ADAPTIVE TESTS FOR BANDEDNESS OF HIGH-DIMENSIONAL COVARIANCE MATRICES

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Supplementary Material

The online Supplementary Materials include the proofs and additional simulation results. Section S1 provides the propositions of U-statistics, Sections S2, S3, and S4 prove Theorems 1, 2, and 3, respectively. Furthermore, all assisted lemmas and propositions needed by Theorems 1, 2, and 3 are proved in Section S5. Sections S6, S7, and S8 prove Propositions 1–3, respectively. Section S9 presents additional simulation study results, and Section S10 presents an example to illustrate the performance of our proposed adaptive estimator in Section 2.6 of the manuscript.

S1. Propositions and relevant proofs

Due to the construction of $\mathcal{U}(a)$, we have the following properties.

Proposition S1.1. *$\mathcal{U}(a)$ is location invariant. That is, for any vector $\mathbf{\Delta} \in \mathbb{R}^p$, if we replace $\{\mathbf{x}_i : i = 1, ..., n\}$ by $\{\mathbf{x}_i + \mathbf{\Delta} : i = 1, ..., n\}$, then $\mathcal{U}(a)$ is not changed.*

Proof. To prove $\mathcal{U}(a)$ is location invariant, we recall

$$\mathcal{U}(a) = \sum_{k<|j_1-j_2|<p} (P_{2a}^n)^{-1} \sum_{1\le i_1\neq\cdots\neq i_{2a}\le n} \prod_{l=1}^{a} (x_{i_{2l-1},j_1}x_{i_{2l-1},j_2} - x_{i_{2l-1},j_1}x_{i_{2l},j_2}).$$

We denote $\mathbf{\Delta} = (\Delta_1, \ldots, \Delta_p)^{\mathrm{T}}$, and consider $a = 1$ first.

$$\begin{aligned} & (x_{i_1,j_1} + \Delta_{j_1})(x_{i_1,j_2} + \Delta_{j_2}) - (x_{i_1,j_1} + \Delta_{j_1})(x_{i_2,j_2} + \Delta_{j_2}) \\ = \;& (x_{i_1,j_1}x_{i_1,j_2} - x_{i_1,j_1}x_{i_2,j_2}) + \Delta_{j_1}(x_{i_1,j_2} - x_{i_2,j_2}). \end{aligned}$$

It follows that

$$\begin{aligned} & \sum_{1\le i_1\neq i_2\le n} [(x_{i_1,j_1} + \Delta_{j_1})(x_{i_1,j_2} + \Delta_{j_2}) - (x_{i_1,j_1} + \Delta_{j_1})(x_{i_2,j_2} + \Delta_{j_2})] \\ & \qquad - \sum_{1\le i_1\neq i_2\le n} (x_{i_1,j_1}x_{i_1,j_2} - x_{i_1,j_1}x_{i_2,j_2}) \\ = \;& \sum_{1\le i_1\neq i_2\le n} \Delta_{j_1}(x_{i_1,j_2} - x_{i_2,j_2}) + \sum_{i=1}^{n} \Delta_{j_1}(x_{i_1,j_2} - x_{i_1,j_2}) \\ = \;& \Delta_{j_1} \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} (x_{i_1,j_2} - x_{i_2,j_2}) \\ = \;& 0. \end{aligned}$$

Thus, $\mathcal{U}(1)$ is location invariant. Next, we consider $a = 2$,

$$\sum_{1\le i_1\neq i_2\le i_3\neq i_4\le n} \Big\{ [(x_{i_1,j_1} + \Delta_{j_1})(x_{i_1,j_2} + \Delta_{j_2}) - (x_{i_1,j_1} + \Delta_{j_1})(x_{i_2,j_2} + \Delta_{j_2})]$$

$$\times [(x_{i_3,j_1}+\Delta_{j_1})(x_{i_3,j_2}+\Delta_{j_2})-(x_{i_3,j_1}+\Delta_{j_1})(x_{i_4,j_2}+\Delta_{j_2})]\Big\}$$

$$-\sum_{1\le i_1\neq i_2\le i_3\neq i_4\le n}\Big\{(x_{i_1,j_1}x_{i_1,j_2}-x_{i_1,j_1}x_{i_2,j_2})$$

$$\times [(x_{i_3,j_1}+\Delta_{j_1})(x_{i_3,j_2}+\Delta_{j_2})-(x_{i_3,j_1}+\Delta_{j_1})(x_{i_4,j_2}+\Delta_{j_2})]\Big\}=0.$$

□

Proposition S1.2. *For any integer a,* $\mathrm{E}[\mathcal{U}(a)]=\sum_{k<|j_1-j_2|<p}\sigma_{j_1j_2}^a=||\mathcal{E}||_a^a$. *Under $H_{k,0}$,* $\mathrm{E}[\mathcal{U}(a)]=0$.

S2. Proof of Theorem 1

Form Proposition S1.1, $\mathcal{U}(a)$ is location invariant. Without loss of generality, we assume $\mathrm{E}(\mathbf{x})=0$ in the entire proof of Theorem 1. Our proof strategy is motivated by that of Theorem 2.1 in He et al. (2021), which studied the problem of testing the diagonality of the covariance matrix, a special case of the considered bandedness test with $k=0$. Technically, we note that the general banded structure of the covariance matrix makes the analysis more involved. To prove Theorem 1, we first derive the variances and covariances of the U-statistics and then make use of the martingale central limit theorem and U-statistic theory to prove the asymptotic joint normality of the U-statistics. The detailed results are as follows in Lemmas 1, 2, 3, 4, 5, and 6.

Lemma 1. *Under the conditions in Theorem 1, for any finite integer a,*

$$\sigma^2(a) = (P_a^n)^{-1} a! \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} (\Pi_{j_1,j_2,j_3,j_4})^a (1+o(1)),$$

which is of order $\Theta(p^2 n^{-a})$. In addition, define $\tilde{\mathcal{U}}^*(a) = \mathcal{U}(a) - \tilde{\mathcal{U}}(a)$, we have $\tilde{\mathcal{U}}^*(a) = o_p(\sigma(a))$.

Proof. See Section S5.1. □

Lemma 2. *Under the conditions in Theorem 1, for any finite integers $a \neq b$, as $n, p \to \infty$,*

$$\mathrm{Cov}\left[\frac{\mathcal{U}(a)}{\sigma(a)}, \frac{\mathcal{U}(b)}{\sigma(b)}\right] = o(1).$$

Proof. See Section S5.2. □

Lemmas 1–2 show that the variance of $[\mathcal{U}(a_1)/\sigma(a_1), \cdots, \mathcal{U}(a_m)/\sigma(a_m)]^{\mathrm{T}}$ converges to I_m. To finish the proof of Theorem 1, we need to prove that the joint asymptotic distribution of the statistic is normal. By Cramér-Wold theorem, it is equivalent to prove that any fixed linear combination of $\left[\frac{\mathcal{U}(a_1)}{\sigma(a_1)}, \cdots, \frac{\mathcal{U}(a_m)}{\sigma(a_m)}\right]^{\mathrm{T}}$ converges to normal. From Lemma 1, it is obvious that $\tilde{\mathcal{U}}^*(a)/\sigma(a) \xrightarrow{p} 0$ for any finite a. By Slutsky's theorem, it is suffices to prove that any fixed linear combination of $\left[\frac{\tilde{\mathcal{U}}(a_1)}{\sigma(a_1)}, \cdots, \frac{\tilde{\mathcal{U}}(a_m)}{\sigma(a_m)}\right]^{\mathrm{T}}$ converges to

normal. That is , we need to prove

$$Z_n := \sum_{r=1}^{m} t_r \frac{\tilde{\mathcal{U}}(a_r)}{\sigma(a_r)} \xrightarrow{D} \mathcal{N}(0,1), \tag{S2.1}$$

with any constants $t_1,\ldots,t_m$ satisfy $\sum_{r=1}^{m} t_r^2 = 1$.

We apply the martingale central limit theorem to prove (S2.1). Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_h = \sigma\{\mathbf{x}_1, ..., \mathbf{x}_h\}$ for $h = 1, \cdots, n$, be a sequence of $\sigma-$field generated by the data sequence. Let $\mathrm{E}_h(\cdot)$ denote the conditional expectation with respect to $\mathcal{F}_h$. Rewrite $Z_n - \mathrm{E}Z_n = \sum_{h=1}^{n} D_{n,h}$ with $D_{n,h} = (\mathrm{E}_h - \mathrm{E}_{h-1})Z_n$. Then $\{D_{n,h}, 1 \le h \le n\}$ is a martingale difference sequence with respect to the $\sigma-$fields $\{\mathcal{F}_t\}_{t=0}^{\infty}$. Let $\pi_{n,h}^2 = \mathrm{E}_{h-1}(D_{n,h}^2)$. To prove (S2.1), it is sufficient to show

$$\frac{\sum_{h=1}^{n} \pi_{n,h}^2}{\mathrm{Var}(Z_n)} \xrightarrow{P} 1, \quad \frac{\sum_{h=1}^{n} \mathrm{E}[D_{n,h}^4]}{\mathrm{Var}^2(Z_n)} \to 0. \tag{S2.2}$$

From Lemmas 1-2 and $\sum_{r=1}^{m} t_r^2 = 1$, it is obvious that $\mathrm{var}(Z_n) \to 1$. Moreover, the following Lemma 3 shows $\mathrm{E}(\sum_{h=1}^{n} \pi_{n,h}^2) = \mathrm{Var}(Z_n)$. Thus, to prove (S2.2), it suffices to show

$$\mathrm{Var}\left(\sum_{h=1}^{n} \pi_{n,h}^2 \right) = o(1) \quad \text{and} \quad \sum_{h=1}^{n} \mathrm{E}(D_{n,h}^4) = o(1). \tag{S2.3}$$

In order to present $\pi_{n,h}^2$ and $D_{n,h}$, we define $A_{n,h,a_r} = (\mathrm{E}_h - \mathrm{E}_{h-1})\{\tilde{\mathcal{U}}(a_r)/\sigma(a_r)\}$,

for $r = 1, \ldots, m$. Then, $D_{n,h}$ and $\pi^2_{n,h}$ can be written as

$$D_{n,h} = \sum_{r=1}^{m} t_r A_{n,h,a_r}, \quad \pi^2_{n,h} = \sum_{r_1=1}^{m} \sum_{r_2=1}^{m} t_{r_1} t_{r_2} \mathrm{E}_{h-1}(A_{n,h,a_{r_1}} A_{n,h,a_{r_2}}).$$

We give the explicit form of $A_{n,h,a}$ in Lemma 4.

Lemma 3. *Under the conditions of Theorem 1,* $\mathrm{E}(\sum_{h=1}^{n} \pi^2_{n,h}) = \mathrm{var}(Z_n)$.

Proof. See Section S5.3. □

Lemma 4. *For finite integer a, we have*

$$A_{n,h,a} = \begin{cases} 0, & h < a \\ \frac{a}{\sigma(a) P_a^n} \sum_{1 \le i_1 \ne \cdots \ne i_{a-1} \le h-1} \sum_{k < |j_1 - j_2| < p} (x_{h,j_1} x_{h,j_2}) \prod_{t=1}^{a-1} x_{i_t,j_1} x_{i_t,j_2}, & h \ge a. \end{cases} \tag{S2.4}$$

Proof. See Section S5.4. □

The forms of $D_{n,h}$ and $\pi^2_{n,h}$ can be obtained with the form of $A_{n,h,a}$ in Lemma 4. With these, we can prove the following Lemmas.

Lemma 5. *Under the conditions in Theorem 1,* $\mathrm{Var}\left(\sum_{h=1}^{n} \pi^2_{n,h} \right) = o(1)$.

Proof. See Section S5.5. □

Lemma 6. *Under the conditions in Theorem 1,* $\sum_{h=1}^{n} \mathrm{E}(D^4_{n,h}) = o(1)$.

Proof. See Section S5.6. □

Based on Lemmas 1–6, by Heyde and Brown (1970), we have as $n, p \to$

∞,

$$\begin{aligned} & \sup_t \left| P(Z_n \le t) - \Phi(t) \right| \\ \le \quad & C\left\{ \mathrm{E}\left[\frac{\sum_{h=1}^n \mathrm{E}_{h-1}(D_{n,h}^2)}{\mathrm{Var}(Z_n) - 1} \right]^2 + \frac{\sum_{h=1}^n \mathrm{E}(D_{n,h}^4)}{\mathrm{Var}^2(Z_n)} \right\}^{1/5} \\ \to \quad & 0. \end{aligned}$$

Then (S2.1) has been proved. That is, Theorem 1 is proved.

S3. Proof of Theorem 2

Without loss of generality, we assume $\mathrm{E}(\mathbf{x}) = \mathbf{0}$, and decompose $\hat{\sigma}^2(a) = \hat{\sigma}_1^2(a) + \hat{\sigma}_2^2(a)$, where

$$\hat{\sigma}_1^2(a) = 2(P_a^n)^{-2} a! \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p \\ |j_1-j_3|\le k, |j_2-j_4|\le k}} \sum_{1\le i_1\ne\cdots\ne i_a\le n} \prod_{l=1}^{a} (x_{i_l,j_1} x_{i_l,j_2} x_{i_l,j_3} x_{i_l,j_4})$$

and $\hat{\sigma}_2^2(a) = \hat{\sigma}^2(a) - \hat{\sigma}_1^2(a)$. We only need to show $\hat{\sigma}_1^2(a)$ is of larger order than $\hat{\sigma}_2^2(a)$. This will be shown in Lemma 7.

Lemma 7. *Under the conditions of Theorem 2,* $\hat{\sigma}_1^2(a)/\mathrm{E}\{\hat{\sigma}_1^2(a)\} \xrightarrow{P} 1$ *and* $\hat{\sigma}_2^2(a)/\mathrm{E}\{\hat{\sigma}_1^2(a)\} \xrightarrow{P} 0$.

Proof. See Section S5.7. □

S4. Proof of Theorem 3

Proof. In order to facilitate the proof, we first introduce some notations. For a given $j_1 \in \{1, \ldots, p\}$, we define the following two-tuple sets:

$$J_{j_1} = \{(j_1, j_2) : \sigma_{j_1 j_2} \neq 0,\ k < |j_1 - j_2| < p, j_2 = 1, \ldots, p\},$$

$$J_{j_1}^c = \{(j_1, j_2) : \sigma_{j_1 j_2} = 0,\ k < |j_1 - j_2| < p, j_2 = 1, \ldots, p\}.$$

Recall the definition of J_A in the main article, we have $J_A = \cup_{j_1=1,\ldots,p} J_{j_1}$. Let $J_A^c = \cup_{j_1=1,\ldots,p} J_{j_1}^c$, then $J_A \cup J_A^c = \{k < |j_1 - j_2| < p, j_1, j_2 = 1, \ldots, p\}$. Furthermore, we denote

$$H(\mathbf{i}_{(a+c)}, j_1, j_2) = F(a,c) \sum_{\mathbf{i}_{(a+c)} \in \mathcal{P}(n,a+c)} \prod_{l=1}^{a-c} (x_{i_l, j_1} x_{i_l, j_2}) \prod_{s=a-c+1}^{a} x_{i_s, j_1} \prod_{t=a+1}^{a+c} x_{i_t, j_2},$$

where $\mathbf{i}_{(a+c)} = \{(i_1, \ldots, i_{a+c}) : 1 \leq i_1 \neq \cdots \neq i_{a+c} \leq n\}$ and $F(a,c) = \binom{a}{c}(-1)^c (P_{a+c}^n)^{-1}$. To complete the proof, we decompose $\mathcal{U}(a) = \mathcal{U}_{1,1}(a) + \mathcal{U}_{1,2}(a) + \mathcal{U}_2(a)$, where

$$\mathcal{U}_{1,1}(a) = \sum_{(j_1,j_2) \in J_A^c} H(\mathbf{i}_{(a)}, j_1, j_2),$$

$$\mathcal{U}_{1,2}(a) = \sum_{(j_1,j_2) \in J_A^c} \sum_{c=1}^{a} H(\mathbf{i}_{(a+c)}, j_1, j_2),$$

$$\mathcal{U}_2(a) \;=\; \sum_{(j_1,j_2)\in J_A} \sum_{c=0}^{a} H(\mathbf{i}_{(a+c)}, j_1, j_2),$$

Lemma 8. *Under Conditions 1, 4 and 5 of Theorem 3, for any finite integer* a*, as* $n, p \to \infty$*,*

$$\sigma_A^2(a) \simeq 2(P_a^n)^{-1} a! \kappa^a \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p \\ |j_1-j_3|\le k, |j_2-j_4|\le k}} \sigma^a_{j_1 j_3} \sigma^a_{j_2 j_4},$$

with order of $\Theta(n^{-a}p^2)$*.*

Proof. See Section S5.8. □

Lemma 9. *Under Conditions 1, 4 and 5 of Theorem 3, for any finite integers* $a \neq b$*, as* $n, p \to \infty$*,*

$$\mathrm{Cov}\left[\frac{\mathcal{U}(a)}{\sigma_A(a)}, \frac{\mathcal{U}(b)}{\sigma_A(b)}\right] = o(1).$$

Proof. See Section S5.9. □

Lemma 8 and Lemma 9 show that the asymptotic variances and covariances of $[\mathcal{U}(a_1), \cdots, \mathcal{U}(a_m)]^{\mathrm{T}}$ under the specific local alternative hypothesis $H_{k,A} : \mathbf{\Sigma} = \mathbf{\Sigma}_A$. Lemma 8 shows that $\mathcal{U}_{1,1}(a)$ is the leading order term in the variance of $\mathcal{U}(a)$. By Cramér-Wold theorem, Slutsky's theorem, Lemma 8 and Lemma 9, it is equivalent to prove that any fixed linear combination of

$[\mathcal{U}_{1,1}(a_1)/\sigma_A(a_1), \cdots, \mathcal{U}_{1,1}(a_m)/\sigma_A(a_m)]^{\mathrm{T}}$ converges to normal distribution. That is, we need to prove

$$Z_n := \sum_{r=1}^{m} t_r \frac{\mathcal{U}_{1,1}(a_r)}{\sigma_A(a_r)} \xrightarrow{D} \mathcal{N}(0,1), \tag{S4.5}$$

with any constants t_1,....,t_m satisfy $\sum_{r=1}^{m} t_r^2 = 1$. Next, we apply the martingale central limit theorem to prove (S4.5). Similarly to Section S2, we define $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_h = \sigma\{\mathbf{x}_1, ..., \mathbf{x}_h\}$ for $h = 1, \cdots, n$, be a sequence of $\sigma-$field generated by the data sequence. Let $\mathrm{E}_h(\cdot)$ denote the conditional expectation with respect to $\mathcal{F}_h$. Rewrite $Z_n - \mathrm{E}Z_n = \sum_{h=1}^{n} D_{n,h}$ with $D_{n,h} = (\mathrm{E}_h - \mathrm{E}_{h-1})Z_n = \sum_{r=1}^{m} t_r A_{n,h,a}$ with

$$A_{n,h,a} = \begin{cases} 0, & h < a \\ \frac{a}{\sigma(a) P_a^n} \sum\limits_{1 \le i_1 \ne \cdots \ne i_{a-1} \le h-1} \sum\limits_{(j_1,j_2) \in J_A^c} (x_{h,j_1} x_{h,j_2}) \prod\limits_{t=1}^{a-1} x_{i_t,j_1} x_{i_t,j_2}, & h \ge a. \end{cases}$$

Now, $\{D_{n,h}, 1 \le h \le n\}$ is a martingale difference sequence with respect to the $\sigma-$fields $\{\mathcal{F}_t\}_{t=0}^{\infty}$. Let $\pi_{n,h}^2 = \mathrm{E}_{h-1}(D_{n,h}^2)$. To prove (S4.5), it is sufficient to prove the two following lemmas.

Lemma 10. *Under the conditions of Theorem 3,* $\mathrm{Var}\left(\sum_{h=1}^{n} \pi_{n,h}^2\right) = o(1)$.

Proof. See Section S5.10. □

Lemma 11. *Under the conditions of Theorem 3,* $\sum_{h=1}^{n} \mathrm{E}(D_{n,h}^4) = o(1)$.

Proof. See Section S5.11. □

□

S5. Proofs of Assisted Lemmas and Proposition in Theorem 1–Theorem 3

In this section, we provide the proofs of all the lemmas used in Section S2–Section S4. We first introduce some notations for facilitate the presentation of the proofs. In the whole paper, we use sign C denotes different constants without loss of generality.

S5.1 Proof of Lemma 1

Proof. Before proving Lemma 1, we define some notations to facilitate discussion. For any $a < n$, let $\mathcal{P}(n, a)$ denotes the collection of a-tuples $\mathbf{i} = (i_1, ..., i_a)$ satisfying $1 \leq i_1 \neq \cdots \neq i_a \leq n$. Given $\mathbf{i} \in \mathcal{P}(n, a)$, we define $\{\mathbf{i}\}$ as the corresponding set containing the elements of $\mathbf{i}$ without order, that is, $\{\mathbf{i}\} = \{i_1, \cdots, i_a\}$. Moreover, $\text{card}(\{\mathbf{i}\})$ denotes the size of the set $\{\mathbf{i}\}$.

Under $H_{k,0}$, $\mathrm{E}\{\mathcal{U}(a)\} = 0$ and $\mathrm{Var}\{\mathcal{U}(a)\} = \mathrm{E}\{\mathcal{U}(a)^2\}$. Recall the

definition of $\mathcal{U}(a)$ in (2.2) of the manuscript, it follows that

$$\begin{aligned}\mathcal{U}^2(a) &= \sum_{k<|j_1-j_2|<p}\sum_{c_1=0}^{a}(-1)^{c_1}\binom{a}{c_1}\frac{1}{P^n_{a+c_1}}\sum_{1\le i_1\neq\cdots\neq i_{a+c_1}\le n}\\ &\quad\sum_{k<|j_3-j_4|<p}\sum_{c_2=0}^{a}(-1)^{c_2}\binom{a}{c_2}\frac{1}{P^n_{a+c_2}}\sum_{1\le \tilde{i}_1\neq\cdots\neq \tilde{i}_{a+c_2}\le n}\\ &\quad\times\prod_{l=1}^{a-c_1}(x_{i_l,j_1}x_{i_l,j_2})\prod_{s=a-c_1+1}^{a}x_{i_s,j_1}\prod_{t=a+1}^{a+c_1}x_{i_t,j_2}\prod_{\tilde{l}=1}^{a-c_2}(x_{\tilde{i}_{\tilde{l}},j_3}x_{\tilde{i}_{\tilde{l}},j_4})\\ &\quad\times\prod_{\tilde{s}=a-c_2+1}^{a}x_{\tilde{i}_{\tilde{s}},j_3}\prod_{\tilde{t}=a+1}^{a+c_2}x_{\tilde{i}_{\tilde{t}},j_4},\end{aligned}$$

and

$$\mathrm{E}[\mathcal{U}^2(a)]=\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}}\sum_{\substack{0\le c_1,c_2\le a\\ \mathbf{i}\in\mathcal{P}(n,a+c_1)\\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}}F(c_1,c_2,a)\times Q(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4),\quad\text{(S5.6)}$$

where

$$F(c_1,c_2,a)=(-1)^{c_1+c_2}\binom{a}{c_1}\binom{a}{c_2}(P^n_{a+c_1}P^n_{a+c_2})^{-1},\quad\text{(S5.7)}$$

$$\begin{aligned}Q(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4) &= \mathrm{E}\Big[\prod_{l=1}^{a-c_1}x_{i_l,j_1}x_{i_l,j_2}\prod_{s=a-c_1+1}^{a}x_{i_s,j_1}\prod_{t=a+1}^{a+c_1}x_{i_t,j_2}\\ &\quad\times\prod_{\tilde{l}=1}^{a-c_2}x_{\tilde{i}_{\tilde{l}},j_3}x_{\tilde{i}_{\tilde{l}},j_4}\prod_{\tilde{s}=a-c_2+1}^{a}x_{\tilde{i}_{\tilde{s}},j_3}\prod_{\tilde{t}=a+1}^{a+c_2}x_{\tilde{i}_{\tilde{t}},j_4}\Big]\quad\text{(S5.8)}\end{aligned}$$

with

$$\begin{aligned}\mathbf{i} &= \{i_1, ..., i_{a-c_1}, i_{a-c_1+1}, ..., i_a, i_{a+1}, ..., i_{a+c_1}\},\\ \tilde{\mathbf{i}} &= \{\tilde{i}_1, ..., \tilde{i}_{a-c_2}, \tilde{i}_{a-c_2+1}, ..., \tilde{i}_a, \tilde{i}_{a+1}, ..., \tilde{i}_{a+c_2}\}.\end{aligned}$$

For evaluating $\mathrm{Var}\{\mathcal{U}(a)\}$, we examine $Q(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4)$ in (S5.8) now.

Case 1.When $\{\mathbf{i}\} \neq \{\tilde{\mathbf{i}}\}$. Without loss of generality, we assume that there exists an index $i \in \{\mathbf{i}\}$ and $i \notin \{\tilde{\mathbf{i}}\}$, then $Q(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4)$ takes one of the two following forms:

$$\begin{aligned}Q(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) &= \mathrm{E}(x_{i,j_1}) \times \mathrm{E}(\text{other terms}) = 0,\\ Q(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) &= \mathrm{E}(x_{i,j_1} x_{i,j_2}) \times \mathrm{E}(\text{other terms}).\end{aligned}$$

Then under $H_{k,0}$,

$$\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}} \sum_{\substack{0\le c_1,c_2\le a\\ \mathbf{i}\in\mathcal{P}(n,a+c_1)\\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1, c_2, a) \times Q(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4)\mathbf{1}_{\{\{\mathbf{i}\}\neq\{\tilde{\mathbf{i}}\}\}} = 0.$$

Case 2. When $\{\mathbf{i}\} = \{\tilde{\mathbf{i}}\}$. It is apparent that $c_1 = c_2$. We denote $c_1 = c_2 = c$, and decompose $\{\mathbf{i}\}$ and $\{\tilde{\mathbf{i}}\}$ into three disjoint subsets for further

discussing. In details, there exists

$$\begin{aligned}
\{\mathbf{i}\}_1 &= \{i_1, ..., i_{a-c}\}, \quad \{\mathbf{i}\}_2 = \{i_{a-c+1}, ..., i_a\}, \quad \{\mathbf{i}\}_3 = \{i_{a+1}, ..., i_{a+c}\}, \\
\{\tilde{\mathbf{i}}\}_1 &= \{\tilde{i}_1, ..., \tilde{i}_{a-c}\}, \quad \{\tilde{\mathbf{i}}\}_2 = \{\tilde{i}_{a-c+1}, ..., \tilde{i}_a\}, \quad \{\tilde{\mathbf{i}}\}_3 = \{\tilde{i}_{a+1}, ..., \tilde{i}_{a+c}\},
\end{aligned}$$

satisfy $\{\mathbf{i}\} = \overset{3}{\underset{k=1}{\cup}} \{\mathbf{i}\}_k$ and $\{\tilde{\mathbf{i}}\} = \overset{3}{\underset{k=1}{\cup}} \{\tilde{\mathbf{i}}\}_k$.

Case 2.1. If $c = 0$, it is obvious that $\{\mathbf{i}\}_1 = \{\tilde{\mathbf{i}}\}_1 \neq \emptyset$ and $\{\mathbf{i}\}_2 = \{\mathbf{i}\}_3 = \{\tilde{\mathbf{i}}\}_2 = \{\tilde{\mathbf{i}}\}_3 = \emptyset$, then

$$Q(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4)\mathbf{1}_{\{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\}, c=0\}} = \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^a.$$

Thus

$$\begin{aligned}
&\sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \sum_{\substack{0\le c_1,c_2\le a \\ \mathbf{i}\in\mathcal{P}(n,a+c_1) \\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1, c_2, a) \times Q(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4)\mathbf{1}_{\{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\}, c=0\}} \\
= \; & (P_a^n)^{-2} \sum_{\mathbf{i}\in\mathcal{P}(n,a)} a! \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^a \\
= \; & (P_a^n)^{-1} a! \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^a. \qquad \text{(S5.9)}
\end{aligned}$$

Now, we will show that

$$\sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^a = \Theta(p^2). \qquad \text{(S5.10)}$$

For calculating the summation over $\{(j_1, j_2, j_3, j_4) : k < |j_1 - j_2| < p, k < |j_3 - j_4| < p\}$, it is suffices to discuss two sub-cases as follows: (a). $|j_s - j_t| \leq k$ and $|j_{s'} - j_{t'}| \leq k$, where $s \neq s^{'} \neq t \neq t^{'} \in \{1, \ldots, 4\}$; (b). $|j_s - j_t| > k$ and $|j_s - j_{t'}| > k$, where $s \neq t \neq t^{'} \in \{1, \ldots, 4\}$.

(a). Without loss of generality, we assume $|j_1 - j_3| \leq k$ and $|j_2 - j_4| \leq k$. We define $\kappa_1 = \min\{|j_1 - j_2|, |j_3 - j_2|, |j_1 - j_4|, |j_3 - j_4|\}$ as the distance between the tuples (j_1, j_3) and (j_2, j_4), and the constant

$$K_0 = -(2 + \epsilon)(4 + \mu)(\log p)/(\epsilon \log \delta),$$

for some small positive constants μ, ϵ and δ in Condition 2. Under Conditions 1-2, when $\kappa_1 > K_0$, by the α-mixing inequality (see Eq. (3.5) in Guyon (1995)), we have

$$\begin{aligned} \mathrm{E}(x_{i,j_1} x_{i,j_2} x_{i,j_3} x_{i,j_4}) &= \mathrm{cov}(x_{i,j_1} x_{i,j_3}, x_{i,j_2} x_{i,j_4}) + \mathrm{E}(x_{i,j_1} x_{i,j_3}) \mathrm{E}(x_{i,j_2} x_{i,j_4}) \\ &\geq \Theta(1) - |\mathrm{cov}(x_{i,j_1} x_{i,j_3}, x_{i,j_2} x_{i,j_4})| \\ &\geq \Theta(1) - C\delta^{\frac{K_0 \epsilon}{2+\epsilon}} = \Theta(1); \end{aligned}$$

when $\kappa_1 \leq K_0$, $\mathrm{E}(x_{i,j_1} x_{i,j_2} x_{i,j_3} x_{i,j_4}) = O(1)$, and the total number of

(j_1, j_2, j_3, j_4) tuples are pK_0k^2. Thus,

$$\begin{aligned}
&\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}} \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^a \mathbf{1}_{\left\{\substack{\{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},c=0,\\ |j_1-j_3|>k,|j_1-j_4|>k\}}\right\}} \\
= &\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}} \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^a \mathbf{1}_{\left\{\substack{\{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},c=0,\\ |j_1-j_3|>k,|j_1-j_4|>k\}}\right\}} (\mathbf{1}_{\{\kappa_1>K_0\}} + \mathbf{1}_{\{\kappa_1\le K_0\}}) \\
= &\ \Theta(p^2k^2) + O(pK_0k^2) = \Theta(p^2). \qquad \text{(S5.11)}
\end{aligned}$$

(b). Without loss of generality, we assume $|j_1 - j_3| > k$ and $|j_1 - j_4| > k$. Under H_0, there is at least one of $\mathrm{E}(x_{i,\tilde{j}_1}x_{i,\tilde{j}_2}) = 0$ or $\mathrm{E}(x_{i,\tilde{j}_3}x_{i,\tilde{j}_4}) = 0$ holds, where $(\tilde{j}_1, \tilde{j}_2, \tilde{j}_3, \tilde{j}_4)$ is the ordered version of (j_1, j_2, j_3, j_4) satisfying $\tilde{j}_1 \le \tilde{j}_2 \le \tilde{j}_3 \le \tilde{j}_4$. Denote $\kappa_2 = \max\{\tilde{j}_2 - \tilde{j}_1, \tilde{j}_3 - \tilde{j}_2, \tilde{j}_4 - \tilde{j}_3\}$. When $\kappa_2 > K_0$, we have

$$\begin{aligned}
|\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})| = |\mathrm{E}(x_{i,\tilde{j}_1}x_{i,\tilde{j}_2}x_{i,\tilde{j}_3}x_{i,\tilde{j}_4})| &= |\mathrm{cov}(x_{i,\tilde{j}_1}, x_{i,\tilde{j}_2}x_{i,\tilde{j}_3}x_{i,\tilde{j}_4})| \\
&= |\mathrm{cov}(x_{i,\tilde{j}_1}x_{i,\tilde{j}_2}, x_{i,\tilde{j}_3}x_{i,\tilde{j}_4})| \\
&= |\mathrm{cov}(x_{i,\tilde{j}_1}x_{i,\tilde{j}_2}x_{i,\tilde{j}_3}, x_{i,\tilde{j}_4})| \\
&\le C\delta^{\frac{K_0\epsilon}{2+\epsilon}} = O(1)p^{-(4+\mu)};
\end{aligned}$$

when $\kappa_2 \le K_0$, $|\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})| = O(1)$, and the total number of

(j_1, j_2, j_3, j_4) tuples are pK_0^3. Thus,

$$\begin{aligned}
&\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}} \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^a \mathbf{1}_{\{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},c=0,|j_1-j_3|>k,|j_1-j_4|>k\}} \\
= &\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}} \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^a \mathbf{1}_{\left\{\substack{\{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},c=0,\\ |j_1-j_3|>k,|j_1-j_4|>k\}}\right\}} (\mathbf{1}_{\{\kappa_2>K_0\}}+\mathbf{1}_{\{\kappa_2\le K_0\}}) \\
= &\ O(1)p^4p^{-(4+\mu)} + O(1)pK_0^3 = o(p^2). \qquad \text{(S5.12)}
\end{aligned}$$

Combining the equations (S5.11) and (S5.12), (S5.10) is proved.

Case 2.2. If $1 \le c \le a-1$, it follows that $\{\mathbf{i}\}_1, \{\mathbf{i}\}_2, \{\mathbf{i}\}_3, \{\tilde{\mathbf{i}}\}_1, \{\tilde{\mathbf{i}}\}_2, \{\tilde{\mathbf{i}}\}_3$ are all nonempty. There are three sub-cases need to be considered under this case.

Case 2.2.1. When $\{\mathbf{i}\}_1 = \{\tilde{\mathbf{i}}\}_1$. As $\{\mathbf{i}\} = \{\tilde{\mathbf{i}}\}$, we have $\{\mathbf{i}\}_2 \bigcup \{\mathbf{i}\}_3 = \{\tilde{\mathbf{i}}\}_2 \bigcup \{\tilde{\mathbf{i}}\}_3$.

