Arc-Sin Transformation for Binomial Sample Proportions  
in Small Area Estimation

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Supplementary Material

The supplemental material provides the proves of Theorems 1-3. For this purpose, we first introduce several results which help to proving to the lemmas and theorems hereafter. We define $Z^* = Z - X\beta \sim N(0, V)$.

\[ \hat{\theta}^B_i = (1 - B_i)\nu_i'Z + B_i x_i'\beta = a_{1i}(A)'Z^* + x_i'\beta, \quad (S0.1) \]
\[ \hat{\theta}^{EB}_i = (1 - \hat{B}_i)\nu_i'Z + \hat{B}_i x_i'\hat{\beta} = (a_{1i}(\hat{A}) + a_{2i}(\hat{A}))'Z^* + x_i'\beta, \quad (S0.2) \]

where $a_{1i}(A) = (1 - B_i)\nu_i$, $a_{2i}(A) = B_i(x_i'(X'V^{-1}X)^{-1}X'V^{-1})'$, $\nu_i$ being the $n$-dimensional vector of which $i$-th component is one while others are zero.

The above $a_{1i}(A)$ and $a_{2i}(A)$ are used throughout our proofs and we have
under the regularity conditions R1–R3, for large $m$,

$$a_1'V a_{1i} = (1 - B_i)^2 \nu_i' V \nu_i = \frac{A^2}{A + D_i} = A - g_{1i}(A), \quad \text{(S0.3)}$$

$$a_2'V a_{2i} = B_i^2 x_i'(X'V^{-1}X)^{-1} x_i = g_{2i}(A) = O(m^{-1}), \quad \text{(S0.4)}$$

$$a_2'V a_{1i} = B_i (1 - B_i) x_i'(X'V^{-1}X)^{-1} x_i = O(m^{-1}), \quad \text{(S0.5)}$$

$$i_u V a_{1i} = i_u A \nu_i, \quad \text{(S0.6)}$$

where $D_i = 1/(4n_i)$ and $i_u = \sqrt{-1}$. Note that regularity conditions are given in the main manuscript.

We also prove two more lemmas with some additional notations for proofs of theorems. Specifically, let

$$\hat{A}_+ = \hat{A}(Z^* + i_u V a_{1i}), \quad \hat{A}_- = \hat{A}(Z^* - i_u V a_{1i}),$$

$$\hat{\theta}_i^B = \hat{\theta}_i^B(Z^* + i_u V a_{1i}), \quad \hat{\theta}_i^B = \hat{\theta}_i^{EB}(Z^* - i_u V a_{1i}),$$

$$\hat{\theta}_i^{EB} = \hat{\theta}_i^{EB}(Z^* + i_u V a_{1i}), \quad \hat{\theta}_i^{EB} = \hat{\theta}_i^{EB}(Z^* - i_u V a_{1i}),$$

where $i_u = \sqrt{-1}$.

**Lemma 2.** Let $Z^* \equiv Z - X \beta \sim N(0, V)$, then we have under the regularity conditions R1-R3,

(i) $E[(\hat{A}_+ - A)^2] = V_A + o(m^{-1}),$

(ii) $E[(\hat{A}_+ - A)(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)] = i_u V_A B_i(1 - B_i) + o(m^{-1}),$


(iii) \( E[\hat{\theta}_{i+}^{EB} - \hat{\theta}_{i+}^{B}] = O(m^{-1}) \),

(iv) \( E[(\hat{\theta}_{i+}^{EB} - \hat{\theta}_{i+}^{B})^2] = g_{2i}(A) + g_{3i}(A) - V_A(1 - B_i)^2B_i^2 + o(m^{-1}) \),

(v) \( E[\hat{A}_+ - A] = O(m^{-1}) \),

where \( E[(\hat{A} - A)^2] = V_A + o(m^{-1}) \).

Lemma 3. Under the regularity conditions R1-R3, we have for large \( m \),

(i) \( E[(x'_i\hat{\beta} - x'_i\beta)^4] = O(m^{-2}) \),

(ii) \( E[x'_i\hat{\beta} - x'_i\beta] = 0 \),

(iii) \( E[(\hat{A} - A)(x'_i\hat{\beta} - x'_i\beta)] = 0 \).

Lemmas 2 and 3 are shown in S1.2 and S1.3, respectively.

S1. PROOFS OF LEMMAS

We now prove Lemmas 1–3 in this section. Lemma 1 is provided in the body of the main manuscript. Hereafter, \( a_{1i}^{(j)}(A) \) and \( a_{2i}^{(j)}(A) \) denote \( \frac{\partial^j a_{1i}}{\partial A^j} \bigg|_A \) and \( \frac{\partial^j a_{2i}}{\partial A^j} \bigg|_A \), respectively.

