On the efficiency of composite likelihood estimation for Gaussian spatial processes

Nelson J. Y. Chua*[1,2], Francis K. C. Hui[1] and A. H. Welsh[1]

[1] *The Australian National University and* [2]*Australian Bureau of Statistics*

This material covers proofs of Theorems 2, 3 and 4 in the main paper, as well as derivations of the Fisher information and sandwich covariance for a constant mean parameter under the exponential covariance model. Note that equations from the main paper are referenced here.

S1 Proof of Theorem 2

Note the following result pertaining to the expectation of fourth-order moments (see p. 109 of Rencher and Schaalje, 2008, for instance):

Lemma 1. *Let* $\boldsymbol{x} = (x_1, ..., x_n)^T$ *follow a Gaussian distribution with zero mean and covariance matrix* $\boldsymbol{\Sigma}$. *Also, let* $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *be* $n \times n$ *symmetric matrices. Then* $E[\boldsymbol{x}^T \boldsymbol{U} \boldsymbol{x} \boldsymbol{x}^T \boldsymbol{V} \boldsymbol{x}] = tr(\boldsymbol{U\Sigma}) tr(\boldsymbol{V\Sigma}) + 2tr(\boldsymbol{U\Sigma V\Sigma})$.

We also note that $\partial \boldsymbol{\Sigma}_m^{-1} / \partial \theta_i = \boldsymbol{\Sigma}_m^{-1} (\partial \boldsymbol{\Sigma}_m / \partial \theta_i) \boldsymbol{\Sigma}_m^{-1}$, which is a useful expression to avoid the direct evaluation of the derivative of a matrix inverse.

By using these two results, we can derive (3.2) as follows:

$$
\begin{aligned}
\{\mathbf{J}(\boldsymbol{\theta})\}_{ij} &= E\left[\frac{\partial}{\partial\theta_i}c\ell(\boldsymbol{\theta};\mathbf{y})\frac{\partial}{\partial\theta_j}c\ell(\boldsymbol{\theta};\mathbf{y})\right] \\
&= \frac{1}{4}\sum_{m=1}^{M}\sum_{l=1}^{M} w_m w_l \Bigg(\operatorname{tr}\left(\boldsymbol{\Sigma}_m^{-1}\frac{\partial\boldsymbol{\Sigma}_m}{\partial\phi_i}\right)\operatorname{tr}\left(\boldsymbol{\Sigma}_l^{-1}\frac{\partial\boldsymbol{\Sigma}_l}{\partial\phi_j}\right) + \operatorname{tr}\left(\boldsymbol{\Sigma}_m^{-1}\frac{\partial\boldsymbol{\Sigma} m}{\partial\theta_i}\right)E\left[\mathbf{y}_l^T\frac{\partial\boldsymbol{\Sigma}_l^{-1}}{\partial\theta_j}\mathbf{y}_l\right] \\
&\qquad + \operatorname{tr}\left(\boldsymbol{\Sigma}_l^{-1}\frac{\partial\boldsymbol{\Sigma}_l}{\partial\theta_j}\right)E\left[\mathbf{y}_m^T\frac{\partial\boldsymbol{\Sigma}_m^{-1}}{\partial\theta_i}\mathbf{y}_m\right] + E\left[\mathbf{y}_m^T\frac{\partial\boldsymbol{\Sigma}_m^{-1}}{\partial\theta_i}\mathbf{y}_m\mathbf{y}_l^T\frac{\partial\boldsymbol{\Sigma}_l^{-1}}{\partial\theta_j}\mathbf{y}_l\right]\Bigg) \\
&= \frac{1}{4}\sum_{m=1}^{M}\sum_{l=1}^{M} w_m w_l \Bigg(-\operatorname{tr}\left(\boldsymbol{\Sigma}_m^{-1}\frac{\partial\boldsymbol{\Sigma}_m}{\partial\theta_i}\right)\operatorname{tr}\left(\boldsymbol{\Sigma}_l^{-1}\frac{\partial\boldsymbol{\Sigma}_l}{\partial\theta_j}\right) \\
&\qquad + E\left[\begin{bmatrix}\mathbf{y}_m^T & \mathbf{y}_l^T\end{bmatrix}\begin{bmatrix}\frac{\partial\boldsymbol{\Sigma}_m^{-1}}{\partial\theta_i} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{y}_m\\ \mathbf{y}_l\end{bmatrix}\begin{bmatrix}\mathbf{y}_m^T & \mathbf{y}_l^T\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{\partial\boldsymbol{\Sigma}_l^{-1}}{\partial\theta_j}\end{bmatrix}\begin{bmatrix}\mathbf{y}_m\\ \mathbf{y}_l\end{bmatrix}\right]\Bigg) \\
&= \frac{1}{2}\sum_{m=1}^{M}\sum_{l=1}^{M} w_m w_l \operatorname{tr}\left(\begin{bmatrix}\frac{\partial\boldsymbol{\Sigma}_m^{-1}}{\partial\theta_i} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}_m & \boldsymbol{\Sigma}_{m,l}\\ \boldsymbol{\Sigma}_{m,l}^T & \boldsymbol{\Sigma}_l\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{\partial\boldsymbol{\Sigma}_l^{-1}}{\partial\theta_j}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}_m & \boldsymbol{\Sigma}_{m,l}\\ \boldsymbol{\Sigma}_{m,l}^T & \boldsymbol{\Sigma}_l\end{bmatrix}\right) \\
&= \frac{1}{2}\sum_{m=1}^{M}\sum_{l=1}^{M} w_m w_l \operatorname{tr}\left(\boldsymbol{\Sigma}_m^{-1}\frac{\partial\boldsymbol{\Sigma}_m}{\partial\theta_i}\boldsymbol{\Sigma}_m^{-1}\boldsymbol{\Sigma}_{m,l}\boldsymbol{\Sigma}_l^{-1}\frac{\partial\boldsymbol{\Sigma}_l}{\partial\theta_j}\boldsymbol{\Sigma}_l^{-1}\boldsymbol{\Sigma}_{m,l}^T\right) \\
&= \frac{1}{2}\sum_{m=1}^{M} w_m^2 \operatorname{tr}\left(\boldsymbol{\Sigma}_m^{-1}\frac{\partial\boldsymbol{\Sigma}_m}{\partial\theta_i}\boldsymbol{\Sigma}_m^{-1}\frac{\partial\boldsymbol{\Sigma}_m}{\partial\theta_j}\right) \\
&\qquad + \frac{1}{2}\sum_{m=1}^{M}\sum_{l\neq m}^{M} w_m w_l \operatorname{tr}\left(\boldsymbol{\Sigma}_m^{-1}\frac{\partial\boldsymbol{\Sigma}_m}{\partial\theta_i}\boldsymbol{\Sigma}_m^{-1}\boldsymbol{\Sigma}_{m,l}\boldsymbol{\Sigma}_l^{-1}\frac{\partial\boldsymbol{\Sigma}_l}{\partial\theta_j}\boldsymbol{\Sigma}_l^{-1}\boldsymbol{\Sigma}_{m,l}^T\right).
\end{aligned}
$$

