2}_{\min_0} \sum_{j=1}^{t-1} O(\kappa_T)\rho_B^{j-1}\|u_{t-j}\| = O(\kappa_T) \sum_{j=1}^{t-1} O(\rho_B^j)\|u_{t-j}\|, \quad (B.11)
\]
where the first inequality holds by (B.6) and the inequality that \(\|A\| \leq \sqrt{n}\rho(A)\), the second inequality holds by (B.9)–(B.10) and the fact that \(\|AB\| \leq \|A\|\rho(B)\), and the third inequality holds by (B.8). Hence,
\[
E\left[\frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\text{tr}(S_t u_t)|\right] \leq \frac{1}{T} \sum_{t=1}^{T} E \sup_{\phi \in \Phi} \|S_t\|\|u_t\|
\leq \frac{1}{T} \sum_{t=1}^{T} O(\kappa_T) \sum_{j=1}^{t-1} O(\rho_B^j)E\|u_{t-j}\|\|u_t\|
\[ \frac{1}{T} \sum_{t=1}^{T} O(\kappa_T) \sum_{j=1}^{t-1} \rho_B^j E \| u_t \|^2 = O(\kappa_T), \]

where the last inequality holds by Hölder’s inequality. By Markov’s inequality and (B.5), we have that

\[ \frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\text{tr}(S_t u_t)| = o_p(1). \]

Similarly, we can show that

\[ \frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\text{tr}(\tilde{G}_t^{-1} \Delta_t) + \text{tr}(S_t \Delta_t)| = o_p(1), \]

hence the result (i) follows.

For (ii) and (iii), by Lemma A.1(x) in Zhou et al. (2021),

\[ n \| \Sigma_t - \tilde{\Sigma}_t \| (\| \tilde{\Sigma}_t^{-1} \| + \| \Sigma_t^{-1} \|) \text{ and } \log \det(G_t \tilde{G}_t^{-1}) \leq n \| G_t - \tilde{G}_t \| (\| G_t^{-1} \| + \| \tilde{G}_t^{-1} \|). \]

Then, the results (ii) and (iii) follow similarly as for the result (i).

[Discontinuous case] For simplicity, we only prove that \( \frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\text{tr}(S_t u_t)| = o_p(1) \) in the case of one discontinuous point \( u_d \in (0, 1) \).

Define \( E_d = \{ [Tu_d] - [Th], \ldots, [Tu_d] + [Th] \} \). Since \( \sup_{t \in E_d} \| \tilde{\Sigma}_t - \Sigma_t \| < \infty \) and \( \sup_{t \in E_d} |\tilde{\Sigma}_t - \Sigma_t| = \sup_{t \in E_d} \left| \frac{1}{T} \sum_{s=1}^{T} K_h((t-s)/T)V_s(t/T) \right| \leq O\left( \sqrt{\frac{\log T}{Th}} \right) \)

by Lemma B.3(ii), it follows that

\[ \| \Delta_t \| \leq C \| u_t \| \text{ for } t \in E_d \quad (B.12) \]

by using Lemma B.2(ii).

Next, for ease of proof, we introduce a truncation lag \( n_T \) such that \( n_T = o(Th) \) and \( n_T \to \infty \) as \( T \to \infty \). Here, \( n_T \) is introduced such that the impact of the discontinuous point on \( S_t \) is small enough for \( t \) away from the \([Tu_d] + [Th]\). Then, for \( t \geq [Tu_d] + [Th] + n_T \), the similar arguments as for
(B.11) entail
\[ \|S_t\| \leq n\lambda_{m_0}^{-2}\|\tilde{G}_t - \tilde{G}_t\| \leq Cn\lambda_{m_0}^{-2} \sum_{j=1}^{t-1} \rho_B^{-1} \|\Delta_{t-j}\| = S_{1t} + S_{2t}, \]
where
\[ S_{1t} = Cn\lambda_{m_0}^{-2} \sum_{j=1}^{t-(\lceil Tu_d \rceil + [Th])} \rho_B^{j-1} \|\Delta_{t-j}\|, \]
\[ S_{2t} = Cn\lambda_{m_0}^{-2} \sum_{j=t-(\lceil Tu_d \rceil + [Th]) + 1}^{t-1} \rho_B^{j-1} \|\Delta_{t-j}\|. \]
Since \( t \geq [Tu_d] + [Th] + n_T \), \( \Delta_{t-j} \) in \( S_{1t} \) behaves similarly as in the continuous case, and hence by (B.8), we have that \( S_{1t} \leq C\kappa_T \sum_{j=1}^{t-(\lceil Tu_d \rceil + [Th])} \rho_B^{j} \|u_{t-j}\| \).

On the other hand, \( S_{2t} \leq C \sum_{j=t-(\lceil Tu_d \rceil + [Th]) + 1}^{t} \rho_B^{j} \|u_{t-j}\| \) by (B.12). Hence, it follows that for \( t \geq [Tu_d] + [Th] + n_T \),
\[ \|S_t\| \leq C\kappa_T \sum_{j=1}^{t-(\lceil Tu_d \rceil + [Th])} \rho_B^{j} \|u_{t-j}\| + C \sum_{j=t-(\lceil Tu_d \rceil + [Th]) + 1}^{t} \rho_B^{j} \|u_{t-j}\|. \quad \text{(B.13)} \]

Third, it is straightforward to see
\[ E\frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} \|\text{tr}(S_tu_t)\| \leq \frac{1}{T} \sum_{t=1}^{T} E \sup_{\phi \in \Phi} \|S_t\| \|u_t\| = SU_1 + SU_2 + SU_3, \]
where \( SU_1 = \frac{1}{T} \sum_{t=1}^{[Tu_d - Th]} \sup_{\phi \in \Phi} \|S_t\| \|u_t\| \), \( SU_2 = \frac{1}{T} \sum_{t=[Tu_d - Th]}^{[Tu_d + Th] + n_T} E \sup_{\phi \in \Phi} \|S_t\| \|u_t\| \), and \( SU_3 = \frac{1}{T} \sum_{t=[Tu_d + Th] + n_T}^{T} E \sup_{\phi \in \Phi} \|S_t\| \|u_t\| \).

For \( SU_1 \) and \( SU_2 \), we have \( SU_1 \leq \frac{1}{T} \sum_{t=1}^{[Tu_d - Th]} O(\kappa_T) = O(\kappa_T) \) and \( SU_2 \leq O\left(\frac{1}{T}\right) \sum_{t=[Tu_d - Th]}^{[Tu_d + Th] + n_T} \sum_{j=1}^{t-1} \rho_B^{j} E \|u_{t-j}\| \|u_t\| = O\left(\frac{2Th + n_T}{T}\right) \). For \( SU_3 \), by (B.13), we have
\[ SU_3 \leq O\left(\frac{\kappa_T}{T}\right) \sum_{t=[Tu_d + Th] + n_T}^{T} \sum_{j=1}^{t-(\lceil Tu_d \rceil + [Th])} \rho_B^{j} E \|u_{t-j}\| \|u_t\|. \]
\[ + O\left(\frac{1}{T}\right) \sum_{t=\lfloor Tu_d+Th\rfloor+n_T}^{T} \sum_{j=\lfloor Tu_d+Th\rfloor+1}^{t} \rho_B^j E\|u_{t-j}\|\|u_t\| \]

\[ = O(\kappa_T) + O\left(\frac{1}{T}\right) \sum_{t=\lfloor Tu_d+Th\rfloor+n_T}^{T} \rho_B^{nt} = O(\kappa_T) + O(\rho_B^{nt}). \]

Hence, \( E\frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\text{tr}(S_t u_t)| \leq O(\kappa_T) + O\left(\frac{2Th+nt}{T}\right) + O(\rho_B^{nt}) = o(1), \)

and the result follows by Markov’s inequality. This completes the proof. \( \square \)

**Proof of Proposition 4.** It suffices to prove

(i) \( \frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\text{tr}([G_t(\phi)^{-1} - \tilde{G}_t(\phi)^{-1}]u_t)| = o_p(1); \)

(ii) \( \frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\log \det \tilde{G}_t(\phi)G_t(\phi)^{-1}| = o_p(1). \)