Let some \( n \) dimensional random vector \( W_n \sim N(0, \Sigma) \) with non-singular matrix \( \Sigma \) and let \( f(W_n) \) be some integrable function such that \( f(W_n) \in \mathbb{R} \). Then
S1.1 Lemma 1

Suppose that $W_n \sim N(0, \Sigma)$ with non-singular matrix $\Sigma$ and let $f(W_n)$ be some integrable function such that $f(W_n) \in \mathbb{R}$. Then, we get

\[
E[\exp (i_u c' W_n) f(W_n)]
= \exp \left( -\frac{c' \Sigma c}{2} \right) \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \int f(W_n) \exp \left\{ -\frac{(W_n - i_u \Sigma c)' \Sigma^{-1} (W_n - i_u \Sigma c)}{2} \right\} dW_n,
= \exp \left( -\frac{c' \Sigma c}{2} \right) E[f(W_n + i_u \Sigma c)],
\]

where $c$ denotes some $n$-dimensional vectors of which components are all constants.

Lemma 1 then follows from the above noting $\cos(x) = (\exp(i_u x) + \exp(-i_u x))/2$ and $\sin(x) = (\exp(i_u x) - \exp(-i_u x))/2i_u$.

S1.2 Lemma 2

From the assumption, (S0.1)–(S0.5) and the dominated convergence theorem, defining $r = r(Z^*, i_u V a_{11})$, it follows that

\[
\hat{A}_+ - A = \hat{A}(Z^*) + r = \hat{A} + r,
\] (S1.1)
\[ \hat{\theta}^{EB}_i - \hat{\theta}^B_i = \{ a_{1i}(\hat{A}_+ - a_{1i}(A) + a_{2i}(\hat{A}_+ - a_{2i}(A) + a_{2i}) \} (Z^* + i_u V a_{1i}) \]

\[ = \left\{ (\hat{A} - A + r)a^{(1)}_{1i}(A) + \frac{1}{2}(\hat{A} - A + r)^2 a^{(2)}_{1i}(A^*) \right\} (Z^* + i_u V a_{1i}) \]

\[ + \{ (\hat{A} - A + r)a^{(1)}_{2i}(A^*) + a_{2i}(A) \}'(Z^* + i_u V a_{1i}) \]

\[ = (\hat{\theta}^{EB}_i - \hat{\theta}^B_i) + (\hat{A} - A)(a^{(1)}_{1i})'(i_u V a_{1i}) + R^*, \quad (S1.2) \]

where \( A^* \) lies between \( A \) and \( \hat{A} \). In the above, \( R^* \) is satisfying that \( E[R^*] = O(m^{-1}) \) and \( E[(R^*)^2] = O(m^{-2}) \) from the Cauchy–Schwartz inequality and the assumption on \( r \).

Using the assumption of \( \hat{A} \), (S1.1) and Cauchy-Schwarz inequality,

\[ E[\hat{A}_+ - A] = E[\hat{A}(Z^*) - A] + E[r] = O(m^{-1}), \quad (S1.3) \]

\[ E[(\hat{A}_+ - A)^2] = E[(\hat{A}(Z^*) - A)^2] + E[r^2] + 2E[(\hat{A}(Z^*) - A)r] \]

\[ = E[(\hat{A}(Z^*) - A)^2] + o(m^{-1}). \quad (S1.4) \]

This leads to parts (i) and (v).

Next, we prove parts (iii) and (iv). To this end, we use (S1.2).

\[ E[\hat{\theta}^{EB}_i - \hat{\theta}^B_i] = E[\hat{\theta}^{EB}_i - \hat{\theta}^B_i + (\hat{A} - A)(a^{(1)}_{1i})'(i_u V a_{1i}) + R^*], \quad (S1.5) \]

\[ E[(\hat{\theta}^{EB}_i - \hat{\theta}^B_i)^2] = E[(\hat{\theta}^{EB}_i - \hat{\theta}^B_i)^2 + \{(\hat{A} - A)(a^{(1)}_{1i})'(i_u V a_{1i}) + R^*\}^2] \]

\[ + 2E[(\hat{\theta}^{EB}_i - \hat{\theta}^B_i)\{(\hat{A} - A)(a^{(1)}_{1i})'(i_u V a_{1i}) + R^*\}], \quad (S1.6) \]
where $R^*$ is such that $E[(R^*)^2] = O(m^{-2})$.

Using the result of Kackar and Harville (1981), the Cauchy–Schwartz inequality and (S0.3), (S1.5) and (S1.6) can be rewritten as

$$
(S1.5) = i_u B_i (1 - B_i) E[\hat{A} - A] + O(m^{-1}) = O(m^{-1}),
$$

$$
(S1.6) = g_{2i}(A) + g_{3i}(A) - V_A B_i^2 (1 - B_i)^2 + o(m^{-1}).
$$

The above equalities follow from the result \( (a_{1i}^{(1)})'(i_u V a_{1i}) = i_u B_i (1 - B_i) \) due to (S0.6) and some results of Prasad and Rao (1990) and Datta and Lahiri (2000).