S2 Proof of Theorem 3

For notational convenience, let $\mathbf{M} \equiv \mathbf{M}(\alpha)$. Following on from (4.3), the first and second-order partial derivatives of the composite log-likelihood are

$$\frac{\partial}{\partial \boldsymbol{\theta}} c\ell(\sigma^2, \alpha; \mathbf{y}) = \begin{bmatrix} -\frac{DF}{2\sigma^2} + \frac{1}{2\sigma^4}\mathbf{y}^T\mathbf{M}\mathbf{y} \\ -\frac{\rho^2}{F}\left(\frac{DF}{1-\rho^2} + \frac{DF-2}{1+\rho^2}\right) - \frac{1}{2\sigma^2}\mathbf{y}^T\mathbf{M}'\mathbf{y} \end{bmatrix} \tag{S2.1}$$

and

$$-\frac{\partial^2}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^T} c\ell(\sigma^2, \alpha; \mathbf{y}) = \begin{bmatrix} -\frac{DF}{2\sigma^4} + \frac{1}{\sigma^6}\mathbf{y}^T\mathbf{M}\mathbf{y} & -\frac{1}{2\sigma^4}\mathbf{y}^T\mathbf{M}'\mathbf{y} \\ -\frac{1}{2\sigma^4}\mathbf{y}^T\mathbf{M}'\mathbf{y} & -\frac{2\rho^2}{F^2}\left(\frac{DF}{(1-\rho^2)^2} + \frac{DF-2}{(1+\rho^2)^2}\right) + \frac{1}{2\sigma^2}\mathbf{y}^T\mathbf{M}''\mathbf{y} \end{bmatrix}, \tag{S2.2}$$

where the first and second derivatives of $\mathbf{M}$ with respect to α are

$$\mathbf{M}' = -\frac{2\rho}{F(1-\rho^2)^2}\Bigg[\left(\rho + \frac{\rho(1+\rho^4)}{(1+\rho^2)^2}\right)\mathbf{A}_1 + 2\rho\mathbf{A}_2 + \left(-\rho + \frac{\rho(1+\rho^4)}{(1+\rho^2)^2}\right)\mathbf{A}_3 - (1+\rho^2)\mathbf{A}_4 + \frac{\rho(1+\rho^4)}{(1+\rho^2)^2}\mathbf{A}_5\Bigg]$$

and

$$\mathbf{M}'' = \frac{4\rho}{F^2(1-\rho^2)^3}\Bigg[\left(\rho(1+\rho^2) + \frac{\rho(1+6\rho^4+\rho^8)}{(1+\rho^2)^3}\right)\mathbf{A}_1 + 2\rho(1+\rho^2)\mathbf{A}_2 + \left(-\rho(1+\rho^2) + \frac{\rho(1+6\rho^4+\rho^8)}{(1+\rho^2)^3}\right)\mathbf{A}_3 - \frac{1}{2}(1+6\rho^2+\rho^4)\mathbf{A}_4 + \frac{\rho(1+6\rho^4+\rho^8)}{(1+\rho^2)^3}\mathbf{A}_5\Bigg].$$

First, to find $\mathbf{H}(\sigma^2, \alpha)$, we note the following trace formulae for the exponential covariance matrix $\boldsymbol{\Sigma}$:

$$\operatorname{tr}(\mathbf{A}_1\boldsymbol{\Sigma}) = \sigma^2 DF, \qquad \operatorname{tr}(\mathbf{A}_2\boldsymbol{\Sigma}) = \sigma^2(DF-2), \qquad \operatorname{tr}(\mathbf{A}_3\boldsymbol{\Sigma}) = \sigma^2(DF-4),$$
$$\operatorname{tr}(\mathbf{A}_4\boldsymbol{\Sigma}) = 2\sigma^2\rho(DF-1), \quad \operatorname{tr}(\mathbf{A}_5\boldsymbol{\Sigma}) = 2\sigma^2\rho^2(DF-2). \quad \text{(S2.3)}$$

Due to the linearity of the trace function, we can use the additive decomposition of $\mathbf{M}$ and its derivatives to show that $E[\mathbf{y}^T\mathbf{M}\mathbf{y}] = \operatorname{tr}(\mathbf{M}\boldsymbol{\Sigma}) = \sigma^2 DF$, $E[\mathbf{y}^T\mathbf{M}'\mathbf{y}] = -4\sigma^2\rho^2[DF-(1-\rho^2)]/[F(1-\rho^4)]$, and $E[\mathbf{y}^T\mathbf{M}''\mathbf{y}] = 4\sigma^2\rho^2[(3+\rho^2+5\rho^4-\rho^6)DF-(1-\rho^2)(3-2\rho^2+3\rho^4)]/[F^2(1-\rho^4)^2]$. Thus, by using (S2.2), we obtain

$$\mathbf{H}(\sigma^2,\alpha) = \begin{bmatrix} \frac{DF}{2\sigma^4} & \frac{2\rho^2}{F\sigma^2(1-\rho^4)}[DF-(1-\rho^2)] \\ \frac{2\rho^2}{F\sigma^2(1-\rho^4)}[DF-(1-\rho^2)] & \frac{2\rho^2}{F^2(1-\rho^4)^2}[(1+\rho^2+3\rho^4-\rho^6)DF-(1-\rho^2)(1+3\rho^4)] \end{bmatrix}.$$

The calculation of $\mathbf{J}(\sigma^2,\alpha)$ is considerably more complicated as it requires finding expressions for fourth-order moments. In order to apply Lemma 1, we require traces of the form $\operatorname{tr}(\mathbf{A}_j\boldsymbol{\Sigma}\mathbf{A}_k\boldsymbol{\Sigma})$ for each pair of the five simple matrices $\mathbf{A}_1$ to $\mathbf{A}_5$. However, due to the cyclical invariance of traces, we only need 15 such expressions rather than 25.

