DYNAMIC PENALIZED SPLINES FOR STREAMING DATA

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Supplementary Material

This is the supplementary material for dynamic penalized splines, which includes auxiliary lemmas and detailed proofs to the main theorems.

S1 Auxiliary Lemmas

In the proofs to our theorems, we need the following lemma from Nirenberg (2011), which is known as Gagliardo-Nirenberg interpolation inequality.

Lemma 1. *Fix $1 \le p, q, r \le \infty,\ s > 0$ and natural number m, j, if there is a real number α such that*

$$\frac{1}{p} = j + \left(\frac{1}{r} - m\right)\alpha + \frac{1-\alpha}{q},\ \frac{j}{m} \le \alpha < 1,$$

then there are constants C_1, C_2 such that for all functions $g : (0,1) \to \mathbb{R}$,

$$\|g^{(j)}\|_p \le C_1 \|g^{(m)}\|_r^{\alpha} \|g\|_q^{1-\alpha} + C_2 \|g\|_s$$

whenever both sides are well defined.

Together with the inequality $x^p \le 1-p+px$ where $p \in (0,1]$ and $x \ge 0$, we have the following corollary.

Corollary 1. *Let q be some positive integer, λ, n be positive numbers, non-negative numbers α, β, γ be such that $\alpha+\beta+\gamma=2$, and $g \in H^q((0,1))$ where $H^q((0,1))$ denote the Sobolev space, then*

$$\begin{aligned} &\|g^{(q)}\|_2^{\alpha}\|g'\|_2^{\beta}\|g\|_2^{\gamma} \\ \le& 2\{C_1 w_1^{w_1}(1-w_1)^{1-w_1}\lambda^{-w_1}n^{w_1-1} \\ &+ C_2 w_2^{w_2}(1-w_2)^{1-w_2}\lambda^{-w_2}n^{w_2-1}\} \times \left\{\lambda\|g^{(m)}\|_2^2 + n\|g\|_2^2\right\}, \end{aligned} \tag{S1.1}$$

where $w_1 = (q\alpha+\beta)/(2q)$ and $w_2 = \alpha/2$.

We also need the next lemma, which is part of Theorem 6.25, Section 6.4 of Schumaker (2007).

Lemma 2. *Let $f \in C^l([0,1])$ with $1 \le q \le \infty$ and $1 \le l \le p$. Let $\kappa = \{0 = \kappa_1 \le \cdots \le \kappa_K = 1\} \subseteq [0,1]$. Then there exists $s \in \mathbb{S}_{\kappa,p+1}$ such that*

$$\left\|f^{(r)} - s^{(r)}\right\|_q \le C\Delta^{l-r}\left\|f^{(l)}\right\|_q, \quad 0 \le r \le l-1$$

and

$$\left\|s^{(p)}\right\|_\infty \le C\omega_1(f^{(p)}, \Delta)_\infty$$

where

$$\omega_1(f^{(p)}, \Delta)_\infty = \sup\{|f^{(p)}(x) - f^{(p)}(y)| \,|\, 0 \le x, y \le 1, |x-y| \le \Delta\},$$

$\Delta = \max_j |\kappa_{j+1} - \kappa_j|$ *and* C *is a constant depending only on* p*.*

S2 Proof of Theorem 1

For simplicity, let C^i, H^i and L^2 denote $C^i([0,1])$, $H^i((0,1))$ and $L^2((0,1))$. The idea of our proof roots from Munteanu (1973). Let Z be the Hilbert space $L^2 \times \mathbb{R}^n$, with inner product defined by

$$\langle (g_1, z_{11}, \ldots, z_{1n}), (g_2, z_{21}, \ldots, z_{2n}) \rangle_Z = \lambda_n \int_0^1 g_1(x) g_2(x) dx + \sum_{i=1}^n z_{1i} z_{2i}.$$