Note \( \|G_t^{-1}\| \leq \|G_t^{-1/2}\|^2 = \text{tr}(G_t^{-1}) \leq \text{tr}(I_n - AA' - BB'). \) Hence, by the compactness of \( \Phi, \)

\[ \sup_{\phi \in \Phi} \|G_t^{-1}\| < \infty, \quad \text{(B.14)} \]

and similarly,

\[ \sup_{\phi \in \Phi} \|\tilde{G}_t^{-1}\| < \infty. \quad \text{(B.15)} \]

In addition, similar to (B.9), given initial values \( \tilde{u}_0, \) we can show \( G_t(\phi) - \tilde{G}_t(\phi) = B^{-1}[A(\tilde{u}_0 - u_0)A' + B(G_0 - G_0)B'](B')^{-1}, \) and hence

\[ \|G_t(\phi) - \tilde{G}_t(\phi)\| \leq K\rho_B^t. \quad \text{(B.16)} \]

Observe that

\[ \frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} |\text{tr}([G_t(\phi)^{-1} - \tilde{G}_t(\phi)^{-1}]u_t)| \]
\[ \leq \frac{1}{T} \sum_{t=1}^{T} \sup_{\phi \in \Phi} \| G_t(\phi)^{-1} \| \| G_t(\phi) - \tilde{G}(\phi) \| \| \tilde{G}_t(\phi)^{-1} \| \| u_t \|. \]

By (B.14)–(B.16), the result (i) follows. Using \(| \log \det(\tilde{G}_t G_t^{-1}) | \leq n \| \tilde{G}_t - G_t \| (\| G_t^{-1} \| + \| \tilde{G}_t^{-1} \|)\), the result (ii) follows similarly.

In order to proceed the proof of Theorem 3.2 (ii), we need Lemmas B.3–B.6 below. We note that the related assumptions for Lemmas B.3–B.6 are the same as those for Theorem 3.2 (ii) unless stated otherwise.

**Lemma B.3.** Let \( m_T \) be a truncation lag such that

\[ m_T = O(T^{\lambda_m}) \] for some \( \lambda_m > 0 \) and \( \lambda_m + \lambda_h < 1/2. \] (B.17)

Then, \( \max_{1 \leq j \leq m_T} \max_{1 \leq t \leq T} \| \Sigma_{t-j}^{3/4} (\hat{\Sigma}_{t-j} - \Sigma_{t-j}) \Sigma_{t-j}^{-1/4} - \Sigma_{t-j}^{-3/4} (\hat{\Sigma}_t - \Sigma_t) \Sigma_t^{-1/4} \| = o\left( \frac{1}{\sqrt{T}} \right) \) a.s.

**Proof of Lemma B.3.** For \( 1 \leq j \leq m_T \) and \( j + 1 \leq t \leq T \), we have

\[ \| \Sigma_{t-j}^{3/4} (\hat{\Sigma}_{t-j} - \Sigma_{t-j}) \Sigma_{t-j}^{-1/4} - \Sigma_{t-j}^{-3/4} (\hat{\Sigma}_t - \Sigma_t) \Sigma_t^{-1/4} \| 
\]
\[ = \| \Sigma_{t-j}^{3/4} \hat{\Sigma}_{t-j} \Sigma_{t-j}^{-1/4} - \Sigma_{t-j}^{-3/4} \hat{\Sigma}_t \Sigma_t^{-1/4} \| 
\]
\[ \leq \| \Sigma_{t-j}^{3/4} \hat{\Sigma}_{t-j} \Sigma_{t-j}^{-1/4} \| + \| \Sigma_{t-j}^{-3/4} \hat{\Sigma}_t \Sigma_t^{-1/4} \| 
\]
\[ \Delta I_1 + I_2. \]

We first consider \( I_1 \). Note that \( |K(s/l) - K(s'/l')| \leq \frac{C |s-s'|}{l'h} \) by Lipschitz condition. Then, since \( K(x) = 0 \) for \( |x| > 1 \), we have that for any \( 0 <
\( s - s' < m_T, \)

\[
\| \tilde{\Sigma}_s - \tilde{\Sigma}_{s'} \| = \frac{1}{T h} \left[ \sum_{t = [s'-Th]}^{[s+Th]} \left| K \left( \frac{s - t}{Th} \right) - K \left( \frac{s' - t}{Th} \right) \right| y_t \right] \\
\leq \frac{1}{T h} \sum_{t = [s'-Th]}^{[s'+m_T+Th]} \left| K \left( \frac{s - t}{Th} \right) - K \left( \frac{s' - t}{Th} \right) \right| \| y_t \| \\
\leq C |s - s'| \frac{1}{T^2 h^2} \sum_{t = [s'-Th]}^{[s'+m_T+Th]} c_u \| u_t \| \\
= C |s - s'| \frac{m_T + 2Th}{T^2 h^2} \frac{1}{m_T + 2Th} \sum_{t = [s'-Th]}^{[s'+m_T+Th]} \| u_t \| \\
\leq C m_T \frac{m_T + 2Th}{T^2 h^2} E \| u_t \| = o \left( \frac{1}{\sqrt{T}} \right) \quad \text{a.s.} \quad (B.18)
\]

Hence, by the boundedness of \( \Sigma_t^{-1} \) and (B.18), it follows that \( I_1 = o \left( \frac{1}{\sqrt{T}} \right) \).

Second, we consider \( I_2 \). Since \( \Sigma_t^{-1} \) is bounded and \( \Sigma_t \) is trice differentiable, we can show that \( \| \Sigma_{t-j} - \Sigma_t \| \leq C \left( \frac{1}{T} \right) = o \left( \frac{1}{\sqrt{T}} \right) \) and \( \| \Sigma_t^{-5/8} (\Sigma_{t-j} - \Sigma_t) \Sigma_t^{-5/8} \| \leq C \left( \frac{1}{T^2} \right) = o \left( \frac{1}{\sqrt{T}} \right) \). Then, by Taylor’s expansion, we have

\[
\Sigma_{t-j}^{1/4} = [\Sigma_t + \Sigma_{t-j} - \Sigma_t]^{-1/4} \\
= \Sigma_t^{-1/8} \left[ I_n + \Sigma_t^{-1/2} (\Sigma_{t-j} - \Sigma_t) \Sigma_t^{-1/2} \right]^{-1/4} \Sigma_t^{-1/8} \\
= \Sigma_t^{-1/8} \left[ I_n - \frac{1}{4} \Sigma_t^{-1/2} (\Sigma_{t-j} - \Sigma_t) \Sigma_t^{-1/2} \right] \Sigma_t^{-1/8} + o \left( \frac{1}{T} \right) \\
= \Sigma_t^{-1/4} - \frac{1}{4} \Sigma_t^{-5/8} (\Sigma_{t-j} - \Sigma_t) \Sigma_t^{-5/8} + o \left( \frac{1}{T} \right) \\
= \Sigma_t^{-1/4} + o \left( \frac{1}{\sqrt{T}} \right) + o \left( \frac{1}{T} \right),
\]

which implies that \( \Sigma_{t-j}^{-1/4} = \Sigma_t^{-1/4} + o \left( \frac{1}{\sqrt{T}} \right) \). Similarly, \( \Sigma_t^{-3/4} = \Sigma_t^{-3/4} + o \left( \frac{1}{\sqrt{T}} \right) \), and hence it is not hard to show that \( I_2 = o \left( \frac{1}{\sqrt{T}} \right) \) in view of that
\[ \| \hat{\Sigma} \| = O(1) \text{ a.s. by Lemma B.1} \] This completes the proof. \[ \square \]

**Lemma B.4.** Under the conditions in Proposition 3.1, \( \{ z_t, \text{vec}(G_t), \frac{\text{vec}(G_t)}{\partial \phi} \} \) is strictly stationary and \( \beta \)-mixing with exponential decay.

**Proof of Lemma B.4.** The proof is omitted, since it is similar to the proof of Proposition 3.1 by noting the recursive representation in (C.1).