To simplify notation, define $u_n \equiv \sum_{k=1}^{n}(n-k)\rho^{2k}$. Also, let "$\circ$" denote the Hadamard (entrywise) product of two matrices. Then as an example,

$$\begin{aligned}
\mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\Sigma}) &= \mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}) = \sum_{i=1}^{DF}\sum_{j=1}^{DF}\{\boldsymbol{\Sigma}\circ\boldsymbol{\Sigma}^T\}_{ij} = \sum_{i=1}^{DF}\sum_{j=1}^{DF}(\sigma^2\rho^{|i-j|})^2 \\
&= \sigma^4\left[\sum_{i=1}^{DF}1 + \sum_{i\neq j}^{DF}\rho^{2|i-j|}\right] = \sigma^4\left[DF + 2\sum_{j=1}^{DF-1}\sum_{i=j+1}^{DF}\rho^{2|i-j|}\right] \\
&= \sigma^4\left[DF + 2\sum_{j=1}^{DF-1}\sum_{k=1}^{DF-j}\rho^{2k}\right] = \sigma^4\left[DF + 2\sum_{k=1}^{DF-1}\sum_{j=1}^{DF-k}\rho^{2k}\right] \\
&= \sigma^4\left[DF + 2\sum_{k=1}^{DF}(DF-k)\rho^{2k}\right] = \sigma^4[DF + 2u_{DF}].
\end{aligned}$$

Similarly, we can obtain all of the following:

$$\begin{aligned}
&\mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\Sigma}) = \sigma^4[DF + 2u_{DF}], && \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_2\boldsymbol{\Sigma}) = \sigma^4[DF - 2 + 2u_{DF-1}], \\
&\mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_3\boldsymbol{\Sigma}) = \sigma^4[(1+2\rho^2)(DF-4) + 2\rho^2 u_{DF-3}], && \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_4\boldsymbol{\Sigma}) = \sigma^4[4\rho(DF-1) + 4\rho u_{DF-1}], \\
&\mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_5\boldsymbol{\Sigma}) = \sigma^4[2\rho^2(DF-2) + 4u_{DF-1}], && \mathrm{tr}(\mathbf{A}_2\boldsymbol{\Sigma}\mathbf{A}_2\boldsymbol{\Sigma}) = \sigma^4[DF - 2 + 2u_{DF-2}], \\
&\mathrm{tr}(\mathbf{A}_2\boldsymbol{\Sigma}\mathbf{A}_3\boldsymbol{\Sigma}) = \sigma^4[DF - 4 + 2u_{DF-3}], && \mathrm{tr}(\mathbf{A}_2\boldsymbol{\Sigma}\mathbf{A}_4\boldsymbol{\Sigma}) = \sigma^4[4\rho(DF-2) + 4\rho u_{DF-2}], \\
&\mathrm{tr}(\mathbf{A}_2\boldsymbol{\Sigma}\mathbf{A}_5\boldsymbol{\Sigma}) = \sigma^4[2\rho^2(DF-2) + 4u_{DF-2}], && \mathrm{tr}(\mathbf{A}_3\boldsymbol{\Sigma}\mathbf{A}_3\boldsymbol{\Sigma}) = \sigma^4[DF - 4 + 2u_{DF-4}], \\
&\mathrm{tr}(\mathbf{A}_3\boldsymbol{\Sigma}\mathbf{A}_4\boldsymbol{\Sigma}) = \sigma^4[4\rho(DF-4) + 4\rho u_{DF-3}], && \mathrm{tr}(\mathbf{A}_3\boldsymbol{\Sigma}\mathbf{A}_5\boldsymbol{\Sigma}) = \sigma^4[2\rho^2(DF-4) + 4u_{DF-3}], \\
&\mathrm{tr}(\mathbf{A}_4\boldsymbol{\Sigma}\mathbf{A}_4\boldsymbol{\Sigma}) = \sigma^4[2(1+\rho^2)(DF-1) + 8u_{DF-1}], && \mathrm{tr}(\mathbf{A}_4\boldsymbol{\Sigma}\mathbf{A}_5\boldsymbol{\Sigma}) = \sigma^4[4\rho(1+\rho^2)(DF-2) + 8\rho u_{DF-2}], \\
&\mathrm{tr}(\mathbf{A}_5\boldsymbol{\Sigma}\mathbf{A}_5\boldsymbol{\Sigma}) = \sigma^4[2(1+\rho^4)(DF-2) + 4\rho^2(1+\rho^2)(DF-3) + 8\rho^2 u_{DF-3}]. && \text{(S2.4)}
\end{aligned}$$

Using the above results, we can then find $\mathrm{tr}(\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}\boldsymbol{\Sigma})$, $\mathrm{tr}(\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}'\boldsymbol{\Sigma})$ and $\mathrm{tr}(\mathbf{M}'\boldsymbol{\Sigma}\mathbf{M}'\boldsymbol{\Sigma})$. For instance, by using (4.4), we have

$$\begin{aligned}
\mathrm{tr}(\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}\boldsymbol{\Sigma}) &= \frac{1}{(1-\rho^2)^2}\left[\left(1+\frac{\rho^2}{1+\rho^2}\right)^2\mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\Sigma}) + 2\left(1+\frac{\rho^2}{1+\rho^2}\right)\times 2\rho^2\mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_2\boldsymbol{\Sigma}) + ...\right] \\
&= \frac{\sigma^4}{(1+\rho^2)^2}[(1+4\rho^2+\rho^4)DF - 2\rho^2 + 4\rho^4]. \qquad \text{(S2.5)}
\end{aligned}$$

The algebra required is lengthy but is made more manageable by use of the recursive relation $u_{n+1} = \rho^2(u_n + n)$. In fact, observe that (S2.5) is a linear function of D, which indicates that all of the non-linear terms u_n that are present in (S2.4) cancel out after repeated application of this relation. The same is also true for the two remaining traces, which are given by

$$\mathrm{tr}(\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}'\boldsymbol{\Sigma}) = -\frac{8\sigma^4\rho^2}{F(1-\rho^2)(1+\rho^2)^3}[(1+\rho^2+\rho^4)DF - 1 + \rho^2 + \rho^6]$$

and

$$\mathrm{tr}(\mathbf{M}'\boldsymbol{\Sigma}\mathbf{M}'\boldsymbol{\Sigma}) = \frac{8\sigma^4\rho^2}{F^2(1-\rho^2)^2(1+\rho^2)^4}[(1+2\rho^2+6\rho^4+2\rho^6+\rho^8)DF - 1+\rho^2-4\rho^4+8\rho^6+\rho^8-\rho^{10}].$$

Finally, using these results alongside (S2.1) and Lemma 1, we find that

$$\mathbf{J}(\sigma^2, \alpha) = \begin{bmatrix} \frac{1}{2\sigma^4(1+\rho^2)^2}\mathbf{j}_1^T(\mathbf{DF})^{(1)} & \frac{4\rho^2}{\sigma^2 F(1-\rho^2)(1+\rho^2)^3}\mathbf{j}_2^T(\mathbf{DF})^{(1)} \\ \frac{4\rho^2}{\sigma^2 F(1-\rho^2)(1+\rho^2)^3}\mathbf{j}_2^T(\mathbf{DF})^{(1)} & \frac{4\rho^2}{F^2(1-\rho^2)^2(1+\rho^2)^4}\mathbf{j}_3^T(\mathbf{DF})^{(1)} \end{bmatrix},$$

where $(\mathbf{DF})^{(k)} \equiv ((DF)^k, (DF)^{k-1}, ..., (DF)^0)^T$, and

$$\mathbf{j}_1 = \begin{bmatrix} 1+4\rho^2+\rho^4 \\ -2\rho^2+4\rho^4 \end{bmatrix}, \quad \mathbf{j}_2 = \begin{bmatrix} 1+\rho^2+\rho^4 \\ -1+\rho^2+\rho^6 \end{bmatrix}, \quad \mathbf{j}_3 = \begin{bmatrix} 1+2\rho^2+6\rho^4+2\rho^6+\rho^8 \\ -1+\rho^2-4\rho^4+8\rho^6+\rho^8-\rho^{10} \end{bmatrix}.$$