Let $L : H^q \to Z$ be the bounded linear map given by

$$Lg = \left(g^{(q)}, g(x_1), \ldots, g(x_n)\right).$$

Note that $L(H^q)$ and $L\mathbb{S}_{\kappa_j, p+1},\ j = 1, \ldots, n$ are closed subspace of Z. Let $h = (0, y_1, \ldots, y_n) \in Z$, then $L\hat{f}_n$ is the orthogonal projection from h to $L\mathbb{S}_{\kappa_n, p+1}$. Let G be the injection from H^q to L^2. We need to give a upper bound for $\left\|Gf - G\hat{f}_n\right\|$. To begin with, see that

$$\left\|Gf_0 - G\hat{f}_n\right\| = \left\|Lf_0 - L\hat{f}_n\right\| \cdot \frac{\left\|Gf_0 - G\hat{f}_n\right\|}{\left\|Lf_0 - L\hat{f}_n\right\|} \le \left\|Lf_0 - L\hat{f}_n\right\| \sup_{g \in H^q} \frac{\|Gg\|}{\|Lg\|}.$$

Firstly consider $\sup_{g\in H^q} \|g\|_2/\|Lg\|$, which is

$$\sup_{g\in H^q} \frac{\|g\|_2}{\left\{\lambda_n \|g^{(q)}\|_2^2 + \sum_{i=1}^n g^2(x_i)\right\}^{1/2}}.$$

Suppose $\lambda_n \left\|g^{(q)}\right\|_2^2 + \sum_{i=1}^n g^2(x_i) = \lambda_n \left\|g^{(q)}\right\|_2^2 + n\int_0^1 g^2(x)dF(x) + \mathbf{I}$, then

$$\begin{aligned}
|\mathbf{I}| =& n\left|\sum_{i=1}^n g^2(x_i) - n\int_0^1 g^2(x)dF(x)\right| \\
=& n\left|\int_0^1 g^2(x)d(F_n - F)(x)\right| \\
=& n\left|\int_0^1 g'(x)g(x)\{F_n(x) - F(x)\}dx\right| \\
\leq& n\left\|F_n - F\right\|_\infty \left\|g'\right\|_2 \left\|g\right\|_2.
\end{aligned}$$

Let $\|g\|_K = \left(\lambda_n \left\|g^{(q)}\right\|_2^2 + n\left\|g\right\|_2^2\right)^{1/2}$, by (S1.1), $|\mathbf{I}| \leq C_3 \left\|F_n - F\right\|_\infty \lambda_n^{-1/(2q)} n^{1/(2q)} \|g\|_K$ for some constant C_3. If $\left\|F_n - F\right\|_\infty \lambda_n^{-1/(2q)} n^{1/(2q)} \leq 1/(2C_3)$ then

$$\frac{1}{2}\min\left\{1, \min_{0\leq x\leq 1} F(x)\right\} \leq \frac{\|Lg\|^2}{\|g\|_K^2} \leq \frac{3}{2}\max\left\{1, \max_{0\leq x\leq 1} F(x)\right\}. \tag{S2.1}$$

Since $\|g\|_K \geq n^{1/2}\|g\|_2$,

$$\left(\sup_{g\in H^q} \frac{\|g\|_2}{\|Lg\|}\right)^2 \leq \frac{2}{n\min\left\{1, \min_{0\leq x\leq 1} F(x)\right\}} \tag{S2.2}$$

with the assumption above.

Now consider $\left\|Lf_0 - L\hat{f}_n\right\|$. Let $Q_1 : Z \to LH^q$ and $Q_2 : Z \to L\mathbb{S}_{\kappa_n, p+1}$

be orthogonal projection, then $L\hat{f}_n = Q_2 h$ and $Q_2 = Q_2 Q_1$. We have that

$$\begin{aligned}\left\|Lf_0 - L\hat{f}\right\|^2 &= \|Lf_0 - Q_2 Lf_0\|^2 + \left\|Q_2 Lf_0 - L\hat{f}_n\right\|^2 \\ &= \|Lf_0 - Q_2 Lf_0\|^2 + \|Q_2 Q_1 Lf_0 - Q_2 Q_1 h\|^2 \\ &\leq \|Lf_0 - Q_2 Lf_0\|^2 + \|Q_1 Lf_0 - Q_1 h\|^2 .\end{aligned}$$

