S1 Proofs

Proof of Theorem 1. Since a DCD is a special MCD, part (a) follows from Proposition 1 of He et al. (2017). For part (b), when $D_1 = OA(n, q, s, 2)$, due to the property (ii) of a DCD that the rows in $D_2$ corresponding to each of the $s^2$ level combinations of any two factors in $D_1$ form an LHD with $n/s^2$ runs. This indicates that for $(z_i, z_j, \tilde{d}_k)$, $1 \leq i \neq j \leq q$ and $1 \leq k \leq p$, each of all possible three-tuples occurs exactly once. \qed
**Proof of Theorem 2.** We first show the necessity. If a DCD($n, s^q, p$) exists, Condition (b) in Theorem 1 reveals that the rows in $\mathbf{D}_1$ corresponding to each level of $\tilde{d}_k$ is an OA($s^2, q, s, 2$). Therefore, for a given $k$, according to the level of $\tilde{d}_k$, $\mathbf{D}_1$ can be partitioned into $n/s^2$ OA($s^2, p, s, 2$)'s, say $\mathbf{E}_1, \ldots, \mathbf{E}_{n/s^2}$. It remains to show that each $\mathbf{E}_i$, $i = 1, \ldots, n/s^2$, is a completely resolvable orthogonal array. From Condition (a) of Theorem 1, the rows in $\mathbf{D}_1$ corresponding to each level of $\tilde{d}_k$ is an OA($s, q, s, 1$). Recall that the relationship $\tilde{d}_k = \lfloor \tilde{d}_k / s \rfloor$, which means each level in $\tilde{d}_k$ corresponds to $s$ levels in $\tilde{d}_k$. Hence, each $\mathbf{E}_i$ can be divided into $s$ OA($s, q, s, 1$)'s, and by definition, it is a CROA($s^2, q, s, 2$).

Now, we show the sufficiency, that is, if $\mathbf{D}_1$ can be partitioned into $n/s^2$ CROA($s^2, q, s, 2$)'s, then a DCD($n, s^q, p$) exists. Note that $\mathbf{D}_1$ can be represented by $(\mathbf{E}_1^T, \ldots, \mathbf{E}_{n/s^2}^T)^T$ with $\mathbf{E}_i = (\mathbf{E}_{i,1}^T, \ldots, \mathbf{E}_{i,s}^T)^T$, where $\mathbf{E}_i$ is a CROA($s^2, q, s, 2$) and $\mathbf{E}_{i,j}$ is an OA($s, q, s, 1$), for $i = 1, \ldots, n/s^2$ and $j = 1, \ldots, s$. Since $\mathbf{E}_i$ is an OA($s^2, q, s, 2$), each level combination occurs once in any two columns of $\mathbf{E}_i$, and thus $n/s^2$ times in any two columns of $\mathbf{D}_1$. Likewise, each level appears once in any column of each $\mathbf{E}_{i,j}$, and thus $n/s$ times in any column of $\mathbf{D}_1$. To show the existence of a DCD, given such a $\mathbf{D}_1$, we construct a $\mathbf{D}_2$. Let $\mathbf{d} = (\mathbf{g}_1^T, \ldots, \mathbf{g}_{n/s^2}^T)^T$, such that $\mathbf{g}_i = (\mathbf{g}_{i,1}^T, \ldots, \mathbf{g}_{i,s}^T)^T$, where $\mathbf{g}_{i,j} = ((i-1)s^2 + (j-1)s, \ldots, (i-1)s^2 + js - 1)^T$, for $i = 1, \ldots, n/s^2$ and $j = 1, \ldots, s$. 
Next, we will explain that \((D_1, d)\) is a DCD\((n, s^q, 1)\). Obviously, \(d = (0, 1, \ldots, n-1)^T\), which is an LH. Since \([g_i/s^2] = (i - 1) \cdot 1_s^2\), where \(1_s^2\) is a column of \(s^2\) ones, the rows in \(\tilde{d}\) corresponding to each level combination of any two factors in \(D_1\) are \((0, 1, \ldots, n/s^2-1)^T\), i.e., an LHD of \(n/s^2\) runs. Moreover, \([g_{i,j}/s] = ((i-1)s + j - 1) \cdot 1_s\), so the rows in \(\tilde{d}\) corresponding to each level of any factor in \(D_1\) are \((0, 1, \ldots, n/s-1)^T\), which is an LHD of \(n/s\) runs. Thus, followed by Definition 1, \((D_1, d)\) is a DCD\((n, s^q, 1)\).