(a). When $\{\mathbf{i}\}_2 \bigcap \{\tilde{\mathbf{i}}\}_2 \neq \emptyset$ and $\{\mathbf{i}\}_2 \bigcap \{\tilde{\mathbf{i}}\}_3 \neq \emptyset$. Without loss of generality, we suppose $\mathrm{card}(\{\mathbf{i}\}_2 \bigcap \{\tilde{\mathbf{i}}\}_2) = q$ for $q = 1, ..., c-1$. It is obvious that $\mathrm{card}(\{\mathbf{i}\}_2 \bigcap \{\tilde{\mathbf{i}}\}_3) = c-q$, $\mathrm{card}(\{\mathbf{i}\}_3 \bigcap \{\tilde{\mathbf{i}}\}_2) = c-q$, and $\mathrm{card}(\{\mathbf{i}\}_3 \bigcap \{\tilde{\mathbf{i}}\}_3) = q$. Then

$$\begin{aligned}
\text{(S5.8)} \ = \ &\{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^{a-c}\{\mathrm{E}(x_{i,j_1}x_{i,j_3})\mathrm{E}(x_{i,j_2}x_{i,j_4})\}^q \\
&\times\{\mathrm{E}(x_{i,j_2}x_{i,j_3})\mathrm{E}(x_{i,j_1}x_{i,j_4})\}^{c-q}.
\end{aligned}$$

As $|j_1 - j_2| > k$ and $|j_3 - j_4| > k$, it is obvious that at least one of

$|j_1 - j_3| > k$, $|j_2 - j_4| > k$, $|j_2 - j_3| > k$, $|j_1 - j_4| > k$ holds. Thus, under H_0, we have

$$\begin{aligned} & \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \sum_{\substack{0\le c_1,c_2\le a \\ \mathbf{i}\in\mathcal{P}(n,a+c_1) \\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1,c_2,a)\times Q(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4) \\ & \qquad\qquad \times \mathbf{1}_{\left\{\substack{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},1\le c\le a-1, \\ \{\mathbf{i}\}_1=\{\tilde{\mathbf{i}}\}_1,\{\mathbf{i}\}_2\bigcap\{\tilde{\mathbf{i}}\}_2\neq\emptyset,\{\mathbf{i}_2\}\bigcap\{\tilde{\mathbf{i}}\}_3\neq\emptyset}\right\}} \\ = \; & 0. \end{aligned} \tag{S5.13}$$

(b). When $\{\mathbf{i}\}_2\bigcap\{\tilde{\mathbf{i}}\}_2 = \emptyset$ or $\{\mathbf{i}\}_2\bigcap\{\tilde{\mathbf{i}}\}_3 = \emptyset$. Without loss of generality, we assume $\{\mathbf{i}\}_3 = \{\tilde{\mathbf{i}}\}_3$, it means that $\{\mathbf{i}\}_2 = \{\tilde{\mathbf{i}}\}_2$ and $\{\mathbf{i}\}_3 = \{\tilde{\mathbf{i}}\}_3$. It follows that

$$(\text{S5.8}) = \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^{a-c}\{\mathrm{E}(x_{i,j_1}x_{i,j_3})\mathrm{E}(x_{i,j_2}x_{i,j_4})\}^{c},$$

which is nonzero when $|j_1 - j_3| \le k$ and $|j_2 - j_4| \le k$. Thus

$$\begin{aligned} & \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \sum_{\substack{0\le c_1,c_2\le a \\ \mathbf{i}\in\mathcal{P}(n,a+c_1) \\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1,c_2,a)\times Q(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\left\{\substack{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},1\le c\le a-1, \\ \{\mathbf{i}\}_1=\{\tilde{\mathbf{i}}\}_1,\{\mathbf{i}\}_2=\{\tilde{\mathbf{i}}\}_2,\{\mathbf{i}\}_3=\{\tilde{\mathbf{i}}\}_3}\right\}} \\ = \; & \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p \\ |j_1-j_3|\le k \\ |j_2-j_4|\le k}} \sum_{\substack{1\le c\le a-1 \\ \mathbf{i}\in\mathcal{P}(n,a+c)}} \binom{a}{c}^2 (P_{a+c}^n)^{-2}(a-c)!c!c! \\ & \qquad \times \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^{a-c}\{\mathrm{E}(x_{i,j_1}x_{i,j_3})\mathrm{E}(x_{i,j_2}x_{i,j_4})\}^{c} \end{aligned}$$

$$= \sum_{c=1}^{a-1} O(1)k^2p^2n^{-(a+c)}$$
$$= o(p^2n^{-a}). \quad \text{(S5.14)}$$

Combining the results in (S5.13) and (S5.14), we have

$$\sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \sum_{\substack{0\le c_1,c_2\le a \\ \mathbf{i}\in\mathcal{P}(n,a+c_1) \\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1,c_2,a)\times Q(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},1\le c\le a-1,\{\mathbf{i}\}_1=\{\tilde{\mathbf{i}}\}_1\}}$$
$$= o(p^2n^{-a}). \quad \text{(S5.15)}$$

Case 2.2.2. When $\{\mathbf{i}\}_1 \neq \{\tilde{\mathbf{i}}\}_1$ and $\{\mathbf{i}\}_1 \bigcap \{\tilde{\mathbf{i}}\}_1 \neq \emptyset$.

We assume that there exists an index $i_1 \in \{\mathbf{i}\}_1 \bigcap \{\tilde{\mathbf{i}}\}_1$, another index $i_2 \in \{\mathbf{i}\}_1$, $i_2 \notin \{\tilde{\mathbf{i}}\}_1$. As $\{\mathbf{i}\} = \{\tilde{\mathbf{i}}\}$, then $i_2 \in \{\tilde{\mathbf{i}}\}_2 \bigcup \{\tilde{\mathbf{i}}\}_3$. Without loss of generality, we assume $i_2 \in \{\tilde{\mathbf{i}}\}_2$, then

$$\text{(S5.8)} = \mathrm{E}(x_{i_1,j_1}x_{i_1,j_2}x_{i_1,j_3}x_{i_1,j_4})\mathrm{E}(x_{i_2,j_1}x_{i_2,j_2}x_{i_2,j_3})\mathrm{E}(\text{other terms}).$$

Similarly to the discussion in Case 2.1, we consider two sub-cases when $k < |j_1 - j_2| < p, k < |j_3 - j_4| < p$: (a). $|j_s - j_t| \le k$ and $|j_{s'} - j_{t'}| \le k$, where $s \neq s^{'} \neq t \neq t^{'} \in \{1,\ldots,4\}$; (b). $|j_s - j_t| > k$ and $|j_s - j_{t'}| > k$, where $s \neq t \neq t^{'} \in \{1,\ldots,4\}$.

(a). Without loss of generality, we assume $|j_1 - j_3| \le k$ and $|j_2 - j_4| \le k$.

Define $\kappa_3 = \min\{|j_1 - j_2|, |j_3 - j_2|\}$, if $\kappa_3 > K_0$,

$$\begin{aligned} |(\text{S5.8})| \leq C|\mathrm{E}(x_{i_2,j_1}x_{i_2,j_2}x_{i_2,j_3})| &= C|\mathrm{cov}(x_{i_2,j_2}, x_{i_2,j_1}x_{i_2,j_3})| \\ &\leq C\delta^{\frac{K_0\epsilon}{2+\epsilon}} = O(1)p^{-(4+\mu)}, \end{aligned}$$

if $\kappa_3 \leq K_0$, then $|(\text{S5.8})| = O(1)$, and the total number of (j_1, j_2, j_3, j_4) tuples are pK_0k^2. Thus,

$$\begin{aligned} &\sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p \\ |j_1-j_3|\leq k \\ |j_2-j_4|\leq k}} \sum_{\substack{0\leq c_1,c_2\leq a \\ \mathbf{i}\in\mathcal{P}(n,a+c_1) \\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1,c_2,a) \times Q(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4)\mathbf{1}_{\left\{\substack{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},1\leq c\leq a-1, \\ \{\mathbf{i}\}_1\neq\{\tilde{\mathbf{i}}\}_1,\{\mathbf{i}\}_1\bigcap\{\tilde{\mathbf{i}}\}_1\neq\emptyset}\right\}} \\ \leq\ & \sum_{c=1}^{a-1} n^{-(a+c)}\{O(1)p^2k^2p^{-(4+\mu)} + O(1)pk^2K_0\} \\ =\ & o(p^2n^{-a}). \end{aligned} \tag{S5.16}$$

(b). Similarly to Case 2.1, we have

$$\begin{aligned} &\sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \sum_{\substack{0\leq c_1,c_2\leq a \\ \mathbf{i}\in\mathcal{P}(n,a+c_1) \\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1,c_2,a) \times Q(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4)\mathbf{1}_{\left\{\substack{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},1\leq c\leq a-1, \\ \{\mathbf{i}\}_1\neq\{\tilde{\mathbf{i}}\}_1,\{\mathbf{i}\}_1\bigcap\{\tilde{\mathbf{i}}\}_1\neq\emptyset}\right\}} \\ =\ & o(p^2n^{-a}). \end{aligned} \tag{S5.17}$$

Based on (S5.16)-(S5.17), if follows that

$$\begin{aligned} & \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \sum_{\substack{0\le c_1,c_2\le a \\ \mathbf{i}\in\mathcal{P}(n,a+c_1) \\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1,c_2,a)\times Q(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\left\{\substack{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},1\le c\le a-1, \\ \{\mathbf{i}\}_1\neq\{\tilde{\mathbf{i}}\}_1,\{\mathbf{i}\}_1\bigcap\{\tilde{\mathbf{i}}\}_1\neq\emptyset}\right\}} \\ = \; & o(p^2n^{-a}). \end{aligned} \tag{S5.18}$$

Case 2.2.3. $\{\mathbf{i}\}_1\bigcap\{\tilde{\mathbf{i}}\}_1=\emptyset$. There must exist $i_1\in\{\mathbf{i}\}_1$ and $i_2\in\{\tilde{\mathbf{i}}\}_1$. As $\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\}$, we have $i_1\in\{\tilde{\mathbf{i}}\}_2\bigcup\{\tilde{\mathbf{i}}\}_3$ and $i_2\in\{\mathbf{i}\}_2\bigcup\{\mathbf{i}\}_3$. Without loss of generality, we consider $i_1\in\{\tilde{\mathbf{i}}\}_2$ and $i_2\in\{\mathbf{i}\}_2$. Then

$$(S5.8)=\mathrm{E}(x_{i_1,j_1}x_{i_1,j_2}x_{i_1,j_3})\mathrm{E}(x_{i_2,j_3}x_{i_2,j_4}x_{i_2,j_1})\mathrm{E}(\text{other terms}).$$

Due to $\mathrm{E}(x_{i,j})=0$, we have

$$\mathrm{E}(x_{i_1,j_1}x_{i_1,j_2}x_{i_1,j_3})=\mathrm{Cov}(x_{i_1,j_1},x_{i_1,j_2}x_{i_1,j_3})=\mathrm{Cov}(x_{i_1,j_2},x_{i_1,j_1}x_{i_1,j_3})=\mathrm{Cov}(x_{i_1,j_3},x_{i_1,j_1}x_{i_1,j_2}),$$

$$\mathrm{E}(x_{i_2,j_3}x_{i_2,j_4}x_{i_2,j_1})=\mathrm{Cov}(x_{i_2,j_3},x_{i_2,j_4}x_{i_2,j_1})=\mathrm{Cov}(x_{i_2,j_4},x_{i_2,j_3}x_{i_2,j_1})=\mathrm{Cov}(x_{i_2,j_1},x_{i_2,j_3}x_{i_2,j_4}).$$

If $\kappa_2=\max\{\tilde{j}_2-\tilde{j}_1,\tilde{j}_3-\tilde{j}_2,\tilde{j}_4-\tilde{j}_3\}>K_0$, by Condition 1, 2 and the $\alpha-$mixing inequality, we have

$$|\mathrm{E}(x_{i_1,j_1}x_{i_1,j_2}x_{i_1,j_3})\mathrm{E}(x_{i_2,j_3}x_{i_2,j_4}x_{i_2,j_1})|\le C\delta^{\frac{K_0\epsilon}{2+\epsilon}}=O(1)p^{-(4+\mu)}=O(1)p^{-(4+\mu)}.$$

Thus,

$$\begin{aligned}
&\Big|\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}}\sum_{\substack{0\le c_1,c_2\le a\\ \mathbf{i}\in\mathcal{P}(n,a+c_1)\\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1,c_2,a)\times Q(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\left\{\substack{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},1\le c\le a-1,\\ \{\mathbf{i}\}_1\neq\{\tilde{\mathbf{i}}\}_1,\{\mathbf{i}\}_1\bigcap\{\tilde{\mathbf{i}}\}_1=\emptyset\right\}}}\Big| \\
\le\ & \sum_{c=1}^{a-1} n^{-(a+c)}\{O(1)p^4p^{-(4+\mu)}+O(1)pK_0^3\} \\
=\ & o(p^2n^{-a}). \qquad \text{(S5.19)}
\end{aligned}$$

Following (S5.15), (S5.18), and (S5.19), we have

$$\begin{aligned}
&\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}}\sum_{\substack{0\le c_1,c_2\le a\\ \mathbf{i}\in\mathcal{P}(n,a+c_1)\\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1,c_2,a)\times Q(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},1\le c\le a-1\}} \\
=\ & o(p^2n^{-a}). \qquad \text{(S5.20)}
\end{aligned}$$

Case 2.3. If $c=a$, it follows that $\{\mathbf{i}\}_1=\{\tilde{\mathbf{i}}\}_1=\emptyset$ and $\{\mathbf{i}\}_2\bigcup\{\mathbf{i}\}_3=\{\tilde{\mathbf{i}}\}_2\bigcup\{\tilde{\mathbf{i}}\}_3$. Following similar arguments in Case 2.2.1., we have

$$\Big|\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}}\sum_{\substack{0\le c_1,c_2\le a\\ \mathbf{i}\in\mathcal{P}(n,a+c_1)\\ \tilde{\mathbf{i}}\in\mathcal{P}(n,a+c_2)}} F(c_1,c_2,a)\times Q(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\{\{\mathbf{i}\}=\{\tilde{\mathbf{i}}\},c=a\}}\Big| = o(p^2n^{-a}).$$

Following the all above results, we have

$$\mathrm{Var}\{\mathcal{U}(a)\} = (P_a^n)^{-1}a!\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}}[\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})]^a + o(p^2n^{-a}). \qquad \text{(S5.21)}$$

By the definition of $\tilde{\mathcal{U}}(a)$, it is obvious that $\tilde{\mathcal{U}}(a) = \mathcal{U}(a)1_{\{c=0\}}$, $\tilde{\mathcal{U}}^*(a) = \mathcal{U}(a)1_{\{1\leq c\leq a\}}$. Under $H_{k,0}$, $\mathrm{E}[\tilde{\mathcal{U}}(a)] = \mathrm{E}[\tilde{\mathcal{U}}^*(a)] = 0$. From the previous discussion, it is obvious that $\mathrm{Var}\{\mathcal{U}(a)\} = (P_a^n)^{-1}a! \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \{\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_3}x_{i,j_4})\}^a + o(p^2n^{-a})$ and $\mathrm{Var}\{\tilde{\mathcal{U}}^*(a)\} = o(\sigma^2(a))$. Now, Lemma 1 has been proved. □

S5.2 Proof of Lemma 2

Proof. It is obvious that $\mathrm{Cov}(\mathcal{U}(a)/\sigma(a), \mathcal{U}(b)/\sigma(b)) = \{\sigma(a)\sigma(b)\}^{-1}\mathrm{E}\{\mathcal{U}(a)\mathcal{U}(b)\}$. Through Lemma 1, we have $\mathrm{Var}(\tilde{\mathcal{U}}^*(a)) = o(\mathrm{Var}(\tilde{\mathcal{U}}(a)))$. By Cauchy-Schwarz inequality, we have

$$\mathrm{Cov}(\mathcal{U}(a)/\sigma(a), \mathcal{U}(b)/\sigma(b)) = \{\sigma(a)\sigma(b)\}^{-1}\mathrm{E}\{\tilde{\mathcal{U}}(a)\tilde{\mathcal{U}}(b)\} + o(1).$$

Recall the definition of $\tilde{\mathcal{U}}(a)$, we have

$$\mathrm{E}\{\tilde{\mathcal{U}}(a)\tilde{\mathcal{U}}(b)\} = \sum_{\substack{k<|j_1-j_2|<p \\ k<|j_3-j_4|<p}} \sum_{\substack{\mathbf{i}\in\mathcal{P}(n,a) \\ \tilde{\mathbf{i}}\in\mathcal{P}(n,b)}} \mathrm{E}\Big\{ \prod_{l=1}^{a}(x_{i_l,j_1}x_{i_l,j_2}) \prod_{\tilde{l}=1}^{b}(x_{\tilde{i}_{\tilde{l}},j_3}x_{\tilde{i}_{\tilde{l}},j_4}) \Big\}.$$

As $a \neq b$, there exists $i \in \{i_1, \ldots, i_a\}$ and $i \notin \{i_1, \ldots, i_b\}$. Thus $\mathrm{E}(x_{i,j_1}x_{i,j_2}) = 0$ under $H_{k,0}$. Lemma 2 is proved. □

S5.3 Proof of Lemma 3

Proof. Without loss of generality, we consider $h_1 < h_2$. Then

$$\begin{aligned}
\mathrm{E}(D_{n,h_1} D_{n,h_2}) &= \mathrm{E}\{(\mathrm{E}_{h_1} Z_n - \mathrm{E}_{h_1-1} Z_n)(\mathrm{E}_{h_2} Z_n - \mathrm{E}_{h_2-1} Z_n)\} \\
&= \mathrm{E}\{\mathrm{E}_{h_1} Z_n \mathrm{E}_{h_2} Z_n - \mathrm{E}_{h_1-1} Z_n \mathrm{E}_{h_2} Z_n - \mathrm{E}_{h_1} Z_n \mathrm{E}_{h_2-1} Z_n + \mathrm{E}_{h_1-1} Z_n \mathrm{E}_{h_2-1} Z_n\} \\
&= \mathrm{E}\{(\mathrm{E}_{h_1} Z_n) Z_n\} - \mathrm{E}\{(\mathrm{E}_{h_1-1} Z_n) Z_n\} - \mathrm{E}\{(\mathrm{E}_{h_1} Z_n) Z_n\} + \mathrm{E}\{(\mathrm{E}_{h_1-1} Z_n) Z_n\} \\
&= 0.
\end{aligned}$$

Thus,

$$\mathrm{E}\bigg(\sum_{h=1}^{n} \pi_{n,h}^2\bigg) = \sum_{h=1}^{n} \mathrm{E}(D_{n,h}^2) = \mathrm{E}\bigg(\sum_{h=1}^{n} D_{n,h}\bigg)^2 = \mathrm{var}(Z_n).$$

□

S5.4 Proof of Lemma 4

Proof. Recall the definition of $\tilde{\mathcal{U}}(a)$, we have

$$\begin{aligned}
A_{n,h,a} &= (\mathrm{E}_h - \mathrm{E}_{h-1})[\tilde{\mathcal{U}}(a)/\sigma(a)] \\
&= \frac{1}{\sigma(a) P_a^n} \sum_{\mathbf{i} \in \mathcal{P}(n,a)} \sum_{k < |j_1 - j_2| < p} (\mathrm{E}_h - \mathrm{E}_{h-1})[x_{i_1,j_1} x_{i_1,j_2} \cdots x_{i_a,j_1} x_{i_a,j_2}].
\end{aligned}$$

Now, we deal with

$$(\mathrm{E}_h - \mathrm{E}_{h-1})[x_{i_1,j_1}x_{i_1,j_2}\cdots x_{i_a,j_1}x_{i_a,j_2}]. \tag{S5.22}$$

Without loss of generality, we assume $i_1 < i_2 < \cdots < i_a$. Consider the position of h with respect to $\{i_1, i_2, ...i_a\}$. When $h \in \{i_1, ..., i_a\}$. If $h = i_a$, then $i_{a-1} \leq h - 1 < i_a$. Under $H_{k,0}$,

$$\begin{aligned}
& (\mathrm{E}_h - \mathrm{E}_{h-1})[x_{i_1,j_1}x_{i_1,j_2}\cdots x_{i_a,j_1}x_{i_a,j_2}] \\
= \;& x_{i_1,j_1}x_{i_1,j_2}\cdots x_{i_a,j_1}x_{i_a,j_2} - x_{i_1,j_1}x_{i_1,j_2}\cdots x_{i_{a-1},j_1}x_{i_{a-1},j_2}\mathrm{E}[x_{i_a,j_1}x_{i_a,j_2}] \\
= \;& x_{i_1,j_1}x_{i_1,j_2}\cdots x_{i_a,j_1}x_{i_a,j_2}.
\end{aligned}$$

If $h = i_m$, $i_1 \leq i_m < i_a$, then $i_{m-1} \leq h - 1 < i_m$ with $i_0 = 0$. Under H_0,

$$\begin{aligned}
& (\mathrm{E}_h - \mathrm{E}_{h-1})[x_{i_1,j_1}x_{i_1,j_2}\cdots x_{i_a,j_1}x_{i_a,j_2}] \\
= \;& x_{i_1,j_1}x_{i_1,j_2}\cdots x_{i_m,j_1}x_{i_m,j_2}\mathrm{E}[x_{i_{m+1},j_1}x_{i_{m+1},j_2}\cdots x_{i_a,j_1}x_{i_a,j_2}] \\
& -x_{i_1,j_1}x_{i_1,j_2}\cdots x_{i_{m-1},j_1}x_{i_{m-1},j_2}\mathrm{E}[x_{i_m,j_1}x_{i_m,j_2}\cdots x_{i_a,j_1}x_{i_a,j_2}] \\
= \;& 0.
\end{aligned}$$

When $h \notin \{i_1, ..., i_a\}$, there are similar discussions on (S5.22). If $i_m < h < i_{m+1}$ for $m = 1, 2, \ldots, a-1$, then $i_m \leq h - 1$. (S5.22) $= 0$. If $h < i_1$,

then $h-1<i_1$, (S5.22) $=0$. If $h>i_a$, then $h-1\geq i_a$, (S5.22) $=0$. In summary, when $h<a$, $A_{n,h}=0$; when $h\geq a$,

$$A_{n,h,a} = \frac{a}{\sigma(a)P_a^n}\sum_{1\leq i_1\neq\cdots\neq i_{a-1}\leq h-1}\sum_{k<|j_1-j_2|<p}(x_{h,j_1}x_{h,j_2})\times\prod_{t=1}^{a-1}x_{i_t,j_1}x_{i_t,j_2}.$$

□

S5.5 Proof of Lemma 5

Proof. First of all, we define some notations to facilitate discussion. For given tuples $\mathbf{i}^{(l)}=(i_1^{(l)},...,i_{a_l-1}^{(l)})\in\mathcal{P}(h-1,a_l-1)$ with $l=1,2$, we define $(\mathbf{i}^{(1)},\mathbf{i}^{(2)})=(i_1^{(1)},...,i_{a_1-1}^{(1)},i_1^{(2)},...,i_{a_2-1}^{(2)})$, and let $\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)})$ be a collection of tuples $(\mathbf{i}^{(1)},\mathbf{i}^{(2)})$. We decompose $\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)})$ into two parts: $\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)})=\bigcup_{\upsilon=1}^{2}\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\upsilon)$, where $\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},1)$ is the collection which contains the tuples $(\mathbf{i}^{(1)},\mathbf{i}^{(2)})$ satisfying $\{\mathbf{i}^{(1)}\}=\{\mathbf{i}^{(2)}\}$, and $\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},2)$ is the collection which contains the tuples $(\mathbf{i}^{(1)},\mathbf{i}^{(2)})$ satisfying $\{\mathbf{i}^{(1)}\}\neq\{\mathbf{i}^{(2)}\}$. Furthermore, we denote the collection $\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)},\upsilon,\upsilon)$ contains the tuples $(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)})$ satisfying $(\mathbf{i}^{(1)},\mathbf{i}^{(2)})\in\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\upsilon)$ and $(\mathbf{i}^{(3)},\mathbf{i}^{(4)})\in\mathcal{I}(\mathbf{i}^{(3)},\mathbf{i}^{(4)},\upsilon)$ for $\upsilon=1,2$. Furthermore, we define

$$\mathcal{J}=\{(j_1,j_2):k<|j_1-j_2|<p\},\tag{S5.23}$$

$$\mathcal{J}_8=\{(j_1,j_2,j_3,j_4,j_5,j_6,j_7,j_8):(j_{2s-1},j_{2s})\in\mathcal{J},s=1,...,4\}\tag{S5.24}$$

Based on Lemma 4, we have

$$\sum_{h=1}^{n}\pi_{n,h}^2=\sum_{h=1}^{n}\mathrm{E}_{h-1}(D_{n,h}^2)=\sum_{h=1}^{n}\sum_{r_1=1}^{m}\sum_{r_2=1}^{m}t_{r_1}t_{r_2}\mathrm{E}_{h-1}(A_{n,h,a_{r_1}}A_{n,h,a_{r_2}}).$$

Denote $c(n,a_r)=(a_r/\sigma(a_r)P_{a_r}^n)^2$. By Cauchy-Schwarz inequality, for some constant C,

$$\mathrm{var}\bigg(\sum_{h=1}^{n}\pi_{n,h}^2\bigg)\le Cn^2\max_{1\le h\le n;1\le r_1,r_2\le m}\mathrm{var}(T_{h,a_{r_1},a_{r_2}}),$$

where

$$\begin{aligned}T_{h,a_{r_1},a_{r_2}}&:=\mathrm{E}_{h-1}(A_{n,h,a_{r_1}}A_{n,h,a_{r_2}})\\&=\sum_{\substack{\mathbf{i}^{(1)}\in\mathcal{P}(h-1,a_1-1)\\ \mathbf{i}^{(2)}\in\mathcal{P}(h-1,a_2-1)}}\sum_{\substack{k<|j_1-j_2|<p\\ k<|j_3-j_4|<p}}\prod_{t=1}^{2}c(n,a_{r_t})^{1/2}\\&\quad\mathrm{E}\bigg(\prod_{t=1}^{4}x_{h,j_t}\bigg)\prod_{t=1}^{a_{r_1}-1}\Big(x_{i_t^{(1)},j_1}x_{i_t^{(1)},j_2}\Big)\prod_{t=1}^{a_{r_2}-1}\Big(x_{i_t^{(2)},j_3}x_{i_t^{(2)},j_4}\Big).\end{aligned}\tag{S5.25}$$

Now, we will proof $\mathrm{var}(T_{h,a_{r_1},a_{r_2}})=o(n^{-2})$.

We rewrite $T_{h,a_1,a_2}=\sum_{\upsilon=1}^{2}T_{h,a_1,a_2,\upsilon}$, where

$$T_{h,a_1,a_2,\nu}=\sum_{\substack{(\mathbf{i}^{(1)},\mathbf{i}^{(2)})\in\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\upsilon);\\ (j_{2s-1},j_{2s})\in\mathcal{J},s=1,2}}\bigg\{\prod_{l=1}^{2}c(n,a_l)\bigg\}^{1/2}\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),(\mathbf{i}^{(2)}|(j_3,j_4),h\},\tag{S5.26}$$

where

$$\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),(\mathbf{i}^{(2)}|(j_3,j_4),(h)\} = \mathrm{E}\Big(\prod_{t=1}^{4} x_{h,j_t}\Big)\prod_{t=1}^{a_1-1}\Big(x_{i_t^{(1)},j_1}x_{i_t^{(1)},j_2}\Big)\prod_{t=1}^{a_2-1}\Big(x_{i_t^{(2)},j_3}x_{i_t^{(2)},j_4}\Big).$$

To prove $\mathrm{var}(T_{h,a_1,a_2}) = o(n^{-2})$, it is equivalent to prove $\mathrm{var}(T_{h,a_{r_1},a_{r_2},\upsilon}) = o(n^{-2})$, for $\upsilon = 1, 2$. It is obvious that

$$\begin{aligned} T^2_{h,a_1,a_2,\nu} = & \sum_{\substack{(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)}) \\ \in\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)},\upsilon,\upsilon); \\ (j_{2s-1},j_{2s})\in\mathcal{J},s=1,\ldots,4}} \Big\{\prod_{l=1}^{2} c(n,a_l)\Big\} \\ & \times\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\}, \quad \text{(S5.27)} \end{aligned}$$

where

$$\begin{aligned} & \mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\} \\ = \ & \mathrm{E}\Big(\prod_{t=1}^{4} x_{h,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{h,j_t}\Big)\prod_{s=1}^{4}\prod_{t=1}^{a_s-1}\Big(x_{i_t^{(s)},j_{2s-1}}x_{i_t^{(s)},j_{2s}}\Big). \quad \text{(S5.28)} \end{aligned}$$

Step 1. At first, we will prove $\mathrm{var}(T_{h,a_1,a_2,2}) = o(n^{-2})$.

When $\{\mathbf{i}^{(1)}\} \neq \{\mathbf{i}^{(2)}\}$, there must exists an index i such that $i \in \{\mathbf{i}^{(1)}\}$ but $i \notin \{\mathbf{i}^{(2)}\}$. Then under H_0, we have

$$\mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),h)] = \mathrm{E}(x_{i,j_1}x_{i,j_2}) \times \mathrm{E}(\text{other terms}) = 0.$$

Thus, $\mathrm{E}(T_{h,a_1,a_2,2}) = 0$. Next, we prove that $\mathrm{E}(T^2_{h,a_1,a_2,2}) = o(n^{-2}p^{-1}\log^5 p)$. To deal with this, we will show that for given $(j_{2s-1}, j_{2s}) \in \mathcal{J}$, $s = 1, ..., 4$,

$$\sum_{\substack{(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)}) \\ \in \mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)},v,v)}} \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2), \mathbf{i}^{(2)}|(j_3,j_4), \mathbf{i}^{(3)}|(j_5,j_6), \mathbf{i}^{(4)}|(j_7,j_8), h\}] = O(n^{a_1+a_2-2}); \tag{S5.29}$$

and for given tuple $(\mathbf{i}^{(1)}, \mathbf{i}^{(2)}, \mathbf{i}^{(3)}, \mathbf{i}^{(4)}) \in \mathcal{I}(\mathbf{i}^{(1)}, \mathbf{i}^{(2)}, \mathbf{i}^{(3)}, \mathbf{i}^{(4)}, 2, 2)$,

$$\sum_{(j_{2s-1},j_{2s}) \in \mathcal{J}, s=1,...,4} \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2), \mathbf{i}^{(2)}|(j_3,j_4), \mathbf{i}^{(3)}|(j_5,j_6), \mathbf{i}^{(4)}|(j_7,j_8), h\}] = O(p^3\log^5 p). \tag{S5.30}$$

Based on (S5.29) and (S5.30), and $c(n, a_l) = \Theta(p^{-2}n^{-a_l})$ for $l = 1, 2$, we obtain $\mathrm{E}(T^2_{h,a_1,a_2,2}) = o(n^{-2}p^{-1}\log^5 p)$. Now, we will prove (S5.29) and (S5.30).

Firstly, we prove (S5.29). If an index i only appear once in tuple $(\mathbf{i}^{(1)}, \mathbf{i}^{(2)}, \mathbf{i}^{(3)}, \mathbf{i}^{(4)})$, we assume without loss of generality that $i \in \{\mathbf{i}^{(1)}\}$ and $i \notin \{\mathbf{i}^{(2)}\} \cup \{\mathbf{i}^{(3)}\} \cup \{\mathbf{i}^{(4)}\}$. Then, under H_0,

$$\begin{aligned} & \mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2), \mathbf{i}^{(2)}|(j_3,j_4), \mathbf{i}^{(3)}|(j_5,j_6), \mathbf{i}^{(4)}|(j_7,j_8), h\} \\ = \; & \mathrm{E}(x_{i,j_1}x_{i,j_2}) \times \mathrm{E}(\text{other terms}) \\ = \; & 0. \end{aligned}$$

It implies that (S5.28)$\neq 0$ unless each of the indexes in the tuple $(\mathbf{i}^{(1)}, \mathbf{i}^{(2)}, \mathbf{i}^{(3)}, \mathbf{i}^{(4)})$ appears at least twice. It follows that $(\mathbf{i}^{(1)}, \mathbf{i}^{(2)}, \mathbf{i}^{(3)}, \mathbf{i}^{(4)})$ contains at most $a_1 + a_2 - 2$ distinct indexes. Thus, for given $(j_{2s-1}, j_{2s}) \in \mathcal{J}$, $s = 1, ..., 4$,

$$\sum_{\substack{(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)}) \\ \in \mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)},\upsilon,\upsilon)}} \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1, j_2), \mathbf{i}^{(2)}|(j_3, j_4), \mathbf{i}^{(3)}|(j_5, j_6), \mathbf{i}^{(4)}|(j_7, j_8), h\}] = O(n^{a_1+a_2-2}).$$

Secondly, we will prove (S5.30).

As $\{\mathbf{i}^{(1)}\} \neq \{\mathbf{i}^{(2)}\}$ and $\{\mathbf{i}^{(3)}\} \neq \{\mathbf{i}^{(4)}\}$, without loss of generality, we assume $\mathrm{card}(\{\mathbf{i}^{(1)}\} \cap \{\mathbf{i}^{(2)}\}) = q_1$ and $\mathrm{card}(\{\mathbf{i}^{(3)}\} \cap \{\mathbf{i}^{(4)}\}) = q_2$ for $q_1 = 0, 1, ..., a_1 - 2, q_2 = 0, 1, ..., a_2 - 2$.

Case 1. When $q_1 \neq q_2$. Without loss of generality, we assume $q_2 > q_1 \geq 0$, then $a_i - 1 - q_2 < a_i - 1 - q_1$ for $i = 1, 2$. To ensure $\mathrm{E}\{(\text{S5.28})\} \neq 0$, we know that any index i must appears at least two times in $\overset{4}{\underset{s=1}{\cup}}\{\mathbf{i}^{(s)}\}$. When $\mathrm{E}\{(\text{S5.28})\} \neq 0$, we assume there exists $a_1 + a_2 - 2q_1 - 2$ elements in $\{\mathbf{i}^{(1)}\} \cup \{\mathbf{i}^{(2)}\} \backslash \{\mathbf{i}^{(1)}\} \cap \{\mathbf{i}^{(2)}\}$ that are consistent with $\{\mathbf{i}^{(3)}\} \cup \{\mathbf{i}^{(4)}\} \backslash \{\mathbf{i}^{(3)}\} \cap \{\mathbf{i}^{(4)}\}$. Then, there must exist $2(q_2 - q_1)$ elements in $\{\mathbf{i}^{(1)}\} \cup \{\mathbf{i}^{(2)}\} \backslash \{\mathbf{i}^{(1)}\} \cap \{\mathbf{i}^{(2)}\}$ belong to $\{\mathbf{i}^{(3)}\} \cap \{\mathbf{i}^{(4)}\}$. We assume that there exists an index i satisfy

$i \in \{\mathbf{i}^{(1)}\} \cap \{\mathbf{i}^{(3)}\} \cap \{\mathbf{i}^{(4)}\}$, thus

$$\begin{aligned} & \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\}] \\ = \; & \mathrm{E}(x_{h,j_1}x_{h,j_2}x_{h,j_3}x_{h,j_4}) \times \mathrm{E}(x_{h,j_5}x_{h,j_6}x_{h,j_7}x_{h,j_8}) \\ & \times \mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,j_5}x_{i,j_6}x_{i,j_7}x_{i,j_8}) \times \mathrm{E}(\text{other terms}). \end{aligned} \tag{S5.31}$$

For further discussing, we consider three sub-cases (a)-(c).