And by (S2.1),

$$\begin{aligned}\|Lf_0 - Q_2 Lf_0\|^2 &= \inf_{g \in H^q} \|Lf_0 - Lg\|^2 \leq \|Lf_0 - Ls\|^2 \\ &= \lambda_n \left\|(f_0 - s)^{(q)}\right\|_2^2 + \sum_{i=1}^{n} (f_0 - s)^2(x_i) \\ &\leq \frac{3 \max_x F(x)}{2} \left\{\lambda_n \left\|(f_0 - s)^{(q)}\right\|_2^2 + n \|f_0 - s\|_2^2\right\}.\end{aligned}$$

By Lemma 2, there is $s \in \mathbb{S}_{\kappa_n, p+1}$ such that

$$\left\|f_0^{(q)} - s^{(q)}\right\|_2 \leq \left\|f_0^{(q)} - s^{(q)}\right\|_\infty \leq C_4 \left\|f_0^{(q)}\right\|_\infty$$

and

$$\|f_0 - s\|_2 \leq C_4 \varDelta_n^{2l_0} \left\|f_0^{(l_0)}\right\|_2$$

for some constant C_4, where $l_0 = \min\{l, p+1\}$. Thus there is some constant C_5 so that

$$\|Lf_0 - Q_2 Lf_0\|^2 \leq C_5 (\lambda_n \left\|f_0^{(q)}\right\|_\infty^2 + n \varDelta_n^{2l_0} \left\|f_0^{(l_0)}\right\|_2^2). \tag{S2.3}$$

Notice that

$$\|Q_1 Lf_0 - Q_1 h\| = \sup_{g \in H^q} \frac{\langle Lg, Lf_0 - h\rangle_Z}{\|Lg\|}.$$

For all $g \in H^q$, we have

$$\langle Lg, Lf_0 - h \rangle_Z = \lambda_n \int_0^1 f_0^{(q)}(x) g^{(q)}(x) dx - \sum_{i=1}^{n} g(x_i)\varepsilon_i$$

and

$$\lambda_n \left| \int_0^1 f_0^{(q)}(x) g^{(q)}(x) dx \right| \leq \lambda_n \left\| f_0^{(q)} \right\|_2 \left\| g^{(q)} \right\|_2 \leq \lambda_n^{1/2} \left\| f_0^{(q)} \right\|_2 \|Lg\|$$

whenever $\|F_n - F\|_\infty \lambda_n^{-1/(2q)} n^{1/(2q)} \leq 1/(2C_3)$. And

$$\begin{aligned} \left| \sum_{i=1}^{n} g(x_i)\varepsilon_i \right| &= \int_0^1 g(x) dE_n(x) = g(1)E(1) - \int_0^1 g'(x) E_n(x) dx \\ &\leq M_n(\|g\|_\infty + \|g'\|_1). \end{aligned} \tag{S2.4}$$

By Lemma 1, there are constants C_6 and C_7 such that

$$\|g\|_\infty + \|g'\|_1 \leq C_6 \left\| g^{(q)} \right\|_2^{\frac{1}{2q}} \|g\|_2^{\frac{2q-1}{2q}} + C_7 \|g\|_2 . \tag{S2.5}$$

And with the assumption of (S2.1), there is constants C_8 such that

$$C_6 \left\| g^{(q)} \right\|_2^{\frac{1}{2q}} \|g\|_2^{\frac{2q-1}{2q}} + C_7 \|g\|_2 \leq C_8 \left(\lambda_n^{-\frac{1}{4q}} n^{-\frac{2q-1}{4q}} + n^{-\frac{1}{2}} \right) \|Lg\| .$$

So

$$\|Q_1 L f_0 - Q_1 h\| \leq \lambda_n^{1/2} \left\| f_0^{(q)} \right\|_2 + C_8 \left(\lambda_n^{-\frac{1}{4q}} n^{-\frac{2q-1}{4q}} + n^{-\frac{1}{2}} \right).$$

Combining this inequality with (S2.2) and (S2.3) completes the proof Theorem 1. □