Finally, we prove part (ii). With a proof similar as above, (S1.1) and (S1.2) yield the following.

$$
E[(\hat{A} + A)(\hat{\theta}_{i+}^{EB} - \theta_i^{EB})]
= E \left[ (\hat{A} - A + r) \left\{ \hat{\theta}_{i+}^{EB} - \theta_i^{EB} + (\hat{A} - A)(a_{1i}^{(1)})'(i_u V a_{1i}) + R^* \right\} \right],
= i_u V_A B_i (1 - B_i) + o(m^{-1}).
$$

(S1.7)

This leads to Lemma 2.
S1.3 Lemma 3

We first prove part (i). Using $Z^* = Z - X\beta \sim N(0, V)$, we obtain

$$E[(x_i'\hat{\beta} - x_i'\beta)^2] = E[(x_i'(X'V^{-1}X)^{-1}X'V^{-1}Z^*)^2]$$

$$\leq C \sum_{\{(k_1,k_2)\in\{0\}\cup\mathbb{Z}_+:k_1+k_2=l\}} (x_i'(X'V^{-1}X)^{-1}x_i)^{k_1+k_2}$$

$$= O(m^{-l}), \quad (S1.8)$$

where $\hat{\beta} = \hat{\beta}(A), \ l \in \mathbb{Z}_+$ and $C$ is some generic positive constants.

From (S1.8), we obtain the following.

$$E \left[ \left( \frac{\partial x_i'\hat{\beta}}{\partial A} \right) \right]_A^8 = E \left[ \left\{ x_i'(X'V^{-1}X)^{-1}X'V^{-2}(I - X(X'V^{-1}X)^{-1}X'V^{-1})Z \right\}^8 \right]_{A=A^*}$$

$$\leq CE \left[ \left\{ x_i'(X'V^{-1}X)^{-1}X'V^{-1}Z \right\}^8 \right]_{A=A^*} = O(m^{-4}),$$

where $A^*$ lies between $A$ and $\hat{A}$.

From the above results and the Cauchy–Schwartz inequality,

$$E[(x_i'\hat{\beta} - x_i'\beta)^4] \leq E[(\hat{A} - A)^8]^{1/2} E \left[ \left\{ \frac{\partial x_i'\hat{\beta}}{\partial A} \right\}^{8}_{A=A^*} \right]^{1/2} = o(m^{-2}),$$

where $A^*$ is lying between $A$ and $\hat{A}$.

Consequently, part (i) follows from

$$E[(x_i'\hat{\beta} - x_i'\beta)^4] \leq C \left\{ E[(x_i'\hat{\beta} - x_i'\beta)^4] + E[(x_i'\hat{\beta} - x_i'\beta)^4] \right\}$$

$$= O(m^{-2}). \quad (S1.9)$$
The remaining two parts (ii) (iii) follow immediately from the fact that
\( x_i'\hat{\beta} - x_i'\beta \) and \( \hat{A} - A \) are odd and even functions of \( Z^* \) respectively.

S2 Proofs of Theorems

S2.1 Theorem 1

Theorem 1 (i)

We first prove Part (i) of Theorem 1.

The unbiasedness of \( \hat{p}_i^B \), that is, \( E[\hat{p}_i^B - p_i] = 0 \), results in

\[
E[\hat{p}^{EB}_i - p_i] = E[\hat{p}_i^{EB} - \hat{p}_i^B] + E[\hat{p}_i^B - p_i],
\]

\( = E[\hat{p}_i^{EB} - \hat{p}_i^B]. \) \( \tag{S2.1} \)

Let \( C_{1i}(A) \) define \( \exp(-g_{1i}(A)/2) \) hereafter. Then,

\[
(S2.1) = \frac{1}{2} E[C_{1i}(A) \sin(\hat{\theta}_i^{EB}) - C_{1i}(A) \sin(\hat{\theta}_i^B)]
\]

\[
= \frac{1}{2} E \left[ (C_{1i}(\hat{A}) - C_{1i}(A)) \sin(\hat{\theta}_i^{EB}) + C_{1i}(A)(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B)) \right],
\]

\( = \frac{1}{2}(J_1 + J_2), \) \( \tag{S2.2} \)

where \( J_1 = E[(C_{1i}(\hat{A}) - C_{1i}(A)) \sin(\hat{\theta}_i^{EB})] \) and \( J_2 = C_{1i}(A)E[\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B)]. \)

For \( J_1 \), from Lemmas 1, 2 (v) and the dominated convergence theorem,
we have,

\[ J_1 = E[(C_{1i}(\hat{A}) - C_{1i}(A))(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B) + \sin(\hat{\theta}_i^B))] \]

\[ = E \left[ \left\{ (\hat{A} - A)C_{1i}^{(1)}(A) + \frac{1}{2}(\hat{A} - A)^2 C_{1i}^{(2)}(A^*) \right\} (\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B)) \right] \]

\[ + E \left[ \left\{ (\hat{A} - A)C_{1i}^{(1)}(A) + \frac{1}{2}(\hat{A} - A)^2 C_{1i}^{(2)}(A^*) \right\} \sin(\hat{\theta}_i^B) \right], \]

\[ = E[(\hat{A} - A)C_{1i}^{(1)}(A) \cos(\hat{\theta}_i^B - x'_i \beta) \sin(x'_i \beta)] + O(m^{-1}) = O(m^{-1}). \]

(S2.3)

where \( A^* \) lies between \( \hat{A} \) and \( A \). In the above, note that \( C_{1i}^{(j)}(A) = \partial^j C_{1i}(A) / \partial A^j \bigg|_A \) for \( j = 1, 2 \).

For the third and fourth equalities in the above, we use the assumption on \( \hat{A} \), the fact that \( \sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B) \) convergences to zero in probability and \( \sin(\hat{\theta}_i^B - x'_i \beta) \) is an odd function of \( Z^* = Z - X\beta \).