(a). When $\mathrm{card}(j_1,j_2,j_3,j_4) = 2$ and $\mathrm{card}(j_5,j_6,j_7,j_8) = 2$. Without loss of generality, we assume $j_1 = j_3$, $j_2 = j_4$, $j_5 = j_7$, $j_6 = j_8$. Then

$$\begin{aligned} (\text{S5.31}) = & \mathrm{E}(x_{h_1,j_1}^2 x_{h_1,j_2}^2) \times \mathrm{E}(x_{h_2,j_5}^2 x_{h_2,j_6}^2) \\ & \times \mathrm{E}(x_{i_1,j_1}x_{i_1,j_2}x_{i_1,j_5}^2 x_{i_1,j_6}^2) \times \mathrm{E}(\text{other terms}). \end{aligned}$$

Take $D_0 = -(2+\epsilon)(8+\mu)(\log p)/(\epsilon \log \delta)$ for some small positive constants μ, ϵ and δ, recall $\{\tilde{j}_1,\tilde{j}_2,\tilde{j}_5,\tilde{j}_6\}$ is the ordered versions of $\{j_1,j_2,j_5,j_6\}$ with $\tilde{j}_1 \le \tilde{j}_2 \le \tilde{j}_5 \le \tilde{j}_6$. When $\kappa_4 = \min\{\tilde{j}_2 - \tilde{j}_1, \tilde{j}_5 - \tilde{j}_2, \tilde{j}_6 - \tilde{j}_5\} > D_0$,

$$\begin{aligned} |(\text{S5.31})| \; & \le \; C|\mathrm{E}(x_{i_1,j_1}x_{i_1,j_2}x_{i_1,j_5}^2 x_{i_1,j_6}^2)| \\ & = \; C|\mathrm{cov}(x_{i_1,j_1}x_{i_1,j_2}, x_{i_1,j_5}^2 x_{i_1,j_6}^2)| \le C\delta^{\frac{D_0\epsilon}{2+\epsilon}} = O(1)p^{-(8+\mu)}. \end{aligned}$$

When $\kappa_4 \leq D_0$, $|(\text{S5.31})| = O(1)$, and the number of distinct indexes in tuple $\mathbf{j} \in \mathcal{J}_8$ which defined in (S5.24) is $O(p^3 D_0)$. Then

$$\sum_{(j_{2s-1},j_{2s})\in\mathcal{J},s=1,\ldots,4} \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\}] \times \mathbf{1}_{\left\{\substack{\mathrm{card}(j_1,j_2,j_3,j_4)=2\\ \mathrm{card}(j_5,j_6,j_7,j_8)=2}\right\}} = O(1)p^8 p^{-(8+\mu)} + O(p^3 D_0).$$

(b). When $\mathrm{card}(j_1,j_2,j_3,j_4) \geq 3$ and $\mathrm{card}(j_5,j_6,j_7,j_8) \geq 3$. Recall $\{\tilde{j}_1,\tilde{j}_2,\tilde{j}_3,\tilde{j}_4\}$ and $\{\tilde{j}_5,\tilde{j}_6,\tilde{j}_7,\tilde{j}_8\}$ as the order versions of $\{j_1,j_2,j_3,j_4\}$ and $\{j_5,j_6,j_7,j_8\}$, which satisfy $\tilde{j}_1 \leq \tilde{j}_2 \leq \tilde{j}_3 \leq \tilde{j}_4$ and $\tilde{j}_5 \leq \tilde{j}_6 \leq \tilde{j}_7 \leq \tilde{j}_8$. Note that

$$\begin{aligned}
|\mathrm{E}(x_{h,j_1}x_{h,j_2}x_{h,j_3}x_{h,j_4})| = |\mathrm{E}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}x_{h,\tilde{j}_3}x_{h,\tilde{j}_4})| &= |\mathrm{cov}(x_{h,\tilde{j}_1}, x_{h,\tilde{j}_2}x_{h,\tilde{j}_3}x_{h,\tilde{j}_4})| \\
&= |\mathrm{cov}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}x_{h,\tilde{j}_3}, x_{h,\tilde{j}_4})|, \\
|\mathrm{E}(x_{h,j_5}x_{h,j_6}x_{h,j_7}x_{h,j_8})| = |\mathrm{E}(x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8})| &= |\mathrm{cov}(x_{h,\tilde{j}_5}, x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8})| \\
&= |\mathrm{cov}(x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}, x_{h,\tilde{j}_8})|.
\end{aligned}$$

For further discussing, we consider tuple $\tilde{\mathbf{j}} = (\tilde{j}_1,\ldots,\tilde{j}_8)$ instead of $\mathbf{j} = (j_1,\ldots,j_8)$.

(b1). If at least one of $(\tilde{j}_{2s-1},\tilde{j}_{2s}) \in \mathcal{J}, s = 1,\ldots,4$ holds, then at least one of $\mathrm{E}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}) = 0$, $\mathrm{E}(x_{h,\tilde{j}_3}x_{h,\tilde{j}_4}) = 0$, $\mathrm{E}(x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}) = 0$ and $\mathrm{E}(x_{h,\tilde{j}_7}x_{h,\tilde{j}_8}) = 0$ is established. It means that at least one of the

following equations is true,

$$|\mathrm{E}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}x_{h,\tilde{j}_3}x_{h,\tilde{j}_4})| = |\mathrm{cov}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}, x_{h,\tilde{j}_3}x_{h,\tilde{j}_4})|,$$

$$|\mathrm{E}(x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8})| = |\mathrm{cov}(x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}, x_{h,\tilde{j}_7}x_{h,\tilde{j}_8})|.$$

Without loss of generality, we assume $\mathrm{E}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}) = 0$. If $\kappa_5 = \max\{\tilde{j}_2 - \tilde{j}_1, \tilde{j}_3 - \tilde{j}_2, \tilde{j}_4 - \tilde{j}_3, \tilde{j}_6 - \tilde{j}_5, \tilde{j}_8 - \tilde{j}_7\} > D_0$,

$$\begin{aligned} |(\mathrm{S5.31})| &\leq C|\mathrm{E}(x_{h,j_1}x_{h,j_2}x_{h,j_3}x_{h,j_4})||\mathrm{E}(x_{h,j_5}x_{h,j_6}x_{h,j_7}x_{h,j_8})| \\ &\leq C\delta^{\frac{D_0\epsilon}{2+\epsilon}} = O(p^{-(8+\mu)}). \end{aligned}$$

If $\kappa_5 \leq D_0$, $|(\mathrm{S5.31})| = O(1)$, and the number of distinct indexes in $\mathbf{j} \in \mathcal{J}_8$ is $O(p^3 D_0^5)$.
(b2). If none of $(\tilde{j}_{2s-1}, \tilde{j}_{2s}) \in \mathcal{J}, s = 1, ..., 4$ holds, thus

$$|\mathrm{E}(x_{i,j_1}x_{i,j_2}x_{i,\tilde{j}_5}x_{i,\tilde{j}_6}x_{i,\tilde{j}_7}x_{i,\tilde{j}_8})| = |\mathrm{cov}(x_{i_1,j_1}x_{i_1,j_2}, x_{i_1,\tilde{j}_5}x_{i_1,\tilde{j}_6}x_{i_1,\tilde{j}_7}x_{i_1,\tilde{j}_8})|$$

under H_0. If $|\tilde{j}_5 - \tilde{j}_4| > D_0$, then $|(\mathrm{S5.31})| \leq C\delta^{\frac{D_0\epsilon}{2+\epsilon}} = O(p^{-(8+\mu)})$. If $|\tilde{j}_5 - \tilde{j}_4| \leq D_0$, $|(\mathrm{S5.31})| = O(1)$, and the number of distinct indexes in

in $\mathbf{j} \in \mathcal{J}_8$ is $O(p^3 D_0 k^4)$. Then

$$\Bigg| \sum_{(j_{2s-1}, j_{2s}) \in \mathcal{J}, s=1,\ldots,4} \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1, j_2), \mathbf{i}^{(2)}|(j_3, j_4), \mathbf{i}^{(3)}|(j_5, j_6), \mathbf{i}^{(4)}|(j_7, j_8), h\}]$$
$$\times \mathbf{1}_{\left\{\substack{\mathrm{card}(j_1,j_2,j_3,j_4)\geq 3 \\ \mathrm{card}(j_5,j_6,j_7,j_8)\geq 3}\right\}} \Bigg| = O(1)p^8 p^{-(8+\mu)} + O(p^3 D_0^5) + O(p^3 D_0).$$

(c). Only one of $\mathrm{card}(j_1, j_2, j_3, j_4) \geq 3$ and $\mathrm{card}(j_5, j_6, j_7, j_8) \geq 3$ is true. Without loss of generality, we assume $\mathrm{card}(j_1, j_2, j_3, j_4) \geq 3$ and $\mathrm{card}(j_5, j_6, j_7, j_8) = 2$. If at least one of $\tilde{j}_2 - \tilde{j}_1 > k$ and $\tilde{j}_4 - \tilde{j}_3 > k$ holds, then the total number of j indexes in $\mathbf{j} \in \mathcal{J}_8$ is $O(p^3 D_0^3)$ when $\kappa_2 = \max\{\tilde{j}_2 - \tilde{j}_1, \tilde{j}_3 - \tilde{j}_2, \tilde{j}_4 - \tilde{j}_3\} \leq D_0$. When $\kappa_2 > D_0$, we konw

$$|(\mathrm{S5.31})| \leq C|\mathrm{E}(x_{h,j_1} x_{h,j_2} x_{h,j_3} x_{h,j_4})| \leq C\delta^{\frac{D_0 \epsilon}{2+\epsilon}} = O(p^{-(8+\mu)}).$$

If both $\tilde{j}_2 - \tilde{j}_1 \leq k$ and $\tilde{j}_4 - \tilde{j}_3 \leq k$ hold, when $\tilde{j}_5 - \tilde{j}_4 > D_0$,

$$\begin{aligned} |(\mathrm{S5.31})| &\leq C|\mathrm{E}(x_{i_1,\tilde{j}_1} x_{i_1,\tilde{j}_2} x_{i_1,\tilde{j}_3} x_{i_1,\tilde{j}_4} x_{i_1,j_5} x_{i_1,j_6})| \\ &\leq C\delta^{\frac{D_0 \epsilon}{2+\epsilon}} = O(p^{-(8+\mu)}). \end{aligned}$$

When $\tilde{j}_5 - \tilde{j}_4 \leq D_0$, the total number of distinct indexes is $O(p^3 D_0 k^2)$.

Then

$$\Big| \sum_{(j_{2s-1},j_{2s})\in\mathcal{J},s=1,...,4} \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\}] \times \mathbf{1}_{\left\{\substack{\mathrm{card}(j_1,j_2,j_3,j_4)\geq 3\\ \mathrm{card}(j_5,j_6,j_7,j_8)=2}\right\}} \Big| = O(1)p^8 p^{-(8+\mu)} + O(p^3 D_0^3) + O(p^3 D_0).$$

Combine the results in sub-cases (a)-(c), we get

$$\begin{aligned} & \Big| \sum_{(j_{2s-1},j_{2s})\in\mathcal{J},s=1,...,4} \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\}]\mathbf{1}_{\{q_1\neq q_2\}} \Big| \\ = \;& O(1)p^8 p^{-(8+\mu)} + O(p^3 D_0^3) + O(p^3 D_0). \end{aligned}$$

Case 2. When $q_1 = q_2$, we assume $q_1 = q_1 = q$ for $q = 0, ..., a-2$. It is obvious that $\mathrm{E}\{(\mathrm{S5.28})\} \neq 0$ when $q > 0$. Thus, we have

$$\begin{aligned} & \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\}] \\ = \;& \mathrm{E}(x_{h,j_1}x_{h,j_2}x_{h,j_3}x_{h,j_4}) \times \mathrm{E}(x_{h,j_5}x_{h,j_6}x_{h,j_7}x_{h,j_8}) \\ & \times \mathrm{E}(x_{i_1,j_1}x_{i_1,j_2}x_{i_1,j_5}x_{i_1,j_6})\mathrm{E}(x_{i_2,j_3}x_{i_2,j_4}x_{i_2,j_7}x_{i_2,j_8})\mathrm{E}(\text{other terms}). \quad (\mathrm{S5.32}) \end{aligned}$$

When $\mathbf{j} \in \tilde{\mathcal{J}}_1 = \{(j_1, ..., j_8) : \text{at least one of } \tilde{j}_2 - \tilde{j}_1 > k, \tilde{j}_4 - \tilde{j}_3 > k, \tilde{j}_6 - \tilde{j}_5 > k, \tilde{j}_8 - \tilde{j}_7 > k\}$, similar with the sub-cases (a)-(c) in Case 1, we have

$$\begin{aligned} & \Big| \sum_{(j_{2s-1},j_{2s})\in\mathcal{J},s=1,...,4} \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\}\mathbf{1}_{\{\mathbf{j}\in\tilde{\mathcal{J}}_1\}}] \Big| \\ = \;& O(1)p^{-(8+\mu)}p^8 + O(p^3 D_0^5). \end{aligned}$$

When $\mathbf{j} \notin \tilde{\mathcal{J}}_1 = \{(j_1, ..., j_8) : \tilde{j}_2 - \tilde{j}_1 \le k, \tilde{j}_4 - \tilde{j}_3 \le k, \tilde{j}_6 - \tilde{j}_5 \le k, \tilde{j}_8 - \tilde{j}_7 \le k\}$, if $\tilde{j}_5 - \tilde{j}_4 > D_0$

$$\begin{aligned} |(\text{S5.32})| &\le C \times |\mathrm{E}(x_{i,j_1} x_{i,j_2} x_{i,j_5} x_{i,j_6})||\mathrm{E}(x_{h,j_3} x_{h,j_4} x_{h,j_7} x_{h,j_8})| \\ &\le C\delta^{\frac{D_0\epsilon}{2+\epsilon}} = O(p^{-(8+\mu)}). \end{aligned}$$

When $\tilde{j}_5 - \tilde{j}_4 < D_0$, the total number of distinct j is $O(p^3 D_0 k^4)$. So

$$\begin{aligned} &\left| \sum_{(j_{2s-1}, j_{2s}) \in \mathcal{J}, s=1,...,4} \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1, j_2), \mathbf{i}^{(2)}|(j_3, j_4), \mathbf{i}^{(3)}|(j_5, j_6), \mathbf{i}^{(4)}|(j_7, j_8), h\}\mathbf{1}_{\{\mathbf{j} \notin \tilde{\mathcal{J}}_1\}} \right| \\ = \; & O(1)p^{-(8+\mu)}p^8 + O(p^3 D_0). \end{aligned}$$

Then,

$$\begin{aligned} &\left| \sum_{(j_{2s-1}, j_{2s}) \in \mathcal{J}, s=1,...,4} \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1, j_2), \mathbf{i}^{(2)}|(j_3, j_4), \mathbf{i}^{(3)}|(j_5, j_6), \mathbf{i}^{(4)}|(j_7, j_8), h\}\mathbf{1}_{\{q_1 = q_2 = q\}} \right| \\ = \; & O(1)p^8 p^{-(8+\mu)} + O(p^3 D_0) + O(p^3 D_0^5). \end{aligned}$$

In summary, (S5.30) has been proved.

Step 2. Secondly, we will prove $\mathrm{Var}(T_{h,a_1,a_2,1}) = o(n^{-2})$. Recall $\mathrm{Var}(T_{h,a_1,a_2,1})$ is constructed with $(\mathbf{i}^{(1)}, \mathbf{i}^{(2)}) \in \mathcal{I}(\{\mathbf{i}^{(1)}\}, \{\mathbf{i}^{(2)}\}, 1)$, where $\{\mathbf{i}^{(1)}\} = \{\mathbf{i}^{(2)}\}$. As $\{\mathbf{i}^{(1)}\} = \{\mathbf{i}^{(2)}\}$ happens only when $a_1 = a_2$, so we consider $a_1 = a_2 = a$ below. Recall the definition in (S5.26), it follows that

$$\mathrm{E}(T_{h,a_1,a_2,1}) = \sum_{\substack{\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},1);\\ (j_{2s-1},j_{2s})\in\mathcal{J},s=1,2}} c(n,a)\Big\{\mathrm{E}\Big(\prod_{t=1}^{4} x_{1,j_t}\Big)\Big\}^a,$$

then

$$\{\mathrm{E}(T_{h,a_1,a_2,1})\}^2 = \sum_{\substack{\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)},1,1);\\ (j_{2s-1},j_{2s})\in\mathcal{J},s=1,\ldots,4}} c(n,a)^2\Big\{\mathrm{E}\Big(\prod_{t=1}^{4} x_{1,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{1,j_t}\Big)\Big\}^a.$$

Recall the definition in (S5.27), we have

$$\mathrm{E}(T^2_{h,a_1,a_2,1}) = \sum_{\substack{\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)},1,1);\\ (j_{2s-1},j_{2s})\in\mathcal{J},s=1,\ldots,4}} c(n,a)^2\mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\}],$$

To facilitate the proof, we denote $\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)},1,1,q)$ as the collection of tuples $(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)})$ which satisfy $\{\mathbf{i}^{(1)}\} = \{\mathbf{i}^{(2)}\}$, $\{\mathbf{i}^{(3)}\} = \{\mathbf{i}^{(4)}\}$, and $\mathrm{card}(\{\mathbf{i}^{(1)}\} \cap \{\mathbf{i}^{(3)}\}) = q$, where $0 \le q \le a-1$. Thus, we rewrite $\mathrm{E}(T^2_{h,a_1,a_2,1}) = \sum_{q=0}^{a-1} \mathrm{E}(T^2_{h,a_1,a_2,1,q})$, where

$$\mathrm{E}(T^2_{h,a_1,a_2,1,q}) = \sum_{\substack{\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)},1,1,q)\\ (j_{2s-1},j_{2s})\in\mathcal{J},s=1,\ldots,4}} c(n,a)^2\mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\}],$$

where

$$\begin{aligned} & \mathrm{E}[\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2),\mathbf{i}^{(2)}|(j_3,j_4),\mathbf{i}^{(3)}|(j_5,j_6),\mathbf{i}^{(4)}|(j_7,j_8),h\}] \\ = \ & \mathrm{E}\Big(\prod_{t=1}^{4} x_{h,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{h,j_t}\Big)\mathrm{E}\Big\{\prod_{s=1}^{4}\prod_{t=1}^{a-1} x_{i_t^{(s)},j_{2s-1}} x_{i_t^{(s)},j_{2s}}\Big\}. \quad \text{(S5.33)} \end{aligned}$$

It is obvious that (S5.33) $= \Big\{\mathrm{E}\Big(\prod_{t=1}^{4} x_{1,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{1,j_t}\Big)\Big\}^{a}$ when $\mathrm{card}(\{\mathbf{i}^{(1)}\}\cap \{\mathbf{i}^{(3)}\}) = 0$. Thus $\mathrm{E}(T^2_{h,a,a,1,0}) = \{\mathrm{E}(T_{h,a,a,1})\}^2$. Therefore,

$$\begin{aligned} \mathrm{Var}(T_{h,a,a,1}) &= \mathrm{E}(T^2_{h,a,a,1}) - \{\mathrm{E}(T_{h,a,a,1})\}^2 \\ &= \sum_{q=0}^{a-1} \mathrm{E}(T^2_{h,a,a,1,q}) - \{\mathrm{E}(T_{h,a,a,1})\}^2 \\ &= \sum_{q=1}^{a-1} \sum_{\mathcal{I}(\mathbf{i}^{(1)},\mathbf{i}^{(2)},\mathbf{i}^{(3)},\mathbf{i}^{(4)},1,1,q)} c(n,a)^2 D_{h,a,a,1,q}, \end{aligned}$$

where

$$\begin{aligned} D_{h,a,a,1,q} := & \sum_{(j_{2s-1},j_{2s})\in\mathcal{J},s=1,\ldots,4} \Big\{\mathrm{E}\Big(\prod_{t=1}^{4} x_{1,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{1,j_t}\Big)\Big\}^{a-q} \\ & \times\Big[\Big\{\mathrm{E}\Big(\prod_{t=1}^{8} x_{1,j_t}\Big)\Big\}^{q} - \Big\{\mathrm{E}\Big(\prod_{t=1}^{4} x_{1,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{1,j_t}\Big)\Big\}^{q}\Big] \end{aligned}$$

To prove $\mathrm{var}(T_{h,a,a,1}) = o(n^{-2})$, it suffices to show $|D_{h,a,a,1,q}| = o(p^4)$. Before proving it, we give the following lemma.

Lemma 12. *(Lemma 3.4.3 in He et al. (2021)) When $|a_i| \le A$ and*

$|b_i| \leq A$, *it follows that*

$$\left|\prod_{i=1}^{q} a_i - \prod_{i=1}^{q} b_i\right| \leq \sum_{i=1}^{q} |a_i - b_i| A^{q-1}.$$

By Lemma 12, for $1 \leq q \leq a-1$,

$$\begin{aligned} |D_{h,a,a,1,q}| \;\leq\; & C \sum_{(j_{2s-1},j_{2s})\in\mathcal{J},s=1,\ldots,4} \left|\mathrm{E}\Big(\prod_{t=1}^{4} x_{1,j_t}\Big)\right| \times \left|\mathrm{E}\Big(\prod_{t=5}^{8} x_{1,j_t}\Big)\right| \\ & \times \left|\mathrm{E}\Big(\prod_{t=1}^{8} x_{1,j_t}\Big) - \mathrm{E}\Big(\prod_{t=1}^{4} x_{1,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{1,j_t}\Big)\right|. \end{aligned}$$

(a). When $\mathrm{card}(j_1, j_2, j_3, j_4) = 2$ and $\mathrm{card}(j_5, j_6, j_7, j_8) = 2$. We assume without loss of generality $j_1 = j_3$, $j_2 = j_3$, $j_5 = j_7$, $j_6 = j_8$. Note that $(\tilde{j}_1, \tilde{j}_2, \tilde{j}_5, \tilde{j}_6)$ is the ordered version of (j_1, j_2, j_5, j_6). When $\kappa_4 = \min\{\tilde{j}_2 - \tilde{j}_1, \tilde{j}_5 - \tilde{j}_2, \tilde{j}_6 - \tilde{j}_5\} > D_0$,

$$\begin{aligned} & \left|\mathrm{E}\Big(\prod_{t=1}^{8} x_{1,j_t}\Big) - \mathrm{E}\Big(\prod_{t=1}^{4} x_{1,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{1,j_t}\Big)\right| \\ = \;& |\mathrm{E}(x_{1,j_1}^2 x_{1,j_2}^2 x_{1,j_5}^2 x_{1,j_6}^2) - \mathrm{E}(x_{1,j_1}^2 x_{1,j_2}^2)\mathrm{E}(x_{1,j_5}^2 x_{1,j_6}^2)| \leq C\delta^{\frac{D_0\epsilon}{2+\epsilon}} = O(p^{-(8+\mu)}). \end{aligned}$$

When $\kappa_4 \leq D_0$, $\left|\mathrm{E}\Big(\prod_{t=1}^{8} x_{1,j_t}\Big) - \mathrm{E}\Big(\prod_{t=1}^{4} x_{1,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{1,j_t}\Big)\right| = O(1)$, and the total number of distinct j indexes is $O(p^3 D_0)$.

(b). When $\mathrm{card}(j_1, j_2, j_3, j_4) \geq 3$ and $\mathrm{card}(j_5, j_6, j_7, j_8) \geq 3$. If $\tilde{j}_2 - \tilde{j}_1 \leq k$,

$\tilde{j}_4 - \tilde{j}_3 \le k$, $\tilde{j}_6 - \tilde{j}_5 \le k$ and $\tilde{j}_8 - \tilde{j}_7 \le k$. If $\tilde{j}_5 - \tilde{j}_4 > D_0$,

$$\begin{aligned} & \left|\mathrm{E}\Big(\prod_{t=1}^{8} x_{1,j_t}\Big) - \mathrm{E}\Big(\prod_{t=1}^{4} x_{1,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{1,j_t}\Big)\right| \\ = \; & |\mathrm{cov}(x_{1,\tilde{j}_1}x_{1,\tilde{j}_2}x_{1,\tilde{j}_3}x_{1,\tilde{j}_4}, x_{1,\tilde{j}_5}x_{1,\tilde{j}_6}x_{1,\tilde{j}_7}x_{1,\tilde{j}_8})| = O(p^{-(8+\mu)}). \end{aligned}$$

If $\tilde{j}_5 - \tilde{j}_4 \le D_0$, the total number of distinct j indexes is $O(p^3 D_0 k^4)$.

(c). When $\mathrm{card}(j_1, j_2, j_3, j_4) \ge 3$ and $\mathrm{card}(j_5, j_6, j_7, j_8) = 2$. If $\tilde{j}_2 - \tilde{j}_1 > k$ or $\tilde{j}_4 - \tilde{j}_3 > k$, when $\kappa_2 = \max\{\tilde{j}_2 - \tilde{j}_1, \tilde{j}_3 - \tilde{j}_2, \tilde{j}_4 - \tilde{j}_3\} \le D_0$, the total number of j indexes is $O(p^3 D_0)$. When $\kappa_2 > D_0$, we know

$$|\mathrm{E}(x_{h,j_1}x_{h,j_2}x_{h,j_3}x_{h,j_4})| \le C\delta^{\frac{D_0\epsilon}{2+\epsilon}} = O(p^{-(8+\mu)}).$$

If $\tilde{j}_2 - \tilde{j}_1 \le k$ and $\tilde{j}_4 - \tilde{j}_3 \le k$. If $\tilde{j}_5 - \tilde{j}_4 > D_0$,

$$|\mathrm{cov}(x_{1,\tilde{j}_1}x_{1,\tilde{j}_2}x_{1,\tilde{j}_3}x_{1,\tilde{j}_4}, x_{1,\tilde{j}_5}x_{1,\tilde{j}_6}x_{1,\tilde{j}_7}x_{1,\tilde{j}_8})| \le C\delta^{\frac{D_0\epsilon}{2+\epsilon}} = O(p^{-(8+\mu)}).$$

If $\tilde{j}_5 - \tilde{j}_4 \le D_0$, the total number of distinct indexes is $O(p^3 D_0 k^2)$.

Then, $|D_{h,a,a,1,q}| = O(p^3 D_0) = o(p^4)$, and $\mathrm{Var}(T_{h,a,a,1}) = o(n^{-2})$ is proved.

□

S5.6 Proof of Lemma 6

Proof. As $D_{n,h} = \sum_{r=1}^{m} t_r A_{n,h,a_r}$, we have

$$\begin{aligned} & \sum_{h=1}^{n} \mathrm{E}(D_{n,h}^4) \\ = & \sum_{h=1}^{n} \sum_{r_1,r_2,r_3,r_4=1}^{m} \Big(\prod_{l=1}^{4} t_{r_l} \Big) \mathrm{E}\Big(\prod_{l=1}^{4} A_{n,h,a_{r_l}} \Big). \end{aligned}$$

To prove Lemma 6, it suffices to show $\sum_{h=1}^{n} \mathrm{E}(\prod_{l=1}^{4} A_{n,h,a_{r_l}}) = o(1)$. Recall the result of Lemma 4 and take use of the previous definitions, we have

$$\begin{aligned} \mathrm{E}(\prod_{l=1}^{4} A_{n,h,a_{r_l}}) = & \sum_{\substack{\mathbf{i}^{(l)} \in \mathcal{P}(h-1,a_l-1), l=1,\ldots,4; \\ (j_{2s-1},j_{2s}) \in \mathcal{J}, s=1,\ldots,4}} \Big\{ \prod_{l=1}^{4} c(n,a_l) \Big\} \\ & \times \mathrm{E}(\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2), \mathbf{i}^{(2)}|(j_3,j_4), \mathbf{i}^{(3)}|(j_5,j_6), \mathbf{i}^{(4)}|(j_7,j_8), h\}), \end{aligned} \tag{S5.34}$$

where

$$\begin{aligned} & \mathrm{E}(\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2), \mathbf{i}^{(2)}|(j_3,j_4), \mathbf{i}^{(3)}|(j_5,j_6), \mathbf{i}^{(4)}|(j_7,j_8), h\}) \\ = & \mathrm{E}\Big(\prod_{l=1}^{4} x_{h,j_{2l-1}} x_{h,j_{2l}} \Big) \mathrm{E}\Big(\prod_{l=1}^{4} \prod_{t=1}^{a_l-1} x_{i_t^{(l)},j_{2l-1}} x_{i_t^{(l)},j_{2l}} \Big). \end{aligned} \tag{S5.35}$$

Note that $\prod_{l=1}^{4} c(n,a_l) = \Theta(p^{-4} n^{-\sum_{l=1}^{4} a_l/2})$. Similar to the previous discussion, we first consider the order of n. Under $H_{k,0}$, (S5.35) $= 0$ when the in-

dex i only appears once in the sets $\{\mathbf{i}^{(l)}\}, l = 1, ..., 4$. Thus, $(\mathbf{i}^{(1)}, \mathbf{i}^{(2)}, \mathbf{i}^{(3)}, \mathbf{i}^{(4)})$ contains at most $(\sum_{l=1}^{4} a_l - 4)/2$ distinct indexes. Then

$$\begin{aligned} & \sum_{h=1}^{n} \sum_{\mathbf{i}^{(l)} \in \mathcal{P}(h-1, a_l - 1), l=1,...,4} \mathrm{E}(\mathbb{X}\{\mathbf{i}^{(1)}|(j_1, j_2), \mathbf{i}^{(2)}|(j_3, j_4), \mathbf{i}^{(3)}|(j_5, j_6), \mathbf{i}^{(4)}|(j_7, j_8), h\}) \\ = & \sum_{h=a}^{n} O(n^{\sum_{l=1}^{4} a_l/2-2}) = O(n^{\sum_{l=1}^{4} a_l/2-1}). \end{aligned}$$

To prove $\sum_{h=1}^{n} \mathrm{E}(\prod_{l=1}^{4} A_{n,h,a_{r_l}}) = o(1)$, it is suffices to show

$$\sum_{(j_{2s-1}, j_{2s}) \in \mathcal{J}, s=1,...,4} \mathrm{E}(\mathbb{X}\{\mathbf{i}^{(1)}|(j_1, j_2), \mathbf{i}^{(2)}|(j_3, j_4), \mathbf{i}^{(3)}|(j_5, j_6), \mathbf{i}^{(4)}|(j_7, j_8), h\}) = O(p^4). \quad \text{(S5.36)}$$

We define an index set

$$E_J = \{(j_1, ...j_8) : \mathrm{E}(\mathbb{X}\{\mathbf{i}^{(1)}|(j_1, j_2), \mathbf{i}^{(2)}|(j_3, j_4), \mathbf{i}^{(3)}|(j_5, j_6), \mathbf{i}^{(4)}|(j_7, j_8), h\}) \neq O(p^{-(8+\mu)})\},$$

with the μ is a small positive constant same as in D_0. Then

$$\sum_{(j_{2s-1}, j_{2s}) \in \mathcal{J}, s=1,...,4} \mathrm{E}(\mathbb{X}\{\mathbf{i}^{(1)}|(j_1, j_2), \mathbf{i}^{(2)}|(j_3, j_4), \mathbf{i}^{(3)}|(j_5, j_6), \mathbf{i}^{(4)}|(j_7, j_8), h\}) \mathbf{1}_{E_J^c} = o(1).$$

Therefore, we only need to prove

$$\sum_{(j_{2s-1}, j_{2s}) \in \mathcal{J}, s=1,...,4} \mathrm{E}(\mathbb{X}\{\mathbf{i}^{(1)}|(j_1, j_2), \mathbf{i}^{(2)}|(j_3, j_4), \mathbf{i}^{(3)}|(j_5, j_6), \mathbf{i}^{(4)}|(j_7, j_8), h\}) \mathbf{1}_{E_J} = O(p^4). \quad \text{(S5.37)}$$

Denote $\tilde{\mathbf{j}}_8 = (\tilde{j}_1, \cdots, \tilde{j}_8)$ is the ordered version of $(j_1, \cdots, j_8)$ with $\tilde{j}_1 \leq \tilde{j}_2 \leq \tilde{j}_3 \leq \tilde{j}_4 \leq \tilde{j}_5 \leq \tilde{j}_6 \leq \tilde{j}_7 \leq \tilde{j}_8$. Before proving, we give a proposition as follows.

Proposition S5.1. *Under Condition 1 and 2, for the indexes in $\tilde{\mathbf{j}}_8$ which satisfy $\mathbf{1}_{E_J} = 1$, we have*

(P1). For the single index $\tilde{j}_k \in \tilde{\mathbf{j}}_8$, at least one of $\tilde{j}_k - \tilde{j}_{k-1} \leq D_0$ and $\tilde{j}_{k+1} - \tilde{j}_k \leq D_0$ is true. For $\tilde{j}_1$ and $\tilde{j}_8$, they satisfy $\tilde{j}_2 - \tilde{j}_1 \leq D_0$ and $\tilde{j}_8 - \tilde{j}_7 \leq D_0$.

(P2). For the pair index $(\tilde{j}_{k-1}, \tilde{j}_k) \in \tilde{\mathbf{j}}_8$ with $\tilde{j}_k - \tilde{j}_{k-1} > k$, there is at least one of $\tilde{j}_{k-1} - \tilde{j}_{k-2} \leq D_0$ and $\tilde{j}_{k+1} - \tilde{j}_k \leq D_0$ is true. For the pair $(\tilde{j}_1, \tilde{j}_2)$ with $\tilde{j}_2 - \tilde{j}_1 > k$ and the pair $(\tilde{j}_7, \tilde{j}_8)$ with $\tilde{j}_8 - \tilde{j}_7 > k$, they satisfy $\tilde{j}_3 - \tilde{j}_2 \leq D_0$ and $\tilde{j}_7 - \tilde{j}_6 \leq D_0$, respectively.

The proof of Proposition S5.1 is given in Section S5.12. Now, we continue to prove (S5.37). We first consider the tuple $(\tilde{j}_1, \tilde{j}_2, \tilde{j}_7, \tilde{j}_8)$, then consider the tuple $(\tilde{j}_3, \tilde{j}_4, \tilde{j}_5, \tilde{j}_6)$.

Case 1. When both $\tilde{j}_2 - \tilde{j}_1 \leq k$ and $\tilde{j}_8 - \tilde{j}_7 \leq k$ are true.

(a). When $\tilde{j}_5 - \tilde{j}_4 > D_0$, applying (P1) in Proposition S5.1 on the single indexes $\tilde{j}_4$ and $\tilde{j}_5$, then $\tilde{j}_4 - \tilde{j}_3 \leq D_0$ and $\tilde{j}_6 - \tilde{j}_5 \leq D_0$ by (P2) in Proposition S5.1. Furthermore, if at least one of $k < \tilde{j}_4 - \tilde{j}_3 \leq D_0$ and $k < \tilde{j}_6 - \tilde{j}_5 \leq D_0$ is true. Without loss of generality, we assume

$k < \tilde{j}_4 - \tilde{j}_3 \leq D_0$, by (P2) in Proposition S5.1, there is $\tilde{j}_3 - \tilde{j}_2 \leq D_0$. Then the summation over indexes $\tilde{\mathbf{j}}_8$ is $O(p^3 D_0^3 k^2)$.

(b). When $k < \tilde{j}_5 - \tilde{j}_4 \leq D_0$, at least one of $\tilde{j}_4 - \tilde{j}_3 \leq D_0$ and $\tilde{j}_6 - \tilde{j}_5 \leq D_0$ is true. The summation over indexes $\tilde{\mathbf{j}}_8$ is $O(p^3 D_0^3 k^2)$.

(c). When $\tilde{j}_5 - \tilde{j}_4 \leq k$, we further consider the tuples $(\tilde{j}_3, \tilde{j}_4)$ and $(\tilde{j}_5, \tilde{j}_6)$. If at least one of $\tilde{j}_4 - \tilde{j}_3 > D_0$ and $\tilde{j}_6 - \tilde{j}_5 > D_0$ is true, assume $\tilde{j}_4 - \tilde{j}_3 > D_0$ and $\tilde{j}_6 - \tilde{j}_5 \leq D_0$. It is obvious that $\tilde{j}_3 - \tilde{j}_2 \leq D_0$, the the summation order of $\tilde{J}_8$ is $O(p^3 D_0^2 k^3)$. If both $\tilde{j}_4 - \tilde{j}_3 \leq D_0$ and $\tilde{j}_6 - \tilde{j}_5 \leq D_0$ is true, the the summation order of $\tilde{\mathbf{j}}_8$ is $O(p^3 D_0^2 k^3)$.

Case 2. When both $\tilde{j}_2 - \tilde{j}_1 > k$ and $\tilde{j}_8 - \tilde{j}_7 > k$ are true. By (P1) and (P2) in Proposition S5.1, the order of $(\tilde{j}_1, \tilde{j}_2, \tilde{j}_3)$ and $(\tilde{j}_6, \tilde{j}_7, \tilde{j}_8)$ are both $O(pD_0^2)$.

(a). When $\tilde{j}_5 - \tilde{j}_4 > D_0$, then $\tilde{j}_4 - \tilde{j}_3 \leq D_0$ and $\tilde{j}_6 - \tilde{j}_5 \leq D_0$. The summation order is $O(p^2 D_0^6)$.