S3 Proof of Theorem 2

Let $\tilde{L} : H^q \to Z$ be the bounded linear map given by

$$\tilde{L}g = \left(g^{(q)}, P_1 g(x_1), \ldots, P_n g(x_n)\right),$$

then $\tilde{L}H^q$ and $\tilde{L}\mathbb{S}_{\kappa_j, p+1}$, $j = 1, \ldots, n$ are closed subspace of Z. Let $h = (0, y_i) \in Z$, then $\tilde{L}\tilde{f}$ is the orthogonal projection from h to $\tilde{L}\mathbb{S}_{\kappa_n, p+1}$. Let G be the injection from H^q to L^2. Again we have

$$\left\|Gf_0 - G\tilde{f}_n\right\| = \left\|\tilde{L}f_0 - \tilde{L}\tilde{f}_n\right\| \cdot \frac{\left\|Gf_0 - G\tilde{f}_n\right\|}{\left\|\tilde{L}f_0 - \tilde{L}\tilde{f}_n\right\|} \leq \left\|\tilde{L}f_0 - \tilde{L}\tilde{f}_n\right\| \sup_{g \in H^q} \frac{\|Gg\|}{\left\|\tilde{L}g\right\|}.$$

First we prove that

$$\sup_{g \in H^q} \frac{\|Gg\|}{\left\|\tilde{L}g\right\|} = O_p(n^{-1/2}). \tag{S3.1}$$

Let $m = \lfloor n/2 \rfloor$. The same proof as (S2.1) yields that

$$\|g\|_2^2 \leq \|g\|_K^2 \leq \frac{1}{n}\left(\lambda_n \left\|g^{(q)}\right\|_2^2 + \sum_{i=m}^{n} g^2(x_i)\right) O_p(1), \tag{S3.2}$$

where the $O_p(1)$ does not depend on g. The inequality (S3.1) holds as long as we prove that

$$\sum_{i=m}^{n} g^2(x_i) - \sum_{i=m}^{n} (P_i g)^2(x_i) = \|g\|_K^2 o_p(1). \tag{S3.3}$$

Put

$$\mathbf{II} = \sum_{j=m}^{n} \int_0^1 \left\{(P_j g)^2(x) - g^2(x)\right\} dF(x)$$

and

$$\mathbf{III} = \sum_{j=m}^{n} (j-m+1) \int_0^1 \left\{ (P_{j+1}g)^2(x) - (P_j g)^2(x) \right\} d \left\{ \tilde{F}_j(x) - F(x) \right\}$$

where $P_{n+1} = I$ and $\tilde{F}_j = (jF_j - mF_m)/(j-m+1)$, then $\sum_{i=m}^{n} g^2(x_i) - \sum_{i=m}^{n} (P_i g)^2(x_i) = \mathbf{II} + \mathbf{III}$. Note that

$$\begin{aligned} |\mathbf{II}| &\leq \max_x F(x) \sum_{j=m}^{n} \int_0^1 (P_j g(x) - g(x))(P_j g(x) + g(x))\, dx \\ &\leq \max_x F(x) \sum_{j=m}^{n} \|g - P_j g\|_2 (\|g\|_2 + \|P_j g\|_2) \\ &\leq \max_x F(x) \sum_{j=m}^{n} \|g - P_j g\|_{H^1} (\|g\|_{H^1} + \|P_j g\|_{H^1}) \\ &\leq 2 \max_x F(x) \sum_{j=m}^{n} \|g - P_j g\|_{H^1} \|g\|_{H^1}. \end{aligned}$$

By Lemma 2,

$$\begin{aligned} \|g - P_j g\|_{H^1}^2 &= \inf_{s \in \mathbb{S}_{\kappa_j, p+1}} \left(\|g' - s'\|_2^2 + \|g - s\|_2^2 \right) \\ &\leq \varDelta_j^{2q-2} \left\| g^{(q)} \right\|_2^2 + \varDelta_j^{2q} \left\| g^{(q)} \right\|_2^2 \\ &\leq 2\varDelta_j^{2q-2} \left\| g^{(q)} \right\|_2^2. \end{aligned} \tag{S3.4}$$