In addition we use Lemma 2 (iii) and (S0.3) for calculation of \( J_2 \).

\[ J_2 = 2C_{1i}(A)E \left[ \cos \left( \frac{\hat{\theta}_i^{EB} + \hat{\theta}_i^B}{2} \right) \sin \left( \frac{\hat{\theta}_i^{EB} - \hat{\theta}_i^B}{2} \right) \right], \]

\[ = -2C_{1i}(A) \sin(x'_i \beta) E \left[ \sin \left( \frac{\hat{\theta}_i^{EB} + \hat{\theta}_i^B}{2} - x'_i \beta \right) \sin \left( \frac{\hat{\theta}_i^{EB} - \hat{\theta}_i^B}{2} \right) \right], \]

\[ = -2C_{1i}(A) \sin(x'_i \beta) \]

\[ \times E \left[ \sin(\hat{\theta}_i^B - x'_i \beta) \left\{ \left( \frac{\hat{\theta}_i^{EB} - \hat{\theta}_i^B}{2} \right) - \frac{1}{6} \left( \frac{\hat{\theta}_i^{EB} - \hat{\theta}_i^B}{2} \right)^3 \sin \left( \frac{\eta(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)}{2} \right) \right\} \right] \]

\[ + O(m^{-1}) = O(m^{-1}), \]

(S2.4)
where $|\eta| < 1$.

Note that $\sin(\hat{\theta}^E_i - \hat{\theta}^B_i)$ is odd function of $Z^* = Z - X\beta$ and $\sin(x) = x - x^3\sin(\eta x)/6$ with $|\eta| < 1$ for the above calculation. Also, we use Liapounov’s inequality with the following result which comes from Lemma 3 (i) and the assumption on $\hat{A}$.

$$E[(\hat{\theta}^E_i - \hat{\theta}^B_i)^4] \leq E[(B_i - \hat{B}_i)^4(z_i - x_i^\prime\beta)^4] + E[(x_i^\prime\beta - x_i^\prime\beta)^4] = O(m^{-2}).$$  

(S2.5)

Combining (S2.2)-(S2.4), one gets

$$(S2.1) = O(m^{-1}).$$

**Theorem 1 (ii)**

First we use the identity

$$E[(\hat{p}_i^E - p_i)^2] = E[(\hat{p}_i^E - p_i)^2] + E[(\hat{p}_i^E - \hat{p}_i^B)^2].$$  

(S2.6)

Next we evaluate $E[(\hat{p}_i^E - p_i)^2]$ in the right hand side of (S2.6). By standard results,

$$E[(\hat{p}_i^B - p_i)^2] = E[V(p_i|z_i)] = \frac{1}{4}E[V(\sin(\theta_i)|z_i)];$$  

(S2.7)

$$V(\sin(\theta_i)|z_i) = E[\sin^2(\theta_i)|z_i] - [E(\sin(\theta_i)|z_i)]^2,$$

$$= \frac{1}{2}E[1 - \cos(2\theta_i)|z_i] - [E(\sin(\theta_i)|z_i)]^2.$$  

(S2.8)
Equation (2.5) and Corollary 1 provide the results:

\[ E[1 - \cos(2\theta_i)|z_i] = 1 - \exp(-2g_{1i}(A))\cos(2\hat{\theta}^B_i); \]  
\[ (S2.9) \]

\[ [E(\sin(\theta_i)|z_i)]^2 = \exp(-g_{1i}(A))\sin^2(\hat{\theta}^B_i) \]
\[ = \frac{1}{2}\exp(-g_{1i}(A))[1 - \cos(2\hat{\theta}^B_i)]. \]  
\[ (S2.10) \]

Hence, we get from (S2.8)-(S2.10) and Corollary 1 again,

\[ V(\sin(\theta_i)|z_i) = \frac{1}{2}\{1 - \exp(-2g_{1i}(A))\cos(2\hat{\theta}^B_i)\} - \frac{1}{2}\exp(-g_{1i}(A))(1 - \cos(2\hat{\theta}^B_i)), \]
\[ (S2.11) \]

\[ E[V(\sin(\theta_i)|z_i)] = \frac{1}{2}(1 - \exp(-g_{1i}(A)))\{1 + \exp(-2A + g_{1i}(A))\cos(2x_i^\prime\beta)\}. \]
\[ (S2.12) \]

In the above calculation, we used the result \( \hat{\theta}^B_i \sim N(x_i^\prime\beta, A(1 - B_i)). \)

Combining (S2.7) and (S2.12), we obtain

\[ E[(\hat{p}^B_i - p_i)^2] = \frac{1}{8}(1 - \exp(-g_{1i}(A)))\{1 + \exp(-2A + g_{1i}(A))\cos(2x_i^\prime\beta)\}. \]
\[ (S2.13) \]