(b). When $\tilde{j}_5 - \tilde{j}_4 \leq D_0$, the summation order over $\tilde{\mathbf{j}}_8$ is $O(p^3 D_0^5)$.

Case 3. When one of $\tilde{j}_2 - \tilde{j}_1 > k$ and $\tilde{j}_8 - \tilde{j}_7 > k$ is true. We assume $\tilde{j}_2 - \tilde{j}_1 > k$ and $\tilde{j}_8 - \tilde{j}_7 \leq k$. The order of $(\tilde{j}_1, \tilde{j}_2, \tilde{j}_3)$ is $O(pD_0^2)$. Still consider the tuple $(\tilde{j}_4, \tilde{j}_5)$ like Case 1.

(a). When $\tilde{j}_5 - \tilde{j}_4 > D_0$, by (P1), we have $\tilde{j}_4 - \tilde{j}_3 \leq D_0$ and $\tilde{j}_6 - \tilde{j}_5 \leq D_0$. The summation order over $\tilde{\mathbf{j}}_8$ is $O(p^3 D_0^3)$.

(b). When $k < \tilde{j}_5 - \tilde{j}_4 \leq D_0$, at least one of $\tilde{j}_4 - \tilde{j}_3 \leq D_0$ and $\tilde{j}_6 - \tilde{j}_5 \leq D_0$

is true. When $\tilde{j}_4 - \tilde{j}_3 \leq D_0$, $\tilde{j}_6 - \tilde{j}_5 > D_0$, we have $\tilde{j}_7 - \tilde{j}_6 \leq D_0$, the summation order is $O(p^2 D_0^5 k)$; when $\tilde{j}_4 - \tilde{j}_3 > D_0$, $\tilde{j}_6 - \tilde{j}_5 \leq D_0$, we have $\tilde{j}_3 - \tilde{j}_2 \leq D_0$, the summation order is $O(p^3 D_0^4 k)$; when $\tilde{j}_4 - \tilde{j}_3 \leq D_0$, $\tilde{j}_6 - \tilde{j}_5 \leq D_0$, the summation order is $O(p^2 D_0^5 k)$.

(c). When $\tilde{j}_5 - \tilde{j}_4 \leq k$, furthermore, we consider the tuple $(\tilde{j}_3, \tilde{j}_4)$ and $(\tilde{j}_5, \tilde{j}_6)$. If at least one of $\tilde{j}_4 - \tilde{j}_3 > D_0$ and $\tilde{j}_6 - \tilde{j}_5 > D_0$ is true, assume $\tilde{j}_4 - \tilde{j}_3 > D_0$ and $\tilde{j}_6 - \tilde{j}_5 \leq D_0$. It is obvious that $\tilde{j}_3 - \tilde{j}_2 \leq D_0$, the the summation order is $O(p^3 D_0^3 k^2)$. If both $\tilde{j}_4 - \tilde{j}_3 \leq D_0$ and $\tilde{j}_6 - \tilde{j}_5 \leq D_0$ is true, the the summation order of $\tilde{J}_8$ is $O(p^2 D_0^4 k^2)$.

Combining the results obtained, we get (S5.37). Thus, Lemma 6 is proved.

□

S5.7 Proof of Lemma 7

Proof. We first prove $\hat{\sigma}_1^2(a)/\mathrm{E}\{\hat{\sigma}_1^2(a)\} \xrightarrow{P} 1$. It is suffices to prove

$$\mathrm{Var}\{\hat{\sigma}_1^2(a)\}/\mathrm{E}^2\{\hat{\sigma}_1^2(a)\} \to 0.$$

We have

$$\mathrm{Var}\{\hat{\sigma}_1^2(a)\} = \mathrm{E}\{[\hat{\sigma}_1^2(a)]^2\} - \mathrm{E}^2\{\hat{\sigma}_1^2(a)\}$$

$$= 2\frac{(2a!)^2}{(P_a^n)^4} \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a)\\(j_{2s-1},j_{2s})\in\mathcal{J},s=1,\ldots,4\\|j_1-j_3|\le k,|j_2-j_4|\le k\\|j_5-j_7|\le k,|j_6-j_8|\le k}} \Bigg[\mathrm{E}\Big\{\prod_{t=1}^{a}\Big(\prod_{c=1}^{4}x_{i_t,j_c}\Big)\Big(\prod_{c=5}^{8}x_{\tilde{i}_t,j_c}\Big)\Big\} - \Big\{\mathrm{E}\Big(\prod_{c=1}^{4}x_{i_t,j_c}\Big)\mathrm{E}\Big(\prod_{c=5}^{8}x_{i_t,j_c}\Big)\Big\}^a\Bigg].$$

We know

$$\Bigg[\mathrm{E}\Big\{\prod_{t=1}^{a}\Big(\prod_{c=1}^{4}x_{i_t,j_c}\Big)\Big(\prod_{c=5}^{8}x_{\tilde{i}_t,j_c}\Big)\Big\} - \Big\{\mathrm{E}\Big(\prod_{c=1}^{4}x_{i,j_c}\Big)\mathrm{E}\Big(\prod_{c=5}^{8}x_{i,j_c}\Big)\Big\}^a\Bigg]$$

is unequal to 0 only when $\mathrm{card}(\{\mathbf{i}\}\cup\{\tilde{\mathbf{i}}\}) \le 2a-1$. Based on the previous discussion, we have $\mathrm{Var}\{\hat{\sigma}_1^2(a)\} = O(p^4 n^{-2a-1})$. Thus, $\mathrm{Var}\{\hat{\sigma}_1^2(a)\}/\mathrm{E}^2\{\hat{\sigma}_1^2(a)\} \to 0$ as $\mathrm{E}\{\hat{\sigma}_1^2(a)\} = \Theta(n^{-a}p^2)$.

We next prove $\hat{\sigma}_2^2(a)/\mathrm{E}\{\hat{\sigma}_1^2(a)\} \xrightarrow{P} 0$. It is equal to prove $\mathrm{E}[\hat{\sigma}_2^2(a)] = o(1)[\mathrm{E}\{\hat{\sigma}_1^2(a)\}]^2 = o(n^{-2a}p^4)$ by Markov's inequality. Given a tuple $\mathbf{i} \in \mathcal{P}(n,a)$, for $c = 1,2,3,4$, let $\mathbf{i}_{(a-r_c)}^{(c)}$ represent a sub-tuple of $\mathbf{i}$ with length $a - r_c$, and define $\mathcal{I}(\mathbf{i}, a - r_c)$ to be the collection of sub-tuples of $\mathbf{i}$ with length $a - r_c$. Besides that, we define $\mathcal{C}(n,a)$ as the the collection of sub-tuples of $\mathbf{i}$ satisfying $1 \le i_1, \ldots, i_a \le n$ without constraining the elements to

be different. For $a = 1$, we have the following equation

$$\begin{aligned}
&\left\{\prod_{t=1}^{4}(x_{i,j_t}-\bar{x}_{j_t})-\prod_{t=1}^{4}x_{i,j_t}\right\}\\
=&\sum_{\substack{0\le s_c+r_c+w_c\le 1,c=1,2\\ s_c+r_c+w_c=0,c=1,2 \text{ can't be held at the same time}}}(x_{i,j_1}x_{i,j_2})^{1-s_1-r_1-w_1}(-x_{i,j_1}\bar{x}_{j_2})^{s_1}(-x_{i,j_2}\bar{x}_{j_1})^{r_1}(\bar{x}_{j_1}\bar{x}_{j_2})^{w_1}\\
&\times(x_{i,j_3}x_{i,j_4})^{1-s_2-r_2-w_2}(-x_{i,j_3}\bar{x}_{j_4})^{s_2}(-x_{i,j_4}\bar{x}_{j_3})^{r_2}(\bar{x}_{j_3}\bar{x}_{j_4})^{w_2}.
\end{aligned}$$

Then, we rewrite $\hat{\sigma}_2^2(a) = \sum_{\substack{0\le s_c+r_c+w_c\le a,c=1,2\\ s_c+r_c+w_c=0,c=1,2 \text{ can't be held at the same time}}} T_{s_c,r_c}$, where

$$\begin{aligned}
T_{s_c,r_c} &= \frac{2a!}{(P_a^n)^2}\sum_{\substack{\mathbf{i}\in\mathcal{P}(n,a)\\ \mathbf{i}^{(c)}_{(a-w_c)}\in\mathcal{I}(\mathbf{i},a-w_c)\\ (j_{2s-1},j_{2s})\in\mathcal{J},s=1,2,\\ |j_1-j_3|\le k,|j_2-j_4|\le k}} C_{s_c,r_c,w_c}(-1)^{\sum_{c=1}^{2}(r_c+s_c)}\bar{x}_{j_1}^{r_1+w_1}\bar{x}_{j_2}^{s_1+w_1}\bar{x}_{j_3}^{r_2+w_2}\bar{x}_{j_4}^{s_2+w_2}\\
&\times\prod_{t_c=1}^{s_1}x_{i^{(1)}_{t_c},j_1}\prod_{t_c=s_1+1}^{s_1+r_1}x_{i^{(1)}_{t_c},j_2}\prod_{t_c=s_1+r_1+1}^{a-w_1}x_{i^{(1)}_{t_c},j_1}x_{i^{(1)}_{t_c},j_2}\\
&\times\prod_{t_c=1}^{s_2}x_{i^{(2)}_{t_c},j_3}\prod_{t_c=s_2+1}^{s_2+r_2}x_{i^{(2)}_{t_c},j_4}\prod_{t_c=s_2+r_2+1}^{a-w_2}x_{i^{(2)}_{t_c},j_3}x_{i^{(2)}_{t_c},j_4}.
\end{aligned}$$

Thus, we have

$$\mathrm{E}[T_{s_c,r_c}^2] = \frac{4a!^2}{(P_a^n)^4}\sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a)\\ \mathbf{i}^{(c)}_{(a-w_c)}\in\mathcal{I}(\mathbf{i},a-w_c)\\ \tilde{\mathbf{i}}^{(c)}_{(a-w_c)}\in\mathcal{I}(\tilde{\mathbf{i}},a-w_c)}}\sum_{\substack{(j_{2s-1},j_{2s}),(\tilde{j}_{2s-1},\tilde{j}_{2s})\in\mathcal{J},s=1,2,\\ |j_1-j_3|\le k,|j_2-j_4|\le k,|\tilde{j}_1-\tilde{j}_3|\le k,|\tilde{j}_2-\tilde{j}_4|\le k}} C_{s_c,r_c,w_c}^2$$

$$\times \prod_{c=1}^{2} \Bigg\{ (\bar{x}_{j_{2c-1}} \bar{x}_{\tilde{j}_{2c-1}})^{r_c+w_c} (\bar{x}_{j_{2c}} \bar{x}_{\tilde{j}_{2c}})^{s_c+w_c} \prod_{t_c=1}^{s_c} x_{i_{t_c}^{(c)}, j_{2c-1}} x_{\tilde{i}_{t_c}^{(c)}, \tilde{j}_{2c-1}} \prod_{t_c=s_c+1}^{s_c+r_c} x_{i_{t_c}^{(c)}, j_{2c}} x_{\tilde{i}_{t_c}^{(c)}, \tilde{j}_{2c}} \prod_{t_c=s_c+r_c+1}^{a-w_c} x_{i_{t_c}^{(c)}, j_{2c-1}} x_{i_{t_c}^{(c)}, j_{2c}} x_{\tilde{i}_{t_c}^{(c)}, \tilde{j}_{2c-1}} x_{\tilde{i}_{t_c}^{(c)}, \tilde{j}_{2c}} \Bigg\}.$$

As $\bar{x}_j = n^{-1} \sum_{i=1}^{n} x_{i,j}/n$, we have

$$\begin{aligned} \mathrm{E}[T^2_{s_c,r_c}] &= \frac{4a!^2}{(P_a^n)^4 n^{\sum_{c=1}^{2} 2(r_c+s_c+2w_c)}} \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}} \in \mathcal{P}(n,a) \\ \mathbf{i}_{(a-w_c)}^{(c)} \in \mathcal{I}(\mathbf{i}, a-w_c) \\ \tilde{\mathbf{i}}_{(a-w_c)}^{(c)} \in \mathcal{I}(\tilde{\mathbf{i}}, a-w_c)}} \sum_{\substack{(j_{2s-1}, j_{2s}), (\tilde{j}_{2s-1}, \tilde{j}_{2s}) \in \mathcal{J}, s=1,2, \\ |j_1-j_3| \le k, |j_2-j_4| \le k, |\tilde{j}_1-\tilde{j}_3| \le k, |\tilde{j}_2-\tilde{j}_4| \le k}} \\ &\quad \times C^2_{s_c,r_c,w_c} \times \mathscr{T}\{\mathbf{i}_{(a-w_c)}^{(c)}, \tilde{\mathbf{i}}_{(a-w_c)}^{(c)}, \mathbf{m}^{(c)}, \tilde{\mathbf{m}}^{(c)}, \mathbf{e}^{(c)}, \tilde{\mathbf{e}}^{(c)}; c = 1, 2\}, \end{aligned}$$

where

$$\mathscr{T}\{\mathbf{i}_{(a-w_c)}^{(c)}, \tilde{\mathbf{i}}_{(a-w_c)}^{(c)}, \mathbf{m}^{(c)}, \tilde{\mathbf{m}}^{(c)}, \mathbf{e}^{(c)}, \tilde{\mathbf{e}}^{(c)}; c = 1, 2\}$$

$$\begin{aligned} &= \mathrm{E}\Bigg\{ \prod_{c=1}^{2} \prod_{\tilde{t}_c=1}^{r_c+w_c} x_{m_{\tilde{t}_c}^{(c)}, j_{2c-1}} x_{\tilde{m}_{\tilde{t}_c}^{(c)}, \tilde{j}_{2c-1}} \prod_{\tilde{t}_c=1}^{s_c+w_c} x_{e_{\tilde{t}_c}^{(c)}, j_{2c}} x_{\tilde{e}_{\tilde{t}_c}^{(c)}, \tilde{j}_{2c}} \\ &\quad \times \prod_{t_c=1}^{s_c} x_{i_{t_c}^{(c)}, j_{2c-1}} x_{\tilde{i}_{t_c}^{(c)}, \tilde{j}_{2c-1}} \prod_{t_c=s_c+1}^{s_c+r_c} x_{i_{t_c}^{(c)}, j_{2c}} x_{\tilde{i}_{t_c}^{(c)}, \tilde{j}_{2c}} \prod_{t_c=s_c+r_c+1}^{a-w_c} x_{i_{t_c}^{(c)}, j_{2c-1}} x_{i_{t_c}^{(c)}, j_{2c}} x_{\tilde{i}_{t_c}^{(c)}, \tilde{j}_{2c-1}} x_{\tilde{i}_{t_c}^{(c)}, \tilde{j}_{2c}} \Bigg\}. \end{aligned}$$

Since $\mathrm{E}(x_{i,j}) = 0$, $\mathscr{T}\{\mathbf{i}_{(a-w_c)}^{(c)}, \tilde{\mathbf{i}}_{(a-w_c)}^{(c)}, \mathbf{m}^{(c)}, \tilde{\mathbf{m}}^{(c)}, \mathbf{e}^{(c)}, \tilde{\mathbf{e}}^{(c)}; c = 1, 2\} \neq 0$ only

when

$$\begin{aligned} & \mathrm{card}\Big(\bigcup_{c=1}^{2}\{\mathbf{i}^{(c)}_{(a-w_c)}\}\cup\{\tilde{\mathbf{i}}^{(c)}_{(a-w_c)}\}\cup\{\mathbf{m}^{(c)}\}\cup\{\tilde{\mathbf{m}}^{(c)}\}\cup\{\mathbf{e}^{(c)}\}\cup\{\tilde{\mathbf{e}}^{(c)}\}\Big) \\ - \quad & \mathrm{card}\Big(\bigcup_{c=1}^{2}\{\mathbf{i}^{(c)}_{(a-w_c)}\}\cup\{\tilde{\mathbf{i}}^{(c)}_{(a-w_c)}\}\Big) \leq \sum_{c=1}^{2}(r_c+s_c+2w_c). \end{aligned}$$

Moreover, $\mathrm{card}\big(\bigcup_{c=1}^{2}\{\mathbf{i}^{(c)}_{(a-w_c)}\}\cup\{\tilde{\mathbf{i}}^{(c)}_{(a-w_c)}\}\big) \leq 2a$ because $\mathbf{i}^{(c)}_{(a-w_c)}, \tilde{\mathbf{i}}^{(c)}_{(a-w_c)}$ are sub-tuples of $\mathbf{i}$ and $\tilde{\mathbf{i}} \in \mathcal{P}(n,a)$ for $c=1,2$. Therefore,

$$\begin{aligned} & \mathrm{card}\Big(\bigcup_{c=1}^{2}\{\mathbf{i}^{(c)}_{(a-w_c)}\}\cup\{\tilde{\mathbf{i}}^{(c)}_{(a-w_c)}\}\cup\{\mathbf{m}^{(c)}\}\cup\{\tilde{\mathbf{m}}^{(c)}\}\cup\{\mathbf{e}^{(c)}\}\cup\{\tilde{\mathbf{e}}^{(c)}\}\Big) \\ \leq \quad & \sum_{c=1}^{2}(r_c+s_c+2w_c)+2a. \end{aligned}$$

By Condition 3 and $\sum_{c=1}^{2}(r_c+s_c+2w_c) \geq 1$, we have

$$\begin{aligned} \mathrm{E}[T^2_{s_c,r_c}] &= O(1)p^4k^4n^{-4a-\sum_{c=1}^{2}2(r_c+s_c+2w_c)+\sum_{c=1}^{2}(r_c+s_c+2w_c)+2a} \\ &= O(1)p^4k^4n^{-2a-\sum_{c=1}^{2}(r_c+s_c+2w_c)} = o(p^4n^{-2a}). \end{aligned}$$

□

S5.8 Proof of Lemma 8

Proof. To finish the proof of Lemma 8, we need to prove the following three results.

1. $\mathrm{Var}[\mathcal{U}_{1,1}(a)] = \Theta(n^{-a}p^2k^2)$. $\mathrm{E}[\mathcal{U}_{1,1}(a)] = 0$ because $(j_1, j_2) \in J_A^c$. Thus, $\mathrm{Var}[\mathcal{U}_{1,1}(a)] = \mathrm{E}[\mathcal{U}_{1,1}(a)^2]$.

$$\mathrm{Var}[\mathcal{U}_{1,1}(a)] = \sum_{(j_1,j_2),(j_3,j_4)\in J_A^c} (P_a^n)^{-2} \sum_{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a)} \mathrm{E}\bigg[\prod_{t=1}^{a} x_{i_t,j_1}x_{i_t,j_2}x_{\tilde{i}_t,j_3}x_{\tilde{i}_t,j_4}\bigg].$$

As $(j_1, j_2), (j_3, j_4) \in J_A^c$, $\mathrm{Var}[\mathcal{U}_{1,1}(a)] \neq 0$ only when $\mathbf{i} = \tilde{\mathbf{i}}$. Thus,

$$\begin{aligned}\mathrm{Var}[\mathcal{U}_{1,1}(a)] &= \sum_{(j_1,j_2),(j_3,j_4)\in J_A^c} (P_a^n)^{-1} a! \bigg[\mathrm{E}\Big(\prod_{s=1}^{4} x_{i,j_s}\Big)\bigg]^a \\ &= \sum_{(j_1,j_2),(j_3,j_4)\in J_A^c} (P_a^n)^{-1} a! [\kappa(\sigma_{j_1j_3}\sigma_{j_2j_4} + \sigma_{j_1j_4}\sigma_{j_2j_3})]^a, \quad \text{(S5.38)}\end{aligned}$$

where the second equation holds under Condition 5. Let

$$B \triangleq \mathrm{card}\{(j_s, j_t) : |j_s - j_l| \le k,\ j_s, j_t = 1, \cdots, p \text{ with } s = 1, 2, l = 3, 4\}$$

denotes the number of the four inequalities $|j_1 - j_3| \le k, |j_2 - j_4| \le k, |j_1 - j_4| \le k$ and $|j_2 - j_3| \le k$ established. It is obvious that $B \neq 4$ as $(j_1, j_2), (j_3, j_4) \in J_A^c$. Now, we evaluate (S5.38) by discussing B.

Case 1. $B = 3$. Without loss of generality, we assume $|j_1 - j_3| \le k, |j_2 - j_4| \le k, |j_1 - j_4| \le k$ and $k < |j_2 - j_3| < p$. Then

$$\begin{aligned}\text{(S5.38)} &\le Cn^{-a} \sum_{(j_1,j_2),(j_3,j_4)\in J_A^c} |\sigma_{j_1j_3}\sigma_{j_2j_4} + \sigma_{j_1j_4}\sigma_{j_2j_3}|^a \times \mathbf{1}_{\{B=3\}} \\ &\le Cn^{-a}pk^3(\Theta(1) + \rho^a) = o(n^{-a}p^2),\end{aligned}$$

since $n^{-a}p\rho^a = n^{-a}|J_A|\rho^a \times p/|J_A| = o(n^{-a}p^2)$ and $|J_A|\rho^a = O(n^{-a/2}p)$ as $\rho = O(|J_A|^{-1/a_t}p^{1/a_t}n^{-1/2})$.

Case 2. $B = 2$. We examine (S5.38) with respect to two different subcases.

Case 2.1. When $|j_1 - j_3| \le k, |j_2 - j_4| \le k$ and $k < |j_1 - j_4| < p, k < |j_2 - j_3| < p$. If $(j_1, j_4) \in J_A^c$ or $(j_2, j_3) \in J_A^c$ holds,

$$\begin{aligned}(\text{S5.38}) &= \sum_{\substack{(j_1,j_2),(j_3,j_4)\in J_A^c \\ |j_1-j_3|\le k, |j_2-j_4|\le k}} (P_a^n)^{-1} a! \kappa^a \sigma_{j_1 j_3}^a \sigma_{j_2 j_4}^a \times \mathbf{1}_{\left\{\{(j_1,j_4)\in J_A^c\}\cup\{(j_2,j_3)\in J_A^c\}\right\}} \\ &= \Theta(n^{-a}p^2k^2).\end{aligned}$$

If $(j_1, j_4), (j_2, j_3) \in J_A$, $|(\text{S5.38})| \le \Theta(1)n^{-a}|J_A|k^2 + n^{-a}|J_A|k^2\rho^{2a} = o(n^{-a}p^2)$ since $|J_A| = o(p^2)$, $|J_A|$ is postive integer and $n^{-a}|J_A|k^2\rho^{2a} = n^{-a}k^2|J_A|^{-1}(|J_A|\rho^a)^2 = o(n^{-a}p^2)$.

Case 2.2. When $|j_1 - j_3| \le k, |j_1 - j_4| \le k$ and $k < |j_2 - j_4| < p, k < |j_2 - j_3| < p$. (S5.38) $\neq 0$ only when $(j_2, j_4), (j_2, j_3) \in J_A$. Thus,

$$\begin{aligned}(\text{S5.38}) &\le \sum_{\substack{(j_1,j_2),(j_3,j_4)\in J_A^c \\ (j_2,j_4),(j_2,j_3)\in J_A \\ |j_1-j_3|\le k, |j_1-j_4|\le k}} (P_a^n)^{-1} a! \kappa^a |\sigma_{j_1 j_3}\sigma_{j_2 j_4} + \sigma_{j_1 j_4}\sigma_{j_2 j_3}|^a \\ &\le O(1)n^{-a}|J_A|k^2\rho^a = o(n^{-a}p^2).\end{aligned}$$

Case 3. $B = 1$. Without of loss generality, we assume $|j_1 - j_3| \le k$. To

ensure (S5.38) $\neq 0$, we discuss this term by three subcases.

Case 3.1. When $(j_2, j_4), (j_1, j_4), (j_2, j_3) \in J_A$,

$$(\text{S5.38}) \quad \leq \quad Cn^{-a}|J_A| \max_{j=1,\ldots,p} |J_j| k(\rho^a + \rho^{2a}) = o(n^{-a}p^2),$$

since

$$\begin{aligned} n^{-a}|J_A| \max_{j=1,\ldots,p} |J_j|\rho^a &= n^{-a} \max_{j=1,\ldots,p} |J_j| \times (|J_A|\rho^a) \\ &= O(n^{-a}p^2) \times n^{-a/2} \times \frac{\max\limits_{j=1,\ldots,p} |J_j|}{p} = o(n^{-a}p^2), \end{aligned}$$

and

$$\begin{aligned} n^{-a}|J_A| \max_{j=1,\ldots,p} |J_j|\rho^{2a} &= n^{-a} \frac{\max\limits_{j=1,\ldots,p} |J_j|}{|J_A|} \times (|J_A|\rho^a)^2 \\ &= O(n^{-a}p^2) \times n^{-a} \times \frac{\max\limits_{j=1,\ldots,p} |J_j|}{|J_A|} = o(n^{-a}p^2). \end{aligned}$$

Case 3.2. When $(j_2, j_4) \in J_A$ and at least one of $(j_1, j_4) \in J_A^c$ and $(j_2, j_3) \in J_A^c$ holds, (S5.38) $\leq O(1)n^{-a}|J_A|pk\rho^a = o(n^{-a}p^2)$.

Case 3.3. When $(j_2, j_4) \in J_A^c$ and $(j_1, j_4), (j_2, j_3) \in J_A$ hold, (S5.38) $\leq O(1)n^{-a}|J_A| \max\limits_{j=1,\ldots,p} |J_j| k\rho^{2a} = o(n^{-a}p^2)$.

Case 4. $B = 0$. In this case, (S5.38) $\neq 0$ when $(j_1, j_3), (j_2, j_4) \in J_A$ or $(j_1, j_4), (j_2, j_3) \in J_A$. Then, (S5.38) $\leq O(n^{-a}|J_A|^2\rho^{2a}) = o(n^{-a}p^2)$.

In summary,

$$\mathrm{Var}[\mathcal{U}_{1,1}(a)] \simeq \sum_{\substack{(j_1,j_2),(j_3,j_4)\in J_A^c \\ (j_1,j_4),(j_2,j_3)\in J_A^c \\ |j_1-j_3|\le k,|j_2-j_4|\le k}} 2(P_a^n)^{-1}a!\kappa^a\sigma^a_{j_1j_3}\sigma^a_{j_2j_4}.$$

2. $\mathrm{Var}[\mathcal{U}_{1,2}(a)] = o(n^{-a}p^2k^2)$. By Cauchy-Schwarz inequality,

$$\mathrm{Var}[\mathcal{U}_{1,2}(a)] \le C\sum_{c=1}^{a}\mathrm{var}\left\{\sum_{(j_1,j_2)\in J_A^c} H(c,j_1,j_2)\right\},$$

where C is some constant. Now, we consider

$$\begin{aligned}\mathrm{Var}\left\{\sum_{(j_1,j_2)\in J_A^c} H(c,j_1,j_2)\right\} &= \mathrm{E}\left[\left\{\sum_{(j_1,j_2)\in J_A^c} H(c,j_1,j_2)\right\}^2\right] \\ &= F^2(a,c)\sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a+c) \\ (j_1,j_2),(j_3,j_4)\in J_A^c}} Q_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4),\end{aligned}$$

where

$$\begin{aligned}Q_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4) &= \mathrm{E}\Bigg[\prod_{l=1}^{a-c}(x_{i_l,j_1}x_{i_l,j_2})\prod_{s=a-c+1}^{a}x_{i_s,j_1}\prod_{t=a+1}^{a+c}x_{i_t,j_2} \\ &\quad\times\prod_{\tilde{l}=1}^{a-c}(x_{\tilde{i}_{\tilde{l}},j_3}x_{\tilde{i}_{\tilde{l}},j_4})\prod_{\tilde{s}=a-c+1}^{a}x_{\tilde{i}_{\tilde{s}},j_3}\prod_{\tilde{t}=a+1}^{a+c}x_{\tilde{i}_{\tilde{t}},j_4}\Bigg].\end{aligned} \tag{S5.39}$$

It is obvious that $Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \neq 0$ only when $\mathbf{i} = \tilde{\mathbf{i}}$. Then,

$$\left| \sum_{\mathbf{i},\tilde{\mathbf{i}} \in \mathcal{P}(n,a+c)} Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \right| = O(n^{a+c}).$$

To facilitate the discussion, we decompose $\{\mathbf{i}\}$ and $\{\tilde{\mathbf{i}}\}$ into three disjoint sets, respectively. That is, $\{\mathbf{i}\} = \cup_{s=1}^{3} \{\mathbf{i}\}_s$ and $\{\tilde{\mathbf{i}}\} = \cup_{s=1}^{3} \{\tilde{\mathbf{i}}\}_s$, where

$$\{\mathbf{i}\}_1 = \{i_1, \cdots, i_{a-c}\},\ \{\mathbf{i}\}_2 = \{i_{a-c+1}, \cdots, i_a\},\ \{\mathbf{i}\}_3 = \{i_{a+1}, \cdots, i_{a+c}\},$$

$$\{\tilde{\mathbf{i}}\}_1 = \{\tilde{i}_1, \cdots, \tilde{i}_{a-c}\},\ \{\tilde{\mathbf{i}}\}_2 = \{\tilde{i}_{a-c+1}, \cdots, \tilde{i}_a\},\ \{\tilde{\mathbf{i}}\}_3 = \{\tilde{i}_{a+1}, \cdots, \tilde{i}_{a+c}\}.$$

Case 1. When $1 \leq c \leq a-1$, $\{\mathbf{i}\}_1 \neq \emptyset$. Without loss of generality, we assume there exists an index $i \in \{\mathbf{i}\}_1$.

Case 1.1. If $i \in \{\tilde{\mathbf{i}}\}_1$, we have $Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) = \mathrm{E}(\prod_{s=1}^{4} x_{i,j_s}) \times \mathrm{E}(\text{other terms})$. Conducting the discussion to (S5.38), then $\left| \sum_{(j_1,j_2),(j_3,j_4) \in J_A^c} Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \right| = O(p^2)$.

Case 1.2. If $i \in \{\tilde{\mathbf{i}}\}_2$ or $\{\tilde{\mathbf{i}}\}_3$, then $Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) = \mathrm{E}(x_{i,j_1} x_{i,j_2} x_{i,j_3}) \times \mathrm{E}(\text{other terms}) = 0$ or $\mathrm{E}(x_{i,j_1} x_{i,j_2} x_{i,j_4}) \times \mathrm{E}(\text{other terms}) = 0$ under Condition 5.

Case 2. When $c = a$, $\{\mathbf{i}\}_1 = \emptyset$ and $\{\mathbf{i}\}_2, \{\mathbf{i}\}_3 \neq \emptyset$. $Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \neq 0$ only when $\{\mathbf{i}\}_2 \cup \{\mathbf{i}\}_3 = \{\tilde{\mathbf{i}}\}_2 \cup \{\tilde{\mathbf{i}}\}_3$. Without loss of generality, we assume $r = \mathrm{card}(\{\mathbf{i}\}_2 \cap \{\tilde{\mathbf{i}}\}_2)$, then $\mathrm{card}(\{\mathbf{i}\}_2 \cap \{\tilde{\mathbf{i}}\}_3) = c - r$, $\mathrm{card}(\{\mathbf{i}\}_3 \cap \{\tilde{\mathbf{i}}\}_2) = c - r$

and $\text{card}(\{\mathbf{i}\}_3 \cap \{\tilde{\mathbf{i}}\}_3) = r$, where $r = 0, \cdots, a$. It follows that

$$\begin{aligned} & Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \\ = \; & [\mathrm{E}(x_{1,j_1}x_{1,j_3})\mathrm{E}(x_{1,j_2}x_{1,j_4})]^r[\mathrm{E}(x_{1,j_1}x_{1,j_4})\mathrm{E}(x_{1,j_2}x_{1,j_3})]^{a-r} \\ = \; & (\sigma_{j_1j_3}\sigma_{j_2j_4})^r(\sigma_{j_2j_3}\sigma_{j_1j_4})^{a-r}. \end{aligned} \tag{S5.40}$$

If $B = 3$, we have

$$\left| \sum_{(j_1,j_2),(j_3,j_4)\in J_A^c} Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \times \mathbf{1}_{\{B=3\}} \right| \leq \Theta(1)pk^3(\rho^r + \rho^{a-r}) = o(p^2),$$

where $p\rho^r = p(|J_A|\rho^a)^{r/a} \times |J_A|^{-r/a} = p^{1+r/a}n^{-r/2}|J_A|^{-r/a} = o(p^2)$ and

$$\begin{aligned} p\rho^{a-r} &= p(|J_A|\rho^a)^{(a-r)/a} \times |J_A|^{-1+r/a} \\ &= p^2 n^{-(a-r)/2} \times \left(\frac{|J_A|}{p^2}\right)^{r/2a} \times |J_A|^{-1+r/2a} = o(p^2), \end{aligned}$$

as $r = 0, \cdots, a$.

If $B = 2$, we have

$$\left| \sum_{(j_1,j_2),(j_3,j_4)\in J_A^c} Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \times \mathbf{1}_{\{B=2\}} \right| \leq C|J_A|k^2(\rho^{2r} + \rho^{2(a-r)} + \rho^a) = o(p^2),$$

since $|J_A|\rho^{2r} = (|J_A|\rho^a)^{2r/a}|J_A|^{1-2r/a} = p^2 n^{-r}(|J_A|/p^2)^{1-r/a}|J_A|^{-r/a} = o(p^2)$,

$$\begin{aligned} |J_A|\rho^{2(a-r)} &= (|J_A|\rho^a)^{2(a-r)/a} \times |J_A|^{1-2(a-r)/a} \\ &= n^{-a+r}p^2 \times p^{-2r/a} \times |J_A|^{1-2(a-r)/a} \\ &= n^{-a+r}p^2 \times (|J_A|/p^2)^{r/a} \times |J_A|^{-1+r/a} = o(p^2), \end{aligned}$$

and $|J_A|\rho^a = O(n^{-a/2}p) = o(p^2)$.

If $B = 1$, we have

$$\left| \sum_{(j_1,j_2),(j_3,j_4)\in J_A^c} Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \times \mathbf{1}_{\{B=1\}} \right| \le C|J_A| k \max_{j=1,\cdots,p} |J_j| (\rho^{2a-r} + \rho^{a+r}) = o(p^2),$$

since

$$\begin{aligned} |J_A| \max_{j=1,\cdots,p} |J_j| \rho^{2a-r} &= (|J_A|\rho^a)^{(2a-r)/a} |J_A|^{1-(2a-r)/a} \max_{j=1,\cdots,p} |J_j| \\ &= n^{-a}p^2 \times n^{r/2} p^{-r/a} \times |J_A|^{-1+r/a} \times \max_{j=1,\cdots,p} |J_j| \\ &= n^{-a+r/2}p^2 \times \frac{\max_{j=1,\cdots,p} |J_j|}{|J_A|} \times \left(\frac{|J_A|}{\max_{j=1,\cdots,p} |J_j|} \frac{\max_{j=1,\cdots,p} |J_j|}{p} \right)^{r/a} \end{aligned}$$

$$= n^{-a+r/2}p^2 \times \left(\frac{\max_{j=1,\cdots,p} |J_j|}{|J_A|} \right)^{1-r/a} \times \left(\frac{\max_{j=1,\cdots,p} |J_j|}{p} \right)^{r/a} = o(p^2),$$

$$\begin{aligned}
|J_A| \max_{j=1,\cdots,p} |J_j| \rho^{a+r} &= n^{-(a+r)/2} p^{1+r/a} \times |J_A|^{-r/a} \times \max_{j=1,\cdots,p} |J_j| \\
&= n^{-(a+r)/2} p^2 \times \frac{\max_{j=1,\cdots,p} |J_j|}{|J_A|} \times \left(\frac{|J_A|}{p} \right)^{1-r/a} \\
&= n^{-(a+r)/2} p^2 \times \left(\frac{\max_{j=1,\cdots,p} |J_j|}{|J_A|} \right)^{r/a} \times \left(\frac{\max_{j=1,\cdots,p} |J_j|}{p} \right)^{1-r/a} \\
&= o(p^2),
\end{aligned}$$

as $\max_{j=1,\cdots,p} |J_j| \leq p$, $\max_{j=1,\cdots,p} |J_j| \leq |J_A|$ and $r = 0, \cdots, a$.