Out of completeness, we provide the expression for $\mathbf{G}(\sigma^2, \alpha)^{-1}$ here, which follows from standard matrix algebra:

$$\begin{aligned}\mathbf{G}(\sigma^2,\alpha)^{-1} &= \mathbf{H}(\sigma^2,\alpha)^{-1}\mathbf{J}(\sigma^2,\alpha)\mathbf{H}(\sigma^2,\alpha)^{-1}\\ &= \frac{1}{(\mathbf{g}_4^T(\mathbf{DF})^{(2)})^2}\begin{bmatrix} \frac{2\sigma^4}{1-\rho^2}\mathbf{g}_1^T(\mathbf{DF})^{(3)} & -2\sigma^2 F\mathbf{g}_2^T(\mathbf{DF})^{(3)} \\ -2\sigma^2 F\mathbf{g}_2^T(\mathbf{DF})^{(3)} & \frac{F^2(1-\rho^2)}{\rho^2}\mathbf{g}_3^T(\mathbf{DF})^{(3)}\end{bmatrix},\end{aligned}$$

where

$$\mathbf{g}_1 = \begin{bmatrix}(1-\rho^2)^3(1+\rho^4) \\ 2(-1+8\rho^2-11\rho^4+15\rho^6-4\rho^8+\rho^{10}) \\ (1-\rho^2)(1-18\rho^2+26\rho^4-42\rho^6+\rho^8) \\ 2\rho^2(1-\rho^2)^2(3-5\rho^2+10\rho^4)\end{bmatrix}, \quad \mathbf{g}_2 = \begin{bmatrix}(1-\rho^2)^3 \\ -2+15\rho^2-17\rho^4+13\rho^6-\rho^8 \\ (1-\rho^2)(1-17\rho^2+13\rho^4-9\rho^6) \\ 2\rho^2(1-\rho^2)^2(3-2\rho^2)\end{bmatrix},$$

$$\mathbf{g}_3 = \begin{bmatrix}(1-\rho^2)^3 \\ -1+12\rho^2-16\rho^4+12\rho^6+\rho^8 \\ -2\rho^2(1-\rho^2)(3-8\rho^2+3\rho^4) \\ 4\rho^4(1-\rho^2)(-1+2\rho^2)\end{bmatrix}, \qquad \mathbf{g}_4 = \begin{bmatrix}(1-\rho^2)^2 \\ -1+8\rho^2-3\rho^4 \\ -4\rho^2(1-\rho^2)\end{bmatrix}.$$

S3 Proof of Theorem 4

Following on from (4.5), the first and second-order partial derivatives of the composite log-likelihood are

$$\frac{\partial}{\partial\boldsymbol{\theta}}c\ell(\sigma^2,\alpha;\mathbf{y}) = \begin{bmatrix} -\frac{DF}{2\sigma^2}+\frac{1}{2\sigma^4}\mathbf{y}^T\mathbf{M}\mathbf{y} \\ -\frac{(DF-B)\rho^2}{F(1-\rho^2)}-\frac{1}{2\sigma^2}\mathbf{y}^T\mathbf{M}'\mathbf{y}\end{bmatrix} \tag{S3.1}$$

and

$$-\frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}c\ell(\sigma^2,\alpha;\mathbf{y}) = \begin{bmatrix} -\frac{DF}{2\sigma^4}+\frac{1}{\sigma^6}\mathbf{y}^T\mathbf{M}\mathbf{y} & -\frac{1}{2\sigma^4}\mathbf{y}^T\mathbf{M}'\mathbf{y} \\ -\frac{1}{2\sigma^4}\mathbf{y}^T\mathbf{M}'\mathbf{y} & -\frac{2(DF-B)\rho^2}{F^2(1-\rho^2)^2}+\frac{1}{2\sigma^2}\mathbf{y}^T\mathbf{M}''\mathbf{y} \end{bmatrix}. \tag{S3.2}$$

In order to find the various traces involving $\mathbf{M}$ (and its derivatives) and $\boldsymbol{\Sigma}$, we can make use of the block-diagonality of $\mathbf{M}$. First, let $\mathbf{M} = \mathbf{M}_{(1)} + \mathbf{M}_{(2)} + ... + \mathbf{M}_{(B)}$, where $\mathbf{M}_{(b)}$ is an $N \times N$ matrix containing only the b-th block of $\mathbf{M}$ (with all other elements set to zero). Also, break down the structure of $\boldsymbol{\Sigma} \in \mathbb{R}^{DF\times DF}$ into blocks of size $W \times W$ as follows:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathbf{S}_{(0)} & \mathbf{S}_{(1)} & \mathbf{S}_{(2)} & \cdots & \mathbf{S}_{(B-2)} & \mathbf{S}_{(B-1)} \\ \mathbf{S}_{(-1)} & \mathbf{S}_{(0)} & \mathbf{S}_{(1)} & \cdots & \mathbf{S}_{(B-3)} & \mathbf{S}_{(B-2)} \\ \mathbf{S}_{(-2)} & \mathbf{S}_{(-1)} & \mathbf{S}_{(0)} & \cdots & \mathbf{S}_{(B-4)} & \mathbf{S}_{(B-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{S}_{(-(B-2))} & \mathbf{S}_{(-(B-3))} & \mathbf{S}_{(-(B-4))} & \cdots & \mathbf{S}_{(0)} & \mathbf{S}_{(1)} \\ \mathbf{S}_{(-(B-1))} & \mathbf{S}_{(-(B-2))} & \mathbf{S}_{(-(B-3))} & \cdots & \mathbf{S}_{(-1)} & \mathbf{S}_{(0)} \end{bmatrix},$$

where

$$\mathbf{S}_{(k)} = \sigma^2 \begin{bmatrix} \rho^{|Wk|} & \rho^{|Wk+1|} & \rho^{|Wk+2|} & \cdots & \rho^{|W(k+1)-2|} & \rho^{|W(k+1)-1|} \\ \rho^{|Wk-1|} & \rho^{|Wk|} & \rho^{|Wk+1|} & \cdots & \rho^{|W(k+1)-3|} & \rho^{|W(k+1)-2|} \\ \rho^{|Wk-2|} & \rho^{|Wk-1|} & \rho^{|Wk|} & \cdots & \rho^{|W(k+1)-4|} & \rho^{|W(k+1)-3|} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^{|W(k-1)+2|} & \rho^{|W(k-1)+3|} & \rho^{|W(k-1)+4|} & \cdots & \rho^{|Wk|} & \rho^{|Wk+1|} \\ \rho^{|W(k-1)+1|} & \rho^{|W(k-1)+2|} & \rho^{|W(k-1)+3|} & \cdots & \rho^{|Wk-1|} & \rho^{|Wk|} \end{bmatrix}.$$