**Lemma B.5.** Let \( \{ c_t \}_{t \in \mathbb{Z}} \) be a sequence of stationary process, and \( \mathcal{F}_t^s = \sigma(c_i, t \leq i \leq s) \) be the sigma-field generated by \( \{ c_i, t \leq i \leq s \} \). Define

\[
S_T = \frac{1}{\sqrt{T}} \sum_{t=1}^T b_t \left\{ \frac{1}{Th} \sum_{s=1-[Th]}^{T+[Th]} K\left( \frac{s-t}{Th} \right) a_s \right\},
\]

where \( a_t = f(c_t) \), \( b_t = g(c_{t-k}) \) for some \( k \leq n_T \), and \( f(\cdot) \) and \( g(\cdot, \cdot) \) are two real-valued functions. Suppose the following conditions hold:

1. \( Ea_t = 0, Eb_t = 0, E|a_t|^{6(1+\delta)} < \infty \) and \( E|b_t|^{3(1+\delta)} < \infty \) for some \( \delta > 0 \);

2. \( c_t \) is \( \beta \)-mixing with mixing coefficients \( \beta(j) \) satisfying \( \sum_{j=1}^{\infty} \beta(j)^{\delta/(1+\delta)} < \infty \);

3. \( K(\cdot) \) satisfies Assumption 2.3 and \( h \) satisfies Assumption 3.5;

4. \( n_T \) is a constant or \( n_T \to \infty \) and \( n_T = o(\sqrt{T \lambda^2}) \) as \( T \to \infty \).

Then,

(i) \( |E S_T| \leq \frac{C n_T}{\sqrt{Th}} \) and \( (ii) E S_T^2 \leq C \max \left\{ \frac{n_T}{\sqrt{Th}}, \frac{1}{Th^2} \right\} \).
Proof of Lemma B.6. The proof is omitted, since it is similar to the one of Proposition A.1 in Jiang et al. (2021) by some minor modifications.

Lemma B.6. Under (3.1),

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} D_n z_t = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \Xi_0 (G_t^{1/2}) \otimes \xi_t + o_p(1).
\]

Proof of Lemma B.6. Write \( u_t - I_n = G_t^{1/2} (e_t - I_n) G_t^{1/2} + G_t - I_n \).

Then,

\[
u_t - I_n = A_0 (u_{t-1} - I_n) A'_0 + B_0 (u_{t-1} - I_n) B'_0 \\
- B_0 G_{t-1}^{1/2} (e_{t-1} - I_n) G_{t-1}^{1/2} B'_0 + G_t^{1/2} (e_t - I_n) G_t^{1/2},
\]

and hence

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} D_n z_t = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (A_0^{\otimes 2} + B_0^{\otimes 2}) D_n z_{t-1} + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (G_t^{1/2}) \otimes \text{vec}(e_t - I_n) \\
- \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (B_0^{\otimes 2}) (G_{t-1}^{1/2}) \otimes \text{vec}(e_{t-1} - I_n).
\]

Recalling that \( \Xi_0 = (I_n^2 - A_0^{\otimes 2} - B_0^{\otimes 2})^{-1} (I_n^2 - B_0^{\otimes 2}) \), the result follows.

Proof of Theorem 3.2(ii). By Assumption 3.4 and Taylor’s expansion, we have

\[
0 = \frac{1}{\sqrt{T}} \frac{\partial \hat{L}_T(\hat{\phi})}{\partial \phi} = \frac{1}{\sqrt{T}} \frac{\partial \hat{L}_T(\hat{\phi}_0)}{\partial \phi} + \frac{1}{T} \frac{\partial \hat{L}_T(\hat{\phi})}{\partial \phi \partial \phi'} \sqrt{T}(\hat{\phi} - \phi_0),
\]
where \( \bar{\phi} \) lies between \( \hat{\phi} \) and \( \phi_0 \). Hence,

\[
\sqrt{T}(\hat{\phi} - \phi_0) = -\left[ \frac{1}{T} \partial^2 \mathcal{L}_T(\bar{\phi}) \right]^{-1} \frac{1}{\sqrt{T}} \partial \mathcal{L}_T(\phi_0).
\]

By letting \( I_n - \hat{u}_t \hat{G}_t^{-1} = I_n - u_t \tilde{G}_t^{-1} - u_t S_t - \Delta_t \tilde{G}_t^{-1}, \partial \tilde{G}_t = (\partial \tilde{G}_t -
\frac{\partial \tilde{G}_t}{\partial \phi_i}) + \frac{\partial \tilde{G}_t}{\partial \phi_i} \) and \( \tilde{G}_t^{-1} = \tilde{G}_t^{-1} + S_t \), we can write

\[
\frac{1}{\sqrt{T}} \partial \mathcal{L}_T(\phi_0) = \sum_{j=1}^{12} R_j,
\]

where

\[
R_1 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ (I_n - u_t \tilde{G}_t^{-1}) \partial \tilde{G}_t \tilde{G}_t^{-1} \right],
\]

\[
R_2 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ (I_n - u_t \tilde{G}_t^{-1}) (\partial \tilde{G}_t - \partial \tilde{G}_t) \tilde{G}_t^{-1} \right],
\]

\[
R_3 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ (I_n - u_t \tilde{G}_t^{-1}) \partial \tilde{G}_t S_t \right],
\]

\[
R_4 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ (I_n - u_t \tilde{G}_t^{-1}) (\partial \tilde{G}_t - \partial \tilde{G}_t) S_t \right],
\]

\[
R_5 = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ u_t S_t \partial \tilde{G}_t \tilde{G}_t^{-1} \right],
\]

\[
R_6 = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ u_t S_t (\partial \tilde{G}_t - \partial \tilde{G}_t) \tilde{G}_t^{-1} \right],
\]

\[
R_7 = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ u_t S_t \partial \tilde{G}_t S_t \right],
\]

\[
R_8 = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ u_t S_t (\partial \tilde{G}_t - \partial \tilde{G}_t) S_t \right],
\]

\[
R_9 = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ \Delta_t \tilde{G}_t^{-1} \partial \tilde{G}_t \tilde{G}_t^{-1} \right],
\]
\[ R_{10} = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr} \left[ \Delta_t \hat{G}_t^{-1} \left( \frac{\partial \hat{G}_t}{\partial \phi_i} - \frac{\partial \hat{G}_t}{\partial \phi_i} \right) \hat{G}_t^{-1} \right], \]

\[ R_{11} = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr} \left[ \Delta_t \hat{G}_t^{-1} \frac{\partial \hat{G}_t}{\partial \phi_i} S_t \right], \]

\[ R_{12} = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr} \left[ \Delta_t \hat{G}_t^{-1} \left( \frac{\partial \hat{G}_t}{\partial \phi_i} - \frac{\partial \hat{G}_t}{\partial \phi_i} \right) S_t \right]. \]

Here, \( \Delta_t \) and \( S_t \) are defined in (B.4) and (B.7), respectively. Furthermore, by Propositions 11-12 below and the central limit theory for mixing process (Hall and Heyde, 2014), we have that

\[ Q_{\phi_0} = \lim_{T \to \infty} \text{Var} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \rho_t \xi_t + \frac{F - E_H}{\sqrt{T}} \sum_{t=1}^{T} \left[ \Sigma_t^{1/4} \otimes \Sigma_t^{-1/4} \right] D_n z_t \right). \]

In view of Proposition 13 below, it remains to show

\[ Q_{\phi_0} = N + \Psi + H + H'. \quad (B.19) \]

First, since \( \rho_t \xi_t \) forms an m.d.s., we have

\[ \lim_{T \to \infty} \text{Var} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \rho_t \xi_t \right) = E[\rho_t \text{Var}(\xi_t) \rho_t^\prime] = N. \]

Second, by similar arguments as for Theorem 31(ii), we can show

\[ \lim_{T \to \infty} \text{Var} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \Upsilon(t/T) D_n z_t \right) = \int_0^1 \Upsilon(x) D_n \sum_{j=-\infty}^{\infty} E z_j^\prime z_{t-j}^\prime D_n^\prime \Upsilon(x) dx. \]

Since \( D_n \sum_{j=-\infty}^{\infty} E z_j^\prime z_{t-j}^\prime D_n^\prime = \lim_{T \to \infty} \text{Var} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} D_n z_t \right) \) and

\[ \lim_{T \to \infty} \text{Var} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} D_n z_t \right) = \Xi_0 E \left[ (G_t^{1/2})^{\otimes 2} \text{Var}(\xi_i)(G_t^{1/2})^{\otimes 2} \right] \Xi_0, \]
by Lemma B.6, the expression of $\Psi$ follows. Third, by similar arguments we have

$$H \triangleq \lim_{T \to \infty} \text{Cov} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \rho_t \xi_t, \frac{F - E \eta_t}{\sqrt{T}} \sum_{t=1}^{T} \left[ \Sigma_t^{1/4} \otimes \Sigma_t^{-1/4} \right] D_n z_t \right)$$

$$= \sum_{j=-\infty}^{\infty} E[\rho_t \xi_t z_{I-j}'] D'_n \int_{0}^{1} \mathcal{Y}'(x) dx.$$

Using Lemma B.6, we obtain

$$\sum_{j=-\infty}^{\infty} E[\rho_t \xi_t z_{I-j}'] D'_n = \lim_{T \to \infty} \text{Cov} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \rho_t \xi_t, \frac{1}{\sqrt{T}} \sum_{t=1}^{T} D_n z_t \right)$$

$$= \lim_{T \to \infty} \text{Cov} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \rho_t \xi_t, \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \Xi_0 (G^{1/2}) \otimes \mathbb{E}_t \right)$$

$$= E \left[ \rho_t \text{Var}(\xi_t) (G^{1/2}) \otimes \mathbb{E}_t \right].$$

Hence, $H = E \left[ \rho_t \text{Var}(\xi_t) (G^{1/2}) \otimes \mathbb{E}_t \right].$ Now, the result (B.19) follows.