If $B = 0$, $Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \neq 0$ only when $(j_1, j_3), (j_2, j_4), (j_1, j_4), (j_2, j_3) \in J_A$. Then, we have

$$\begin{aligned}
&\left| \sum_{(j_1,j_2),(j_3,j_4) \in J_A^c} Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \times \mathbf{1}_{\{B=0\}} \right| \\
\leq\ & C \sum_{\substack{(j_1,j_3),(j_2,j_4) \in J_A \\ (j_1,j_4),(j_2,j_3) \in J_A}} \rho^{2a} = O(|J_A|^2 \rho^{2a}) = o(p^2).
\end{aligned}$$

In summary, we have

$$\left| F^2(a,c) \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}} \in \mathcal{P}(n,a+c) \\ (j_1,j_2),(j_3,j_4) \in J_A^c}} Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \right| = O(1) n^{-2(a+c)} \times n^{a+c} \times o(p^2) = o(n^{-a} p^2).$$

3. $\mathrm{Var}[\mathcal{U}_2(a)] = o(n^{-a} p^2 k^2)$. By Cauchy-Schwarz inequality,

$$\mathrm{Var}[\mathcal{U}_2(a)] \leq C \sum_{c=0}^{a} \mathrm{var} \left\{ \sum_{(j_1,j_2) \in J_A} H(\mathbf{i}, j_1, j_2) \right\}.$$

We only need to prove $\text{var}\left\{\sum_{(j_1,j_2)\in J_A} H(\mathbf{i},j_1,j_2)\right\} = o(n^{-a}p^2)$. Note that,

$$\begin{aligned}
&\mathrm{E}\Big[\Big\{\sum_{(j_1,j_2)\in J_A} H(\mathbf{i},j_1,j_2)\Big\}^2\Big] \\
= \;& F^2(a,c) \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a+c)\\(j_1,j_2),(j_3,j_4)\in J_A}} Q_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4),
\end{aligned}$$

where $F(a,c) = O(n^{-(a+c)})$ and $Q_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)$ was defined in (S5.39). Let

$$\mathrm{Var}[\mathcal{U}_2(a)] \le C\sum_{c=0}^{a} F^2(a,c) \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a+c)\\(j_1,j_2),(j_3,j_4)\in J_A}} \tilde{Q}_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4),$$

where

$$\tilde{Q}_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4) = \begin{cases} Q_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4) & \text{when } 1\le c\le a \\ Q_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4) - (\sigma_{j_1j_2}\sigma_{j_3j_4})^a & \text{when } c=0 \end{cases}$$

Consider an index $i\in\{\mathbf{i}\}_2$, if $i\notin\{\tilde{\mathbf{i}}\}$ or $i\in\{\tilde{\mathbf{i}}\}_1$, under Condition 5, we have $Q_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4) = \mathrm{E}(x_{1,j_1})\mathrm{E}(\text{other terms}) = 0$ or $\mathrm{E}(x_{1,j_1}x_{1,j_3}x_{1,j_4})\mathrm{E}$ (other terms) $= 0$. For an index $i\in\{\mathbf{i}\}_3$, $Q_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4) = 0$ if $i\notin\{\tilde{\mathbf{i}}\}$ or $i\in\{\tilde{\mathbf{i}}\}_1$. It means that, $Q_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\neq 0$ only when $\{\mathbf{i}\}_2\cup\{\mathbf{i}\}_3 = \{\tilde{\mathbf{i}}\}_2\cup\{\tilde{\mathbf{i}}\}_3$. Without loss of generality, we assume $r = \mathrm{card}(\{\mathbf{i}\}_2\cap\{\tilde{\mathbf{i}}\}_2)$, then $\mathrm{card}(\{\mathbf{i}\}_2\cap\{\tilde{\mathbf{i}}\}_3) = c-r$, $\mathrm{card}(\{\mathbf{i}\}_3\cap\{\tilde{\mathbf{i}}\}_2) = c-r$ and $\mathrm{card}(\{\mathbf{i}\}_3\cap\{\tilde{\mathbf{i}}\}_3) = r$.

Furthermore, we assume $\mathrm{card}(\{\mathbf{i}\}_1 \cap \{\tilde{\mathbf{i}}\}_1) = t_c$ with $0 \le t_c \le a - c$.

$$\begin{aligned} & Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \\ = & \left\{ \mathrm{E}\Big(\prod_{s=1}^{4} x_{i,j_s} \Big) \right\}^{t_c} \left\{ \mathrm{E}(x_{1,j_1} x_{1,j_2}) \mathrm{E}(x_{1,j_3} x_{1,j_4}) \right\}^{a-c-t_c} \\ & \times [\mathrm{E}(x_{1,j_1} x_{1,j_3}) \mathrm{E}(x_{1,j_2} x_{1,j_4})]^r [\mathrm{E}(x_{1,j_1} x_{1,j_4}) \mathrm{E}(x_{1,j_2} x_{1,j_3})]^{c-r}. \quad \text{(S5.41)} \end{aligned}$$

Now, we discuss (S5.41) with respect to different c and t_c, where $0 \le c \le a$ and $0 \le t_c \le a - c$.

Case 1. $c = 0$ and $t_c = 0$. In this case, $Q_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) = \sigma^a_{j_1 j_2} \sigma^a_{j_3 j_4}$. Then, $|\tilde{Q}_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \mathbf{1}_{\{c=0, t_c=0\}}| = 0$.

Case 2. $c = 0$ and $1 \le t_c \le a$. Note that, $0 \le r \le c$, then $r = 0$. Thus, we have

$$\begin{aligned} |\text{(S5.41)}| &= \left| \left\{ \mathrm{E}\Big(\prod_{s=1}^{4} x_{i,j_s} \Big) \right\}^{t_c} \left\{ \mathrm{E}(x_{1,j_1} x_{1,j_2}) \mathrm{E}(x_{1,j_3} x_{1,j_4}) \right\}^{a-t_c} \right| \\ &\le C |\sigma_{j_1 j_2} \sigma_{j_3 j_4} + \sigma_{j_1 j_3} \sigma_{j_2 j_4} + \sigma_{j_1 j_4} \sigma_{j_2 j_3}|^{t_c} |\sigma_{j_1 j_2} \sigma_{j_3 j_4}|^{a-t_c}. \quad \text{(S5.42)} \end{aligned}$$

When $B = 3$, $\Big| \sum_{(j_1,j_2),(j_3,j_4) \in J_A} \text{(S5.42)} \Big| \le C \max_{j=1,\cdots,p} |J_j| k^3 (\rho^{2a} + \rho^{2a-t_c} + \rho^{2a-2t_c})$; when $B = 2$, $\Big| \sum_{(j_1,j_2),(j_3,j_4) \in J_A} \text{(S5.42)} \Big| \le C |J_A| k^2 (\rho^{2a} + \rho^{2a-t_c})$; when $B = 1$, $| \sum_{(j_1,j_2),(j_3,j_4) \in J_A} \text{(S5.42)} | \le C |J_A| \max_{j=1,\cdots,p} |J_j| k (\rho^{2a} + \rho^{2a-t_c})$; if $B = 0$, we have $| \sum_{(j_1,j_2),(j_3,j_4) \in J_A} \text{(S5.42)} | \le C |J_A|^2 \rho^{2a}$. As $\mathrm{card}(\{\mathbf{i}\} \cup \{\tilde{\mathbf{i}}\}) \le 2a - t_c$,

then

$$\begin{aligned}
&\left|F^2(a,c)\sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a+c)\\(j_1,j_2),(j_3,j_4)\in J_A}}\tilde{Q}_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\{c=0,1\le t_c\le a\}}\right|\\
\le\ & C\sum_{t_c=1}^{a}n^{-t_c}\{\max_{j=1,\cdots,p}|J_j|k^3(\rho^{2a}+\rho^{2a-t_c}+\rho^{2a-2t_c})\\
&\quad+|J_A|^2\rho^{2a}+|J_A|k^2\rho^{2a-t_c}+|J_A|k^2\rho^{2a}\\
&\quad+|J_A|\max_{j=1,\cdots,p}|J_j|k\rho^{2a}+|J_A|\max_{j=1,\cdots,p}|J_j|k\rho^{2a-t_c}\}=o(n^{-a}p^2),
\end{aligned}$$

where

$$\begin{aligned}
&n^{-t_c}\max_{j=1,\cdots,p}|J_j|k^3(\rho^{2a}+\rho^{2a-t_c}+\rho^{2a-2t_c})\\
=\ & n^{-t_c}k^3\frac{\max_{j=1,\cdots,p}|J_j|}{|J_A|}\{|J_A|^{-1}(|J_A|\rho^a)^2+|J_A|^{-1+t_c/a}(|J_A|\rho^a)^{(2a-t_c)/a}+|J_A|^{-1+2t_c/a}(|J_A|\rho^a)^{(2a-2t_c)/a}\}\\
=\ & o(n^{-a}p^2)+O(1)n^{-a-t_c/2}p^2|J_A|^{-1}\times\left(\frac{\max_{j=1,\cdots,p}|J_j|}{|J_A|}\right)^{1-t_c/a}\times\left(\frac{\max_{j=1,\cdots,p}|J_j|}{p}\right)^{t_c/a}\\
&+O(1)n^{-a}p^2\times\frac{\max_{j=1,\cdots,p}|J_j|}{|J_A|}\left(\frac{|J_A|}{p^2}\right)^{t_c/a}\times|J_A|^{-1+t_c/a}=o(n^{-a}p^2).\\
&n^{-t_c}\{|J_A|k^2\rho^{2a-t_c}+|J_A|k^2\rho^{2a}\}\\
=\ & n^{-t_c}k^2\{|J_A|^{t_c/a-1}(|J_A|\rho^a)^{(2a-t_c)/a}+|J_A|^{-1}(|J_A|\rho^a)^2\}\\
=\ & O(1)n^{-a-t_c/2}p^2\times p^{-t_c/a}|J_A|^{t_c/a-1}+O(1)n^{-a-t_c}p^2|J_A|^{-1}\\
=\ & o(n^{-a}p^2).
\end{aligned}$$

$$
\begin{aligned}
& n^{-t_c}|J_A| \max_{j=1,\cdots,p} |J_j| k(\rho^{2a} + \rho^{2a-t_c}) \\
= \; & n^{-t_c} k \frac{\max_{j=1,\cdots,p} |J_j|}{|J_A|} \left\{ |J_A|^2 \rho^{2a} + (|J_A|\rho^a)^{(2a-t_c)/a} \times |J_A|^{-2+t_c/a} \right\} \\
= \; & o(n^{-a}p^2) + O(1) n^{-a} p^2 n^{-t_c/2} \frac{\max_{j=1,\cdots,p} |J_j|}{|J_A|} \times \left(\frac{|J_A|}{\max_{j=1,\cdots,p} |J_j|} \times \frac{\max_{j=1,\cdots,p} |J_j|}{p} \right)^{t_c/a} \\
= \; & o(n^{-a}p^2) + O(1) n^{-a} p^2 n^{-t_c/2} \left(\frac{\max_{j=1,\cdots,p} |J_j|}{|J_A|} \right)^{1-t_c/a} \left(\frac{\max_{j=1,\cdots,p} |J_j|}{p} \right)^{t_c/a} \\
= \; & o(n^{-a}p^2),
\end{aligned}
$$

and $n^{-t_c}|J_A|^2\rho^{2a} = o(n^{-a}p^2)$ as $|J_A|^2\rho^{2a} = O(n^{-a}p^2)$, $|J_A| = o(p^2)$, $\max_{j=1,\cdots,p} |J_j| \leq p$ and $\max_{j=1,\cdots,p} |J_j| \leq |J_A|$. **Case 3.** $1 \leq c \leq a$ and $0 \leq t_c \leq a - c$. We have $\mathrm{card}(\{\mathbf{i}\} \cup \{\tilde{\mathbf{i}}\}) \leq 2a - t_c$.

$$
\begin{aligned}
|(\text{S5.41})| \; = \; & |\kappa^{t_c}(\sigma_{j_1j_2}\sigma_{j_3j_4} + \sigma_{j_1j_3}\sigma_{j_2j_4} + \sigma_{j_1j_4}\sigma_{j_2j_3})^{t_c} \\
& \times (\sigma_{j_1j_2}\sigma_{j_3j_4})^{a-c-t_c} (\sigma_{j_1j_3}\sigma_{j_2j_4})^r (\sigma_{j_1j_4}\sigma_{j_2j_3})^{c-r}| \\
\leq \; & C|\sigma_{j_1j_2}\sigma_{j_3j_4}|^{a-c} |\sigma_{j_1j_3}\sigma_{j_2j_4}|^r |\sigma_{j_1j_4}\sigma_{j_2j_3}|^{c-r} \\
& + C|\sigma_{j_1j_2}\sigma_{j_3j_4}|^{a-c-t_c} |\sigma_{j_1j_3}\sigma_{j_2j_4}|^{t_c+r} |\sigma_{j_1j_4}\sigma_{j_2j_3}|^{c-r} \\
& + C|\sigma_{j_1j_2}\sigma_{j_3j_4}|^{a-c-t_c} |\sigma_{j_1j_3}\sigma_{j_2j_4}|^r |\sigma_{j_1j_4}\sigma_{j_2j_3}|^{c-r+t_c}. \quad \text{(S5.43)}
\end{aligned}
$$

If $B = 3$, without loss of generality, we assume $|j_1 - j_3| \leq k$, $|j_2 - j_4| \leq k$

and $|j_1 - j_4| \leq k$, then

$$\begin{aligned}
&\left| F^2(a,c) \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}} \in \mathcal{P}(n,a+c) \\ (j_1,j_2),(j_3,j_4) \in J_A}} \tilde{Q}_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4) \mathbf{1}_{\{1 \leq c \leq a, 0 \leq t_c \leq a-c, B=3\}} \right| \\
\leq \; & C \sum_{c=1}^{a} \sum_{t_c=0}^{a-c} n^{-2c-t_c} \max_{j=1,\cdots,p} |J_j| k^3 \{ \rho^{2(a-c)+c-r} + \rho^{2(a-c-t_c)+c-r} + \rho^{2(a-c-t_c)+c-r+t_c} \}
\end{aligned}$$

Since

$$\begin{aligned}
n^{-2c-t_c} \max_{j=1,\cdots,p} |J_j| k^3 \rho^{2(a-c)+c-r} &= n^{-2c-t_c} k^3 \frac{\max_{j=1,\cdots,p} |J_j|}{|J_A|} (|J_A| \rho^a)^{(2a-c-r)/a} |J_A|^{-1+(c+r)/a} \\
&= O(1) n^{-a} p^2 n^{-2c-t_c+(r+c)/2} \times p^{-(c+r)/a} |J_A|^{-1+(c+r)/a} \\
&= O(1) n^{-a} p^2 n^{-2c-t_c+(r+c)/2} \times \left(\frac{|J_A|}{p^2} \right)^{\frac{c+r}{2a}} |J_A|^{-1+(c+r)/2a} \\
&= o(n^{-a} p^2),
\end{aligned}$$

as $2c + t_c - (r+c)/2 \geq 2c + t_c - c > 0$ and $c + r < 2a$; and

$$\begin{aligned}
n^{-2c-t_c} \max_{j=1,\cdots,p} |J_j| k^3 \rho^{2(a-c-t_c)+c-r} &= n^{-2c-t_c} k^3 \frac{\max_{j=1,\cdots,p} |J_j|}{|J_A|} (|J_A| \rho^a)^{(2a-c-r-2t_c)/a} \times |J_A|^{-1+(c+r+2t_c)/a} \\
&= O(1) n^{-a} p^2 n^{-2c-t_c+(r+c+2t_c)/2} \times p^{-(c+r+2t_c)/a} |J_A|^{-1+(c+r+2t_c)/a} \\
&= O(1) n^{-a} p^2 n^{(r-3c)/2} \times \left(\frac{|J_A|}{p^2} \right)^{(c+r+2t_c)/2a} |J_A|^{-1+(c+r+2t_c)/2a} \\
&= o(n^{-a} p^2),
\end{aligned}$$

as $c + r + 2t_c < 2a$. Using the similar technique, it is obvious that

$n^{-2c-t_c}\max_{j=1,\cdots,p}|J_j|k^3\rho^{2(a-c-t_c)+c-r+t_c}=o(n^{-a}p^2)$. Then

$$\left|F^2(a,c)\sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a+c)\\(j_1,j_2),(j_3,j_4)\in J_A}}\tilde{Q}_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\{1\le c\le a,0\le t_c\le a-c,B=3\}}\right|=o(n^{-a}p^2).$$

If $B=2$,

$$\begin{aligned}
&\left|F^2(a,c)\sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a+c)\\(j_1,j_2),(j_3,j_4)\in J_A}}\tilde{Q}_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\{1\le c\le a,0\le t_c\le a-c,B=2\}}\right|\\
\le\ & C\sum_{c=1}^{a}\sum_{t_c=0}^{a-c}n^{-2c-t_c}|J_A|k^2\{\rho^{2(a-c)}\times(\rho^{2r}+\rho^{2(c-r)}+\rho^{c})\\
&+\rho^{2(a-c-t_c)}\times(\rho^{2(t_c+r)}+\rho^{2(c-r)}+\rho^{t_c+c})\\
&+\rho^{2(a-c-t_c)}\times(\rho^{2r}+\rho^{2(c-r+t_c)}+\rho^{t_c+c})\}\\
=\ & C\sum_{c=1}^{a}\sum_{t_c=0}^{a-c}n^{-2c-t_c}|J_A|k^2\{\rho^{2(a-c+r)}+\rho^{2(a-r)}+\rho^{2a-c}\\
&+\rho^{2(a-t_c-r)}+\rho^{2a}+\rho^{2(a-c-t_c+r)}+\rho^{2a-c-t_c}\}. \qquad \text{(S5.44)}
\end{aligned}$$

Before investigating (S5.44), we first give a fact. For any w which satisfy $0\le w<\min\{2a,4c+2t_c\}$, we have

$$\begin{aligned}
n^{-2c-t_c}|J_A|k^2\rho^{2a-w} &= n^{-2c-t_c}k^2\times(|J_A|\rho^a)^{(2a-w)/a}\times|J_A|^{-1+w/a}\\
&= O(1)n^{-a}p^2n^{-2c-t_c+w/2}p^{-w/a}\times|J_A|^{-1+w/a}\\
&= O(1)n^{-a}p^2n^{-2c-t_c+w/2}\left(\frac{|J_A|}{p^2}\right)^{w/2a}\times|J_A|^{-1+w/2a}\\
&= o(n^{-a}p^2).
\end{aligned}$$

Since $2(c-r), 2r, c, 2(t_c+r), 0, 2(c+t_c-r), c+t_c < \min\{2a, 4c+2t_c\}$, then

$$\left| F^2(a,c) \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a+c)\\(j_1,j_2),(j_3,j_4)\in J_A}} \tilde{Q}_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4)\mathbf{1}_{\{1\le c\le a, 0\le t_c\le a-c, B=2\}} \right| = o(n^{-a}p^2).$$

If $B = 1$,

$$\begin{aligned}
&\left| F^2(a,c) \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a+c)\\(j_1,j_2),(j_3,j_4)\in J_A}} \tilde{Q}_c(\mathbf{i}, j_1, j_2, \tilde{\mathbf{i}}, j_3, j_4)\mathbf{1}_{\{1\le c\le a, 0\le t_c\le a-c, B=1\}} \right| \\
\le\ & C\sum_{c=1}^{a}\sum_{t_c=0}^{a-c} n^{-2c-t_c}|J_A| \max_{j=1,\cdots,p}|J_j| k\{\rho^{2(a-c)} \times (\rho^{2c-r} + \rho^{c+r}) \\
&\qquad + \rho^{2(a-c-t_c)} \times (\rho^{2c-r+t_c} + \rho^{2t_c+c+r}) \\
&\qquad + \rho^{2(a-c-t_c)} \times (\rho^{2c-r+2t_c} + \rho^{t_c+c+r})\} \\
=\ & C\sum_{c=1}^{a}\sum_{t_c=0}^{a-c} n^{-2c-t_c}|J_A| \max_{j=1,\cdots,p}|J_j| k\{\rho^{2a-r} + \rho^{2a-c+r} + \rho^{2a-r-t_c} + \rho^{2a-c-t_c+r}\}.
\end{aligned}$$

For any $0 \le w < \min(a, 4c+t_c)$, we have

$$\begin{aligned}
& n^{-2c-t_c}|J_A| \max_{j=1,\cdots,p}|J_j| k\rho^{2a-w} \\
=\ & n^{-2c-t_c} \max_{j=1,\cdots,p}|J_j| k \times (|J_A|\rho^a)^{(2a-w)/a} \times |J_A|^{-1+w/a} \\
=\ & O(1)n^{-a}p^2 n^{-2c-t_c+w/2} p^{-w/a} \times |J_A|^{-1+w/a} \times \max_{j=1,\cdots,p}|J_j| \\
=\ & O(1)n^{-a}p^2 n^{-2c-t_c+w/2} \frac{\max_{j=1,\cdots,p}|J_j|}{|J_A|} \left(\frac{|J_A|}{p}\right)^{w/a}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)n^{-a}p^2 n^{-2c-t_c+w/2}\frac{\max_{j=1,\cdots,p}|J_j|}{|J_A|}^{1-w/a}\left(\frac{\max_{j=1,\cdots,p}|J_j|}{p}\right)^{w/a}\\
&= o(n^{-a}p^2).
\end{aligned}
$$

Since $r, c-r, t_c+r, c+t_c-r < \min\{a, 4c+2t_c\}$, thus

$$
\left|F^2(a,c)\sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a+c)\\(j_1,j_2),(j_3,j_4)\in J_A}}\tilde{Q}_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\{1\le c\le a, 0\le t_c\le a-c, B=1\}}\right| = o(n^{-a}p^2).
$$

If $B=0$, we have

$$
\begin{aligned}
&\left|F^2(a,c)\sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,a+c)\\(j_1,j_2),(j_3,j_4)\in J_A}}\tilde{Q}_c(\mathbf{i},j_1,j_2,\tilde{\mathbf{i}},j_3,j_4)\mathbf{1}_{\{1\le c\le a, 0\le t_c\le a-c, B=1\}}\right|\\
\le\ & Cn^{-2c-t_c}|J_A|^2\rho^{2a} = o(n^{-a}p^2).
\end{aligned}
$$

In summary, $\mathrm{Var}[\mathcal{U}_2(a)] = o(n^{-a}p^2)$. Thus,

$$
\sigma_A^2(a) = 2(P_a^n)^{-1}a!\kappa^a\sum_{\substack{k<|j_1-j_2|<p\\k<|j_3-j_4|<p\\|j_1-j_3|\le k,|j_2-j_4|\le k}}\sigma^a_{j_1j_3}\sigma^a_{j_2j_4} + o(n^{-a}p^2).
$$

□

S5.9 Proof of Lemma 9

Proof. As shown in Lemma 8, $\mathrm{Var}\{\mathcal{U}(a)\} = \mathrm{Var}\{\mathcal{U}_{1,1}(a)\} + o(\mathrm{Var}\{\mathcal{U}_{1,1}(a)\})$. Then,

$$\mathrm{Cov}\left[\frac{\mathcal{U}(a)}{\sigma(a)}, \frac{\mathcal{U}(b)}{\sigma(b)}\right] = \frac{\mathrm{E}[\mathcal{U}_{1,1}(a)\mathcal{U}_{1,1}(b)]}{\sigma(a)\sigma(b)} + o(1),$$

as $\mathrm{E}[\mathcal{U}_{1,1}(a)] = \mathrm{E}[\mathcal{U}_{1,1}(b)] = 0$.

$$\mathrm{E}[\mathcal{U}_{1,1}(a)\mathcal{U}_{1,1}(b)] = (P_a^n P_b^n)^{-1} \sum_{\substack{\mathbf{i}\in\mathcal{P}(n,a), \tilde{\mathbf{i}}\in\mathcal{P}(n,b) \\ (j_1,j_2),(j_3,j_4)\in J_A^c}} \mathrm{E}\Big(\prod_{t=1}^{a} x_{i_t,j_1} x_{i_t,j_2} \prod_{\tilde{t}=1}^{a} x_{\tilde{i}_{\tilde{t}},j_3} x_{\tilde{i}_{\tilde{t}},j_4} \Big)$$

It is obvious that $\{\mathbf{i}\} \neq \{\tilde{\mathbf{i}}\}$ as $a \neq b$. Assume without loss of generality that $a < b$ and index $i \in \{\mathbf{i}\}$ but $i \notin \{\tilde{\mathbf{i}}\}$. We have

$$\mathrm{E}[\mathcal{U}_{1,1}(a)\mathcal{U}_{1,1}(b)] = \begin{cases} \mathrm{E}(x_{1,j_1})\mathrm{E}(\text{other terms}), & i \in \{\mathbf{i}\}_1 \\ \mathrm{E}(x_{1,j_2})\mathrm{E}(\text{other terms}), & i \in \{\mathbf{i}\}_2 \\ \mathrm{E}(x_{1,j_1}x_{1,j_2})\mathrm{E}(\text{other terms}), & i \in \{\mathbf{i}\}_3. \end{cases}$$

Since $\sigma_{j_1 j_2} = 0$ for $(j_1, j_2) \in J_A^c$, we have $\mathrm{E}[\mathcal{U}_{1,1}(a)\mathcal{U}_{1,1}(b)] = 0$. Then, Lemma 9 is proved. □

S5.10 Proof of Lemma 10

Proof. By Cauchy-Schwarz inequality, for some constant C,

$$\mathrm{Var}\bigg(\sum_{h=1}^{n}\pi_{n,h}^2\bigg) \le Cn^2 \max_{1\le h\le n; 1\le r_1,r_2\le m} \mathrm{Var}(T_{h,a_{r_1},a_{r_2}}),$$

where

$$\begin{aligned} T_{h,a_1,a_2} &= \mathrm{E}_{h-1}(A_{n,h,a_1}A_{n,h,a_2}) \\ &= \sum_{\substack{\mathbf{i}^{(l)}\in\mathcal{P}(h-1,a_l-1)\\ (j_{2l-1},j_{2l})\in J_A^c, l=1,2}} \prod_{l=1}^{2} c(n,a_l)^{1/2}\mathcal{H}(h,\mathbf{i}^{(l)},j_{2l-1},j_{2l};l=1,2), \quad \text{(S5.45)} \end{aligned}$$

where

$$\mathcal{H}(h,\mathbf{i}^{(l)},j_{2l-1},j_{2l};l=1,2) = \mathrm{E}\bigg(\prod_{t=1}^{4}x_{h,j_t}\bigg)\bigg(\prod_{l=1}^{2}\prod_{t=1}^{a_l-1}x_{i_t^{(l)},j_{2l-1}}x_{i_t^{(l)},j_{2l}}\bigg).$$

Next, it is sufficient to proof $\mathrm{Var}(T_{h,a_1,a_2}) = o(n^{-2})$ for any $1 \le r_1, r_2 \le m$.

$$T_{h,a_1,a_2}^2 = \sum_{\substack{\mathbf{i}^{(l)}\in\mathcal{P}(h-1,a_l-1)\\ (j_{2l-1},j_{2l})\in J_A^c, l=1,2,3,4}} \prod_{t=1}^{2} c(n,a_l)\mathcal{H}(h,\mathbf{i}^{(l)},j_{2l-1},j_{2l};l=1,2,3,4), \quad \text{(S5.46)}$$

where

$$\begin{aligned} & \mathcal{H}(h, \mathbf{i}^{(l)}, j_{2l-1}, j_{2l}; l = 1, 2, 3, 4) \\ = \; & \mathrm{E}\Big(\prod_{t=1}^{4} x_{h,j_t} \Big) \mathrm{E}\Big(\prod_{t=5}^{8} x_{h,j_t} \Big) \Big(\prod_{l=1}^{4} \prod_{t=1}^{a_l - 1} x_{i_t^{(l)}, j_{2l-1}} x_{i_t^{(l)}, j_{2l}} \Big) \end{aligned} \tag{S5.47}$$

and $c(n, a_l) = \{a_l / \sigma_A(a_l) P_{a_l}^n\}^2 = \Theta(n^{-a_l} p^{-2})$. Then,

$$\mathrm{Var}(T_{h,a_1,a_2}) \;=\; \sum_{\substack{\mathbf{i}^{(l)} \in \mathcal{P}(h-1, a_l - 1) \\ (j_{2l-1}, j_{2l}) \in J_A^c, l = 1,2,3,4}} \prod_{t=1}^{2} c(n, a_l) \mathbb{H}(h, \mathbf{i}^{(l)}, j_{2l-1}, j_{2l}; l = 1, 2, 3, 4),$$

where

$$\begin{aligned} & \mathbb{H}(h, \mathbf{i}^{(l)}, j_{2l-1}, j_{2l}; l = 1, 2, 3, 4) \qquad\qquad (\text{S5.48}) \\ = \; & \mathrm{E}\{\mathcal{H}(h, \mathbf{i}^{(l)}, j_{2l-1}, j_{2l}; l = 1, 2, 3, 4)\} \\ & - \mathrm{E}\{\mathcal{H}(h, \mathbf{i}^{(l)}, j_{2l-1}, j_{2l}; l = 1, 2)\} \times \mathrm{E}\{\mathcal{H}(h, \mathbf{i}^{(l)}, j_{2l-1}, j_{2l}; l = 3, 4)\}, \end{aligned}$$

with

$$\mathcal{H}(h, \mathbf{i}^{(l)}, j_{2l-1}, j_{2l}; l = 3, 4) = \mathrm{E}\Big(\prod_{t=5}^{8} x_{h,j_t} \Big) \Big(\prod_{l=3}^{4} \prod_{t=1}^{a_l - 1} x_{i_t^{(l)}, j_{2l-1}} x_{i_t^{(l)}, j_{2l}} \Big).$$

Now, we examine (S5.48). If $(\{\mathbf{i}^{(1)}\} \cup \{\mathbf{i}^{(2)}\}) \cap (\{\mathbf{i}^{(3)}\} \cup \{\mathbf{i}^{(4)}\}) = \emptyset$, then (S5.48) $= 0$. Without loss of generality, we let $a_3 = a_1$ and $a_4 = a_2$. If one of the index only appears once in $\bigcup_{l=1}^{4} \{\mathbf{i}^l\}$, $\mathrm{card}(\bigcup_{l=1}^{4} \{\mathbf{i}^l\}) > a_1 + a_2 - 2$, in

this case (S5.48) $= 0$ as $(j_{2l-1}, j_{2l}) \in J_A^c$ for $l = 1, 2, 3, 4$. Thus, (S5.48) $\neq 0$ when $(\{\mathbf{i}^{(1)}\} \cup \{\mathbf{i}^{(2)}\}) \cap (\{\mathbf{i}^{(3)}\} \cup \{\mathbf{i}^{(4)}\}) \neq \emptyset$ and $\text{card}(\underset{l=1}{\overset{4}{\cup}}\{\mathbf{i}^l\}) \leq a_1 + a_2 - 2$.

Case 1. When $\text{card}(\underset{l=1}{\overset{4}{\cup}}\{\mathbf{i}^l\}) = a_1 + a_2 - 2$, we assume $\text{card}(\{\mathbf{i}^1\} \cup \{\mathbf{i}^2\}) = s$, where $s \leq a_1 - 1$. Then, $\text{card}(\{\mathbf{i}^3\} \cup \{\mathbf{i}^4\}) = s$. For further discussion, we assume $\text{card}(\{\mathbf{i}^1\} \cup \{\mathbf{i}^3\}) = s_1$, then $\text{card}(\{\mathbf{i}^2\} \cup \{\mathbf{i}^3\}) = a_1 - 1 - s - s_1$, $\text{card}(\{\mathbf{i}^1\} \cup \{\mathbf{i}^4\}) = a_1 - 1 - s - s_1$ and $\text{card}(\{\mathbf{i}^2\} \cup \{\mathbf{i}^4\}) = a_2 - a_1 + s_1$. Then $\text{card}(\underset{l=1}{\overset{4}{\cup}}\{\mathbf{i}^l\}) = a_1 + a_2 - 2 - 2s$. It follows that (S5.48) $\neq 0$ when $a_1 + a_2 - 2 - 2s > 0$, that is $s < (a_1 + a_2 - 2)/2$. Now, we rewrite the expression

$$\begin{aligned}(\text{S5.48}) \;=\; & \Big\{\mathrm{E}\Big(\prod_{t=1}^{4} x_{h,j_t}\Big)\mathrm{E}\Big(\prod_{t=5}^{8} x_{h,j_t}\Big)\Big\}^{s+1} \\ & \times\Big\{\mathrm{E}\Big(\prod_{t=1,2,5,6} x_{h,j_t}\Big)\Big\}^{s_1}\Big\{\mathrm{E}\Big(\prod_{t=3,4,7,8} x_{h,j_t}\Big)\Big\}^{a_2-a_1+s_1} \\ & \times\Big\{\mathrm{E}\Big(\prod_{t=1,2,7,8} x_{h,j_t}\Big)\mathrm{E}\Big(\prod_{t=3,4,5,6} x_{h,j_t}\Big)\Big\}^{a_1-1-s-s_1}. \quad (\text{S5.49})\end{aligned}$$

The above formula is the product of 6 terms. For the sake of expression, we denote

$$T_{l_1,l_2,l_3,l_4} \triangleq \mathrm{E}\Big(\prod_{t=l_1,l_2,l_3,l_4} x_{h,j_{l_t}}\Big),$$

$$B_{l_1,l_2,l_3,l_4} \triangleq \text{card}\{(j_s, j_{\tilde{s}}) : |j_s - j_{\tilde{s}}| \leq k,\ j_s, j_{\tilde{s}} = 1, \ldots, p \text{ with } s = l_1, l_2, \tilde{s} = l_3, l_4\} \quad (\text{S5.50})$$

for $l_1, l_2, l_3, l_4 = 1, \ldots, 8$. Since $(j_{2l-1}, j_{2l}) \in J_A^c$ for $l = 1, 2, 3, 4$, we have $T_{1,2,3,4} = \sigma_{j_1 j_3}\sigma_{j_2 j_4} + \sigma_{j_1 j_4}\sigma_{j_2 j_3}$ and $T_{5,6,7,8} = \sigma_{j_5 j_7}\sigma_{j_6 j_8} + \sigma_{j_5 j_8}\sigma_{j_6 j_7}$. We first discuss $T_{1,2,3,4}$, then similar results to $T_{5,6,7,8}, T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ hold. By Holder inequality, $O(1)$ is the leading term compared to ρ and ρ^2. Then

$$T_{1,2,3,4} = \begin{cases} O(1), & B_{1,2,3,4} = 3 \text{ or } B_{1,2,3,4} = 2 \\ O(\rho), & B_{1,2,3,4} = 2 \text{ or } B_{1,2,3,4} = 1 \\ O(\rho^2), & B_{1,2,3,4} = 0. \end{cases}$$

We further discuss (S5.48) by considering $T_{1,2,3,4}$ and $T_{5,6,7,8}$.