Next, we find an asymptotic expansion of the second term \( E[(\hat{p}^B_i - p_i)^2] \) in the right hand side of (S2.6), correct up to the order \( O(m^{-1}) \) for
large $m$. Let $C_{1i}(A)$ continue to define $\exp(-g_{1i}(A)/2)$,

$$E[(\hat{p}_i^{EB} - \hat{p}_i^B)^2] = \frac{1}{4} E \left[ (C_{1i}(\hat{A}) \sin(\hat{\theta}_i^{EB}) - C_{1i}(A) \sin(\hat{\theta}_i^B))^2 \right],$$

$$= \frac{1}{4} E \left\{ (C_{1i}(\hat{A}) - C_{1i}(A)) \sin(\hat{\theta}_i^{EB}) + C_{1i}(A)(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B)) \right\}^2,$$

$$= \frac{1}{4} (I + II + III), \quad (S2.14)$$

where

$$I = E \left[ (C_{1i}(\hat{A}) - C_{1i}(A))^2 \sin^2(\hat{\theta}_i^{EB}) \right],$$

$$II = (C_{1i}(A))^2 E \left[ (\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B))^2 \right],$$

$$III = 2C_{1i}(A) E \left[ (C_{1i}(\hat{A}) - C_{1i}(A)) \sin(\hat{\theta}_i^{EB})(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B)) \right].$$

We first calculate $I$ using Lemma 1 and Lemma 2 (i):

$$I = E \left\{ (\hat{A} - A)(C_{1i}^{(1)}(A) + \frac{1}{2}(\hat{A} - A)^2 C_{1i}^{(2)}(A^*))^2 \sin^2(\hat{\theta}_i^{EB}) \right\}$$

$$= \frac{1}{2} (C_{1i}^{(1)}(A))^2 E \left[ (\hat{A} - A)^2 (1 - \cos(2\hat{\theta}_i^B)) \right] + o(m^{-1}),$$

$$= \frac{B_i^4}{8} V_A \exp(-g_{1i}(A))(1 - \cos(2x_i \beta) \exp(-2A + 2g_{1i}(A))) + o(m^{-1}),$$

(S2.15)

where $A^*$ lies between $A$ and $\hat{A}$. We note that the second equality holds due to the dominated convergence theorem, the assumption on $\hat{A}$ and the result that $\sin^2(\hat{\theta}_i^{EB}) - \sin^2(\hat{\theta}_i^B)$ converges to zero in probability.
Next, we prove II;

$$E \left[ \left\{ \sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B) \right\}^2 \right] = 4E \left[ \cos^2 \left( \frac{\hat{\theta}_i^{EB} + \hat{\theta}_i^B}{2} \right) \sin^2 \left( \frac{\hat{\theta}_i^{EB} - \hat{\theta}_i^B}{2} \right) \right],$$

$$= E[(1 + \cos(\hat{\theta}_i^{EB} + \hat{\theta}_i^B))(1 - \cos(\hat{\theta}_i^{EB} - \hat{\theta}_i^B))],$$

$$= E \left[ (1 + \cos(\hat{\theta}_i^{EB} + \hat{\theta}_i^B)) \left\{ \frac{(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2}{2} + \frac{(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^4}{24} \cos(\eta(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)) \right\} \right].$$

(S2.16)

In the above calculation, the third equality follows from the fact that $1 - \cos(x) = x^2/2 - x^4 \cos(\eta x)/24$ with $|\eta| < 1$.

The results (2.2) and (2.3), given in the main paper, for the untransformed case remind us that $g_{2i}(A) + g_{3i}(A)$ are second-order approximations of $E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2]$, and recalling (S2.5). Also, note that $\sin(2(\hat{\theta}_i^{EB} - x'_i\beta))$ is odd function of $Z^* = Z - X\beta$ while $\cos(2(\hat{\theta}_i^B - x'_i\beta))$ is even function of $Z^* = Z - X\beta$. Moreover, $\cos(\hat{\theta}_i^{EB} + \hat{\theta}_i^B) - \cos(2\hat{\theta}_i^{EB} - \hat{\theta}_i^B)$ converges to zero in probability. These above results provide the following.

(S2.16) $= \frac{1}{2} \left\{ g_{2i}(A) + g_{3i}(A) + E \left[ \cos(\hat{\theta}_i^{EB} + \hat{\theta}_i^B)(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2 \right] \right\} + o(m^{-1}),$

$= \frac{1}{2} \left\{ g_{2i}(A) + g_{3i}(A) + E \left[ \cos(2\hat{\theta}_i^B - 2x'_i\beta + 2x'_i\beta)(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2 \right] \right\} + o(m^{-1}),$

$= \frac{1}{2} \left\{ g_{2i}(A) + g_{3i}(A) + \cos(2x'_i\beta)E \left[ \cos(2\hat{\theta}_i^B - 2x'_i\beta)(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2 \right] \right\} + o(m^{-1}).$

(S2.17)

The third equality follows from the result that $\sin(2\hat{\theta}_i^B - 2x'_i\beta)(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2$ is a odd function of $Z^*$ with zero mean.
Using Lemma 1 and (S0.3),

\[(S2.17) = \frac{1}{2} (g_{2i}(A) + g_{3i}(A))
+ \frac{\exp(-2A + 2g_{1i}(A))}{4} \cos(2x_i' \beta) \left\{ E[(\hat{\theta}^{EB}_{i+} - \hat{\theta}_{i+}^{EB})^2] + E[(\hat{\theta}_{i-}^{EB} - \hat{\theta}_{i-}^{EB})^2] \right\}
+ o(m^{-1}). \quad (S2.18)\]