We note that Propositions 10-13 are all proved under the conditions of Theorem 3.2(ii).

**Proposition 10.** $R_j = o_p(1)$ for $j = 4, 6, 7, 8, 10, 11, 12.$

**Proposition 11.** $R_j = o_p(1)$ for $j = 2, 3.$

**Proposition 12.** For each $i$,

$$R_1 + R_5 + R_9 = - \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \rho_{t,i} \xi_t - \frac{F_i}{\sqrt{T}} \sum_{t=1}^{T} \left[ \Sigma_t^{1/4} \otimes \Sigma_t^{-1/4} \right] D_n z_t$$
\[ + \frac{E \eta_i}{\sqrt{T}} \sum_{t=1}^{T} |\Sigma_t^{1/4} \otimes \Sigma_t^{-1/4}| D_n z_t + o_p(1). \]

**Proposition 13.** \( \frac{1}{T} \frac{\partial^2 \tilde{\mathcal{F}}_T(\varphi)}{\partial \phi \partial \phi'} \rightarrow J_{\phi_0} \) in probability.

**Proof of Proposition 10.** We only give the proof for \( R_{11} \), since the proofs for other terms are similar. First, note \( \| \Delta_t \| \leq O(\kappa_T) \| u_t \| \) by (B.8), \( \| \tilde{G}_t^{-1} \| \leq \lambda_{\min}^{-1} \) by (B.10), \( \| \partial \tilde{G}_t \| \leq \sum_{m=0}^{t-1} O(\rho_B^m) + O(\rho_B^m) \| u_{t-m} \| + O(\rho_B^m) \| u_0 \| \) as for (C.2), and \( \| S_t \| \leq O(\kappa_T) \sum_{j=1}^{t-1} O(\rho_B^j) \| u_{t-j} \| \) by (B.11). Then, it follows that

\[
\| R_{11} \| \leq \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \| \Delta_t \| \| \tilde{G}_t^{-1} \| \| \partial \tilde{G}_t \| \| S_t \|

\leq \frac{1}{\sqrt{T}} \sum_{t=1}^{T} O(\kappa_T) \| u_t \| \sqrt{h} \lambda_{\min}^{-1} \sum_{m=0}^{t-1} O(\rho_B^m) + O(\rho_B^m) \| u_{t-m} \|

\times O(\kappa_T) \sum_{j=1}^{t-1} O(\rho_B^j) \| u_{t-j} \|,
\]

which implies that \( E \| R_{11} \| \leq O(\sqrt{T} \kappa_T^2) \) by Hölder’s inequality and (B.10). Under Assumption 3.2, we have that \( \sup_t (\tilde{\Sigma}_t - \Sigma_t) = O(h^2) \), which entails \( O(\kappa_T^2) = o(T^{-1/2}) \) and hence \( E \| R_{11} \| = o(1) \). Finally, the result follows by using Markov’s inequality.

**Proof of Proposition 11.** We only give the proof for \( R_3 \), since the proof for \( R_2 \) is similar. By (B.7) and (B.10), it is not hard to see

\[
R_3 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr} \left[ (I_n - u_t G_t^{-1}) \frac{\partial G_t}{\partial \phi} S_t \right] + o_p(1),
\]
Let $\varphi_t = G_t^{-1/2}(I_n - e_t)G_t^{-1/2}\partial G_t / \partial \phi_t G_t^{-1}(G_t - \tilde{G}_t)$. By using Lemma B.2(iv) and (B.3), we have

$$R_3 = - \frac{1}{\sqrt{T}} \sum_{t=1}^T \sum_{j=1}^{t-1} \text{tr}[B_0^{j-1}A_0\Delta_{t-j}A_0'(B_0')^{-1} \varphi_t] + o_p(1)$$

$$= - \frac{1}{\sqrt{T}} \sum_{j=1}^{T-1} \sum_{t=j+1}^T \text{tr}[B_0^{j-1}A_0\Delta_{t-j}A_0'(B_0')^{-1} \varphi_t] + o_p(1)$$

$$= - \frac{1}{\sqrt{T}} \sum_{j=1}^{T-1} \sum_{t=j+1}^T \text{vec}(\varphi_t)'[B_0^{j-1}A_0 \otimes B_0^{-1}A_0]\text{vec}[\Delta_{t-j}] + o_p(1)$$

$$= - \frac{1}{\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^T \text{vec}(\varphi_t)'[B_0^{j-1}A_0 \otimes B_0^{-1}A_0]\text{vec}[\Delta_{t-j}] + R_{34} + o_p(1)$$

$$= R_{31} + R_{32} + R_{33} + R_{34},$$

where

$$R_{31} = - \frac{1}{2\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^T \text{vec}(\varphi_t)'[B_0^{j-1}A_0 \otimes B_0^{-1}A_0]$$

$$\times \text{vec}[\Sigma_{t-j}^{-3/4} (\hat{\Sigma}_{t-j} - \Sigma_{t-j})\Sigma_{t-j}^{-1/4} u_{t-j}],$$

$$R_{32} = - \frac{1}{2\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^T \text{vec}(\varphi_t)'[B_0^{j-1}A_0 \otimes B_0^{-1}A_0]$$

$$\times \text{vec}[u_{t-j} \Sigma_{t-j}^{-1/4} (\hat{\Sigma}_{t-j} - \Sigma_{t-j})\Sigma_{t-j}^{-3/4}],$$

$$R_{33} = O(\kappa_T^2) \frac{1}{\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^T \text{vec}(\varphi_t)'[B_0^{j-1}A_0 \otimes B_0^{-1}A_0]\text{vec}(y_{t-j}),$$
\[ R_{34} = \frac{1}{\sqrt{T}} \sum_{j=m+1}^{T} \sum_{t=j+1}^{T} \text{vec}(\varphi_t)'[B_0^{j-1} A_0 \otimes B_0^{j-1} A_0] \text{vec}[\Delta_{t-j}]. \]

Here, \( m_T \) is defined in (B.17). The remaining is to show that (i) \( R_{31} = o_p(1) \), (ii) \( R_{32} = o_p(1) \), (iii) \( R_{33} = o_p(1) \), and (iv) \( R_{34} = o_p(1) \).

(i) Note that

\[ R_{31} = -\frac{1}{2\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^{T} \text{vec}(\varphi_t)'[B_0^{j-1} A_0 \otimes B_0^{j-1} A_0] \]

\[ \times \text{vec}[\Sigma_t^{-3/4} (\hat{\Sigma}_t - \Sigma_t) \Sigma_t^{-1/4} u_{t-j}] + o_p(1) \]

\[ = -\frac{1}{2\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^{T} \text{vec}(\varphi_t)'[B_0^{j-1} A_0 \otimes B_0^{j-1} A_0] \]

\[ \times [u_{t-j} \Sigma_t^{-1/4} \otimes \Sigma_t^{-3/4}] \text{vec}[(\hat{\Sigma}_t - \Sigma_t)] + o_p(1), \]

where the first equality holds by Lemma B.3, and the second equality holds by the property of vec operator. Since \( \text{vec}(\hat{\Sigma}_t - \Sigma_t) = \frac{1}{T h} \sum_{s=1}^{T} K\left(\frac{t-s}{T h}\right) (\Sigma_1^{1/2})^{\otimes 2} \text{vec}(u_s - I_n) + O_p\left(\frac{\log T}{h^2} \right) \) by Lemma B.1, it follows that

\[ R_{31} = -\sum_{j=1}^{m_T} R_{31j} + o_p(1), \]

where

\[ R_{31j} = -\frac{1}{2\sqrt{T}} \sum_{t=j+1}^{T} \text{vec}(\varphi_t)'[B_0^{j-1} A_0 \otimes B_0^{j-1} A_0] \]

\[ \times [u_{t-j} \Sigma_t^{-1/4} \otimes \Sigma_t^{-3/4}] \frac{1}{T h} \sum_{s=1}^{T} K\left(\frac{t-s}{T h}\right) (\Sigma_1^{1/2})^{\otimes 2} \text{vec}(u_s - I_n). \]