Case 1.1. When $T_{1,2,3,4} = O(1)$ and $T_{5,6,7,8} = O(1)$. If at least one of $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ is ρ. The order of $(j_1, \cdots, j_8)$ is $O(p^2|J_A|k^4) = o(p^4)$, then $\prod_{t=1}^{2} c(n, a_l) n^{a_1+a_2-2} p^2 |J_A| k^4 \times$ (S5.49) $= o(n^{-2})$. If there is no one term among $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ is ρ, it means that all of the four terms is ρ^2. In this case, (S5.49) achieves the order $O(\rho^{2(a_1+a_2)-4(s+1)})$ and the order of $(j_1, \cdots, j_8)$ is $O(|J_A|^2 k^4)$. Thus,

$$\begin{aligned} & \prod_{t=1}^{2} c(n, a_l) n^{a_1+a_2-2} |J_A|^2 k^4 \rho^{2(a_1+a_2)-4(s+1)} \\ = \; & n^{-2} p^{-4} k^4 \times \left(|J_A| \rho^{a_1} |J_A| \rho^{a_2}\right)^{\frac{2(a_1+a_2-2-2s)}{a_1+a_2}} \times |J_A|^{2-\frac{4(a_1+a_2-2-2s)}{a_1+a_2}} \end{aligned}$$

$$
\begin{aligned}
&= O(1)n^{-2-(a_1+a_2-2-2s)}p^{-\frac{8(1+s)}{a_1+a_2}} \times |J_A|^{-2+\frac{8(1+s)}{a_1+a_2}} \\
&= O(1)n^{-2-(a_1+a_2-2-2s)}|J_A|^{-2+\frac{4(1+s)}{a_1+a_2}} \left(\frac{|J_A|}{p^2}\right)^{\frac{4(1+s)}{a_1+a_2}} \\
&= o(n^{-2}),
\end{aligned}
$$

as $2(1+s) \leq a_1 + a_2$ and $1 + s > 0$.

Case 1.2. When $T_{1,2,3,4} = O(\rho)$ and $T_{5,6,7,8} = O(\rho)$. If at least one of $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ is $O(1)$, then the order of $(j_1, \cdots, j_8)$ is $O(|J_A|^2k^4) = o(p^4)$, it follows that $\prod_{t=1}^{2} c(n, a_l)n^{a_1+a_2-2}|J_A|^2k^4 \times$ (S5.49) $= o(n^{-2})$. If at least two of $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ are $O(\rho)$, the order of $(j_1, \cdots, j_8)$ is $O(|J_A|^2k^2) = o(p^4)$. If there exists only one among $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ is $O(\rho)$, and the remaining three terms are $O(\rho^2)$. For any finite non-negative integer $\omega \leq a_1 + a_2$, we have

$$
\begin{aligned}
&\prod_{t=1}^{2} c(n, a_l)n^{a_1+a_2-2}|J_A|^2k^3 \max_{j=1,\cdots,p} |J_j|\rho^{2(a_1+a_2)-\omega} \\
&= n^{-2}p^{-4}k^3 \times (|J_A|\rho^{a_1}|J_A|\rho^{a_2})^{\frac{2(a_1+a_2)-\omega}{a_1+a_2}} \times |J_A|^{2-\frac{4(a_1+a_2)-2\omega}{a_1+a_2}} \max_{j=1,\cdots,p} |J_j| \\
&= O(1)n^{-2-(a_1+a_2)+\omega/2}p^{-\frac{2\omega}{a_1+a_2}} \times |J_A|^{-2+\frac{2\omega}{a_1+a_2}} \max_{j=1,\cdots,p} |J_j| \\
&= O(1)n^{-2-(a_1+a_2)+\omega/2}|J_A|^{-1+\frac{\omega}{a_1+a_2}} \left(\frac{|J_A|}{p^2}\right)^{\frac{\omega}{a_1+a_2}} \frac{\max_{j=1,\cdots,p} |J_j|}{|J_A|} \\
&= o(n^{-2}),
\end{aligned}
$$

with $|J_A| = o(p^2)$ and $\max_{j=1,\cdots,p} |J_j| \leq |J_A|$. As $2s + 2 + s_1, a_2 - a_1 + s_1 + 2s +$

$2, a_1+1+s-s_1 \le a_1+a_2$, we have

$$\prod_{t=1}^{2} c(n,a_l) n^{a_1+a_2-2}|J_A|^2 k^3 \max_{j=1,\cdots,p}|J_j|\{\rho^{2(a_1+a_2)-(2s+2+s_1)} + \rho^{2(a_1+a_2)-(a_2-a_1+s_1+2s+2)} + \rho^{2(a_1+a_2)-(a_1+1+s-s_1)}\} = o(n^{-2}).$$

If there is no one term among $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ is ρ, it means that all of the four terms is ρ^2. Then,

$$\begin{aligned}
&\prod_{t=1}^{2} c(n,a_l) n^{a_1+a_2-2}|J_A|^3 k^2 \rho^{2(a_1+a_2)-2(s+1)} \\
= \;& n^{-2}p^{-4}k^2 \times \left(|J_A|\rho^{a_1}|J_A|\rho^{a_2}\right)^{\frac{2(a_1+a_2)-2(s+1)}{a_1+a_2}} \times |J_A|^{3-\frac{4(a_1+a_2)-4(s+1)}{a_1+a_2}} \\
= \;& O(1) n^{-2-(a_1+a_2)+(s+1)} p^{-\frac{4(s+1)}{a_1+a_2}} \times |J_A|^{-1+\frac{4(s+1)}{a_1+a_2}} \\
= \;& O(1) n^{-2-(a_1+a_2)+(s+1)} |J_A|^{-1+\frac{2(s+1)}{a_1+a_2}} \left(\frac{|J_A|}{p^2}\right)^{\frac{2(s+1)}{a_1+a_2}} \\
= \;& o(n^{-2}),
\end{aligned}$$

since $2(s+1) \le a_1+a_2$.

Case 1.3. When $T_{1,2,3,4} = O(\rho^2)$ and $T_{5,6,7,8} = O(\rho^2)$. We further consider (S5.49) into sub-cases.

(a). If at least two of $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ are $O(1)$, then the order of $(j_1, \cdots, j_8)$ is $O(|J_A|^2 k^4) = o(p^4)$, $\prod_{t=1}^{2} c(n,a_l) n^{a_1+a_2-2}|J_A|^2 k^4 \times$ (S5.49) $= o(n^{-2})$.

(b). If only one of $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ is $O(1)$, the order of

$(j_1, \cdots, j_8)$ is $O(|J_A|^3 k^2)$. For any finite non-negative integer $\omega \leq a_1 + a_2$, we have

$$\begin{aligned}
&\prod_{t=1}^{2} c(n, a_l) n^{a_1+a_2-2} |J_A|^3 k^2 \rho^{2(a_1+a_2)-\omega} \\
= \;& n^{-2} p^{-4} k^2 \times \left(|J_A| \rho^{a_1} |J_A| \rho^{a_2}\right)^{\frac{2(a_1+a_2)-\omega}{a_1+a_2}} \times |J_A|^{3-\frac{4(a_1+a_2)-2\omega}{a_1+a_2}} \\
= \;& O(1) n^{-2-(a_1+a_2)+\omega/2} p^{-\frac{2\omega}{a_1+a_2}} \times |J_A|^{-1+\frac{2\omega}{a_1+a_2}} \\
= \;& O(1) n^{-2-(a_1+a_2)+\omega/2} |J_A|^{-1+\frac{\omega}{a_1+a_2}} \left(\frac{|J_A|}{p^2}\right)^{\frac{\omega}{a_1+a_2}} \\
= \;& o(n^{-2}).
\end{aligned}$$

As $a_2 - a_1 + 3s_1, a_1 + s_1 - s - 1, 2(a_1 - s - 1), a_1 + a_2 - s_1 - 2s, 2(a_2 - a_1 + s_1) + s_1, 2(a_2 - a_1 + s_1) + (a_1 - 1 - s - s_1), 2(a_2 - a_1 + s_1), a_2 + s_1 - s - 1, 2(a_2 - 1 - s - s_1) + s_1, 0 \leq a_1 + a_2$, we have

$$\begin{aligned}
&\prod_{t=1}^{2} c(n, a_l) n^{a_1+a_2-2} |J_A|^3 k^2 \{\rho^{2(a_1+a_2)-(a_2-a_1+3s_1)} \\
&+\rho^{2(a_1+a_2)-(a_1+s_1-s-1)} + \rho^{2(a_1+a_2)-2(a_1-s-1)} \\
&+\rho^{2(a_1+a_2)-(a_1+a_2-s_1-2s)} + \rho^{2(a_1+a_2)-2(a_2-a_1+s_1)-s_1} \\
&+\rho^{2(a_1+a_2)-2(a_2-a_1+s_1)-(a_1-1-s-s_1)} + \rho^{2(a_1+a_2)-2(a_2-a_1+s_1)} \\
&+\rho^{2(a_1+a_2)-(a_2+s_1-s-1)} + +\rho^{2(a_1+a_2)-2(a_2-1-s-s_1)-s_1} \\
&+\rho^{2(a_1+a_2)}\} = o(n^{-2}).
\end{aligned}$$

(c). There is no term among $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ are $O(1)$ and all of them are $O(\rho^2)$. It follows that

$$\prod_{t=1}^{2} c(n,a_l) n^{a_1+a_2-2} |J_A|^3 \rho^{2(a_1+a_2)} = o(n^{-2}),$$

as $|J_A|\rho_a = O(n^{-a/2}p)$.

(d). There is no term among $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ are $O(1)$ and only one is $O(\rho)$. It follows that

$$\prod_{t=1}^{2} c(n,a_l) n^{a_1+a_2-2} |J_A|^3 k \max_{j=1,\cdots,p} |J_j| \{\rho^{2(a_1+a_2)-s_1} + \rho^{2(a_1+a_2)-(a_2-a_1+s_1)} + \rho^{2(a_1+a_2)-(a_1-1-s-s_1)}\} = o(n^{-2}).$$

(d). There is no term among $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ are $O(1)$ and two are $O(\rho)$. It follows that

$$\prod_{t=1}^{2} c(n,a_l) n^{a_1+a_2-2} |J_A|^3 k^2 \{\rho^{2(a_1+a_2)-(a_2-a_1+2s_1)} + \rho^{2(a_1+a_2)-(a_1-1+s)} + \rho^{2(a_1+a_2)-(a_2-1-s)}\} = o(n^{-2}).$$

(e). There is no term among $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ are $O(1)$ and

three are $O(\rho)$. It follows that

$$\prod_{t=1}^{2} c(n,a_l)n^{a_1+a_2-2}|J_A|^2k^3 \max_{j=1,\cdots,p}|J_j|\{\rho^{2(a_1+a_2)-(a_2+s_1-s-1)} +\rho^{2(a_1+a_2)-(a_1+a_2-s_1-2s-2)}+\rho^{2(a_1+a_2)-(2a_1-2s-s_1-2)}\}=o(n^{-2}).$$

(f). There is no term among $T_{1,2,5,6}, T_{3,4,7,8}, T_{1,2,7,8}$ and $T_{3,4,5,6}$ are $O(1)$ and all of them are $O(\rho)$ and the order of $(j_1,\cdots,j_8)$ is $O(|J_A|^2k^2)=o(p^4)$.

In summary,

$$\begin{aligned}&\Big|\sum_{\substack{\mathbf{i}^{(l)}\in\mathcal{P}(h-1,a_l-1)\\(j_{2l-1},j_{2l})\in J_A^c,l=1,2,3,4}}\prod_{t=1}^{2}c(n,a_l)\mathcal{H}(h,\mathbf{i}^{(l)},j_{2l-1},j_{2l};l=1,2,3,4)\mathbf{1}_{\{\mathrm{card}(\bigcup_{l=1}^{4}\{\mathbf{i}^l\})=a_1+a_2-2\}}\Big|\\ =\;& o(n^{-2}).\end{aligned}$$

Case 2. When $\mathrm{card}(\bigcup_{l=1}^{4}\{\mathbf{i}^l\})<a_1+a_2-2$, we can discuss different cases of (S5.49) as above. We have $\sum_{(j_{2l-1},j_{2l})\in J_A^c,l=1,2,3,4}(\text{S5.49})=O(p^4)$. If follows that

$$\begin{aligned}&\Big|\sum_{\substack{\mathbf{i}^{(l)}\in\mathcal{P}(h-1,a_l-1)\\(j_{2l-1},j_{2l})\in J_A^c,l=1,2,3,4}}\prod_{t=1}^{2}c(n,a_l)\mathcal{H}(h,\mathbf{i}^{(l)},j_{2l-1},j_{2l};l=1,2,3,4)\mathbf{1}_{\{\mathrm{card}(\bigcup_{l=1}^{4}\{\mathbf{i}^l\})\le a_1+a_2-2\}}\Big|\\ =\;& O\{\prod_{l=1}^{2}c(n,a_l)p^4n^{a_1+a_2-3}\}=o(n^{-2}).\end{aligned}$$

Then, Lemma 10 has been proved. □

S5.11 Proof of Lemma 11

Proof. Similarly to the proof skeletons of Lemma 6, it suffices to show $\mathrm{E}(\prod_{l=1}^{4} A_{n,h,a_{r_l}}) = o(n^{-1})$. By Cauchy-Schwarz inequality, we only need to prove $\mathrm{E}(\prod_{l=1}^{4} A_{n,h,a}) = o(n^{-1})$ for any finite integer a. When $h < a$, we have $A_{n,h,a} = 0$; when $h \geq a$,

$$\begin{aligned}\mathrm{E}(\prod_{l=1}^{4} A_{n,h,a}) \;=& \sum_{\substack{\mathbf{i}^{(l)} \in \mathcal{P}(h-1,a-1), l=1,\ldots,4; \\ (j_{2s-1}, j_{2s}) \in J_A^c, s=1,\ldots,4}} \Big\{ \prod_{l=1}^{4} c(n,a) \Big\} \\ & \times \mathrm{E}(\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2), \mathbf{i}^{(2)}|(j_3,j_4), \mathbf{i}^{(3)}|(j_5,j_6), \mathbf{i}^{(4)}|(j_7,j_8), h\}),\end{aligned}$$

where

$$\begin{aligned}& \mathrm{E}(\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2), \mathbf{i}^{(2)}|(j_3,j_4), \mathbf{i}^{(3)}|(j_5,j_6), \mathbf{i}^{(4)}|(j_7,j_8), h\}) \\ =\;& \mathrm{E}\Big(\prod_{l=1}^{4} x_{h,j_{2l-1}} x_{h,j_{2l}} \Big) \mathrm{E}\Big(\prod_{l=1}^{4} \prod_{t=1}^{a_l - 1} x_{i_t^{(l)}, j_{2l-1}} x_{i_t^{(l)}, j_{2l}} \Big). \qquad \text{(S5.51)}\end{aligned}$$

As $\prod_{l=1}^{4} c(n, a_l) = \Theta(p^{-4} n^{-2a})$, to prove $\mathrm{E}(\prod_{l=1}^{4} A_{n,h,a}) = o(n^{-1})$, it suffices to show

$$\sum_{\substack{\mathbf{i}^{(l)} \in \mathcal{P}(h-1,a-1), l=1,\ldots,4; \\ (j_{2s-1}, j_{2s}) \in J_A^c, s=1,\ldots,4}} \text{(S5.51)} = o(p^4 n^{2a-1}). \qquad \text{(S5.52)}$$

Since $\sigma_{j_1,j_2} = 0$ when $(j_1, j_2) \in J_A^c$, it follows that (S5.35) $= 0$ when the index i only appears once in the sets $\bigcup_{l=1,\ldots,4} \{\mathbf{i}^{(l)}\}$. Thus,

$$\sum_{\mathbf{i}^{(l)} \in \mathcal{P}(h-1,a-1), l=1,\ldots,4} (\text{S5.51}) = O(n^{2a-2}).$$

To prove (S5.52), it suffices to show

$$\sum_{(j_{2s-1},j_{2s}) \in J_A^c, s=1,\ldots,4} (\text{S5.51}) = O(p^4). \tag{S5.53}$$

By Condition 5, (S5.51) can be written as certain linear combination of $\prod_{t=1}^{4a} (\sigma_{j_{g_{2t-1}} j_{g_{2t}}})$, where $g_{2t-1} \neq g_{2t}$ and $(g_1, \cdots, g_{8a})$ contain a number of $1, \cdots, 8$ respectively. Now, we prove (S5.53) by discussing $B_{1,2,3,4}$ which is defined in (S5.10), to be noticed that, acccording to symmetry, $B_{5,6,7,8} = B_{1,2,3,4}$. Then

$$\begin{aligned}
& \left| \sum_{(j_{2s-1},j_{2s}) \in J_A^c, s=1,\ldots,4} \mathrm{E}(\mathbb{X}\{\mathbf{i}^{(1)}|(j_1,j_2), \mathbf{i}^{(2)}|(j_3,j_4), \mathbf{i}^{(3)}|(j_5,j_6), \mathbf{i}^{(4)}|(j_7,j_8), h\}) \right| \\
\leq\ & C \sum_{(j_{2s-1},j_{2s}) \in J_A^c, s=1,\ldots,4} \sigma^a_{j_1 j_3} \sigma^a_{j_2 j_4} \sigma^a_{j_5 j_7} \sigma^a_{j_6 j_8} \{ \mathbf{1}_{\{B_{5,6,7,8}=B_{1,2,3,4}=3\}} + \mathbf{1}_{\{B_{5,6,7,8}=B_{1,2,3,4}=2\}} \\
& \qquad + \mathbf{1}_{\{B_{5,6,7,8}=B_{1,2,3,4}=1\}} + \mathbf{1}_{\{B_{5,6,7,8}=B_{1,2,3,4}=0\}} \} \\
=\ & O(p^2k^6) + O(p^4k^4) + O(|J_A|^3 k^2 \rho^{3a}) + O(|J_A|^4 \rho^{4a}) = o(p^4).
\end{aligned}$$

□

S5.12 Proof of Proposition S5.1

Proof. (P1). When the $\tilde{j}_k$ has two neighbors $\tilde{j}_{k-1}$ and $\tilde{j}_{k+1}$. Without loss of generality, we assume $k = 3$. Suppose $\tilde{j}_3 - \tilde{j}_2 > D_0$ and $\tilde{j}_4 - \tilde{j}_3 > D_0$.

$$\begin{aligned}
& |\mathrm{E}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}x_{h,\tilde{j}_3}x_{h,\tilde{j}_4}x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8})| \\
= \; & |\mathrm{cov}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}x_{h,\tilde{j}_3}, x_{h,\tilde{j}_4}x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8}) \\
& \quad +\mathrm{E}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}x_{h,\tilde{j}_3})\mathrm{E}(x_{h,\tilde{j}_4}x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8})| \\
\leq \; & C\delta^{\frac{D_0\epsilon}{2+\epsilon}} + C|\mathrm{cov}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}, x_{h,\tilde{j}_3})| \\
= \; & O(p^{-(8+\mu)}).
\end{aligned}$$

For $\tilde{j}_1$ and $\tilde{j}_8$, we use the same similar proof method, which will not be repeated here.

(P2). Without loss of generality, we take $k = 5$, i.e. the pair $(\tilde{j}_{k-1}, \tilde{j}_k) = (\tilde{j}_4, \tilde{j}_5)$ with $\tilde{j}_5 - \tilde{j}_4 > k$. Suppose $\tilde{j}_4 - \tilde{j}_3 > D_0$ and $\tilde{j}_6 - \tilde{j}_5 > D_0$.

$$\begin{aligned}
& |\mathrm{E}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}x_{h,\tilde{j}_3}x_{h,\tilde{j}_4}x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8})| \\
= \; & |\mathrm{cov}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}x_{h,\tilde{j}_3}, x_{h,\tilde{j}_4}x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8}) + \mathrm{E}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}x_{h,\tilde{j}_3})\mathrm{E}(x_{h,\tilde{j}_4}x_{h,\tilde{j}_5}x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8})| \\
\leq \; & C\delta^{\frac{D_0\epsilon}{2+\epsilon}} + |\mathrm{E}(x_{h,\tilde{j}_1}x_{h,\tilde{j}_2}x_{h,\tilde{j}_3})||\mathrm{cov}(x_{h,\tilde{j}_4}x_{h,\tilde{j}_5}, x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8}) + \mathrm{E}(x_{h,\tilde{j}_4}x_{h,\tilde{j}_5})\mathrm{E}(x_{h,\tilde{j}_6}x_{h,\tilde{j}_7}x_{h,\tilde{j}_8})| \\
= \; & O(p^{-(8+\mu)}).
\end{aligned}$$

For the pairs $(\tilde{j}_1, \tilde{j}_2)$, suppose $\tilde{j}_2 - \tilde{j}_1 > k$ with $\tilde{j}_3 - \tilde{j}_2 > D_0$.

$$
\begin{aligned}
& |\mathrm{E}(x_{h,\tilde{j}_1} x_{h,\tilde{j}_2} x_{h,\tilde{j}_3} x_{h,\tilde{j}_4} x_{h,\tilde{j}_5} x_{h,\tilde{j}_6} x_{h,\tilde{j}_7} x_{h,\tilde{j}_8})| \\
= \; & |\mathrm{cov}(x_{h,\tilde{j}_1} x_{h,\tilde{j}_2}, x_{h,\tilde{j}_3} x_{h,\tilde{j}_4} x_{h,\tilde{j}_5} x_{h,\tilde{j}_6} x_{h,\tilde{j}_7} x_{h,\tilde{j}_8}) + \mathrm{E}(x_{h,\tilde{j}_1} x_{h,\tilde{j}_2}) \mathrm{E}(x_{h,\tilde{j}_3} x_{h,\tilde{j}_4} x_{h,\tilde{j}_5} x_{h,\tilde{j}_6} x_{h,\tilde{j}_7} x_{h,\tilde{j}_8})| \\
= \; & O(p^{-(8+\mu)}).
\end{aligned}
$$

□

S6. Proof of Proposition 1

Proof. To find the minimum of $\rho_a = (Mp'/|J_A|)^{1/a}(a!)^{1/2a}\kappa^{1/2}\nu n^{-1/2}$, it suffices to find the minimum of $f_a = (Mp'/|J_A|)^{1/a}(a!)^{1/2a}$.

(i). When $|J_A| \geq Mp'$, it is obvious that ρ_a achieves the minimum value at $a = 1$.

(ii). When $|J_A| \geq Mp'$,

$$
\begin{aligned}
\frac{f_{a+1}}{f_a} &= \frac{(Mp'/|J_A|)^{1/(a+1)}[(a+1)!]^{\frac{1}{2(a+1)}}}{(Mp'/|J_A|)^{1/a}[(a)!]^{\frac{1}{2a}}} \\
&= \left[\frac{(Mp'/|J_A|)^{2a}[(a+1)!]^a}{(Mp'/|J_A|)^{2(a+1)}[(a)!]^{a+1}}\right]^{\frac{1}{2a(a+1)}} \\
&= \left[(Mp'/|J_A|)^{-2}\frac{(a+1)^a}{a!}\right]^{\frac{1}{2a(a+1)}}
\end{aligned}
$$

To be noticed that $h_a = \frac{(a+1)^a}{a!}$ is a strictly increasing function of a

because

$$\frac{h_{a+1}}{h_a} = \frac{\frac{(a+2)^{a+1}}{(a+1)!}}{\frac{(a+1)^a}{a!}} = \left(\frac{a+2}{a+1}\right)^{a+1} > 1.$$

Thus, if $h_a > (Mp'/|J_A|)^{-2}$, then $h_{a+1} > (Mp'/|J_A|)^{-2}$. It means that if $f_{a+1}/f_a \geq 1$, then $f_{a+2}/f_{a+1} \geq 1$. In another words, $f(a)$ will strictly increase after some point. For example, if a_0 is the first integer such that $h_{a_0} > (Mp'/|J_A|)^{-2}$, it means that f_a is decreasing when $a < a_0$, and increasing when $a > a_0$, thus, a_0 achieves the minimum of f_a.

□

S7. Proof of Proposition 2

Proof. To prove (i) and (ii), we use the following equation

$$\begin{aligned}
&\sum_{1\leq i_1\neq\ldots\neq i_b\leq n}\prod_{l=1}^{b} x_{i_l,j_1}^{r_{l,1}} x_{i_l,j_2}^{r_{l,2}} \\
=\ &\left(\sum_{1\leq i_1\neq\ldots\neq i_{b-1}\leq n}\prod_{l=1}^{b-1} x_{i_l,j_1}^{r_{l,1}} x_{i_l,j_2}^{r_{l,2}}\right)\left(\sum_{i=1}^{n} x_{i,j_1}^{r_{b,1}} x_{i,j_2}^{r_{b,2}}\right) \\
&-\sum_{m=1}^{b-1}\sum_{1\leq i_1\neq\ldots\neq i_{b-1}\leq n}\left(\prod_{l=1}^{b-1} x_{i_l,j_1}^{r_{l,1}} x_{i_l,j_2}^{r_{l,2}}\right) x_{i_m,j_1}^{r_{b,1}} x_{i_m,j_2}^{r_{b,2}}, \qquad \text{(S7.54)}
\end{aligned}$$

where $a \leq b \leq 2a$, and $r_{l,1}, r_{l,2} \geq 0$ and $r_{l,1} + r_{l,2} \geq 1$ holds. It can reduce the computation cost of $\sum_{1\leq i_1\neq\ldots\neq i_b\leq n}\prod_{l=1}^{b} x_{i_l,j_1}^{r_{l,1}} x_{i_l,j_2}^{r_{l,2}}$ from $O(n^b)$ to $O(n)$ with

several iterative steps. Now, we show how to improve the computation efficiency of $\mathcal{U}(a)$ with different order a.

(i). When $a = 1$, we have

$$\begin{aligned}\mathcal{U}(1) &= \sum_{k<|j_1-j_2|<p}\Big\{n^{-1}\sum_{i=1}^{n}x_{i,j_1}x_{i,j_2}-(P_2^n)^{-1}\sum_{1\le i_1\neq i_2\le n}x_{i_1,j_1}x_{i_2,j_2}\Big\}\\ &= \sum_{k<|j_1-j_2|<p}\Big\{n^{-1}\sum_{i=1}^{n}x_{i,j_1}x_{i,j_2}-(P_2^n)^{-1}\Big(\sum_{i_1=1}^{n}x_{i_1,j_1}\sum_{i_2=1}^{n}x_{i_2,j_2}-\sum_{i=1}^{n}x_{i,j_1}x_{i,j_2}\Big)\Big\}.\end{aligned}$$

(ii). When $a = 2$, we denote

$$\mathcal{U}(2) = \sum_{k<|j_1-j_2|<p}\sum_{c=0}^{a}\binom{a}{c}(-1)^c(P_{a+c}^n)^{-1}U_c(2),$$

where

$$U_0(2) = \sum_{1\le i_1\neq i_2\le n}x_{i_1,j_1}x_{i_1,j_2}x_{i_2,j_1}x_{i_2,j_2}, \tag{S7.55}$$

$$U_1(2) = \sum_{1\le i_1\neq i_2\neq i_3\le n}x_{i_1,j_1}x_{i_1,j_2}x_{i_2,j_1}x_{i_3,j_2}, \tag{S7.56}$$

$$U_2(2) = \sum_{1\le i_1\neq i_2\neq i_3\neq i_4\le n}x_{i_1,j_1}x_{i_2,j_1}x_{i_3,j_2}x_{i_4,j_2}. \tag{S7.57}$$

Then, (S7.55)-(S7.57) can be rewritten as follows.

$$\begin{aligned}U_0(2) &= \Big(\sum_{i=1}^{n}x_{i,j_1}x_{i,j_2}\Big)^2-\sum_{i=1}^{n}(x_{i,j_1}x_{i,j_2})^2,\\ U_1(2) &= \Big(\sum_{i=1}^{n}x_{i,j_1}x_{i,j_2}\Big)\Big(\sum_{1\le i_1\neq i_2\le n}x_{i_1,j_1}x_{i_2,j_2}\Big)\end{aligned}$$

$$- \sum_{1\le i_1\neq i_2\le n} x_{i_1,j_1}^2 x_{i_1,j_2} x_{i_2,j_2} - \sum_{1\le i_1\neq i_2\le n} x_{i_1,j_1} x_{i_1,j_2}^2 x_{i_2,j_2}.$$

Furthermore,

$$\begin{aligned}
U_{11}(2) &\overset{def}{=} \sum_{1\le i_1\neq i_2\le n} x_{i_1,j_1} x_{i_2,j_2} = \Big(\sum_{i=1}^n x_{i,j_1}\Big)\Big(\sum_{i=1}^n x_{i,j_2}\Big) - \sum_{i=1}^n x_{i,j_1} x_{i,j_2},\\
U_{12}(2) &\overset{def}{=} \sum_{1\le i_1\neq i_2\le n} x_{i_1,j_1}^2 x_{i_1,j_2} x_{i_2,j_2} = \Big(\sum_{i=1}^n x_{i,j_1}^2 x_{i,j_2}\Big)\Big(\sum_{i=1}^n x_{i,j_2}\Big) - \sum_{i=1}^n x_{i,j_1}^2 x_{i,j_2}^2,\\
U_{13}(2) &\overset{def}{=} \sum_{1\le i_1\neq i_2\le n} x_{i_1,j_1} x_{i_1,j_2}^2 x_{i_2,j_2} = \Big(\sum_{i=1}^n x_{i,j_1} x_{i,j_2}^2\Big)\Big(\sum_{i=1}^n x_{i,j_2}\Big) - \sum_{i=1}^n x_{i,j_1}^2 x_{i,j_2}^2.
\end{aligned}$$

Then,

$$\begin{aligned}
U_2(2) &= \Big(\sum_{1\le i_1\neq i_2\le n} x_{i_1,j_1} x_{i_2,j_1}\Big)\Big(\sum_{1\le i_3\neq i_4\le n} x_{i_3,j_2} x_{i_4,j_2}\Big) - 2U_0(2) - 4U_1(2)\\
&= \prod_{s=1}^2 \Big\{\Big(\sum_{i=1}^n x_{i,j_s}\Big)^2 - \Big(\sum_{i=1}^n x_{i,j_s}^2\Big)\Big\} - 2U_0(2) - 4U_1(2).
\end{aligned}$$

(iii). When $a \ge 3$, the closed form is hard to derive by the iterative method. So we propose to substitute $\mathcal{U}(a)$ with $\mathcal{U}_c(a)$. Let $\mathcal{U}_{c,2}(a) = \mathcal{U}_c(a) - \mathcal{U}_{c,1}(a)$, where $\mathcal{U}_{c,1}(a) = \tilde{\mathcal{U}}(a)$. In Lemma 1, we have proved that $\tilde{\mathcal{U}}(a)/\sigma(a) \xrightarrow{D} \mathcal{N}(0,1)$. To establish Proposition 1.(iii), it suffices to prove $\mathcal{U}_{c,2}(a) = o_p(\sigma^2(a))$. As $\sigma^2(a) = \Theta(p^2 n^{-a})$, we only need to prove $E\{\mathcal{U}_{c,2}^2(a)\} = o(p^2 n^{-a})$. For a given tuple $\mathbf{i} \in \mathcal{P}(n,a)$, let $\mathbf{i}_{(r_1+r_2+r_3)}$ represent a sub-tuple of $\mathbf{i}$ with length $r_1+r_2+r_3$, and define $\mathcal{I}(\mathbf{i}, r_1+r_2+r_3)$

to be the collection of sub-tuples of $\mathbf{i}$ with length $r_1 + r_2 + r_3$. Besides that, we define $\mathcal{C}(n,k)$ as the the collection of sub-tuples of $\mathbf{w}$ satisfying $1 \leq w_1, ..., w_k \leq n$ without constraining the elements to be different.

$$\begin{aligned}
\mathcal{U}_{c,2}(a) &= (P_a^n)^{-1} \sum_{\substack{\mathbf{i}\in\mathcal{P}(n,a);\\ k<|j_1-j_2|<p}} \Big\{ \prod_{l=1}^{a} (x_{i_l,j_1}x_{i_l,j_2} - x_{i_l,j_1}\bar{x}_{j_2} - \bar{x}_{j_1}x_{i_l,j_2} + \bar{x}_{j_1}\bar{x}_{j_2}) \\
&\qquad\qquad - \prod_{l=1}^{a}(x_{i_l,j_1}x_{i_l,j_2}) \Big\} \\
&= (P_a^n)^{-1} \sum_{\substack{\mathbf{i}\in\mathcal{P}(n,a);\\ k<|j_1-j_2|<p}} \sum_{\substack{0\leq r_1,r_2\leq a;\\ 0\leq r_3<a}} \sum_{\mathbf{i}_{r_1+r_2+r_3}\in\mathcal{I}(\mathbf{i},r_1+r_2+r_3)} \Big\{ (-\bar{x}_{j_2})^{r_1} \prod_{l=1}^{r_1} x_{i_l,j_1} \Big\} \\
&\qquad \times \Big\{ (-\bar{x}_{j_1})^{r_2} \prod_{l=r_1+1}^{r_1+r_2} x_{i_l,j_2} \Big\} \Big\{ \prod_{l=r_1+r_2+1}^{r_1+r_2+r_3} x_{i_l,j_1}x_{i_l,j_2} \Big\} (\bar{x}_{j_1}\bar{x}_{j_2})^{a-r_1-r_2-r_3} \\
&= \sum_{\substack{0\leq r_1,r_2\leq a;\\ 0\leq r_3<a}} C_{r_1,r_2,r_3} \mathscr{T}_{r_1,r_2,r_3},
\end{aligned}$$

where C_{r_1,r_2,r_3} are some constants depend on r_1, r_2, r_3 and a,

$$\begin{aligned}
\mathscr{T}_{r_1,r_2,r_3} &= (P_{r_1+r_2+r_3}^n)^{-1} \sum_{\substack{\mathbf{i}\in\mathcal{P}(n,r_1+r_2+r_3);\\ k<|j_1-j_2|<p}} \\
&\qquad \times \Big\{ \prod_{l=r_1+r_2+1}^{r_1+r_2+r_3} x_{i_l,j_1}x_{i_l,j_2} \Big\} (\bar{x}_{j_1}\bar{x}_{j_2})^{a-r_1-r_2-r_3}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{E}[\mathscr{T}^2_{r_1,r_2,r_3}] &= (P^n_{r_1+r_2+r_3})^{-2} \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,r_1+r_2+r_3);\\ k<|j_1-j_2|<p,k<|j_3-j_4|<p}} \Big\{(\bar{x}_{j_2}\bar{x}_{j_4})^{r_1}\prod_{l=1}^{r_1} x_{i_l,j_1}x_{\tilde{i}_l,j_3}\Big\} \\
&\quad\times\Big\{(\bar{x}_{j_1}\bar{x}_{j_3})^{r_2}\prod_{l=r_1+1}^{r_1+r_2} x_{i_l,j_2}x_{\tilde{i}_l,j_4}\Big\} \\
&\quad\times\Big\{\prod_{l=r_1+r_2+1}^{r_1+r_2+r_3} x_{i_l,j_1}x_{i_l,j_2}x_{\tilde{i}_l,j_3}x_{\tilde{i}_l,j_4}\Big\}(\bar{x}_{j_1}\bar{x}_{j_2}\bar{x}_{j_3}\bar{x}_{j_4})^{a-r_1-r_2-r_3} \\
&= (P^n_{r_1+r_2+r_3})^{-2} \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,r_1+r_2+r_3);\\ k<|j_1-j_2|<p,k<|j_3-j_4|<p}} \Big\{\prod_{l=1}^{r_1} x_{i_l,j_1}x_{\tilde{i}_l,j_3}\Big\}\Big\{\prod_{l=r_1+1}^{r_1+r_2} x_{i_l,j_2}x_{\tilde{i}_l,j_4}\Big\} \\
&\quad\times\Big\{\prod_{l=r_1+r_2+1}^{r_1+r_2+r_3} x_{i_l,j_1}x_{i_l,j_2}x_{\tilde{i}_l,j_3}x_{\tilde{i}_l,j_4}\Big\}(\bar{x}_{j_1}\bar{x}_{j_3})^{a-r_1-r_3}(\bar{x}_{j_2}\bar{x}_{j_4})^{a-r_2-r_3} \\
&= \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,r_1+r_2+r_3);\\ \mathbf{w},\tilde{\mathbf{w}}\in\mathcal{C}(n,2a-r_1-r_2-2r_3);\\ k<|j_1-j_2|<p,k<|j_3-j_4|<p}} C_{n,r_1,r_2,r_3}\mathrm{E}\{\mathscr{D}(\mathbf{i},\tilde{\mathbf{i}},\mathbf{w},\tilde{\mathbf{w}},\mathbf{j})\},
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{D}(\mathbf{i},\tilde{\mathbf{i}},\mathbf{w},\tilde{\mathbf{w}},\mathbf{j}) &= \Big\{\prod_{l=1}^{r_1} x_{i_l,j_1}x_{\tilde{i}_l,j_3}\Big\}\Big\{\prod_{l=r_1+1}^{r_1+r_2} x_{i_l,j_2}x_{\tilde{i}_l,j_4}\Big\}\Big\{\prod_{l=r_1+r_2+1}^{r_1+r_2+r_3} x_{i_l,j_1}x_{i_l,j_2}x_{\tilde{i}_l,j_3}x_{\tilde{i}_l,j_4}\Big\} \\
&\quad\times\Big\{\prod_{l=1}^{a-r_1-r_3} x_{w_l,j_1}x_{\tilde{w}_l,j_3}\Big\}\Big\{\prod_{l=a-r_1-r_3+1}^{2a-r_1-r_2-2r_3} x_{w_l,j_2}x_{\tilde{w}_l,j_4}\Big\},
\end{aligned}$$