Lemma 2 (iv) yields

\[(S2.18) = \frac{1}{2} (g_{2i}(A) + g_{3i}(A))
+ \frac{\exp(-2A + 2g_{1i}(A))}{2} \cos(2x_i' \beta) \{g_{2i}(A) + g_{3i}(A) - V_A(1 - B_i)^2 B_i^2\}
+ o(m^{-1}), \quad (S2.19)\]

where \(E[(\hat{A} - A)^2] = V_A + o(m^{-1}).\)

Hence,

\[II = \frac{1}{2} \exp(-g_{1i}(A))(g_{2i}(A) + g_{3i}(A))(1 + \cos(2x_i' \beta) \exp(-2A + 2g_{1i}(A)))
- \frac{1}{2} \cos(2x_i' \beta) \exp(-2A + g_{1i}(A))V_A(1 - B_i)^2 B_i^2 + o(m^{-1}). \quad (S2.20)\]

We finally calculate III.

\[III = 2C_{1i}(A)E \left\{(C_{1i}(\hat{A}) - C_{1i}(A))(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^{EB})) \right\},
= 2C_{1i}(A)E \left\{\left((\hat{A} - A)C_{1i}^{(1)}(A) + \frac{1}{2}(\hat{A} - A)^2 C_{1i}^{(2)}(A^*)\right) (\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^{EB}))^2 \right\}
+ 2C_{1i}(A)E \left\{\left((\hat{A} - A)C_{1i}^{(1)}(A) + \frac{1}{2}(\hat{A} - A)^2 C_{1i}^{(2)}(A^*)\right) \sin(\hat{\theta}_i^B)(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^{EB})) \right\}
= 2C_{1i}(A)C_{1i}^{(1)}(A)E \left\{(\hat{A} - A) \sin(\hat{\theta}_i^B)(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^{EB})) \right\} + o(m^{-1}). \quad (S2.21)\]
Using the results that $(\hat{\theta}^E_i - \hat{\theta}^B_i)$ and $\cos(2(\hat{\theta}^B_i - x'_i \beta))$ are respectively odd and even functions of $Z^* = Z - X \beta$, we obtain

$$(S2.21) = 2C_1(A)C_{i_1}^{(1)}(A)$$

$$\times E \left[ (\hat{A} - A) \sin(\hat{\theta}^B_i) \left\{ (\hat{\theta}^E_i - \hat{\theta}^B_i) \cos(\hat{\theta}^B_i) - \frac{1}{2}(\hat{\theta}^E_i - \hat{\theta}^B_i)^2 \sin(\eta(\hat{\theta}^E_i - \hat{\theta}^B_i)) \right\} \right] + o(m^{-1}),$$

$$= C_1(A)C_{i_1}^{(1)}(A) \cos(2x'_i \beta) E[ (\hat{A} - A) \sin(2(\hat{\theta}^B_i - x'_i \beta))(\hat{\theta}^E_i - \hat{\theta}^B_i)] + o(m^{-1}),$$

$$= C_1(A)C_{i_1}^{(1)}(A) \cos(2x'_i \beta) \frac{1}{2i_{i_1}} \exp(-2A + 2g_{i_1}(A))$$

$$\times \left\{ E \left[ (\hat{A} - A)(\hat{\theta}^E_i - \hat{\theta}^B_i) \right] - E \left[ (\hat{A} - A)(\hat{\theta}^E_i - \hat{\theta}^B_i) \right] \right\} + o(m^{-1}),$$

$$(S2.22)$$

where $|\eta| < 1$. For the last equality, Lemma 1 is used.

From Lemma 2 (ii) and (S2.22), we can rewrite (S2.21) as

$$-\frac{1}{2} V_A \cos(2x'_i \beta) \exp(-2A + g_{i_1}(A)) B_i^3 (1 - B_i) + o(m^{-1}).$$

$$(S2.23)$$

From (S2.15), (S2.20), and (S2.23), (S2.14) can be approximated up to the order of $O(m^{-1})$ as

$$(S2.14) = \frac{1}{8} \exp(-g_{i_1}(A)) \left\{ g_{2i}(A) + g_{3i}(A) + \frac{B_i^4}{4} V_A \right\}$$

$$+ \frac{1}{8} \cos(2x'_i \beta) \exp(-2A + g_{i_1}(A)) \left\{ g_{2i}(A) + g_{3i}(A) - \frac{B_i^2(B_i - 2)^2}{4} V_A \right\}$$

$$+ o(m^{-1}).$$

$$(S2.24)$$
S2.2 Theorem 2

We first prove part (i).

\[ E[\hat{M}_{1i}] = \frac{1}{8} E[1 - \exp(-g_{1i}(\hat{A}))] + \frac{1}{8} E[\cos(2x_i'\hat{\beta}) \exp(-2\hat{A} + g_{1i}(\hat{A}))] \]
\[ - \frac{1}{8} E[\cos(2x_i'\hat{\beta}) \exp(-2\hat{A})], \]
\[ = \frac{1}{8} - \frac{1}{8} \exp(-g_{1i}(A)) E[\exp\{-g_{1i}(\hat{A}) - g_{1i}(A)\}] \]
\[ + \frac{1}{8} \exp(-2A + g_{1i}(A)) E[\cos(2x_i'\hat{\beta}) \exp\{-2(\hat{A} - A) + (g_{1i}(\hat{A}) - g_{1i}(A))\}] \]
\[ - \frac{1}{8} \exp(-2A) E[\cos(2x_i'\hat{\beta}) \exp\{-2(\hat{A} - A)\}], \]
\[ = \frac{1}{8} - \frac{1}{8} \{\exp(-g_{1i}(A))T_1 + \exp(-2A + g_{1i}(A))T_2 - \exp(-2A)T_3\}, \]