Then in order to compute $E[\mathbf{y}^T\mathbf{M}\mathbf{y}]$ for instance, we have

$$\mathrm{tr}(\mathbf{M}\boldsymbol{\Sigma}) = \sum_{b=1}^{B} \mathrm{tr}(\mathbf{M}_{(b)}\boldsymbol{\Sigma}) = \sum_{b=1}^{B} \mathrm{tr}(\mathbf{Q}\mathbf{S}_{(0)}) = B\,\mathrm{tr}(\mathbf{Q}\mathbf{S}_{(0)}).$$

This reduces the problem down to using the additive decomposition $\mathbf{Q} = (\mathbf{A}_1+\rho^2\mathbf{A}_2-\rho\mathbf{A}_4)/(1-\rho^2)$ alongside similar expressions to (S2.3): $\mathrm{tr}(\mathbf{A}_1\mathbf{S}_{(0)}) = \sigma^2 W$, $\mathrm{tr}(\mathbf{A}_2\mathbf{S}_{(0)}) = \sigma^2(W-2)$ and $\mathrm{tr}(\mathbf{A}_4\mathbf{S}_{(0)}) = 2\sigma^2\rho(W-1)$. We can therefore evaluate the expectation of (S3.2) to obtain

$$\mathbf{H}(\sigma^2, \alpha) = \begin{bmatrix} \frac{DF}{2\sigma^4} & \frac{\rho^2(DF-B)}{F\sigma^2(1-\rho^2)} \\ \frac{\rho^2(DF-B)}{F\sigma^2(1-\rho^2)} & \frac{\rho^2(1+\rho^2)(DF-B)}{F^2(1-\rho^2)^2} \end{bmatrix}.$$

Next, for the traces of the four-matrix products $\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}\boldsymbol{\Sigma}$, $\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}'\boldsymbol{\Sigma}$ and $\mathbf{M}'\boldsymbol{\Sigma}\mathbf{M}'\boldsymbol{\Sigma}$, it is useful to observe that $\mathbf{S}_{(k)} = \rho^{W(k-1)}\mathbf{S}_{(1)}$ and $\mathbf{S}_{(-k)} =$

$\rho^{W(k-1)}\mathbf{S}_{(-1)}$ for $k \geq 1$. Then, for instance,

$$\begin{aligned}
\operatorname{tr}(\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}'\boldsymbol{\Sigma}) &= \sum_{b=1}^{B}\sum_{c=1}^{B} \operatorname{tr}(\mathbf{M}_{(b)}\boldsymbol{\Sigma}\mathbf{M}'_{(c)}\boldsymbol{\Sigma}) = \sum_{b=1}^{B}\sum_{c=1}^{B} \operatorname{tr}(\mathbf{Q}\mathbf{S}_{(c-b)}\mathbf{Q}'\mathbf{S}_{(b-c)}) \\
&= B\operatorname{tr}(\mathbf{Q}\mathbf{S}_{(0)}\mathbf{Q}'\mathbf{S}_{(0)}) + \sum_{1\leq b<c\leq B} \operatorname{tr}(\mathbf{Q}\mathbf{S}_{(c-b)}\mathbf{Q}'\mathbf{S}_{(b-c)}) + \sum_{1\leq c<b\leq B} \operatorname{tr}(\mathbf{Q}\mathbf{S}_{(c-b)}\mathbf{Q}'\mathbf{S}_{(b-c)}) \\
&= B\operatorname{tr}(\mathbf{Q}\mathbf{S}_{(0)}\mathbf{Q}'\mathbf{S}_{(0)}) + \sum_{a=1}^{B-1}(B-a)\operatorname{tr}(\mathbf{Q}\mathbf{S}_{(a)}\mathbf{Q}'\mathbf{S}_{(-a)}) + \sum_{a=1}^{B-1}(B-a)\operatorname{tr}(\mathbf{Q}\mathbf{S}_{(-a)}\mathbf{Q}'\mathbf{S}_{(a)}) \\
&= B\operatorname{tr}(\mathbf{Q}\mathbf{S}_{(0)}\mathbf{Q}'\mathbf{S}_{(0)}) + \big(\operatorname{tr}(\mathbf{Q}\mathbf{S}_{(1)}\mathbf{Q}'\mathbf{S}_{(-1)}) + \operatorname{tr}(\mathbf{Q}\mathbf{S}_{(-1)}\mathbf{Q}'\mathbf{S}_{(1)})\big)\sum_{a=1}^{B-1}(B-a)\rho^{2W(a-1)} \\
&= B\operatorname{tr}(\mathbf{Q}\mathbf{S}_{(0)}\mathbf{Q}'\mathbf{S}_{(0)}) + \big(\operatorname{tr}(\mathbf{Q}\mathbf{S}_{(1)}\mathbf{Q}'\mathbf{S}_{(-1)}) + \operatorname{tr}(\mathbf{Q}\mathbf{S}_{(-1)}\mathbf{Q}'\mathbf{S}_{(1)})\big)\frac{1}{1-\rho^{2W}}\left(B - \frac{1-\rho^{2DF}}{1-\rho^{2W}}\right) \\
&= B\operatorname{tr}(\mathbf{Q}\mathbf{S}_{(0)}\mathbf{Q}'\mathbf{S}_{(0)}) + \operatorname{tr}(\mathbf{Q}\mathbf{S}_{(1)}\mathbf{Q}'\mathbf{S}_{(-1)})\frac{2}{1-\rho^{2W}}\left(B - \frac{1-\rho^{2DF}}{1-\rho^{2W}}\right).
\end{aligned}$$

The final line can be obtained by observing that $\mathbf{Q}$ and $\mathbf{Q}'$ are rotationally symmetric by 180 degrees; that is, for the antidiagonal identity matrix $\mathbf{R}$ (a square matrix with a diagonal of 1s from the top-right to the bottom-left), we have $\mathbf{Q} = \mathbf{RQR}$ and $\mathbf{Q}' = \mathbf{RQ}'\mathbf{R}$. Additionally, $\mathbf{S}_{(1)}$ and $\mathbf{S}_{(-1)}$ are 180-degree rotations of each other, such that $\mathbf{S}_{(1)} = \mathbf{R}\mathbf{S}_{(-1)}\mathbf{R}$ and $\mathbf{S}_{(-1)} = \mathbf{R}\mathbf{S}_{(1)}\mathbf{R}$. The equality $\operatorname{tr}(\mathbf{Q}\mathbf{S}_{(1)}\mathbf{Q}'\mathbf{S}_{(-1)}) = \operatorname{tr}(\mathbf{Q}\mathbf{S}_{(-1)}\mathbf{Q}'\mathbf{S}_{(1)})$ then follows from cyclical invariance. Note that the traces of $\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}\boldsymbol{\Sigma}$ and $\mathbf{M}'\boldsymbol{\Sigma}\mathbf{M}'\boldsymbol{\Sigma}$ may be derived similarly, but the rotational symmetry argument can be replaced with a direct use of cyclical invariance to prove the equality.