Then

$$|\mathbf{II}| \leq 4 \max_x F(x) \sqrt{2} \left\| g^{(q)} \right\|_2 (\|g\|_2 + \|g'\|_2) \sum_{j=m}^{n} \varDelta_j^{q-1}.$$

By (S1.1) and the assumption that $D_1 n^{1/(2q-1)} \leq \lambda_n \leq D_2 n^{1/(2q-1)}$ and $\varDelta_n = O_p(n^{-\nu})$ for some $\nu > (2q-1)/\{(2q+1)(2q-3)\}$, we have $|\mathbf{II}| = \|g\|_K^2 o_p(1)$.

Next we prove $|\mathbf{III}| = \|g\|_K^2 o_p(1)$. Let $u_j = (j - m + 1)\|\tilde{F}_j - F\|_\infty$, one has $u_j \le j\|F_j - F\|_\infty + m\|F_m - F\|_\infty$. Then $Eu_j^2 = O(n)$. Integration by parts yields that

$$|\mathbf{III}| = \sum_{j=m}^{n} u_j \int_0^1 \left| \frac{d}{dx} \left\{(P_{j+1}g)^2(x) - (P_j g)^2(x)\right\}^2 \right| dx.$$

Let $\delta_j = 1$ for $\kappa_{j+1} \ne \kappa_j$ and $\delta_j = 0$ for $\kappa_{j+1} = \kappa_j$, and

$$\begin{aligned}
A_1 &= \sum_{j=m}^{n} \delta_j u_j^2 \int_0^1 \left\{P_{j+1}g(x) + P_j g(x)\right\}^2 dx,\\
A_2 &= \sum_{j=m}^{n} \delta_j u_j^2 \int_0^1 \left\{(P_{j+1}g)'(x) + (P_j g)'(x)\right\}^2 dx,\\
A_3 &= \sum_{j=m}^{n} \int_0^1 \left\{P_{j+1}g(x) - P_j g(x)\right\}^2 dx,\\
A_4 &= \sum_{j=m}^{n} \int_0^1 \left\{(P_{j+1}g)'(x) - (P_j g)'(x)\right\}^2 dx.
\end{aligned}$$

By the Cauchy–Schwartz inequality,

$$|\mathbf{III}| \le (A_1 A_4)^{1/2} + (A_2 A_3)^{1/2}.$$

Now

$$A_1 \le \sum_{j=m}^{n} \delta_j u_j^2 \left(\|P_j g\|_2 + \|P_{j+1}g\|_2\right)^2.$$

Since $\|P_j g\|_2^2 \le \|P_j g\|_{H^1}^2 \le \|g\|_{H^1}^2$, we have $A_1 \le \|g\|_{H^1}^2 \sum_{j=m}^{n} \delta_j u_j^2$. Similarly, with $\|(P_j g)'\|_2^2 \le \|P_j g\|_{H^1}^2 \le \|g\|_{H^1}^2$, we have $A_2 \le \|g\|_{H^1}^2 \sum_{j=m}^{n} \delta_j u_j^2$.

Also by (S3.4),

$$A_3 = \sum_{j=m}^{n} \|P_{j+1}g - P_j g\|_2^2 \le \sum_{j=m}^{n} \|P_{j+1}g - P_j g\|_{H^1}^2$$

$$= \|g - P_m g\|_{H^1}^2 \le 2\Delta_m^{2q-2} \left\|g^{(q)}\right\|_2^2,$$

and

$$A_4 = \sum_{j=m}^{n} \|(P_{j+1}g)' - (P_j g)'\|_2^2 \le \sum_{j=m}^{n} \|P_{j+1}g - P_j g\|_{H^1}^2$$

$$= \|g - P_m g\|_{H^1}^2 \le 2\Delta_m^{2q-2} \left\|g^{(q)}\right\|_2^2.$$

So

$$|\mathbf{III}| \le 2\sqrt{2} \left(\sum_{j=m}^{n} \delta_j u_j^2\right)^{1/2} \Delta_m^{q-1} \|g\|_{H^1} \left\|g^{(q)}\right\|_2.$$

Assumption 5 gives that $\Delta_n = O_p\left(n^{-\nu}\right)$ and $\sum_{j=m}^{n} \delta_j u_j^2 = o_p\left(n^{(2q-2)\nu + 2q/(2q+1)}\right)$. By inequality (S1.1) we have $|\mathbf{III}| = \|g\|_K^2 o_p(1)$. Thus we have proved (S3.1).