$C_{n,r_1,r_2,r_3} = (P^n_{r_1+r_2+r_3})^{-2}n^{-(4a-2r_1-2r_2-4r_3)} = n^{2r_3-4a}$. Now, we discuss

$\mathscr{D}(\mathbf{i},\tilde{\mathbf{i}},\mathbf{w},\tilde{\mathbf{w}},\mathbf{j})$. Let $\mathscr{D}(\mathbf{i},\tilde{\mathbf{i}},\mathbf{w},\tilde{\mathbf{w}},\mathbf{j})=\mathscr{D}_{j_1}\mathscr{D}_{j_2}\mathscr{D}_{j_3}\mathscr{D}_{j_4}$, where

$$\begin{aligned}
\mathscr{D}_{j_1} &= \prod_{l=1}^{r_1} x_{i_l,j_1} \prod_{l=r_1+r_2+1}^{r_1+r_2+r_3} x_{i_l,j_1} \prod_{l=1}^{a-r_1-r_3} x_{w_l,j_1},\\
\mathscr{D}_{j_2} &= \prod_{l=r_1+1}^{r_1+r_2+r_3} x_{i_l,j_2} \prod_{l=a-r_1-r_3+1}^{2a-r_1-r_2-2r_3} x_{w_l,j_2},\\
\mathscr{D}_{j_3} &= \prod_{l=1}^{r_1} x_{\tilde{i}_l,j_3} \prod_{l=r_1+r_2+1}^{r_1+r_2+r_3} x_{\tilde{i}_l,j_3} \prod_{l=1}^{a-r_1-r_3} x_{\tilde{w}_l,j_3},\\
\mathscr{D}_{j_4} &= \prod_{l=r_1+1}^{r_1+r_2+r_3} x_{\tilde{i}_l,j_4} \prod_{l=a-r_1-r_3+1}^{2a-r_1-r_2-2r_3} x_{\tilde{w}_l,j_4}.
\end{aligned}$$

For convenience, we denote following sets

$$S_{j_1}=S_{j_1,\mathbf{i}}\cup S_{j_1,\mathbf{w}},\quad S_{j_2}=S_{j_2,\mathbf{i}}\cup S_{j_2,\mathbf{w}},$$

$$S_{j_3}=S_{j_3,\tilde{\mathbf{i}}}\cup S_{j_3,\tilde{\mathbf{w}}},\quad S_{j_4}=S_{j_4,\tilde{\mathbf{i}}}\cup S_{j_4,\tilde{\mathbf{w}}},$$

where

$$S_{j_1,\mathbf{i}}=\{i_1,...,i_{r_1},i_{r_1+r_2+1},...,i_{r_1+r_2+r_3}\},\quad S_{j_1,\mathbf{w}}=\{w_1,...,w_{a-r_1-r_3}\},$$

$$S_{j_2,\mathbf{i}}=\{i_{r_1+1},...,,i_{r_1+r_2+r_3}\},\quad S_{j_2,\mathbf{w}}=\{w_{a-r_1-r_3+1},...,w_{2a-r_1-r_2-2r_3}\},$$

$$S_{j_3,\tilde{\mathbf{i}}}=\{\tilde{i}_1,...,\tilde{i}_{r_1},\tilde{i}_{r_1+r_2+1},...,\tilde{i}_{r_1+r_2+r_3}\},\quad S_{j_3,\tilde{\mathbf{w}}}=\{\tilde{w}_1,...,\tilde{w}_{a-r_1-r_3}\},$$

$$S_{j_4,\tilde{\mathbf{i}}}=\{\tilde{i}_{r_1+1},...,,\tilde{i}_{r_1+r_2+r_3}\},\quad S_{j_4,\tilde{\mathbf{w}}}=\{\tilde{w}_{a-r_1-r_3+1},...,\tilde{w}_{2a-r_1-r_2-2r_3}\}.$$

Without loss of generality, we assume $\mathrm{E}(x_{ij})=0$. $\mathrm{E}(\mathscr{D}_{j_1})\neq 0$ only when

$S_{j_1,\mathbf{i}} \subset S_{j_1,\mathbf{w}}$ and

$$\text{card}(S_{j_1}) \leq r_1 + r_3 + [(a - 2r_1 - 2r_3)/2] = [a/2], \tag{S7.58}$$

where $\text{card}(\cdot)$ denotes the number of the elements in a certain set, and $[\cdot]$ denotes the integer part of a number. Similarly, $\text{E}(\mathscr{D}_{j_2}) \neq 0$ only when $S_{j_2,\mathbf{i}} \subset S_{j_2,\mathbf{w}}$ and

$$\text{card}(S_{j_2}) \leq r_2 + r_3 + [(a - 2r_2 - 2r_3)/2] = [a/2]. \tag{S7.59}$$

Notice that $\text{card}(S_{j_1} \cap S_{j_2}) \geq r_3$, then we can conduct that if $\text{E}(\mathscr{D}_{j_1}) \neq 0$ and $\text{E}(\mathscr{D}_{j_2}) \neq 0$,

$$\text{card}(S_{j_1} \cup S_{j_2}) \leq 2[a/2] - r_3. \tag{S7.60}$$

Moreover, we know that $\text{E}(\mathscr{D}_{j_3}) \neq 0$ only when $S_{j_3,\tilde{\mathbf{i}}} \subset S_{j_3,\tilde{\mathbf{w}}}$ and

$$\text{card}(S_{j_3}) \leq [a/2]; \tag{S7.61}$$

$\text{E}(\mathscr{D}_{j_4}) \neq 0$ only when $S_{j_4,\tilde{\mathbf{i}}} \subset S_{j_4,\tilde{\mathbf{w}}}$ and

$$\text{card}(S_{j_4}) \leq [a/2]. \tag{S7.62}$$

If $\mathrm{E}(\mathscr{D}_{j_3}) \neq 0$ and $\mathrm{E}(\mathscr{D}_{j_4}) \neq 0$,

$$\mathrm{card}(S_{j_3} \cup S_{j_4}) \leq 2[a/2] - r_3. \tag{S7.63}$$

We now consider $\mathrm{E}\{\mathscr{D}(\mathbf{i}, \tilde{\mathbf{i}}, \mathbf{w}, \tilde{\mathbf{w}}, \mathbf{j})\}$ by discussing (j_1, j_2, j_3, j_4). We first consider $\mathrm{card}(j_1, j_2, j_3,$

$j_4) = 4$, the other cases can be discussed like similar analysis used in Theorem 1. We assume $j_1 < j_2 < j_3 < j_4$, and define $\kappa_1 = j_2 - j_1$, $\kappa_2 = j_3 - j_2$, $\kappa_3 = j_4 - j_3$, $K_0 = -(2+\epsilon)(4+\mu)(\log p)/(\epsilon \log \delta)$. If $\kappa_m = \max\{\kappa_1, \kappa_2, \kappa_3\} < K_0$, it is obvious that

$$\begin{aligned} \Bigg| \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}} \in \mathcal{P}(n, r_1+r_2+r_3); \\ \mathbf{w},\tilde{\mathbf{w}} \in \mathcal{P}(n, 2a-r_1-r_2-2r_3); \\ k<|j_1-j_2|<p, k<|j_3-j_4|<p}} C_{n,r_1,r_2,r_3} \mathrm{E}\{\mathscr{D}(\mathbf{i}, \tilde{\mathbf{i}}, \mathbf{w}, \tilde{\mathbf{w}}, \mathbf{j})\} \Bigg| &= O(1) \sum_{r_3=0}^{a-1} n^{2r_3-4a} n^{2a-r_3} p K_0^3 \\ &= o(p^2 n^{-a}), \end{aligned}$$

If $\kappa_m \geq K_0$, we denote

$$|\mathrm{E}\{\mathscr{D}(\mathbf{i}, \tilde{\mathbf{i}}, \mathbf{w}, \tilde{\mathbf{w}}, \mathbf{j})\}| \leq C\delta^{K_0\epsilon/(2+\epsilon)} + \mathscr{M}_{j_1,j_2,j_3,j_4}. \tag{S7.64}$$

Consider the first term to the right of the inequality (S7.64), we know $\mathrm{E}\{\mathscr{D}(\mathbf{i}, \tilde{\mathbf{i}}, \mathbf{w}, \tilde{\mathbf{w}}, \mathbf{j})\} \neq 0$ when $\mathrm{card}(\{\mathbf{i}\} \cup \{\tilde{\mathbf{i}}\} \cup \{\mathbf{w}\} \cup \{\tilde{\mathbf{w}}\}) \leq 2a - r_3$. It

follows that

$$\begin{aligned}\Big|\sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,r_1+r_2+r_3);\\ \mathbf{w},\tilde{\mathbf{w}}\in\mathcal{P}(n,2a-r_1-r_2-2r_3);\\ k<|j_1-j_2|<p,k<|j_3-j_4|<p}} C_{n,r_1,r_2,r_3}\delta^{K_0\epsilon/(2+\epsilon)}\Big| &= C\sum_{r_3=0}^{a-1} n^{2r_3-4a}n^{2a-r_3}p^4\delta^{K_0\epsilon/(2+\epsilon)}\\ &= o(n^{-(a+1)}),\end{aligned}$$

because $C_{n,r_1,r_2,r_3}=O(n^{2r_3-4a})$ and $\delta^{K_0\epsilon/(2+\epsilon)}=O(1)p^{-(4+\mu)}$.

Now, we consider $\mathscr{M}_{j_1,j_2,j_3,j_4}$.

Case 1. When $\kappa_1,\kappa_2,\kappa_3>K_0$.

$$\mathscr{M}_{j_1,j_2,j_3,j_4}=|\mathrm{E}(\mathscr{D}_{j_1})\mathrm{E}(\mathscr{D}_{j_2})\mathrm{E}(\mathscr{D}_{j_3})\mathrm{E}(\mathscr{D}_{j_4})|.$$

If $\mathscr{M}_{j_1,j_2,j_3,j_4}\neq 0$, when $\mathrm{E}(\mathscr{D}_{j_1})$, $\mathrm{E}(\mathscr{D}_{j_2})$, $\mathrm{E}(\mathscr{D}_{j_3})$, $\mathrm{E}(\mathscr{D}_{j_4})\neq 0$, we have (S7.60) and (S7.63) hold. By Condition 3, it follows that

$$\begin{aligned}&\Big|\sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,r_1+r_2+r_3);\\ \mathbf{w},\tilde{\mathbf{w}}\in\mathcal{P}(n,2a-r_1-r_2-2r_3);\\ k<|j_1-j_2|<p,k<|j_3-j_4|<p}} C_{n,r_1,r_2,r_3}\mathscr{M}_{j_1,j_2,j_3,j_4}\mathbf{1}_{\{\kappa_1,\kappa_2,\kappa_3>K_0\}}\Big|\\ \leq\ & O(1)n^{2r_3-4a}p^4n^{4[a/2]-2r_3}=O(1)p^4n^{4[a/2]-4a}. \qquad \text{(S7.65)}\end{aligned}$$

When a is even, (S7.63)$=O(p^4n^{-2a})=o(p^2n^{-a})$; when a is odd, (S7.63)$=O(p^4n^{-2a-2})=o(p^2n^{-a})$.

Case 2.1. When $\kappa_1 \le K_0$, $\kappa_2, \kappa_3 > K_0$.

$$\mathscr{M}_{j_1,j_2,j_3,j_4} = |\mathrm{E}(\mathscr{D}_{j_1}\mathscr{D}_{j_2})\mathrm{E}(\mathscr{D}_{j_3})\mathrm{E}(\mathscr{D}_{j_4})|.$$

$\mathrm{E}(\mathscr{D}_{j_1}\mathscr{D}_{j_2}) \neq 0$ only when $S_{j_1,\mathbf{i}} \cup S_{j_2,\mathbf{i}} \subset S_{j_1,\mathbf{w}} \cup S_{j_2,\mathbf{w}}$, and $\mathrm{card}(S_{j_1} \cup S_{j_2}) \le [(2a - r_3)/2]$. $\mathrm{E}(\mathscr{D}_{j_3})$, $\mathrm{E}(\mathscr{D}_{j_4}) \neq 0$ when (S7.63) hold. Thus,

$$\begin{aligned}
&\Bigg| \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}} \in \mathcal{P}(n, r_1+r_2+r_3);\\ \mathbf{w},\tilde{\mathbf{w}} \in \mathcal{P}(n, 2a-r_1-r_2-2r_3);\\ k<|j_1-j_2|<p, k<|j_3-j_4|<p}} C_{n,r_1,r_2,r_3} \mathscr{M}_{j_1,j_2,j_3,j_4} \mathbf{1}_{\{\kappa_1 \le K_0, \kappa_2, \kappa_3 > K_0\}} \Bigg| \\
\le & \sum_{r_3=0}^{a-1} O(1) n^{2r_3-4a} n^{2[a/2]-r_3+[(2a-r_3)/2]} p^3 K_0 \\
\le & \sum_{r_3=0}^{a-1} O(1) n^{-2a+r_3/2} p^3 K_0, \\
= & O(1) n^{-2a+(a-1)/2} p^3 K_0. \qquad \text{(S7.66)}
\end{aligned}$$

as $2[a/2] - r_3 + [(2a - r_3)/2] + 2r_3 - 4a \le -2a + r_3/2$. When a is even, then (S7.66)$= o(p^2 n^{-a})$; when a is odd, then (S7.66)$= o(p^2 n^{-a})$.

Case 2.2. When $\kappa_1, \kappa_2 > K_0$, $\kappa_3 \le K_0$.

Similar to Case 2.1., we can get the same results.

Case 2.3. When $\kappa_1, \kappa_3 > K_0$, $\kappa_2 \le K_0$.

$$\mathscr{M}_{j_1,j_2,j_3,j_4} = |\mathrm{E}(\mathscr{D}_{j_1})\mathrm{E}(\mathscr{D}_{j_2}\mathscr{D}_{j_3})\mathrm{E}(\mathscr{D}_{j_4})|,$$

Recall

$$\mathscr{D}_{j_2}\mathscr{D}_{j_3} = \prod_{l=r_1+1}^{r_1+r_2+r_3} x_{i_l,j_2} \prod_{l=a-r_1-r_3+1}^{2a-r_1-r_2-2r_3} x_{w_l,j_2} \prod_{l=1}^{r_1} x_{\tilde{i}_l,j_3} \prod_{l=r_1+r_2+1}^{r_1+r_2+r_3} x_{\tilde{i}_l,j_3} \prod_{l=1}^{a-r_1-r_3} x_{\tilde{w}_l,j_3},$$

To ensure $\mathrm{E}(\mathscr{D}_{j_2}\mathscr{D}_{j_3}) \neq 0$, $\mathrm{card}(S_{j_2} \cup S_{j_3}) \leq a$ holds. If $\mathrm{E}(\mathscr{D}_{j_1})$, $\mathrm{E}(\mathscr{D}_{j_4}) \neq 0$, we know (S7.58) and (S7.62) hold. Since $\mathrm{card}((S_{j_2} \cup S_{j_3}) \cap (S_{j_1} \cup S_{j_4})) = 2r_3$, then we have $\mathrm{card}((S_{j_2} \cup S_{j_3}) \cup (S_{j_1} \cup S_{j_4})) \leq a + 2[a/2] - 2r_3$. Thus,

$$\begin{aligned} & \Bigg| \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}} \in \mathcal{P}(n, r_1+r_2+r_3); \\ \mathbf{w},\tilde{\mathbf{w}} \in \mathcal{P}(n, 2a-r_1-r_2-2r_3); \\ k<|j_1-j_2|<p, k<|j_3-j_4|<p}} C_{n,r_1,r_2,r_3} \mathscr{M}_{j_1,j_2,j_3,j_4} \mathbf{1}_{\{\kappa_2 \leq K_0, \kappa_1, \kappa_3 > K_0\}} \Bigg| \\ \leq \; & O(1) n^{2[a/2]-3a} p^3 K_0. \end{aligned} \tag{S7.67}$$

When a is even, we have (S7.67)$= O(p^3 K_0 n^{-2a}) = o(p^2 n^{-a})$; when a is odd, (S7.64)$= O(p^3 K_0 n^{-2a-1}) = o(p^2 n^{-a})$.

Case 3. When Only one of $\kappa_1, \kappa_2, \kappa_3 > K_0$. $\mathrm{E}(\mathscr{M}_{j_1,j_2,j_3,j_4}) \neq 0$ only when $\mathrm{card}(S_{j_2} \cup S_{j_3} \cup S_{j_1} \cup S_{j_4}) \leq 2a - r_3$. It follows that

$$\begin{aligned} & \Bigg| \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}} \in \mathcal{P}(n, r_1+r_2+r_3); \\ \mathbf{w},\tilde{\mathbf{w}} \in \mathcal{P}(n, 2a-r_1-r_2-2r_3); \\ k<|j_1-j_2|<p, k<|j_3-j_4|<p}} C_{n,r_1,r_2,r_3} \mathscr{M}_{j_1,j_2,j_3,j_4} \mathbf{1}_{\left\{ \substack{\mathrm{card}(j_1,j_2,j_3,j_4)=4, \\ \text{only one of } \kappa_1,\kappa_2,\kappa_3 > K_0 \text{ holds}} \right\}} \Bigg| \\ \leq \; & \sum_{r_3=0}^{a-1} O(1) n^{2r_3-4a} n^{2a-r_3} p^2 K_0^2 \\ = \; & O(1) p^2 K_0^2 n^{-a-1} = o(p^2 n^{-a}). \end{aligned}$$

Case 4. $\mathrm{card}(j_1, j_2, j_3, j_4) = 2$ or 3. Using similar analysis and techniques, we know that

$$\begin{aligned} & \Bigg| \sum_{\substack{\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{P}(n,r_1+r_2+r_3);\\ \mathbf{w},\tilde{\mathbf{w}}\in\mathcal{P}(n,2a-r_1-r_2-2r_3);\\ k<|j_1-j_2|<p, k<|j_3-j_4|<p}} C_{n,r_1,r_2,r_3} \mathrm{E}\{\mathscr{D}(\mathbf{i},\tilde{\mathbf{i}},\mathbf{w},\tilde{\mathbf{w}},\mathbf{j})\}\mathbf{1}_{\{\mathrm{card}(j_1,j_2,j_3,j_4)=2,3\}} \Bigg| \\ = \; & o(n^{-(a+1)}) + o(p^2 n^{-a}) \\ = \; & o(p^2 n^{-a}). \end{aligned}$$

Then, Proposition 2 has been proved. □

S8. Proof of Proposition 3

For a given finite order a, we rewrite $\mathcal{T}_{a,k} = \mathcal{T}_{a,k,1} + \mathcal{T}_{a,k,2}$, where

$$\mathcal{T}_{a,k,1} = n^{-1}\frac{\mathcal{U}_{a,k} - \mu_{a,k}}{\sigma_{a,k}}\frac{\sigma_{a,k}}{\hat{\sigma}_{a,k}} \quad \text{and} \quad \mathcal{T}_{a,k,2} = n^{-1}\frac{\mu_{a,k}}{\sigma_{a,k_0}}\frac{\sigma_{a,k_0}}{\sigma_{a,k}}\frac{\sigma_{a,k}}{\hat{\sigma}_{a,k}}$$

Based on the discussion in Section 2.6 of the main text, we have $n^{\delta}\mathcal{T}_{a,k,1} = O_p(n^{\delta-1}) \to 0$ for any $\delta \in (0,1)$. Besides that, we have

$$n^{\delta}(\mathcal{T}_{a,k,2} - \mathcal{T}_{a,k+1,2}) = n^{\delta-1}\frac{\mu_{a,k} - \mu_{a,k+1}}{\sigma_{a,k_0}}(C + o_p(1)), \tag{S8.68}$$

with

$$\frac{\mu_{a,k}-\mu_{a,k+1}}{\sigma_{a,k_0}} = \frac{\sum_{k<|j_1-j_2|<p}\sigma^a_{j_1j_2} - \sum_{k+1<|j_1-j_2|<p}\sigma^a_{j_1j_2}}{n[(P^n_a)^{-1}a!\sum_{\substack{k_0<|j_1-j_2|<p\\ k_0<|j_3-j_4|<p}}(\Pi_{j_1,j_2,j_3,j_4})^a]^{1/2}} \tag{S8.69}$$

$$= \frac{\sum_{|j_1-j_2|=k+1}\sigma^a_{j_1j_2}}{n[(P^n_a)^{-1}a!\sum_{\substack{k_0<|j_1-j_2|<p\\ k_0<|j_3-j_4|<p}}(\Pi_{j_1,j_2,j_3,j_4})^a]^{1/2}} \tag{S8.70}$$

and C is a constant which is determined by $\sigma_{a,k_0}/\sigma_{a,k}$. For any finite integer a and $k < k_0$, it follows that $\mu_{a,k} - \mu_{a,k+1} = O(p)$ and $\sigma_{a,k_0} = \Theta(n^{-a/2}p)$. Furthermore, since $\liminf_n\{\inf_{k<k_0}(\mu_{a,k} - \mu_{a,k+1})\} > 0$, it follows that $n^\delta(\mathcal{T}_{a,k,2} - \mathcal{T}_{a,k+1,2}) \to \infty$ in probability. For $k \geq k_0$, $d^\delta_{a,k} = o_p(1)$. Therefore, for any $\theta > 0$, as $n \to \infty$, there are $P(|d^\delta_{a,k}| > \theta) \to 1$ for $k < k_0$ and $P(|d^\delta_{a,k}| > \theta) \to 0$ for $k \geq k_0$. Then, for any $\theta > 0$ and $\epsilon > 0$, for each fixed bandwidth parameter k, there exsits a positive integer N_k such that for all $n \geq N_k$,

$$P(|d^\delta_{a,k}| < \theta) < \epsilon/(M+1) \qquad \text{for any} \quad k < k_0$$

$$P(|d^\delta_{a,k}| \geq \theta) < \epsilon/(M+1) \qquad \text{for any} \quad k_0 \leq k \leq M,$$

where M is a given positive constant. We set an $N = \max\{N_k, k = 0, ..., p-1\}$ such that the above are satisfied. We further define $B_{nk} \triangleq \{|d^\delta_{a,k}| < \theta\}$ and $B_n \triangleq (\bigcap_{i=0}^{k_0-1} B^c_{ni}) \cup (\bigcap_{i=k_0}^{M} B_{ni})$ for $n > N$. It follows that, for any $\omega \in B_n$,

$\hat{k}_{\delta,\theta}(\omega) = k_0,\ P(B_n^c) \leq \sum_{i=0}^{k_0-1} P(B_{ni}) + \sum_{i=k_0}^{M} P(B_{ni}^c) \leq \epsilon$. Hence, for any $0 \leq \delta < 1$ and $\theta > 0$, k_0, then $\hat{k}_{a,\delta,\theta} - k_0 \xrightarrow{p} 0$ with a given finite integer a. It is obvious that $\hat{k}_{\delta,\theta} - k_0 \xrightarrow{p} 0$ as $|\mathcal{I}|$ is finite.

S9. Additional Simulation Results

S9.1 Adaptive Testing Methods

In this section, we present additional extensive simulation studies under the Settings 1–3. Note that, for multivariate t population, we exclude "QC" test and "XW" test because they can't control the type I error well under these settings.

Table 1: Empirical sizes under Setting 2 for $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ and $n = 100$ (in percentage).

p	50	100	200	400	600	800	1000
adpUmin	6.60	5.10	6.10	5.50	6.50	5.90	5.30
adpUf	8.40	6.40	5.90	6.20	7.40	6.70	6.50
$\mathcal{U}(1)$	5.20	5.20	6.10	6.20	5.90	5.60	4.80
$\mathcal{U}(2)$	5.50	5.80	4.40	4.40	5.20	4.70	4.80
$\mathcal{U}(3)$	5.00	4.60	6.30	5.50	5.30	6.50	6.20
$\mathcal{U}(4)$	5.20	4.00	5.50	6.50	5.70	6.60	5.00
$\mathcal{U}(5)$	5.10	4.30	6.20	4.90	6.10	5.30	5.60
$\mathcal{U}(6)$	4.50	4.00	4.80	4.70	5.50	5.00	4.40
QC	4.50	3.90	4.90	5.00	4.90	5.90	6.20
XW	4.80	3.60	3.80	4.30	4.40	4.50	3.90

S9.2 Adaptive Bandwidth Estimator

In this subsection, we present the bandwidth estimates of considered estimators of multivaraite t_7 random vector which generated based on the

Table 2: Empirical sizes under Setting 3 for $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ and $n = 100$ (in percentage).

p	50	100	200	400	600	800	1000
adpUmin	5.40	6.50	6.60	6.60	5.80	6.50	5.80
adpUf	7.00	8.10	6.50	7.10	6.60	6.00	6.20
$\mathcal{U}(1)$	6.00	5.70	4.90	5.00	6.20	6.20	5.30
$\mathcal{U}(2)$	5.40	5.10	5.20	4.20	5.10	4.70	4.10
$\mathcal{U}(3)$	5.40	5.50	5.60	6.30	4.30	4.80	5.30
$\mathcal{U}(4)$	4.30	4.80	6.20	6.10	5.00	4.90	5.70
$\mathcal{U}(5)$	4.10	5.90	5.30	5.90	5.20	5.70	5.10
$\mathcal{U}(6)$	3.50	4.40	5.40	4.70	5.50	5.50	5.20
QC	3.50	4.20	4.50	3.40	4.30	4.00	4.80
XW	3.40	4.00	4.40	4.50	4.80	4.30	5.00

Table 3: Empirical sizes under Setting 2 for multivariate $t_7(\mathbf{0}, \mathbf{\Sigma})$ and $n = 100$ (in percentage).

p	50	100	200	400	600	800	1000
adpUmin	6.80	7.70	6.60	6.40	8.70	6.20	5.90
adpUf	6.80	7.30	5.90	6.90	7.90	6.08	5.70
$\mathcal{U}(1)$	3.50	5.00	3.90	5.10	6.10	5.10	4.20
$\mathcal{U}(2)$	6.20	6.90	5.70	6.00	5.80	5.50	6.40
$\mathcal{U}(3)$	7.20	5.00	4.70	5.50	4.70	5.00	4.60
$\mathcal{U}(4)$	4.90	6.60	4.90	5.20	6.40	4.80	5.10
$\mathcal{U}(5)$	3.10	4.70	5.10	6.50	5.50	5.60	5.00
$\mathcal{U}(6)$	3.80	4.90	5.60	4.40	6.10	5.24	5.20
QC	18.5	18.4	20.6	20.6	18.5	19.5	19.5
XW	1.20	1.40	1.30	0.70	0.90	1.00	0.70

Table 4: Empirical sizes under Setting 3 for multivariate $t_7(\mathbf{0}, \mathbf{\Sigma})$ and $n = 100$ (in percentage).

p	50	100	200	400	600	800	1000
adpUmin	5.90	5.80	6.00	7.40	6.40	6.30	7.30
adpUf	7.60	7.40	7.40	6.90	6.00	6.50	6.40
$\mathcal{U}(1)$	3.80	5.10	4.10	5.00	6.20	5.00	4.50
$\mathcal{U}(2)$	5.40	5.30	6.30	6.00	5.30	5.10	6.30
$\mathcal{U}(3)$	4.90	5.30	5.90	5.20	3.50	4.20	4.80
$\mathcal{U}(4)$	5.20	6.20	5.90	4.70	5.80	4.40	5.80
$\mathcal{U}(5)$	4.30	5.60	4.60	6.50	5.60	5.60	5.20
$\mathcal{U}(6)$	4.60	5.80	5.30	5.90	3.70	4.40	5.50
QC	13.4	15.2	19.0	19.0	17.6	18.4	19.4
XW	1.00	1.10	0.50	1.10	0.60	0.90	0.30

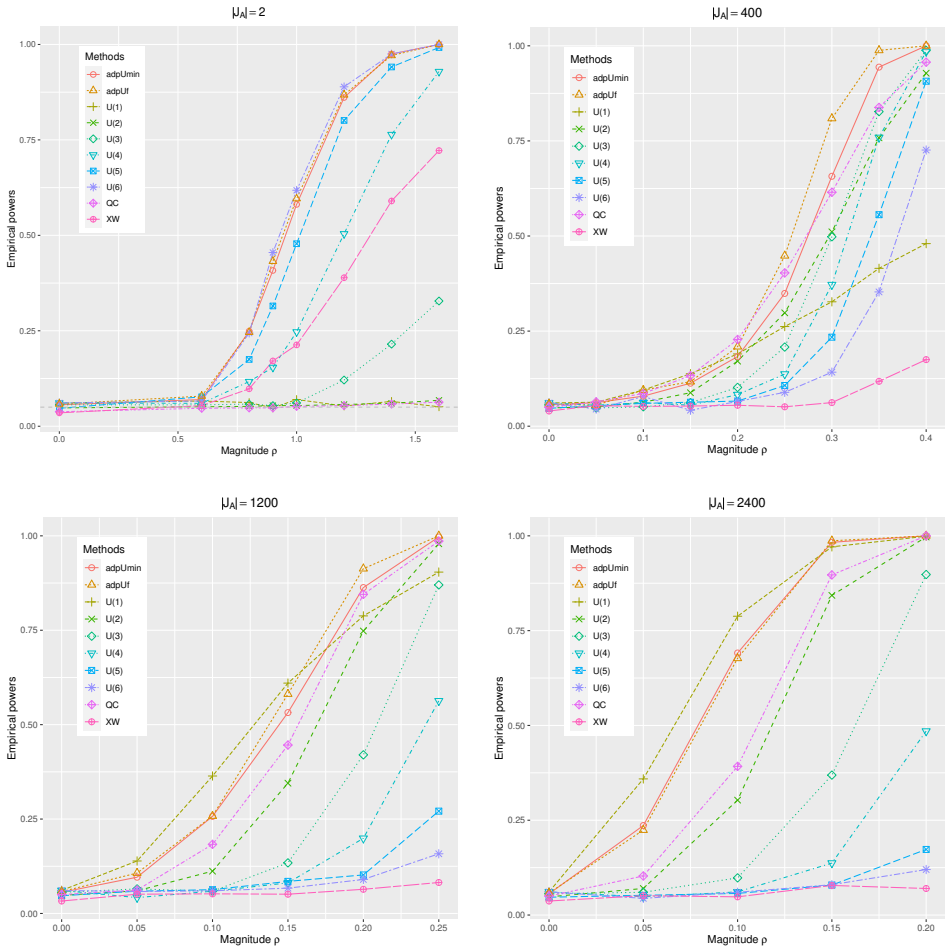

Figure 1: Empirical power comparison under Setting 1 for multivariate normal distribution: $n = 100$, $p = 600$.

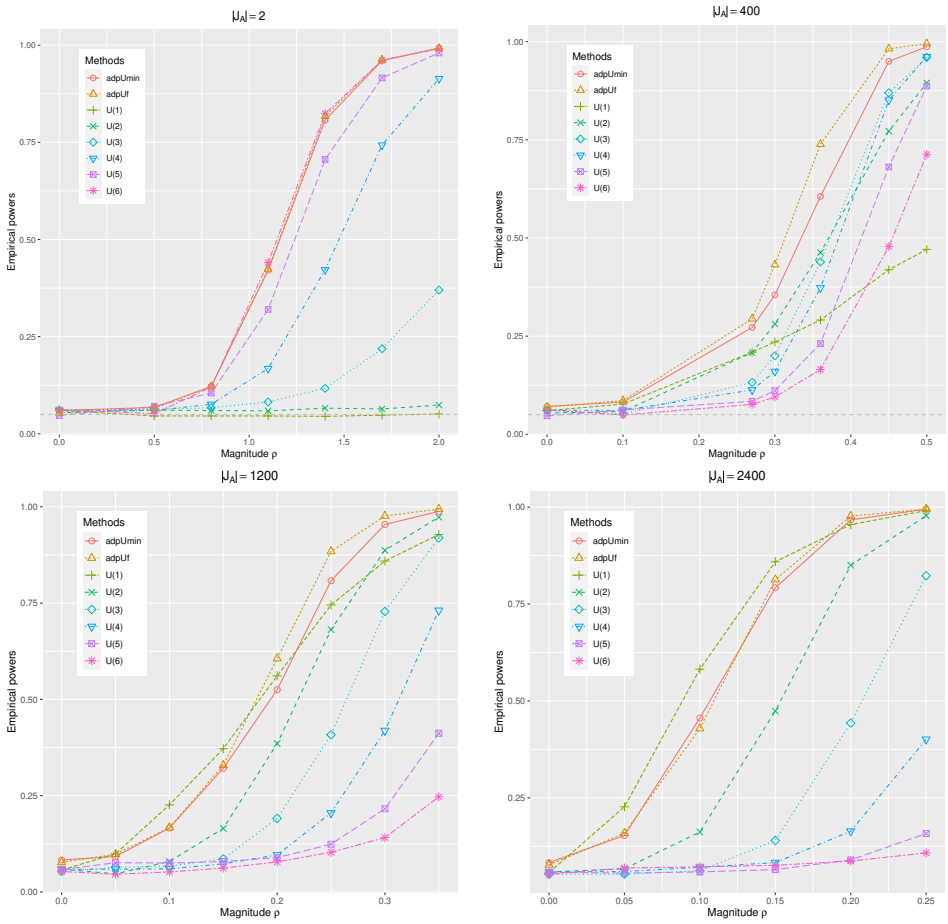

Figure 2: Empirical power comparison under Setting 1 for multivariate t distribution: $n = 100$, $p = 600$.