By once again using the additive decomposition of $\mathbf{Q}$, we now require traces of four-matrix products involving $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_4$ and $\mathbf{S}_{(0)}$ or $\mathbf{S}_{(1)}$. These are summarised below with use of the notation $u_n \equiv \sum_{k=1}^{n}(n-k)\rho^{2k}$, $v_n \equiv \sum_{k=1}^{2n-1}(n-|n-k|)\rho^{2k}$ and $q_n \equiv \sum_{k=1}^{n}\rho^{2k}$:

$$\mathrm{tr}(\mathbf{A}_1\mathbf{S}_{(0)}\mathbf{A}_1\mathbf{S}_{(0)}) = \sigma^4[W+2u_W], \qquad \mathrm{tr}(\mathbf{A}_1\mathbf{S}_{(0)}\mathbf{A}_2\mathbf{S}_{(0)}) = \sigma^4[W-2+2u_{W-1}],$$

$$\mathrm{tr}(\mathbf{A}_1\mathbf{S}_{(0)}\mathbf{A}_4\mathbf{S}_{(0)}) = 4\sigma^4[\rho(W-1)+\rho u_{W-1}], \qquad \mathrm{tr}(\mathbf{A}_2\mathbf{S}_{(0)}\mathbf{A}_2\mathbf{S}_{(0)}) = \sigma^4[W-2+2u_{W-2}],$$

$$\mathrm{tr}(\mathbf{A}_2\mathbf{S}_{(0)}\mathbf{A}_4\mathbf{S}_{(0)}) = 4\sigma^4[\rho(W-2)+\rho u_{W-2}], \qquad \mathrm{tr}(\mathbf{A}_4\mathbf{S}_{(0)}\mathbf{A}_4\mathbf{S}_{(0)}) = \sigma^4[2(1+\rho^2)(W-1)+8u_{W-1}],$$

$$\mathrm{tr}(\mathbf{A}_1\mathbf{S}_{(1)}\mathbf{A}_1\mathbf{S}_{(-1)}) = \sigma^4 v_W, \qquad \mathrm{tr}(\mathbf{A}_1\mathbf{S}_{(1)}\mathbf{A}_2\mathbf{S}_{(-1)}) = \sigma^4[\rho^2 v_{W-1}-\rho^{2W}],$$

$$\mathrm{tr}(\mathbf{A}_1\mathbf{S}_{(1)}\mathbf{A}_4\mathbf{S}_{(-1)}) = 2\sigma^4[\rho v_W - \rho^{2W-1}q_W], \qquad \mathrm{tr}(\mathbf{A}_2\mathbf{S}_{(1)}\mathbf{A}_2\mathbf{S}_{(-1)}) = \sigma^4\rho^4 v_{W-2},$$

$$\mathrm{tr}(\mathbf{A}_2\mathbf{S}_{(1)}\mathbf{A}_4\mathbf{S}_{(-1)}) = 2\sigma^4[\rho^3 v_{W-1} - \rho^{2W-1}q_{W-1}], \qquad \mathrm{tr}(\mathbf{A}_4\mathbf{S}_{(1)}\mathbf{A}_4\mathbf{S}_{(-1)}) = 4\sigma^4\rho^2 v_{W-1}.$$

Expressions for the traces of $\mathbf{QS}_{(0)}\mathbf{QS}_{(0)}$, $\mathbf{QS}_{(0)}\mathbf{Q}'\mathbf{S}_{(0)}$ and $\mathbf{Q}'\mathbf{S}_{(0)}\mathbf{Q}'\mathbf{S}_{(0)}$ can be obtained by repeatedly applying $u_{n+1} = \rho^2(u_n+n)$ to cancel out the u_n terms in a similar manner to (S2.5). For the traces of $\mathbf{QS}_{(1)}\mathbf{QS}_{(1)}$, $\mathbf{QS}_{(1)}\mathbf{Q}'\mathbf{S}_{(1)}$ and $\mathbf{Q}'\mathbf{S}_{(1)}\mathbf{Q}'\mathbf{S}_{(1)}$, we can instead make use of the relation $v_{n+1} = v_n + 2\rho^{2n}q_n + \rho^{2(2n+1)}$ to cancel out the v_n terms. This results in the expressions $\mathrm{tr}(\mathbf{M\Sigma M'\Sigma}) = -2\sigma^4\rho^2(DF-B)/[F(1-\rho^2)]$, $\mathrm{tr}(\mathbf{M'\Sigma M'\Sigma}) = 2\sigma^4\rho^2(1+\rho^2)(DF-B)/[F^2(1-\rho^2)^2]$ and

$$\mathrm{tr}(\mathbf{M\Sigma M\Sigma}) = \sigma^4\left[DF + \frac{2\rho^2}{1-\rho^{2W}}\left(B - \frac{1-\rho^{2DF}}{1-\rho^{2W}}\right)\right].$$

Hence, using (S3.1) and applying Lemma 1 gives

$$\mathbf{J}(\sigma^2,\alpha) = \begin{bmatrix} \frac{1}{2\sigma^4}\left[DF + \frac{2\rho^2}{1-\rho^{2W}}\left(B - \frac{1-\rho^{2DF}}{1-\rho^{2W}}\right)\right] & \frac{\rho^2}{\sigma^2 F(1-\rho^2)}(DF-B) \\ \frac{\rho^2}{\sigma^2 F(1-\rho^2)}(DF-B) & \frac{\rho^2(1+\rho^2)}{F^2(1-\rho^2)^2}(DF-B) \end{bmatrix}$$

$$= \mathbf{H}(\sigma^2,\alpha) + \begin{bmatrix} \frac{\rho^2}{\sigma^4(1-\rho^{2W})}\left(B - \frac{1-\rho^{2DF}}{1-\rho^{2W}}\right) & 0 \\ 0 & 0 \end{bmatrix}.$$

We observe that there is a small perturbation term between $\mathbf{J}(\sigma^2,\alpha)$ and $\mathbf{H}(\sigma^2,\alpha)$, which is consistent with (3.1) and (3.2) in the general Gaussian case for unweighted composite likelihood functions. Also note that if $B = 1$ (and $W = DF$), then $\mathbf{J}(\sigma^2,\alpha) = \mathbf{H}(\sigma^2,\alpha)$, which is as expected for the full likelihood.

Following standard matrix algebra, it can be shown that

$$\mathbf{G}(\sigma^2,\alpha)^{-1} = \frac{1}{(1-\rho^2)DF + 2\rho^2 B}\begin{bmatrix} 2(\sigma^2)^2(1+\rho^2) & -2\sigma^2 F(1-\rho^2) \\ -2\sigma^2 F(1-\rho^2) & F^2\frac{DF(1-\rho^2)^2}{(DF-B)\rho^2} \end{bmatrix}$$
$$+ \frac{4\rho^2}{(1-\rho^{2W})((1-\rho^2)DF + 2\rho^2 B)^2}\left(B - \frac{1-\rho^{2DF}}{1-\rho^{2W}}\right)\begin{bmatrix} (\sigma^2)^2(1+\rho^2)^2 & -\sigma^2 F(1-\rho^4) \\ -\sigma^2 F(1-\rho^4) & F^2(1-\rho^2)^2 \end{bmatrix}. \tag{S3.3}$$

S4 Asymptotics for a Constant Mean Parameter

In this section, we extend our results for the one-dimensional Gaussian exponential covariance process to allow for a constant mean μ. Due to the orthogonality between the mean parameter and the covariance parameters under a Gaussian process, the asymptotic relative efficiency for μ can be derived separately from σ^2 and α.