Finally, we are going to show

\[ \text{Var}(R_{31j}) \leq o(\hat{\mu}_B), \quad (B.20) \]
where \( o(1) \) holds uniformly in \( j \). Note that \( \|B_0^{-1}A_0 \otimes B_0^{-1}A_0\| = O(\rho_B^j) \), and \( \sup_t \|\Sigma_t\| \leq c_u \) and \( \sup_t \|\Sigma_t^{-1}\| \leq c_l^{-1} \) by Assumption 2.1. Then, (B.20) holds if

\[
\Var \left\{ \frac{1}{\sqrt{T}} \sum_{t=j+1}^T \text{vec}(\varphi_t)'[u_{t-j} \otimes I_n] \frac{1}{Th} \sum_{s=1}^T K\left( \frac{t-s}{Th} \right) \text{vec}(u_s - I_n) \right\} = o(1).
\]

(B.21)

To prove (B.21), we apply Lemma B.5 with \( a_t = \text{vec}(u_t - I_n) \), \( b_t = \text{vec}(\varphi_t)'[u_{t-j} \otimes I_n] \) and \( c_t = \{ z_t, \text{vec}(G_t), \frac{\partial \text{vec}(G_t)}{\partial \phi_t} \} \). It remains to verify Conditions (1)--(4) in Lemma B.5. First, by Hölder’s inequality, (B.14) and the fact that \( E \|u_t\|^{6(1+2\delta)} < \infty \), we have

\[
E \|b_t\|^{3(1+2\delta)} \leq CE \left\| G_t^{-1/2} G_t^{-1/2} \frac{\partial G_t}{\partial \phi_t} G_t^{-1}[u_{t-j} \otimes I_n] \right\|^{3(1+2\delta)} E \|I_n - e_t\|^{3(1+2\delta)}
\]

\[
\leq CE \left\| \frac{\partial G_t}{\partial \phi_t} \right\|^{6(1+2\delta)} E \|u_{t-j} - I_n\|^{6(1+2\delta)} E \|I_n - e_t\|^{3(1+2\delta)} < \infty,
\]

and hence Condition (1) holds. Second, Lemma B.4 ensures Condition (2). Third, Conditions (3)--(4) hold by Assumption 2.3, 3.5 and (B.17). Therefore, (B.21) holds.

By the Cauchy-Schwarz inequality and (B.20),

\[
\Var \left( \sum_{j=1}^{m_T} R_{31j} \right) \leq \sum_{j=1}^{m_T} [\Var^{1/2}(R_{31j})]^2 = o(1),
\]

and hence Chebyshev’s inequality implies that \( R_{31} = o_p(1) \).

(ii) By the similar arguments as for \( R_{31} \), we can show that \( R_{32} = o_p(1) \).
(iii) Note that \( E[\|\text{vec}(\varphi_t)\| \|y_t\|] < \infty \) by Hölder’s inequality. Since \( \kappa_T^2 = o(T^{-1/2}) \), by (B.11) we have
\[
E|R_{33}| \leq \frac{O(\kappa_T^2)}{\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^{T} \|B_0^{j-1} A_0 \otimes B_0^{j-1} A_0\| E[\|\text{vec}(\varphi_t)\| \|y_t\|]
\leq \frac{O(\kappa_T^2)}{\sqrt{T}} \sum_{j=1}^{m_T} \sum_{t=j+1}^{T} \rho_T E[\|\text{vec}(\varphi_t)\| \|y_t\|] = o(1).
\]

Then, we obtain that \( R_{33} = o_p(1) \) by Markov’s inequality.

(iv) Note that \( E[\|\text{vec}(\varphi_t)\| \sup_s \|\Delta_s\|] < \infty \) by Hölder’s inequality.

Then, we can show that \( E|R_{34}| \leq O(\rho_T^{m_T})O\left(\sqrt{\log T}\right) = o(1) \) as for (iii).

Next, it follows that \( R_{34} = o_p(1) \) by Markov’s inequality. This completes the proof.

\[\square\]

**Proof of Proposition 12.** First, we consider \( R_5 \). Write \( R_5 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr} [u_t \tilde{G}_t^{-1} (\tilde{G}_t - \tilde{G}_i) \tilde{G}_t^{-1} \frac{\partial \tilde{G}_t}{\partial \varphi_i} \tilde{G}_t^{-1}] \) by (B.17), and then using similar arguments as for \( R_{11} \) in Proposition 10, we can show
\[
R_5 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr} [u_t G_t^{-1} (\tilde{G}_t - \tilde{G}_i) G_t^{-1} \frac{\partial G_t}{\partial \varphi_i} G_t^{-1}] + o_p(1).
\]

Further, by letting \( u_t G_t^{-1} = (u_t G_t^{-1} - I_n) + I_n \), it follows that
\[
R_5 = \hat{R}_5 + o_p(1), \tag{B.22}
\]

where \( \hat{R}_5 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr} [(\tilde{G}_t - \tilde{G}_i) G_t^{-1} \frac{\partial G_t}{\partial \varphi_i} G_t^{-1}] \), and we have used the fact that
\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr} [(u_t G_t^{-1} - I_n)(\tilde{G}_t - \tilde{G}_i) G_t^{-1} \frac{\partial G_t}{\partial \varphi_i} G_t^{-1}] = o_p(1)
\]
by Proposition 11.

Since $\kappa_T^2 = o(T^{-1/2})$, by Lemma B.2(iv) and (B.9), we have

\[
\hat{R}_5 = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr} \left[ \sum_{j=1}^{t-1} B_0^{j-1} A_0 \Sigma_{t-j}^{-3/4} (\hat{\Sigma}_{t-j} - \Sigma_{t-j}) \Sigma_{t-j}^{-1/4} u_{t-j} \right]
\times A_0' (B_0')^{j-1} G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1} + o_p(1).
\]

By interchanging the summations, we obtain

\[
\hat{R}_5 = -\frac{1}{\sqrt{T}} \sum_{j=1}^{T} \sum_{t=j+1}^{T} \text{tr} \left[ B_0^{j-1} A_0 \Sigma_{t-j}^{-3/4} (\hat{\Sigma}_{t-j} - \Sigma_{t-j}) \Sigma_{t-j}^{-1/4} u_{t-j} \right]
\times A_0' (B_0')^{j-1} G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1} + o_p(1). \tag{B.23}
\]

Let $m_T$ satisfy (B.17). By Lemma B.1, Lemma B.3 and similar arguments as for Proposition 11, it is not hard to see

\[
\hat{R}_5 = -\frac{1}{\sqrt{T}} \sum_{j=1}^{m_T} R_{5j} + o_p(1),
\]

where

\[
R_{5j} = \frac{1}{\sqrt{T}} \sum_{t=j+1}^{T} \text{tr} \left[ B_0^{j-1} A_0 \Sigma_{t-j}^{-3/4} (\hat{\Sigma}_{t-j} - \Sigma_{t-j}) \Sigma_{t-j}^{-1/4} u_{t-j} A_0' (B_0')^{j-1} G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1} \right]
\]

\[
= \frac{1}{\sqrt{TTh}} \sum_{t=j+1}^{T} \text{vec}(B_0^{j-1} A_0)' \left[ u_{t-j} \otimes G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1} \right]
\times [\Sigma_{t-j}^{1/4} \otimes B_0^{j-1} A_0 \Sigma_{t-j}^{-1/4}] \sum_{s=1}^{T} K(t - s) D_n z_s + o_p(1).
\]