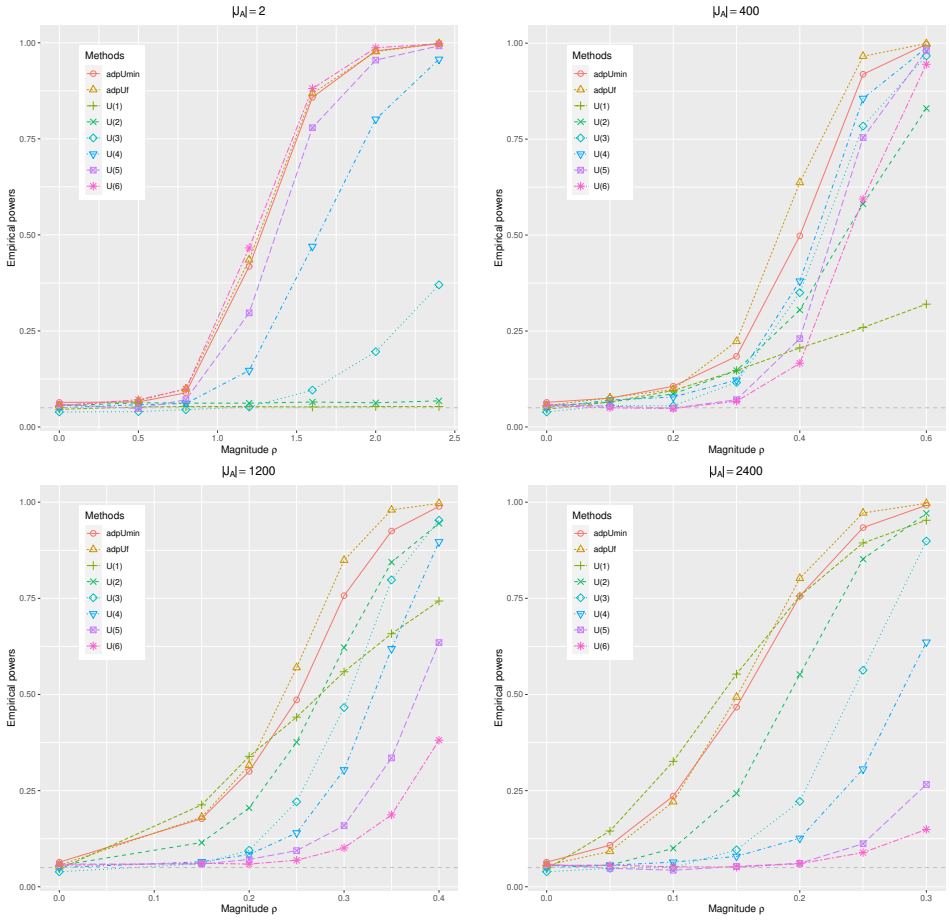

Figure 3: Empirical power comparison under Setting 1 for multivariate t distribution: $n = 100$, $p = 1000$.

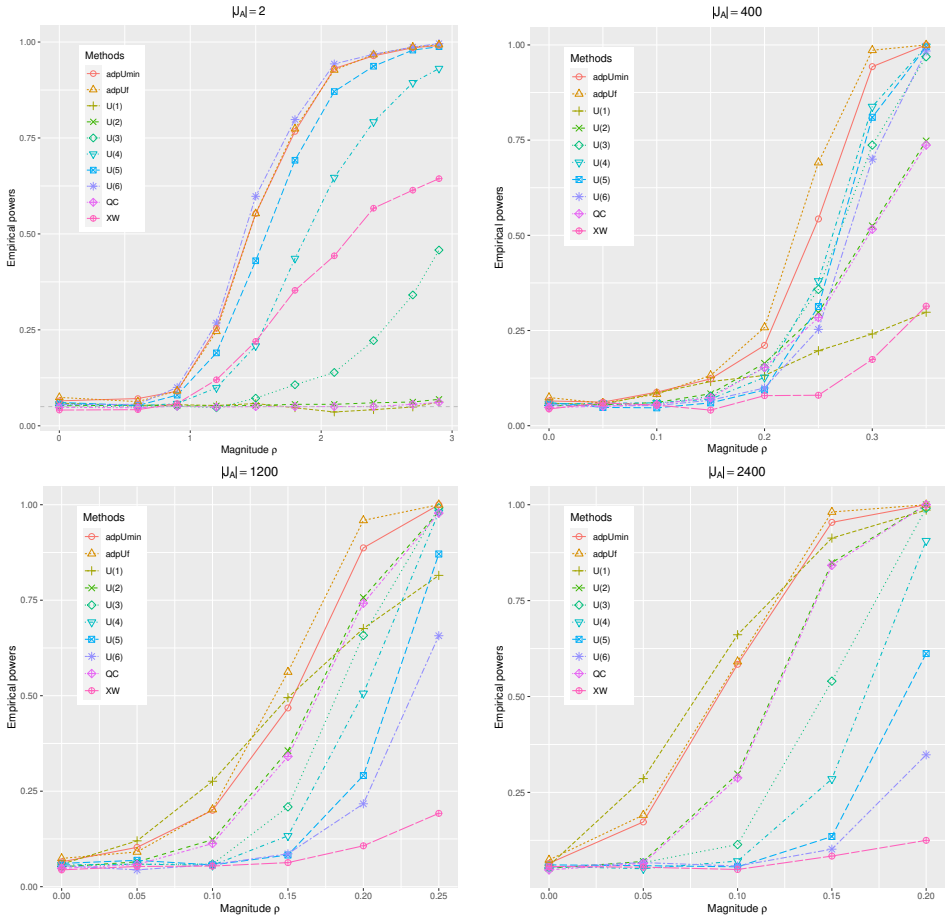

Figure 4: Empirical power comparison under Setting 2 for multivariate normal distribution: $n = 100$, $p = 600$.

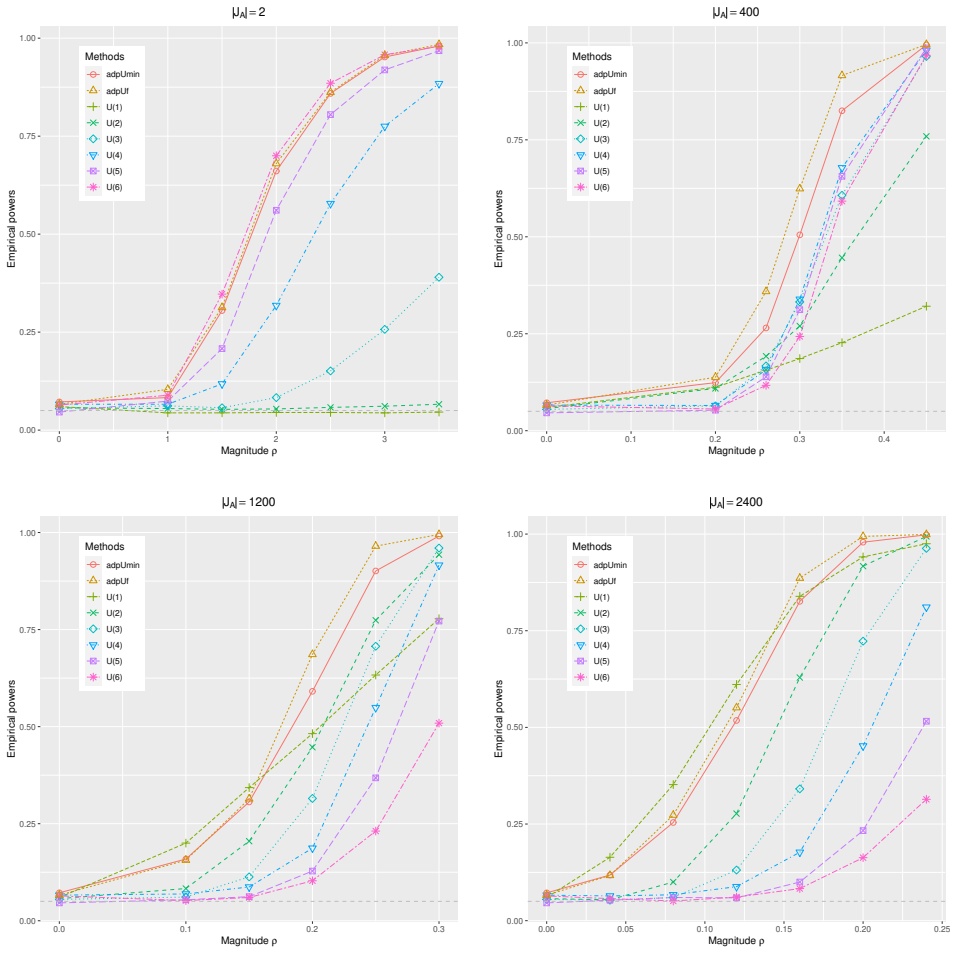

Figure 5: Empirical power comparison under Setting 2 for multivariate t distribution: $n = 100$, $p = 600$.

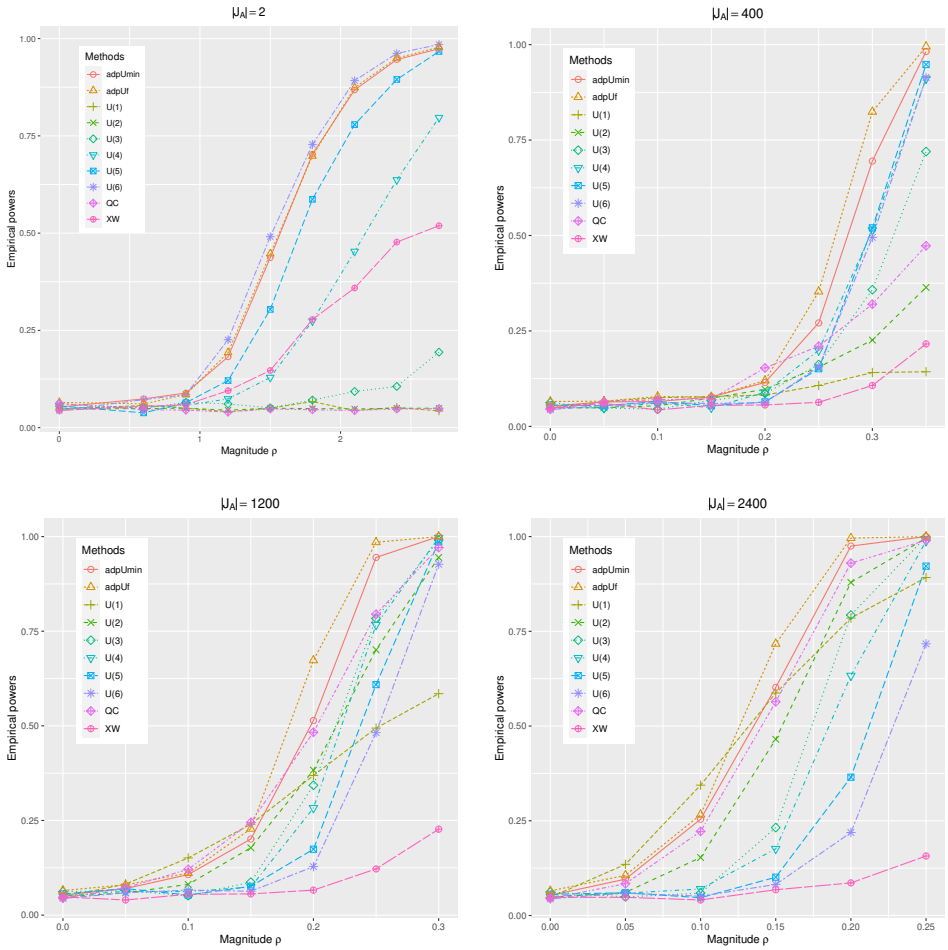

Figure 6: Empirical power comparison under Setting 2 for multivariate normal distribution: $n = 100$, $p = 1000$.

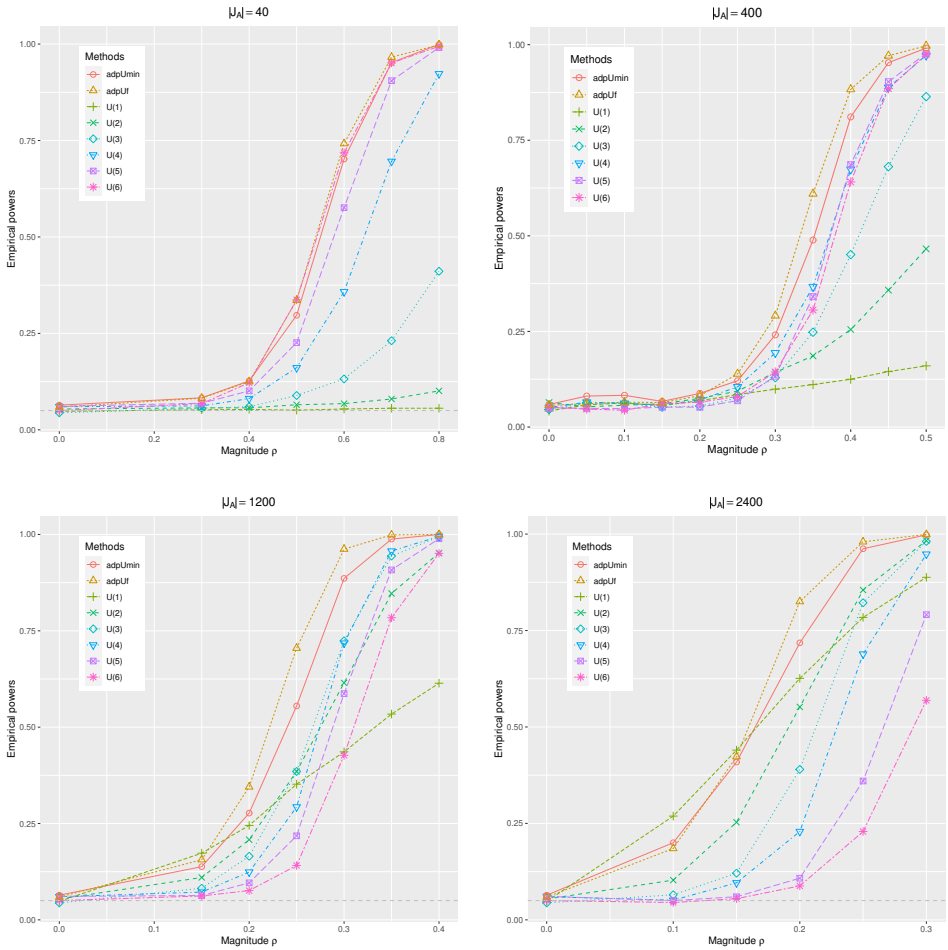

Figure 7: Empirical power comparison under Setting 2 for multivariate t distribution: $n = 100$, $p = 1000$.

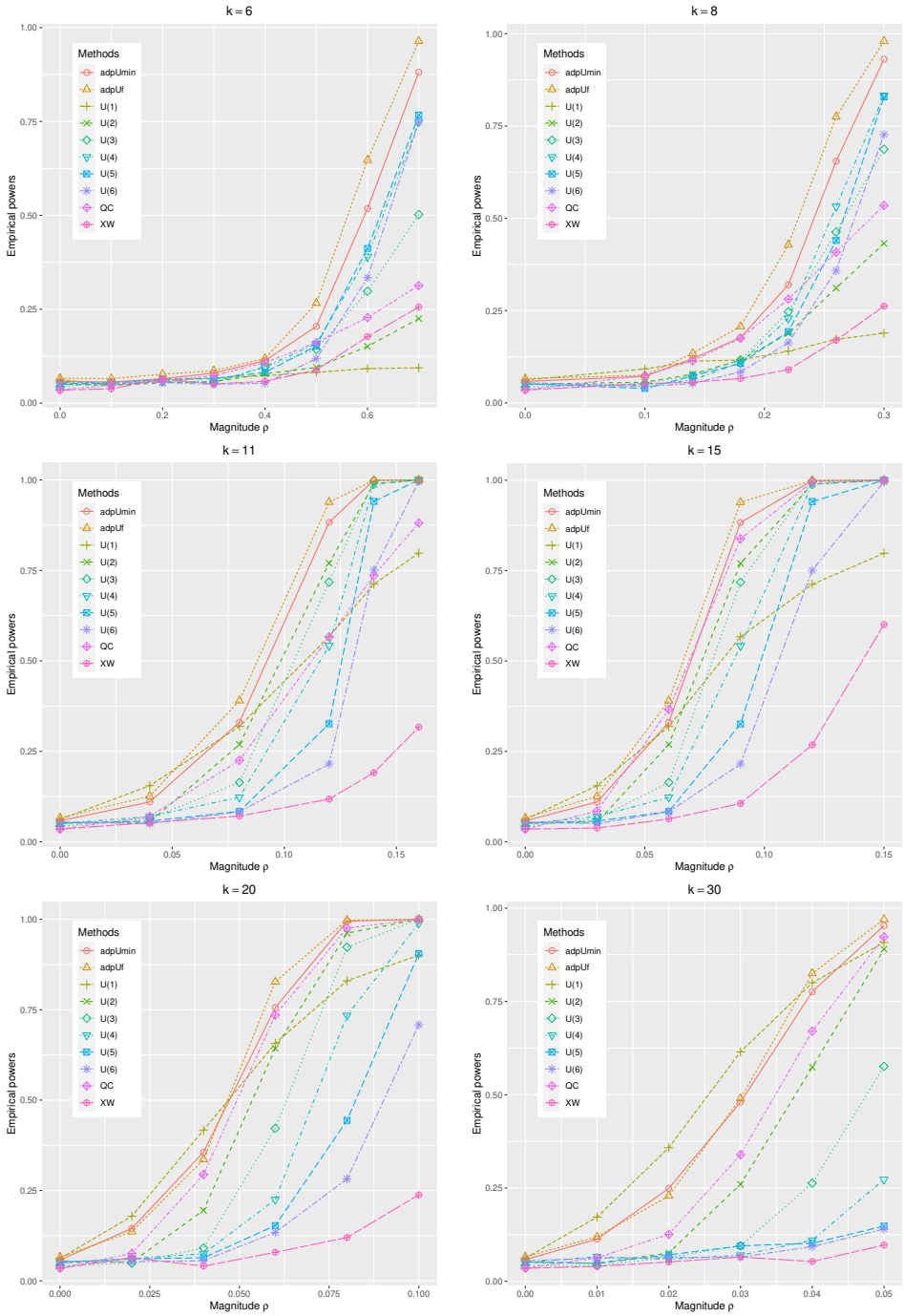

Figure 8: Empirical power comparison under Setting 3 for multivariate normal distribution: $n = 100$, $p = 600$.

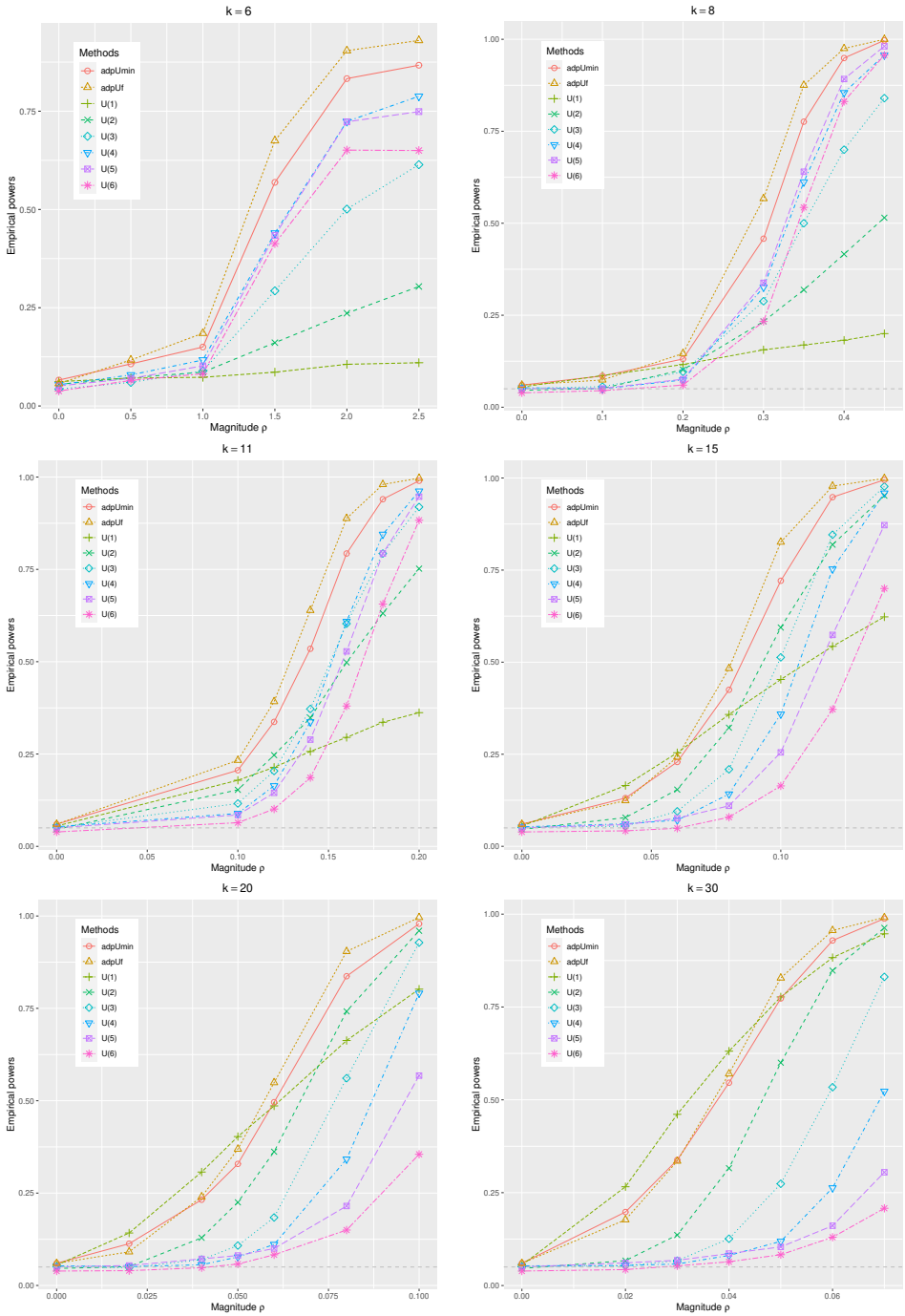

Figure 9: Empirical power comparison under Setting 3 for multivariate t distribution: $n = 100$, $p = 600$.

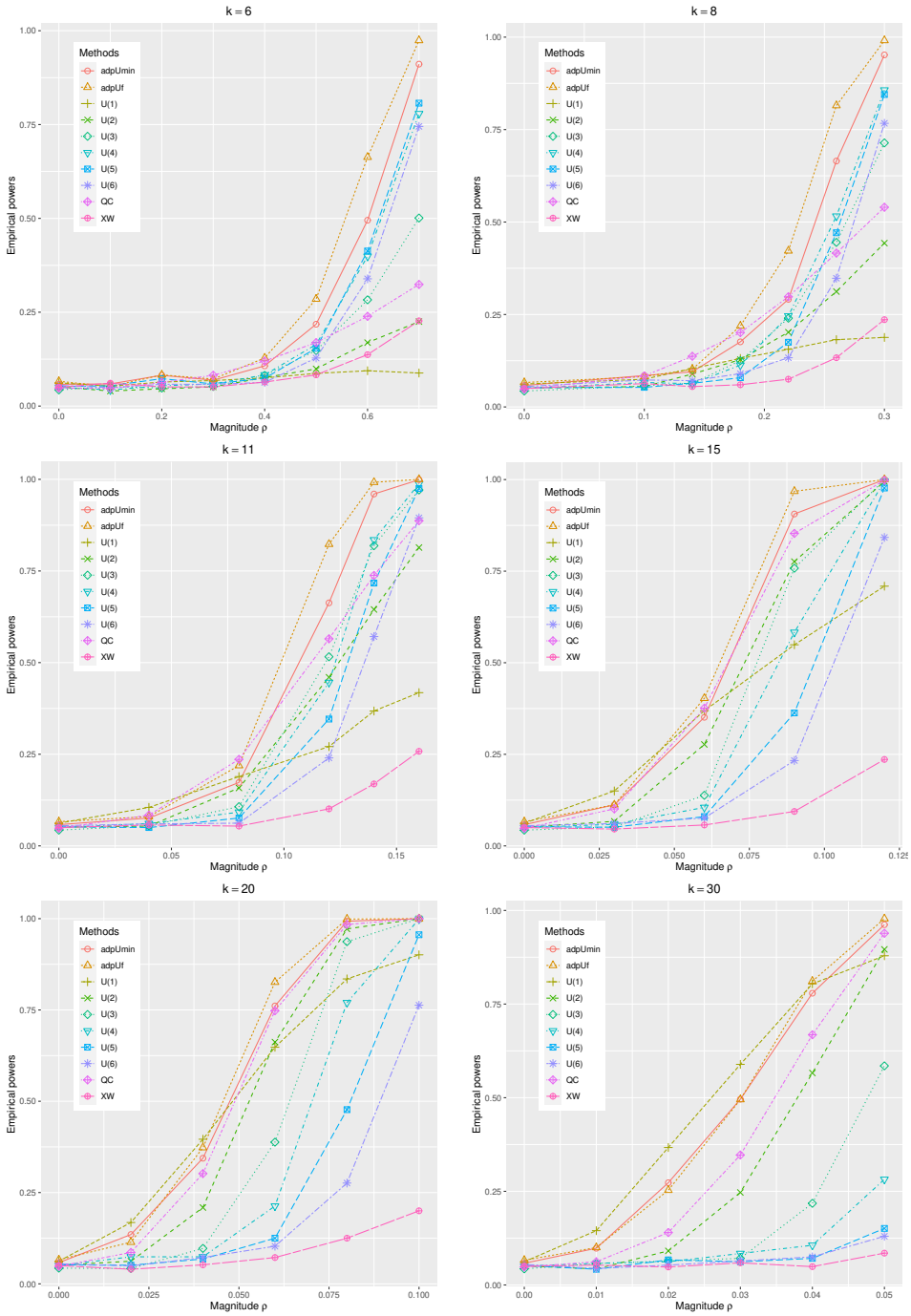

Figure 10: Empirical power comparison under Setting 3 for multivariate normal distribution: $n = 100$, $p = 1000$.

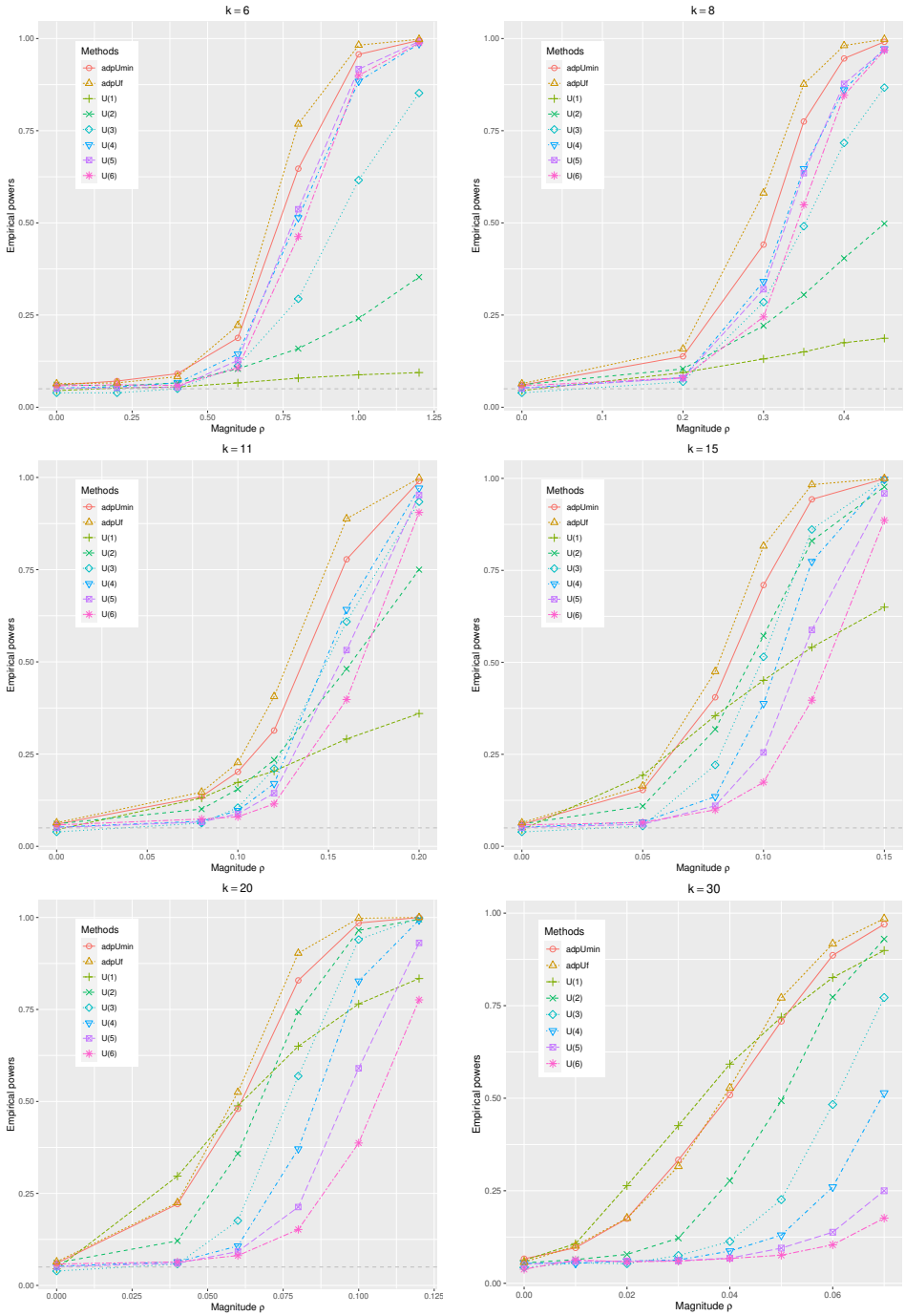

Figure 11: Empirical power comparison under Setting 3 for multivariate t distribution: $n = 100$, $p = 1000$.

setting in Section 3.2 of the main file, please see Table 5.

S10. Example

We also give an example to illustrate the motivation of $\hat{k}_{\delta,\theta}$ by a simulation study. We generate 100 data from the multivariate normal distribution under model 2 in Section 3.2. In this case, the true bandwidth $k_0 = 5$. Figure 12 presents the box-plots of the adjusted first order differences $d^{\delta}_{a,k}$. It shows that the corresponding curve given different order a decays as k increases, and $d^{\delta}_{a,k}$ are significantly larger than 0 when $k < k_0$, and close to 0 when $k \leq k_0$.

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Heyde, C. and Brown, B. (1970). On the departure from normality of a

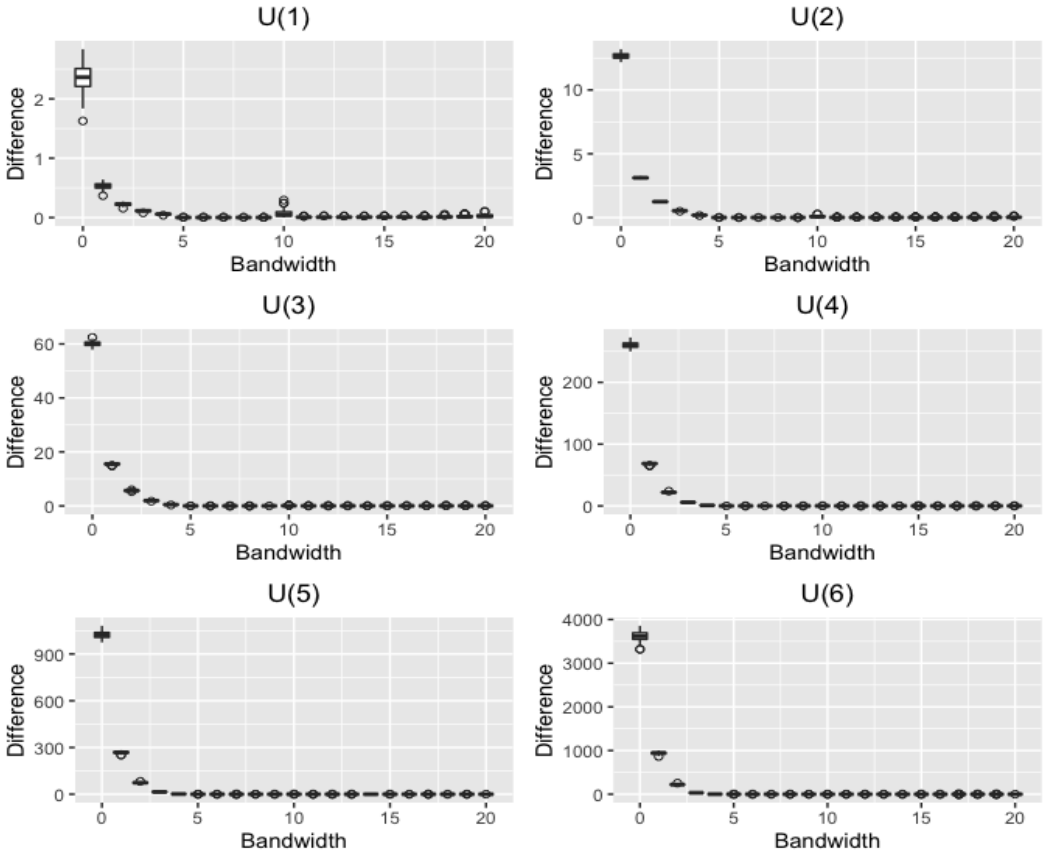

Figure 12: Box-plots of the adjusted first order differences $d^{\delta}_{a,k}$ of the simulated data, and the true bandwidth k_0 is 5.

certain class of martingales. *Annals of Mathematical Statistics*, **41**:129–133.

Qiu, Y. M. and Chen, S. X. (2012). Test for bandedness of high-dimensional covariance matrices and bandwidth estimation. *The Annals of Statistics*, **40**(**3**):1285–1314.

Table 5: Average empirical bias and standard deviation in parentheses of three bandwidth estimators with t_7 innovations: our proposed adaptive bandwidth estimator with $\delta = 0.5$ and $\theta = 0.06$, the estimators proposed in Bickel and Levina (2008) (BLa) and Qiu and Chen (2012) (QC).

			Bandwidth			
n	p	Methods	2	5	10	15
100	50	Adaptive	0(0)	-0.05(0.261)	-0.03(0.300)	-0.18(0.479)
		BLa	-0.05(0.609)	-0.75(0.892)	-2.11(1.979)	-3.04(3.275)
		QC	0.10(0.333)	0(0)	0.01(0.100)	-0.07(0.256)
	200	Adaptive	0(0)	-0.01(0.10)	0(0)	-0.21(0.433)
		BLa	1.65(2.583)	0.70(2.452)	-0.90(2.769)	-2.04(3.905)
		QC	0(0)	0(0)	0(0)	0(0)
	400	Adaptive	0(0)	0(0)	0(0)	-0.19(0.394)
		BLa	3.67(4.397)	2.49(3.442)	1.05(3.647)	-0.22(3.897)
		QC	0(0)	0(0)	0(0)	0(0)
	600	Adaptive	0(0)	0(0)	0(0)	-0.04(0.197)
		BLa	5.20(6.779)	3.96(6.530)	2.95(7.136)	1.91(6.264)
		QC	0(0)	0(0)	0(0)	0(0)
	1000	Adaptive	0(0)	0(0)	0(0)	0(0)
		BLa	9.98(12.87)	9.54(13.108)	6.79(11.89)	6.22(13.42)
		QC	0(0)	0(0)	0(0)	0(0)
200	50	Adaptive	0(0)	-0.01(0.100)	0(0)	0.01(0.100)
		BLa	0.09(0.429)	-0.23(0.930)	-0.31(1.212)	-0.37(2.177)
		QC	0.06(0.371)	0(0)	0(0)	0(0)
	200	Adaptive	0(0)	0(0)	0(0)	0(0)
		BLa	1.31(1.807)	0.88(1.754)	-0.12(1.929)	-0.60(2.070)
		QC	0(0)	0(0)	0(0)	0(0)
	400	Adaptive	0(0)	0(0)	0(0)	0(0)
		BLa	3.18(3.534)	2.34(3.163)	1.11(3.387)	0.56(3.540)
		QC	0(0)	0(0)	0(0)	0(0)
	600	Adaptive	0(0)	0(0)	0(0)	0(0)
		BLa	5.30(5.506)	5.24(5.429)	2.32(4.822)	1.79(5.192)
		QC	0(0)	0(0)	0(0)	0(0)
	1000	Adaptive	0(0)	0(0)	0(0)	0(0)
		BLa	9.14(9.517)	7.04(8.665)	5.76(7.803)	5.54(8.822)
		QC	0(0)	0(0)	0(0)	0(0)