To begin, note that the full/composite log-likelihood can be written in the form $c\ell(\mu; \sigma^2, \alpha, \mathbf{y}) = q(\sigma^2, \alpha) - (\mathbf{y} - \mu\mathbf{1}_{DF})^T\mathbf{M}(\mathbf{y} - \mu\mathbf{1}_{DF})/(2\sigma^2)$, where $\mathbf{M}$ is equal to $\sigma^2\mathbf{\Sigma}^{-1}$ (with $\mathbf{\Sigma}^{-1}$ as defined in (4.1)) for the full likelihood, (4.4) for the composite full conditional likelihood and (4.5) for the composite marginal block likelihood. This emits the following first and second-order derivatives:

$$\frac{\partial}{\partial\mu} c\ell(\mu; \sigma^2, \alpha, \mathbf{y}) = \frac{1}{\sigma^2}\mathbf{1}_{DF}^T\mathbf{M}(\mathbf{y} - \mu\mathbf{1}_{DF})$$

and

$$-\frac{\partial^2}{\partial\mu^2} c\ell(\mu; \sigma^2, \alpha, \mathbf{y}) = \frac{1}{\sigma^2}\mathbf{1}_{DF}^T\mathbf{M}\mathbf{1}_{DF}$$

S4.1 Full Likelihood

The inverse of the Fisher information is given by

$$I(\mu)^{-1} = \left\{\mathbb{E}\left[-\frac{\partial^2}{\partial\mu^2}\ell(\mu; \sigma^2, \alpha, \mathbf{y})\right]\right\}^{-1} = \left\{\mathbf{1}_{DF}^T\mathbf{\Sigma}^{-1}\mathbf{1}_{DF}\right\}^{-1} = \frac{\sigma^2(1+\rho)}{(1-\rho)DF + 2\rho}.$$

It should be noted that $\hat{\mu}_{\mathrm{ML}}$ is not consistent under infill when σ^2 and α are known, because $I(\mu)^{-1} = \mathrm{var}(\hat{\mu}_{\mathrm{ML}})$ (i.e., it is the exact finite sample variance), and $\lim_{F\to\infty} I(\mu)^{-1} = 2\sigma^2/(\alpha D + 2) > 0$.

S4.2 Composite Full Conditional Likelihood

To derive the sandwich covariance $G(\mu)^{-1} = H(\mu)^{-1}J(\mu)H(\mu)^{-1}$, we begin by evaluating $H(\mu)$ as follows:

$$\begin{aligned} H(\mu) &= \mathbb{E}\left[-\frac{\partial^2}{\partial\mu^2} c\ell(\mu;\sigma^2,\alpha,\mathbf{y})\right] = \frac{1}{\sigma^2}\mathbf{1}_{DF}^T\mathbf{M}\mathbf{1}_{DF} \\ &= \frac{1}{\sigma^2(1-\rho^2)}\left[\frac{1+2\rho^2}{1+\rho^2}DF + 2\rho^2(DF-2) - \frac{\rho^4}{1+\rho^2}(DF-4)\right. \\ &\qquad\qquad\qquad\qquad \left. - 2\rho\times 2(DF-1) + \frac{\rho^2}{1+\rho^2}2(DF-2)\right] \\ &= \frac{1-\rho}{\sigma^2(1+\rho)(1+\rho^2)}[(1-\rho^2)DF + 4\rho]. \end{aligned}$$

Next, note that

$$J(\mu) = \mathbb{E}\left[\left(\frac{\partial}{\partial\mu} c\ell(\mu;\sigma^2,\alpha,\mathbf{y})\right)^2\right] = \frac{1}{\sigma^4}\mathbf{1}_{DF}^T\mathbf{M}\boldsymbol{\Sigma}\mathbf{M}\mathbf{1}_{DF}.$$

Our approach for evaluating $J(\mu)$ is to derive the vector quantity $\mathbf{M}\mathbf{1}_{DF}$ and then subsequently exploit the simple structure of this vector to compute the quadratic form $(\mathbf{M}\mathbf{1}_{DF})^T\boldsymbol{\Sigma}(\mathbf{M}\mathbf{1}_{DF})$. In particular, it can be shown that $\mathbf{M}\mathbf{1}_{DF} = (m_1, m_2, \underbrace{m_3, ..., m_3}_{DF-4 \text{ times}}, m_2, m_1)^T/[(1+\rho)(1+\rho^2)]$, where $m_1 =$

$1-\rho+2\rho^2$, $m_2 = 1-3\rho+2\rho^2-2\rho^3$ and $m_3 = (1-\rho)^3$. As such,

$$\begin{aligned}
J(\mu) &= \frac{1}{\sigma^4}\sum_{i=1}^{DF}\sum_{j=1}^{DF}\Sigma_{ij}(\mathbf{M1}_{DF})_i(\mathbf{M1}_{DF})_j \\
&= \frac{1}{\sigma^2(1+\rho)^2(1+\rho^2)^2}\Bigg[m_1^2\sum_{i\in\{1,DF\}}\sum_{j\in\{1,DF\}}\rho^{|i-j|} + 2m_1m_2\sum_{i\in\{1,DF\}}\sum_{j\in\{2,DF-1\}}\rho^{|i-j|} \\
&\quad + 2m_1m_3\sum_{i\in\{1,DF\}}\sum_{j=3}^{DF-2}\rho^{|i-j|} + m_2^2\sum_{i\in\{2,DF-1\}}\sum_{j\in\{2,DF-1\}}\rho^{|i-j|} \\
&\quad + 2m_2m_3\sum_{i\in\{2,DF-1\}}\sum_{j=3}^{DF-2}\rho^{|i-j|} + m_3^2\sum_{i=3}^{DF-2}\sum_{j=3}^{DF-2}\rho^{|i-j|}\Bigg] \\
&= \frac{1}{\sigma^2(1+\rho)^2(1+\rho^2)^2}\Bigg[2m_1^2(1+\rho^{DF-1}) + 4m_1m_2(\rho+\rho^{DF-2}) \\
&\quad + 4m_1m_3\rho^2\frac{1-\rho^{DF-4}}{1-\rho} + 2m_2^2(1+\rho^{DF-3}) \\
&\quad + 4m_2m_3\rho\frac{1-\rho^{DF-4}}{1-\rho} + m_3^2\left(DF-4+2\sum_{k=1}^{DF-4}(DF-4-k)\rho^k\right)\Bigg].
\end{aligned}$$