Decompose $R_{5j} = R_{5j1} + R_{5j2}$, where

\[
R_{5j1} = \frac{1}{\sqrt{TTh}} \sum_{t=j+1}^{T} \text{vec}(B_0^{j-1} A_0)' \left( [u_{t-j} \otimes G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1}] - M_j \right)
\]

\[
R_{5j2} = \frac{1}{\sqrt{TTh}} \sum_{t=j+1}^{T} [\Sigma_{t-j}^{1/4} \otimes B_0^{j-1} A_0 \Sigma_{t-j}^{-1/4}] \sum_{s=1}^{T} K(t - s) D_n z_s + o_p(1).
\]
\[
\times [\Sigma_t^{1/4} \otimes B_0^{j-1} A_t \Sigma_t^{-1/4}] \sum_{s=1}^{T} K(\frac{t-s}{T h}) D_n z_s.
\]

\[R_{5j2} = \frac{1}{\sqrt{T} T h} \sum_{t=j+1}^{T} \text{vec}(B_0^{j-1} A_0)' M_j' [\Sigma_t^{1/4} \otimes B_0^{j-1} A_0 \Sigma_t^{-1/4}] \sum_{s=1}^{T} K(\frac{t-s}{T h}) D_n z_s,
\]

and let \( R_{51} = \sum_{j=1}^{m_T} R_{5j1} \) and \( R_{52} = \sum_{j=1}^{m_T} R_{5j2} \). Then, by similar arguments as for Lemma B.5, we have that \( R_{51} = o_p(1) \), and hence \( R_5 = -R_{52} + o_p(1) \), where

\[R_{52} = \frac{1}{\sqrt{T}} \sum_{s=1}^{T} \sum_{j=1}^{m_T} \frac{1}{T h} \text{vec}(B_0^{j-1} A_0)' M_j' [\Sigma_t^{1/4} \otimes B_0^{j-1} A_0 \Sigma_t^{-1/4}] D_n z_s
\]

by the continuity of \( \Sigma(x) \). Since \( \frac{1}{T h} \sum_{t=j+1}^{T} K(\frac{t-s}{T h}) = 1 + O(\frac{m_T}{T h^2}) \) for any fixed \( s \), and \( \frac{m_T}{T h^2} \to 0 \), we can show

\[R_{52} = \frac{1}{\sqrt{T}} \sum_{s=1}^{T} \sum_{j=1}^{m_T} \text{vec}(B_0^{j-1} A_0)' M_j' [\Sigma_t^{1/4} \otimes B_0^{j-1} A_0 \Sigma_t^{-1/4}] D_n z_s + o_p(1).
\]

(B.24)

Moreover, since \( \sqrt{T} m_T = o(1) \), it follows that

\[R_5 = -\frac{1}{\sqrt{T}} \sum_{s=1}^{T} \sum_{j=1}^{\infty} \text{vec}(B_0^{j-1} A_0)' M_j' [I_n \otimes B_0^{j-1} A_0] [\Sigma_s^{1/4} \otimes \Sigma_s^{-1/4}] D_n z_s + o_p(1)
\]

\[= -\frac{F_i}{\sqrt{T}} \sum_{s=1}^{T} [\Sigma_s^{1/4} \otimes \Sigma_s^{-1/4}] D_n z_s + o_p(1).
\]

(B.25)

Next, we consider \( R_9 \). Write \( R_9 = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(\Delta_t G_t^{-1} \partial G_t / \partial \phi_t) G_t^{-1} -
\]
\[ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(\Delta_t S_t \frac{\partial G_t}{\partial \phi_t} G_t^{-1}) \] by (B.7). Since \[ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(\Delta_t S_t \frac{\partial G_t}{\partial \phi_t} G_t^{-1}) = o_p(1) \] by using similar arguments as for \( R_{11} \) in Proposition 11, it follows that \( R_9 = o_p(1) \). Then, by the similar arguments as for (B.23), \( R_9 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ \Sigma_t^{-3/4} (\tilde{\Sigma}_t - \Sigma_t) \Sigma_t^{-1/4} G_t^{-1/2} (e_t - I_n) G_t^{-1/2} \frac{\partial G_t}{\partial \phi_t} G_t^{-1} \right] + \tilde{R}_9 + o_p(1) \), where

\begin{align*}
\tilde{R}_9 &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ \Sigma_t^{-3/4} (\tilde{\Sigma}_t - \Sigma_t) \Sigma_t^{-1/4} \frac{\partial G_t}{\partial \phi_t} G_t^{-1} \right].
\end{align*}

Moreover, since \[ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left[ \Sigma_t^{-3/4} (\tilde{\Sigma}_t - \Sigma_t) \Sigma_t^{-1/4} G_t^{1/2} (e_t - I_n) G_t^{-1/2} \frac{\partial G_t}{\partial \phi_t} G_t^{-1} \right] = o_p(1) \) by similar arguments as for Proposition 11, it entails that \( R_9 = \tilde{R}_9 + o_p(1) \).

Recall \( \eta_{t,i} = \text{vec}(G_t^{-1} \frac{\partial G_t}{\partial \phi_t})' \). By Lemma B.31, we can show

\[ \tilde{R}_9 = \frac{1}{\sqrt{T}} \sum_{s=1}^{T} \frac{1}{T} \sum_{t=1}^{T} K_h\left( \frac{t-s}{T} \right) \eta_{t,i} [\Sigma_t^{1/4} \otimes \Sigma_t^{-1/4}] D_n z_s + o_p(1), \]

using the property of trace operator. Since \[ \text{Var}\left( \frac{1}{T_h} \sum_{t=1:s \leq T_h} K\left( \frac{t-s}{T_h} \right) \eta_{t,i} \right) = O\left( \frac{1}{T_h} \right) \] by Davydov’s inequality and \( \int K(x)dx = 1 \), it follows that

\[ R_9 = \frac{E \eta_{t,i}}{\sqrt{T}} \sum_{s=1}^{T} [\Sigma_s^{1/4} \otimes \Sigma_s^{-1/4}] D_n z_s + o_p(1), \quad (B.26) \]

by using similar arguments as for (B.24).

Finally, by (B.16), it is straightforward to see that \( R_1 = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \rho_{t,i} \xi_t + o_p(1) \), and then the conclusion follows by (B.25) and (B.26). \( \Box \)

**Proof of Proposition 13.** By Theorem 3.1 in Ling and McAleer (2003), the conclusion holds by the following two arguments:
(i) \[ \sup_{\phi \in \Phi} \left| \frac{1}{T} \frac{\partial^2 \hat{\phi}_t(\phi)}{\partial \phi_i \partial \phi_j} - \frac{1}{T} \frac{\partial^2 \tilde{\phi}_t(\phi)}{\partial \phi_i \partial \phi_j} \right| = o_p(1); \]

(ii) \[ E \sup_{\phi \in \Phi} \left| \frac{\partial^2 \tilde{\phi}_t(\phi)}{\partial \phi_i \partial \phi_j} \right| < \infty. \]

For (i), we first note that

\[
\frac{\partial^2 \hat{\phi}_t(\phi)}{\partial \phi_i \partial \phi_j} = \text{tr} \left[ \frac{\partial^2 \hat{G}_t^{-1}}{\partial \phi_i \partial \phi_j} \hat{G}_t^{-1} - \frac{\partial G_t^{-1}}{\partial \phi_i} \frac{\partial G_t^{-1}}{\partial \phi_j} \hat{G}_t^{-1} \right.
\]
\[
\left. + \frac{\partial \hat{G}_t^{-1}}{\partial \phi_i} \frac{\partial \hat{G}_t^{-1}}{\partial \phi_j} \hat{G}_t^{-1} + \frac{\partial \hat{G}_t^{-1}}{\partial \phi_i} \frac{\partial \hat{G}_t^{-1}}{\partial \phi_j} \hat{G}_t^{-1} \right].
\]

By using the similar arguments as for Propositions 4 and 5, we can show

(a) \[ \frac{1}{T} \sum_{t=1}^T \sup_{\phi \in \Phi} \left| \frac{\partial^2 \hat{G}_t^{-1}}{\partial \phi_i \partial \phi_j} \hat{G}_t^{-1} - \frac{\partial G_t^{-1}}{\partial \phi_i} \frac{\partial G_t^{-1}}{\partial \phi_j} \hat{G}_t^{-1} \right| = o_p(1); \]

(b) \[ \frac{1}{T} \sum_{t=1}^T \sup_{\phi \in \Phi} \left| \frac{\partial \hat{G}_t^{-1}}{\partial \phi_i} \frac{\partial \hat{G}_t^{-1}}{\partial \phi_j} \hat{G}_t^{-1} \right| = o_p(1); \]

(c) \[ \frac{1}{T} \sum_{t=1}^T \sup_{\phi \in \Phi} \left| \frac{\partial \hat{G}_t^{-1}}{\partial \phi_i} \frac{\partial \hat{G}_t^{-1}}{\partial \phi_j} \hat{G}_t^{-1} \right| = o_p(1); \]

(d) \[ \frac{1}{T} \sum_{t=1}^T \sup_{\phi \in \Phi} \left| \frac{\partial \hat{G}_t^{-1}}{\partial \phi_i} \frac{\partial \hat{G}_t^{-1}}{\partial \phi_j} \hat{G}_t^{-1} \right| = o_p(1). \]

Hence, it follows that (i) holds.