Besides, it can be easily checked that randomly permuting \(g_i\) in \(d\), randomly permuting \(g_{i,j}\) in \(g_i\) and randomly permuting the entries in \(g_{i,j}\) can generate additional columns of \(D_2\). 

\[\text{Proof of Theorem 3.}\] Let \(D_1 = (z_1, \ldots, z_q)\) be an OA\((n, q, 2)\), \(B = (b_1, \ldots, b_p)\) be an OA\((n, p, n/s^2, 1)\), and \(C = (c_1, \ldots, c_p)\) be an OA\((n, p, s, 1)\), such that for any \(1 \leq i \neq j \leq q\) and \(1 \leq k \leq p\), both \((z_i, z_j, b_k)\) and \((z_i, c_k, b_k)\) are OA\((n, 3, s^2 (n/s^2), 3)'s\). Let \(\tilde{D}_2 = sB + C\). To prove that the resulting design \(D = (D_1, D_2)\) is a DCD, it suffices to check the requirements in Theorem 1. As \((z_i, c_k, b_k)\) is an OA\((n, 3, s^2 (n/s^2), 3)\), it easily follows that \((z_i, \tilde{d}_k) = (z_i, sb_k + c_k)\) is an OA\((n, 2, (n/s), 2)\). Finally, note that \((z_i, z_j, \tilde{d}_k)\) is an OA\((n, 3, s^2 (n/s^2), 3)\), since \(\tilde{d}_k = b_k\).

Conversely, if \((D_1, D_2)\) is a DCD\((n, s^q, p)\), where \(\tilde{D}_2 = sB + C\), i.e., \(\tilde{d}_k = sb_k + c_k\) and \(\tilde{d}_k = b_k\), we will show that there exist three arrays, \(D_1 = (z_1, \ldots, z_q)\) is an
OA(n, q, s, 2), \(B = (b_1, \ldots, b_p)\) is an OA(n, p, n/s^2, 1), and \(C = (c_1, \ldots, c_p)\) is an OA(n, p, s, 1), such that both \((z_i, z_j, b_k)\) and \((z_i, c_k, b_k)\) are OA(n, 3, s^2(n/s^2), 3)’s.

Since \((D_1, D_2)\) is a DCD, we have that \(D_1\) is an OA(n, q, s, 2) and \(D_2\) is an LH(n, p), which means that \(B = \bar{D}_2\) is an OA(n, p, n/s^2, 1). Moreover, according to Theorem 1, \((z_i, \tilde{d}_k)\) forms an OA(n, 2, s(n/s), 2) and \((z_i, z_j, \tilde{d}_k)\) is an OA(n, 3, s^2(n/s^2), 3), then it can be easily obtained that \((z_i, c_k, b_k)\) is an OA(n, 3, s^2(n/s^2), 3), \(C\) is an OA(n, p, s, 1) and \((z_i, z_j, b_k)\) is an OA(n, 3, s^2(n/s^2), 3).

Proof of Proposition 2. Stack \(\lambda A_1\)’s row by row, and denote the resulting design by \(A_1^* = (a_1^*, \ldots, a_{q+1}^*)\), which is an OA(\(\lambda s^2, q + 1, s, 2\)). As \(D_1\) is from \(A_1^*\) by deleting the last column, we have that \(D_1 = (z_1, \ldots, z_q)\) is an OA(\(\lambda s^2, q, s, 2\)) and \(z_i = a_i^*\), for 1 \(\leq i \leq q\). From Step 2, it is easy to find that \(B\) is an OA(\(\lambda s^2, p, \lambda, 1\)) and \((a_i^*, a_j^*, b_k)\) is an OA(\(\lambda s^2, 3, s^2 \times \lambda, 3\)), for any 1 \(\leq i \neq j \leq q + 1\) and 1 \(\leq k \leq p\). In Step 3, note that \(w_k \otimes 1_s\) can be regarded as a level permutation of \(x_{q+1}\), where \(x_{q+1}\) represents the last column of \(A_1\), and thus \(c_k\) is a level permutation of \(a_{q+1}^*\). Clearly, \(C\) should be an OA(n, p, s, 1). It is well-known that level permutations of any factor do not change the orthogonality, so \((a_i^*, c_k, b_k)\) is also an OA(\(\lambda s^2, 3, s^2 \times \lambda, 3\)), for any 1 \(\leq i \neq j \leq q\) and 1 \(\leq k \leq p\). Hence, \(D_1\), \(B\) and \(C\) are the needed arrays in Theorem 3.