(S2.25)

where

\[ T_1 = E[\exp\{-g_{1i}(\hat{A}) - g_{1i}(A)\}]; \]
\[ T_2 = E[\cos(2x_i'\hat{\beta}) \exp\{-2(\hat{A} - A) + (g_{1i}(\hat{A}) - g_{1i}(A))\}]; \]
\[ T_3 = E[\cos(2x_i'\hat{\beta}) \exp\{-2(\hat{A} - A)\}]. \]

The results of Prasad and Rao (1990) and Datta and Lahiri (2000) lead to the following:

\[ T_1 = E \left[ 1 - (g_{1i}(\hat{A}) - g_{1i}(A)) + \frac{1}{2}(g_{1i}(\hat{A}) - g_{1i}(A))^2 \right] \]
\[ - E \left[ \frac{1}{6}(g_{1i}(\hat{A}) - g_{1i}(A))^3 \exp\{\eta(g_{1i}(\hat{A}) - g_{1i}(A))\} \right], \]
\[ S2. \text{ PROOFS OF THEOREMS} \]

\[ 1 + g_3(A) - b_A B_i^2 + \frac{B_i^4}{2} V_A + o(m^{-1}), \quad (S2.26) \]

where \( b_A = E[\hat{A} - A] + o(m^{-1}) \) and \( |\eta| < 1 \).

In the above calculation, we use

\[ E[(g_{1i}(\hat{A}) - g_{1i}(A))^2] = E[(\hat{A} - A)^2 B_i^4] + o(m^{-1}) = B_i^4 V_A + o(m^{-1}) \]

and

\[ E[(g_{1i}(\hat{A}) - g_{1i}(A))^3] = o(m^{-1}), \]

the latter following from the dominated convergence theorem.

We next evaluate \( T_2 \). Consider some integrable functions \( f_1(\cdot) \) and \( f_2(\cdot) \).

Then Lemma 3 (ii) and (iii) yield

\[ E[f_1(\hat{A} - A)f_2(x_i^\beta) \cdot x_i^\beta] = \text{Cov}(f_1(\hat{A} - A), f_2(x_i^\beta - x_i^\beta)) \]

\[ + \ E[f_1(\hat{A} - A)]E[f_2(x_i^\beta - x_i^\beta)], \]

\[ = E[f_1(\hat{A} - A)]E[f_2(x_i^\beta - x_i^\beta)]. \quad (S2.27) \]

Using (S2.27), we obtain as

\[ T_2 = \cos(2x_i^\beta) E[\cos(2(x_i^\beta \cdot x_i^\beta))] E[\exp\{ -2(\hat{A} - A) + (g_{1i}(\hat{A}) - g_{1i}(A)) \}]. \]

On the set \( S = \{ \hat{A} : |\hat{A} - A| < \delta \} \) with some constant value \( \delta > 0 \), by
Corollary 1, $T_2$ reduces to

$$T_2 = \cos(2x_i'\beta) E[\cos(2(x_i'\hat{\beta} - x_i'\beta))]$$

$$\times E[1 - 2(\hat{A} - A) + (g_{1i}(\hat{A}) - g_{1i}(A)) + 2(\hat{A} - A)^2 + \frac{1}{2}(g_{1i}(\hat{A}) - g_{1i}(A))^2 : \mathcal{S}]$$

$$- \cos(2x_i'\beta) E[\cos(2(x_i'\hat{\beta} - x_i'\beta))] E[2(\hat{A} - A)(g_{1i}(\hat{A}) - g_{1i}(A)) : \mathcal{S}]$$

$$+ \frac{1}{6} \cos(2x_i'\beta) E[\cos(2(x_i'\hat{\beta} - x_i'\beta))]$$

$$\times E[-2(\hat{A} - A) + (g_{1i}(\hat{A}) - g_{1i}(A))]^3 \exp\{\eta(-2(\hat{A} - A) + (g_{1i}(\hat{A}) - g_{1i}(A)))\} : \mathcal{S},$$

$$= \cos(2x_i'\beta) \left\{ 1 - \frac{2g_{2i}(A)}{B_i^2} \right\} \left( 1 - g_{3i}(A) + b_A(B_i^2 - 2) + \frac{(B_i^2 - 2)^2 V_A}{2} \right) + o(m^{-1}),$$

$$= \cos(2x_i'\beta) \left\{ 1 - \frac{2g_{2i}(A)}{B_i^2} - g_{3i}(A) + b_A(B_i^2 - 2) + \frac{(B_i^2 - 2)^2 V_A}{2} \right\} + o(m^{-1}),$$

(S2.28)

where $|\eta| < 1$. Note that, in the above calculation, some terms of odd function with zero mean vanish.