After working through the algebra and using the fact that

$$\sum_{k=1}^{K} k\rho^k = \frac{\rho}{1-\rho}\left(\frac{1-\rho^K}{1-\rho} - K\rho^K\right), \tag{S4.4}$$

the expression for $J(\mu)$ can be simplified to

$$J(\mu) = \frac{1-\rho}{\sigma^2(1+\rho)(1+\rho^2)^2}[(1-\rho)^4DF + 2\rho(3-4\rho+3\rho^2+2\rho^3)].$$

Thus,

$$G(\mu)^{-1} = \frac{\sigma^2(1+\rho)[(1-\rho)^4DF + 2\rho(3-4\rho+3\rho^2+2\rho^3)]}{(1-\rho)[(1-\rho)^4(DF)^2 + 8\rho(1-\rho)^2DF + 16\rho^2]}.$$

The expanding domain asymptotic relative efficiency of the composite full conditional likelihood estimator for μ is 1 for $\rho < 1$. However, under infill asymptotics ($F \to \infty$), $G(\mu)^{-1}$ diverges to ∞. This once again highlights the structural instability of the composite full conditional likelihood under this model.

S4.3 Composite Marginal Block Likelihood

Noting that we can express $\mathbf{M} = \text{diag}(\underbrace{\mathbf{Q}, ..., \mathbf{Q}}_{B \text{ times}})$, where $\mathbf{Q}$ is a matrix of size $W \times W$, we have

$$\begin{aligned} H(\mu) &= \mathbb{E}\left[-\frac{\partial^2}{\partial \mu^2} c\ell(\mu; \sigma^2, \alpha, \mathbf{y})\right] = \frac{1}{\sigma^2}\mathbf{1}_{DF}^T \mathbf{M} \mathbf{1}_{DF} \\ &= \frac{B}{\sigma^2}\mathbf{1}_W^T \mathbf{Q} \mathbf{1}_W = \frac{1}{\sigma^2(1+\rho)}[(1-\rho)DF + 2\rho B]. \end{aligned}$$

The approach that we take to derive $J(\mu)$ is to rewrite $\mathbf{M}$ in the form $\mathbf{M} = \sigma^2 \mathbf{\Sigma}^{-1} - \mathbf{T}$, where

$$\mathbf{T}_{ij} = \frac{1}{1-\rho^2} \times \begin{cases} \rho^2, & j = i \in \{Wk, Wk+1\}, \\ -\rho, & j = i+1,\ i = Wk, \\ -\rho, & i = j+1,\ j = Wk, \\ 0, & \text{otherwise.} \end{cases}$$

and $k \in \{1, 2, ..., B-1\}$. Using this expression, we obtain

$$\begin{aligned} J(\mu) &= \frac{1}{\sigma^4} \mathbf{1}_{DF}^T (\sigma^2 \mathbf{\Sigma}^{-1} - \mathbf{T}) \mathbf{\Sigma} (\sigma^2 \mathbf{\Sigma}^{-1} - \mathbf{T}) \mathbf{1}_{DF} \\ &= \frac{1}{\sigma^4} (\sigma^4 \mathbf{1}_{DF}^T \mathbf{\Sigma}^{-1} \mathbf{1}_{DF} - 2\sigma^2 \mathbf{1}_{DF}^T \mathbf{T} \mathbf{1}_{DF} + \mathbf{1}_{DF}^T \mathbf{T} \mathbf{\Sigma} \mathbf{T} \mathbf{1}_{DF}) \\ &= \frac{(1-\rho)DF + 2\rho}{\sigma^2(1+\rho)} + \frac{4(B-1)\rho}{\sigma^2(1+\rho)} + \frac{\rho^2}{\sigma^4(1+\rho)^2} \boldsymbol{\tau}^T \mathbf{\Sigma} \boldsymbol{\tau}, \end{aligned}$$

where $\boldsymbol{\tau}$ is a DF-dimensional vector with $\tau_i = 1$ for $i \in \{Wk, Wk+1\}$, $k \in \{1, 2, ..., B-1\}$ and $\tau_i = 0$ otherwise. In a descriptive sense, the quantity $\boldsymbol{\tau}^T \mathbf{\Sigma} \boldsymbol{\tau}$ can be calculated by partitioning the matrix $\mathbf{\Sigma}$ into B^2 blocks of size $W \times W$ and only summing up the entries that are located in the corners where any four blocks intersect. As such,

$$\begin{aligned} \boldsymbol{\tau}^T \mathbf{\Sigma} \boldsymbol{\tau} &= \sigma^2 \left[2(B-1)(1+\rho) + \sum_{k=1}^{B-2} 2(B-1-k)(\rho^{Wk-1} + 2\rho^{Wk} + \rho^{Wk+1}) \right] \\ &= \sigma^2 \left[2(B-1)(1+\rho) + 2\rho^{-1}(1+\rho)^2 \sum_{k=1}^{B-2} (B-1-k)\rho^{Wk} \right]. \end{aligned}$$

Following further algebraic manipulation and using (S4.4), we obtain

$$J(\mu) = \frac{1}{\sigma^2} \left[\frac{1}{1+\rho} [(1-\rho)DF + 2\rho B] + \frac{2\rho}{1-\rho^W} \left(B - \frac{1-\rho^{DF}}{1-\rho^W} \right) \right].$$

Thus,

$$G(\mu)^{-1} = \frac{\sigma^2(1+\rho)}{(1-\rho)DF + 2\rho B} \left[1 + \frac{2\rho(1+\rho)[B - (1-\rho^{DF})/(1-\rho^W)]}{(1-\rho^W)[(1-\rho)DF + 2\rho B]} \right].$$

Interestingly, we observe that $2\sigma^2 I(\mu)^{-1}|_{\rho=x} = \{\mathbf{I}(\sigma^2, \alpha)^{-1}\}_{11}|_{\rho=\sqrt{x}}$, and $2\sigma^2 G(\mu)^{-1}|_{\rho=x} = \{\mathbf{G}(\sigma^2, \alpha)^{-1}\}_{11}|_{\rho=\sqrt{x}}$ as per (S3.3). Hence, assuming a fixed W, the expanding domain asymptotic relative efficiency of the composite marginal block likelihood estimator for μ is given by

$$\text{EDARE}(\hat{\mu}_{\text{CL}}, \hat{\mu}_{\text{ML}}) \equiv \lim_{D\to\infty} \frac{I(\mu)^{-1}}{G(\mu)^{-1}} = \lim_{D\to\infty} \left. \frac{\{\mathbf{I}(\sigma^2, \alpha)^{-1}\}_{11}}{\{\mathbf{G}(\sigma^2, \alpha)^{-1}\}_{11}} \right|_{\rho=\sqrt{\rho}}$$

$$= \frac{(1-\rho+2\rho/W)/(1-\rho)}{1+[2\rho(1+\rho)]/[W(1-\rho^W)(1-\rho+2\rho/W)]}.$$

References

Rencher, A. C. and Schaalje, G. B. (2008). *Linear models in statistics.* Hoboken, NJ: John Wiley and Sons.