For (ii), it suffices to prove that \[ E \sup_{\phi \in \Phi} \left| \frac{\partial^2 \tilde{\phi}_t(\phi)}{\partial \phi_i \partial \phi_j} \right| < \infty. \] Note that

\[
\frac{\partial^2 \tilde{\phi}_t(\phi)}{\partial \phi_i \partial \phi_j} = \text{tr} \left[ \frac{\partial^2 G_t^{-1}}{\partial \phi_i \partial \phi_j} G_t^{-1} - \frac{\partial G_t^{-1}}{\partial \phi_i} \frac{\partial G_t^{-1}}{\partial \phi_j} G_t^{-1} \right.
\]
\[
\left. + \frac{\partial G_t^{-1}}{\partial \phi_i} \frac{\partial G_t^{-1}}{\partial \phi_j} G_t^{-1} + \frac{\partial G_t^{-1}}{\partial \phi_i} \frac{\partial G_t^{-1}}{\partial \phi_j} G_t^{-1} \right].
\]  

(B.27)

To facilitate our proofs, we first claim that

\[
E \sup_{\phi \in \Phi} \left\| \frac{\partial^2 G_t}{\partial \phi_i \partial \phi_j} \right\|_3^3 < \infty \quad \text{and} \quad E \sup_{\phi \in \Phi} \left\| \frac{\partial G_t}{\partial \phi_i} \right\|_3^3 < \infty, \quad (B.28)
\]

where the preceding results hold by Minkowski’s inequality, \( (B.11) \), \( (B.2) \)–\( (C.3) \), the fact that \( E \| u_t \|_3^3 < \infty \), and some standard arguments.
Next, for the first term in (B.27), we have
\[
E \sup_{\phi \in \Phi} \left| \text{tr} \left[ u_t G_t^{-1} \frac{\partial G_t}{\partial \phi_j} G_t^{-1} \frac{\partial G_t}{\partial \phi_i} G_t^{-1} \right] \right| \leq C \left[ E \sup_{\phi \in \Phi} \left\| u_t \right\| \left\| \frac{\partial G_t}{\partial \phi_j} \right\| \left\| \frac{\partial G_t}{\partial \phi_i} \right\| \right]
\]
\[
\leq C \left[ E \sup_{\phi \in \Phi} \left\| u_t \right\|^3 \right]^{1/3} \left[ E \sup_{\phi \in \Phi} \left\| \frac{\partial G_t}{\partial \phi_j} \right\|^3 \right]^{1/3} \left[ E \sup_{\phi \in \Phi} \left\| \frac{\partial G_t}{\partial \phi_i} \right\|^3 \right]^{1/3} < \infty,
\]
where the first inequality holds by (B.6), Assumption 3.1 and the property that \( \text{tr}(AB) \leq \|A\| \|B\| \), and the second inequality holds by Hölder’s inequality and (B.28). Similarly, we can show the proofs for other terms in (B.27), and consequently, the result follows.

**Proof of Theorem 3.3 (i).** Denote \( l_t(\gamma) = \nu \ell_t(\phi) + c(y_t, \nu) \) and \( \hat{l}_t(\gamma) = \nu \hat{\ell}_t(\phi) + c(y_t, \nu) \). Then, some straightforward calculations give
\[
\frac{\partial l_t(\gamma)}{\partial \nu} = \text{tr}(\Omega_t^{-1}(\phi)y_t) + \log \det \Omega_t(\phi) - \log \det y_t + n \log(2)
\]
\[
+ \sum_{i=1}^{n} \left( \nu + 1 - i \right) - n \log(\nu) - n.
\]

From the proof of Theorem 3.2, we have shown that \( \frac{1}{T} \sum_{t=1}^{T} \left[ \text{tr}(\hat{e}_t) - \text{tr}(e_t) \right] = o_p(1) \), and similarly, \( \frac{1}{T} \sum_{t=1}^{T} \left[ \log \det(\hat{e}_t) - \log \det(e_t) \right] = o_p(1) \). Therefore, by Theorem 4.1.1 in Amemiya (1985) and the strong law of large numbers, it suffices to show that \( \nu_0 \) is the unique solution to
\[
E[\text{tr}(e_t)] - E \log \det(e_t) + n \log(2) + \sum_{i=1}^{n} \psi \left( \frac{\nu + 1 - i}{2} \right) - n - n \log(\nu) = 0.
\]
Note that by the property of Wishart distribution, we have \( E[\text{tr}(e_t)] = n \) and \( E \log \det(e_t) = n \log(2) + \sum_{i=1}^{n} \psi \left( \frac{\nu_0 + 1 - i}{2} \right) - n \log(\nu_0) \). So, clearly \( \nu_0 \) is
the solution of the above equation. Therefore, it suffices to show
\[ f(\nu) \triangleq \sum_{i=1}^{n} \psi \left( \frac{\nu + 1 - i}{2} \right) - n \log(\nu) \]
is monotonic in \( \nu > n \).

By Alzer and Batir (2007), we have that for \( x > 0 \),
\[ \psi(x) - \log(x) + \frac{1}{2} \psi'(x) > 0 \text{ and } \log(x) - \frac{1}{2x} - \psi(x) > 0. \]
Therefore, when \( \nu > n \),
\[ f'(\nu) = \sum_{i=1}^{n} \left[ \frac{1}{2} \psi' \left( \frac{\nu + 1 - i}{2} \right) - \frac{1}{\nu} \right] \]
\[ > \sum_{i=1}^{n} \left[ \log \left( \frac{\nu + 1 - i}{2} \right) - \psi \left( \frac{\nu + 1 - i}{2} \right) - \frac{1}{\nu} \right] \]
\[ > \sum_{i=1}^{n} \left[ \frac{1}{\nu + 1 - i} - \frac{1}{\nu} \right] \geq 0, \]
which implies the monotonicity of \( f(\nu) \).

(ii) Note that
\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\partial \hat{I}_t(\gamma_0)}{\partial \nu} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\partial l_t(\gamma_0)}{\partial \nu} = P_1 + P_2, \tag{B.29}
\]
where \( P_1 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \log \det(\Omega_t \hat{\Omega}_t^{-1}) \) and \( P_2 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}((\hat{G}_t^{-1} - G_t^{-1})u_t + \hat{G}_t^{-1} \Delta_t) \).

For \( P_1 \), we write it as \( P_1 = P_{11} + P_{12} \), where \( P_{11} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \log \det(\Sigma_t \hat{\Sigma}_t^{-1}) \) and \( P_{12} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \log \det(G_t \hat{G}_t^{-1}) \). On one hand, we can show
\[ P_{11} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(\log[I_n + (\Sigma_t - \hat{\Sigma}_t)\hat{\Sigma}_t^{-1}]) \]
\[
\begin{align*}
&= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left((\Sigma_t - \hat{\Sigma}_t)\hat{\Sigma}_t^{-1} + O_p(\kappa_T^2)\right) \\
&= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left((\Sigma_t - \hat{\Sigma}_t)\hat{\Sigma}_t^{-1}\right) + o_p(1), \quad (B.30)
\end{align*}
\]
where the first equality holds by the identity \(\log \det(I_n + \epsilon) = \text{tr} \log(I_n + \epsilon)\), the second equality holds by Taylor’s expansion that \(\log(I_n + \epsilon) = \epsilon + O(\epsilon^2)\) and Lemma 14, and the third equality holds since \(\kappa_T = o(T^{-1/2})\). Further, by similar arguments as for Lemma 14(iii), we have
\[
\hat{\Omega}(x)^{-1} = \Sigma(x)^{-1} - \Sigma(x)^{-1}(\hat{\Omega}(x) - \Sigma(x))\Sigma(x)^{-1} + O(\kappa_T^2)
\]
holds uniformly for all \(x\), and then by (B.30), it is not hard to see
\[
P_{11} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}\left((\Sigma_t - \hat{\Sigma}_t)\hat{\Sigma}_t^{-1}\right) + o_p(1). \quad (B.31)
\]
On the other hand, by similar arguments as for (B.30), \(P_{12} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(G_t - \hat{G}_t)\hat{G}_t^{-1}) + o_p(1)\), and then
\[
P_{12} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(G_t - \hat{G}_t)\hat{G}_t^{-1}) + o_p(1) \quad (B.32)
\]
by similar arguments as for Proposition 11.