Proof of Proposition 3. Since \(D_1\) is a sub-array of \(G\), part (a) follows directly.
S1. PROOFS

For (b), as \( b_k, a^* \) and \( b'_k \) are three different columns of \( G \), i.e., \( (b_k, a^*, b'_k) \) forms an OA\((s^3, 3, s, 3)\), which implies that \( (sb_k + a^*, b_k') \) is an OA\((s^3, 2, s^2 \times s, 2)\). Besides, level permutations of any factor do not alter the orthogonality. Thus, \( (\tilde{d}_k, \tilde{d}_{k'}) = (sb_k + c_k, sb_{k'} + c_{k'}) \) achieves \( s^2 \times s \) and \( s \times s^2 \) grids stratification. Finally, \( \tilde{D}_2 = B = (b_1, \ldots, b_p) \), the proof of (c) is similar to that of (a). \( \Box \)

**Proof of Lemma 2.** For any \( k \in \{1, \ldots, (u - 2)s^2\} \), there always exists a pair of integers \( v \) and \( f \), such that \( k = (f - 1)(u - 2) + v \), where \( 1 \leq v \leq u - 2 \) and \( 1 \leq f \leq s^2 \). Then, \( (a_i, a_j, b_k) = (a_i, a_j, (r_{1,f}, \ldots, r_{u-2,f})t_v) \) is an OA\((s^u, 3, s^2(s^u-2), 3)\) due to Lemma 1(c). Part (b) is straightforward. \( \Box \)

**Proof of Proposition 4.** The resulting \( D = (D_1, D_2) \) is a DCD from Lemma 2 and Theorem 4. For (a), if \( \lfloor (i - 1)/(u - 2) \rfloor = \lfloor (i' - 1)/(u - 2) \rfloor \), \( b_i \) and \( b_{i'} \) come from the same \( B_{\lfloor (i-1)/(u-2) \rfloor + 1} \). According to Equation (4.4) and Lemma 1(b), \( b_i \) and \( b_{i'} \) achieve the stratification on \( s \times s \) grids. If \( \lfloor (i - 1)/(u - 2) \rfloor \neq \lfloor (i' - 1)/(u - 2) \rfloor \), \( b_i \) and \( b_{i'} \) are from \( B_{\lfloor (i-1)/(u-2) \rfloor + 1} \) and \( B_{\lfloor (i'-1)/(u-2) \rfloor + 1} \), respectively. Similarly, part (b) follows from Lemma 1(d). \( \Box \)
S2 Tables of the DCDs

This section lists some examples of the DCDs generated by the proposed constructions. We omit the DCDs by Construction 2 since the results are very similar to that of Construction 1, i.e., Table 3, except that the number of columns of the quantitative factors for Constructions 1 and 2 is $(s!)^{\lambda_s} \cdot (s!)^{\lambda} \cdot \lambda!$ and $(s!)^{\lambda_s} \cdot s! \cdot (\lambda!)^{s^2}$, respectively.