In a similar way on $\mathcal{S}$, $T_3$ reduces to

$$T_3 = \cos(2x_i'\beta) E[\cos(2(x_i'\hat{\beta} - x_i'\beta))]$$

$$\times E \left[ 1 - 2(\hat{A} - A) + 2(\hat{A} - A)^2 - \frac{4}{3}(\hat{A} - A)^3 \exp\{2\eta(\hat{A} - A)\} : \mathcal{S} \right],$$

$$= \cos(2x_i'\beta) \left( 1 - \frac{2g_{2i}(A)}{B_i^2} \right) \left( 1 - 2b_A + 2V_A \right) + o(m^{-1}),$$

$$= \cos(2x_i'\beta) \left( 1 - \frac{2g_{2i}(A)}{B_i^2} - 2b_A + 2V_A \right) + o(m^{-1}).$$

(S2.29)
From (S2.25), (S2.26), (S2.28) and (S2.29), we find on \( \mathcal{S} \),

\[
E[\hat{M}_{1i} - M_{1i} : \mathcal{S}] = -\frac{1}{8} \exp(-g_{1i}(A)) \left( g_{3i}(A) - b_A B_i^2 + \frac{B_i^4}{2} V_A \right) \\
- \frac{1}{8} \exp(-2A + g_{1i}(A)) \cos(2x_i' \beta) \left\{ \frac{2g_{2i}(A)}{B_i^2} + g_{3i}(A) - b_A (B_i^2 - 2) - \frac{(B_i^2 - 2)^2}{2} V_A \right\} \\
- \frac{1}{8} \exp(-2A) \cos(2x_i' \beta) \left( 2V_A - 2b_A - \frac{2g_{2i}(A)}{B_i^2} \right) \\
+ o(m^{-1}). \tag{S2.30}
\]

Then, Part (i) is obtained from (S2.30) with \( 0 < s < 1 \) using a proof similar to that of Das, et al. (2004). Specifically,

\[
|E[\hat{M}_{1i} - M_{1i} - b_M(\lambda)]| = |E[\hat{M}_{1i} - M_{1i} - b_M(\lambda) : \mathcal{S}] + E[\hat{M}_{1i} - M_{1i} : \mathcal{S}^c]|, \\
\leq o(m^{-1}) + C m^s \frac{E[(\hat{A} - A)^4]}{\delta^4} \\
= o(m^{-1}). \tag{S2.31}
\]

We next prove part (ii). From the regularity conditions, \( M_{2i}(\lambda) \) and \( b_M(\lambda) \) are of the order \( O(m^{-1}) \) for large \( m \) and these are every bounded continuous functions with a finite \( \lambda \). Continuous mapping theorem and dominated convergence theorem provide the following with \( s < 1 \):

\[
|E[M_{2i}(\hat{\lambda}) - M_{2i}(\lambda)]| = |E[M_{2i}(\hat{\lambda}) - M_{2i}(\lambda) : \mathcal{S}] + E[M_{2i}(\hat{\lambda}) - M_{2i}(\lambda) : \mathcal{S}^c]| \\
\leq o(m^{-1}) + C m^s \frac{E[(\hat{A} - A)^4]}{\delta^4} = o(m^{-1}).
\]

where \( C \) is some positive constants.
Similarly, we get

$$E[b_M(\hat{\lambda}) - b_M(\lambda)] = o(m^{-1}).$$

The results follow.

S2.3 Theorem 3 (i)

From Lemma 3 (i) and assumption on $\hat{A}$, with some constant value $s > 0$,

$$E[(\hat{M}_i^0(\hat{\lambda}) - M_i(\lambda))^4] = E[(\hat{M}_i^0(\hat{\lambda}) - M_i(\lambda))^4 : \mathcal{S}] + E[(\hat{M}_i^0(\hat{\lambda}) - M_i(\lambda))^4 : \mathcal{S}^c]$$

$$\leq O(m^{-2}) + Cm^4s \frac{E[(\hat{A} - A)^8]}{\delta^8} = O(m^{-2v_4(s-1)}).$$  \hspace{1cm} (S2.32)

where $C(>0)$ is some constants.

We then get from (S2.32),

$$P(\hat{M}_i^0(\hat{\lambda}) \leq 0) \leq P(|\hat{M}_i^0(\hat{\lambda}) - M_i| \geq M_i),$$

$$\leq \frac{E[(\hat{M}_i^0(\hat{\lambda}) - M_i)^4]}{M_i^4} = O(m^{-2v_4(s-1)}).$$  \hspace{1cm} (S2.33)

We now let $\mathcal{M}$ define a set such that $\{\hat{M}_i^0 > 0\}$. Then the result (S2.33) and Theorem 2 lead to the following result with $0 < s < 3/5$.

$$|E[\hat{M}_i - M_i]| = |E[\hat{M}_i - M_i : \mathcal{M}] + E[\hat{M}_i - M_i : \mathcal{M}^c]|$$

$$\leq |E[\hat{M}_i^0(\hat{\lambda}) - M_i]| + Cm^8P(\hat{M}_i^0(\hat{\lambda}) \leq 0),$$

$$= o(m^{-1}) + O(m^{(s-2)\sqrt{(5s-4)}}) = o(m^{-1}).$$

We thus get part (i).
Bibliography