For \(P_2\), we write it as \(P_2 = P_{21} + P_{22} + P_{23}\), where \(P_{21} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(S_t u_t)\), \(P_{22} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(G_t^{-1}\Delta_t)\), and \(P_{23} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(S_t \Delta_t)\). Here, \(\Delta_t\) and \(S_t\) are defined as in (B.4) and (B.7), respectively. By similar arguments as for Proposition 11, we can show that \(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}[G_t(\hat{G}_t - G_t)(e_t - I_n)] = o_p(1)\),
and hence by (B.32), it follows that

\[
P_{21} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(G_t^{-1}(G_t - \hat{G}_t)) - \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(G_t(\hat{G}_t - G_t)(e_t - I_n))
\]

\[= P_{12} + o_p(1).
\]

Further, by Lemma B.2(iv), similar arguments as for \(P_{21}\), and (B.31), it is not hard to see

\[
P_{22} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(G_t^{-1/2} \Sigma_t^{-3/4}(\Sigma_t - \hat{\Sigma}_t)\Sigma_t^{-1/4} G_t^{1/2} e_t) + o_p(1)
\]

\[= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \text{tr}(G_t^{-1/2} \Sigma_t^{-3/4}(\Sigma_t - \hat{\Sigma}_t)\Sigma_t^{-1/4} G_t^{1/2}) + o_p(1)
\]

\[= P_{11} + o_p(1).
\]

Note that \(P_{23} = o_p(1)\) by using similar arguments as for Proposition 10. Therefore, it follows that \(P_2 = P_1 + o_p(1)\), and then by (B.29),

\[
\frac{1}{T} \sum_{t=1}^{T} \frac{\partial^2 \hat{L}_t(\gamma_0)}{\partial \nu^2} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\partial \hat{L}_t(\gamma_0)}{\partial \nu} + o_p(1).
\]  \hfill (B.33)

By similar arguments as for Proposition 13,

\[
\frac{1}{T} \sum_{t=1}^{T} \frac{\partial^2 \hat{L}_t(\gamma_0)}{\partial \nu^2} = E\left[\frac{\partial^2 L_t(\gamma_0)}{\partial \nu^2}\right] + o_p(1) = J_{\nu_0} + o_p(1),
\]  \hfill (B.34)

where we have used the fact that \(E\left[\frac{\partial L_t(\gamma_0)}{\partial \nu}\right]^2 = 2J_{\nu_0}\) and \(E\left[\frac{\partial^2 L_t(\gamma_0)}{\partial \nu^2}\right] = \frac{1}{2}E\left[\frac{\partial L_t(\gamma_0)}{\partial \nu}\right]^2\) by standard arguments for Fisher Information. Finally, by (B.33) \(\Rightarrow\) (B.34), the conclusion follows. \qed
C Derivatives

Let $J_{ij}$ be an $n \times n$ matrix zeros everywhere except for a one at the $(i,j)$th entry. Then,

$$\frac{\partial G_t}{\partial A_{ij}} = J_{ij}(u_{t-1} - I_n)A' + A(u_{t-1} - I_n)J_{ji} + B \frac{\partial G_{t-1}}{\partial A_{ij}} B',$$

$$\frac{\partial G_t}{\partial B_{ij}} = J_{ij}(G_{t-1} - I_n)B' + B(G_{t-1} - I_n)J_{ji} + B \frac{\partial G_{t-1}}{\partial B_{ij}} B'.$$ (C.1)

Therefore, the first order derivative of $G_t(\phi)$ w.r.t $\phi$ is given by

$$\frac{\partial G_t}{\partial A_{ij}} = \sum_{m=0}^{\infty} B^m \{ J_{ij}(u_{t-m-1} - I_n)A' + A(u_{t-m-1} - I_n)J_{ji} \} (B')^m,$$

$$\frac{\partial G_t}{\partial B_{ij}} = \sum_{m=0}^{\infty} \frac{\partial B^m}{\partial B_{ij}} \left( I_n - AA' - BB' + A(u_{t-m-1}A') \right) (B')^m$$

$$\quad + B^m(I_n - AA' - BB' + A(u_{t-m-1}A')) \frac{\partial (B')^m}{\partial B_{ij}}$$

$$\quad - B^m J_{ij}(B')^{m+1} - B^{m+1}J_{ji}(B')^m,$$ (C.2)

where $\frac{\partial B^m}{\partial B_{ij}} = \sum_{n=0}^{m-1} B^n J_{ij} B^{m-1-n}$. The second order derivative of $G_t(\phi)$ is given by

$$\frac{\partial^2 G_t}{\partial A_{ij} \partial A_{kl}} = N_1 + N_1';$$

$$\frac{\partial^2 G_t}{\partial A_{ij} \partial B_{kl}} = N_2 + N_2';$$

$$\frac{\partial^2 G_t}{\partial B_{ij} \partial B_{kl}} = \sum_{q=3}^{8} (N_q + N_q'),$$ (C.3)

where

$$N_1 = \sum_{m=0}^{\infty} B^m [J_{ij}(u_{t-m-1} - I_n)J_{lk}](B')^m,$$
\[ N_2 = \sum_{m=0}^{\infty} \frac{\partial B^m}{\partial B_{kl}} [J_{ij}(u_{t-m-1} - I_n)A' + A(u_{t-m-1} - I_n)J_{ji}] (B')^m, \]

\[ N_3 = \sum_{m=0}^{\infty} \frac{\partial^2 B^m}{\partial B_{ij} \partial B_{kl}} (I_n - AA' - BB' + Au_{t-m-1}A') (B')^m, \]

\[ N_4 = \sum_{m=0}^{\infty} \frac{\partial B^m}{\partial B_{ij}} (I_n - AA' - BB' + Au_{t-m-1}A') \frac{\partial (B')^m}{\partial B_{kl}}, \]

\[ N_5 = \sum_{m=0}^{\infty} \frac{\partial B^m}{\partial B_{ij}} (-J_{kl}B' - BJ_{lk}) (B')^m, \]

\[ N_6 = \sum_{m=0}^{\infty} \frac{\partial B^m}{\partial B_{kl}} (-J_{ij}B' - BJ_{ji}) (B')^m, \]

\[ N_7 = \sum_{m=0}^{\infty} -\frac{\partial B^m}{\partial B_{kl}} J_{ij} (B')^{m+1}, \]

\[ N_8 = \sum_{m=0}^{\infty} -B^m J_{ij} \frac{\partial (B')^{m+1}}{\partial B_{kl}}. \]

### D Some numerical evidences

In this appendix, we generate one data sample from model (2.1) with sample size \( T = 5000 \), where \( u_t \) follows model (3.1) with

\[ A_0 = \begin{pmatrix} 0.5 & 0.4 \\ 0 & 0.2 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0.6 & 0 \\ 0.2 & 0.3 \end{pmatrix}, \quad \text{and} \quad \Sigma(x) = \begin{pmatrix} 1 + 1.5x^2 & 1.1x^2 \\ 1.1x^2 & 1 + 1.5x^3 \end{pmatrix}, \]

and \( e_t \) is a sequence of independent and identically distributed (i.i.d.) Wishart random matrices from \( \nu_0^{-1} \text{Wishart}(\nu_0, I_2) \) with \( \nu_0 = 10 \). To make a comparison, we also generate another data sample from model (2.1) under the same settings except \( \Sigma(x) = I_2 \). Fig. [D.1] plots the \( \rho_{rs}(j) \) for each gener-
ated data sample, where $\rho_{rs}(j)$ is the autocorrelation function of $y_{rs,t}$ at lag $j$, and $y_{rs,t}$ is the $(r,s)$th element of $y_t$. From this figure, we find that when $\Sigma_t$ is time variant (or invariant), $\rho_{rs}(j)$ decays slowly (or fast) with respect to $j$, exhibiting long memory (or short memory) patterns. This implies that the data sample of $y_t$ may exhibit a spurious long memory phenomenon, resulting from the structural change.

Figure D.1: Top panels: the plot of $\rho_{rs}(j)$ when $\Sigma_t$ is time variant. Bottom panels: the plot of $\rho_{rs}(j)$ when $\Sigma_t$ is time invariant.

Bibliography