Table 1: DCDs produced by Construction 1, for $s \leq 11$ and a positive integer $\lambda$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$OA(4\lambda, 2, 2, 2)$</td>
<td>$LH(4\lambda, p)$</td>
<td>$p \leq 2^{3\lambda} \cdot \lambda!$</td>
</tr>
<tr>
<td>3</td>
<td>$OA(9\lambda, 3, 3, 2)$</td>
<td>$LH(9\lambda, p)$</td>
<td>$p \leq 6^{4\lambda} \cdot \lambda!$</td>
</tr>
<tr>
<td>4</td>
<td>$OA(16\lambda, 4, 4, 2)$</td>
<td>$LH(16\lambda, p)$</td>
<td>$p \leq (4!)^{5\lambda} \cdot \lambda!$</td>
</tr>
<tr>
<td>5</td>
<td>$OA(25\lambda, 5, 5, 2)$</td>
<td>$LH(25\lambda, p)$</td>
<td>$p \leq (5!)^{6\lambda} \cdot \lambda!$</td>
</tr>
<tr>
<td>6</td>
<td>$OA(36\lambda, 6, 6, 2)$</td>
<td>$LH(36\lambda, p)$</td>
<td>$p \leq (6!)^{7\lambda} \cdot \lambda!$</td>
</tr>
<tr>
<td>7</td>
<td>$OA(49\lambda, 7, 7, 2)$</td>
<td>$LH(49\lambda, p)$</td>
<td>$p \leq (7!)^{8\lambda} \cdot \lambda!$</td>
</tr>
<tr>
<td>8</td>
<td>$OA(64\lambda, 8, 8, 2)$</td>
<td>$LH(64\lambda, p)$</td>
<td>$p \leq (8!)^{9\lambda} \cdot \lambda!$</td>
</tr>
<tr>
<td>9</td>
<td>$OA(81\lambda, 9, 9, 2)$</td>
<td>$LH(81\lambda, p)$</td>
<td>$p \leq (9!)^{10\lambda} \cdot \lambda!$</td>
</tr>
<tr>
<td>10</td>
<td>$OA(100\lambda, 3, 10, 2)$</td>
<td>$LH(100\lambda, p)$</td>
<td>$p \leq (10!)^{11\lambda} \cdot \lambda!$</td>
</tr>
<tr>
<td>11</td>
<td>$OA(121\lambda, 11, 11, 2)$</td>
<td>$LH(121\lambda, p)$</td>
<td>$p \leq (11!)^{12\lambda} \cdot \lambda!$</td>
</tr>
</tbody>
</table>
Table 2: DCDs produced by Case 1 of Construction 3, for $s \leq 10$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$OA(8, q, 2, t)$</td>
<td>$LH(8, p)$</td>
<td>$q + p \leq 3$</td>
</tr>
<tr>
<td>3</td>
<td>$OA(27, q, 3, t)$</td>
<td>$LH(27, p)$</td>
<td>$q + p \leq 3$</td>
</tr>
<tr>
<td>4</td>
<td>$OA(64, q, 4, t)$</td>
<td>$LH(64, p)$</td>
<td>$q + p \leq 5$</td>
</tr>
<tr>
<td>5</td>
<td>$OA(125, q, 5, t)$</td>
<td>$LH(125, p)$</td>
<td>$q + p \leq 5$</td>
</tr>
<tr>
<td>6</td>
<td>$OA(216, q, 6, t)$</td>
<td>$LH(216, p)$</td>
<td>$q + p \leq 3$</td>
</tr>
<tr>
<td>7</td>
<td>$OA(343, q, 7, t)$</td>
<td>$LH(343, p)$</td>
<td>$q + p \leq 7$</td>
</tr>
<tr>
<td>8</td>
<td>$OA(512, q, 8, t)$</td>
<td>$LH(512, p)$</td>
<td>$q + p \leq 9$</td>
</tr>
<tr>
<td>9</td>
<td>$OA(729, q, 9, t)$</td>
<td>$LH(729, p)$</td>
<td>$q + p \leq 9$</td>
</tr>
<tr>
<td>10</td>
<td>$OA(1000, q, 10, t)$</td>
<td>$LH(1000, p)$</td>
<td>$q + p \leq 3$</td>
</tr>
</tbody>
</table>

The $t$ refers to the strength of an orthogonal array. If $q < 3$, $t = q$, and if $q \geq 3$, $t = 3$. 
Table 3: DCDs produced by Case 2 of Construction 3, for $s \leq 5$ and $n \leq 256$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$u$</th>
<th>$D_1$</th>
<th>$D_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>$OA(8, 2, 2, 2)$</td>
<td>$LH(8, 4)$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$OA(16, 2, 2, 2)$</td>
<td>$LH(16, 8)$</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>$OA(32, 2, 2, 2)$</td>
<td>$LH(32, 12)$</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>$OA(64, 2, 2, 2)$</td>
<td>$LH(64, 16)$</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>$OA(128, 2, 2, 2)$</td>
<td>$LH(128, 20)$</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>$OA(256, 2, 2, 2)$</td>
<td>$LH(256, 24)$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>$OA(27, 3, 3, 2)$</td>
<td>$LH(27, 9)$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$OA(81, 3, 3, 2)$</td>
<td>$LH(81, 18)$</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>$OA(243, 3, 3, 2)$</td>
<td>$LH(243, 27)$</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>$OA(64, 4, 4, 2)$</td>
<td>$LH(64, 16)$</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>$OA(256, 4, 4, 2)$</td>
<td>$LH(256, 32)$</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>$OA(125, 5, 5, 2)$</td>
<td>$LH(125, 25)$</td>
</tr>
</tbody>
</table>
Bibliography