Now we prove that $\left\|\tilde{L} f_0 - \tilde{L}\tilde{f}\right\| = O_p(n^{1/(4q+2)})$.

Let $\tilde{Q}_1 : Z \to \tilde{L}H^q$ and $\tilde{Q}_2 : Z \to \tilde{L}\mathbb{S}_{\kappa_n, p+1}$ be orthogonal projection, then $\tilde{L}\tilde{f} = \tilde{Q}_2 h$ and $\tilde{Q}_2 = \tilde{Q}_2\tilde{Q}_1$. We have that

$$\left\|\tilde{L} f_0 - L\tilde{f}\right\|^2 = \left\|\tilde{L} f_0 - \tilde{Q}_2\tilde{L} f_0\right\|^2 + \left\|\tilde{Q}_2\tilde{L} f_0 - \tilde{Q}_2 h\right\|^2$$

$$\le \left\|\tilde{L} f_0 - \tilde{Q}_2\tilde{L} f_0\right\|^2 + \left\|\tilde{Q}_1\tilde{L} f_0 - \tilde{Q}_1 h\right\|^2.$$

By Lemma 2 there exist a $s \in \mathbb{S}_{\kappa_n, p+1}$ and constant C_9 such that

$$\left\|f_0^{(k)} - s^{(k)}\right\|_2 \le C_9 \Delta_n^{q+1-k} \left\|f_0^{(q+1)}\right\|_2, \; k = 0, 1, q.$$

Then we have

$$\left\|\tilde{L}f_0 - \tilde{Q}_2\tilde{L}f_0\right\|^2 = \inf_{g\in H^q}\left\|\tilde{L}f_0 - \tilde{L}g\right\|^2 \le \left\|\tilde{L}f_0 - \tilde{L}s\right\|^2$$
$$= \lambda_n \left\|(f_0 - s)^{(q)}\right\|_2^2 + \sum_{i=1}^n (P_if_0 - P_is)^2(x_i)$$
$$\le \lambda_n \left\|(f_0 - s)^{(q)}\right\|_2^2 + \sum_{i=1}^n \|P_if_0 - P_is\|_\infty^2 .$$

Lemma 1 implies that $\|g\|_\infty^2 \le C_{10}\|g\|_{H^1}^2$ for some constant C_{10} and all $g \in H^1$. So

$$\left\|\tilde{L}f_0 - \tilde{Q}_2\tilde{L}f_0\right\|^2 \le \lambda_n \left\|(f_0 - s)^{(q)}\right\|_2^2 + C_{10}\sum_{i=1}^n \|P_if_0 - P_is\|_{H^1}^2$$
$$\le \lambda_n \left\|(f_0 - s)^{(q)}\right\|_2^2 + C_{10}n\|f_0 - s\|_{H^1}^2$$
$$\le \left(C_9\lambda_n\Delta_n + C_9C_{10}n\Delta_n^{2q} + C_9C_{10}n\Delta_n^{2q+2}\right)\left\|f_0^{(q+1)}\right\|_2^2$$
$$= O_p(n^{1/(2q+1)}).$$

It remains to show that

$$\left\|\tilde{Q}_1\tilde{L}f_0 - \tilde{Q}_1h\right\| = \sup_{g\in H^q}\frac{\left(\tilde{L}g, \tilde{L}f_0 - h\right)_Z}{\left\|\tilde{L}g\right\|} = O_p(n^{1/(4q+2)}).$$

Put

$$\mathbf{IV} = \lambda_n\int_0^1 f_0^{(q)}(x)g^{(q)}(x)dx,\ \mathbf{V} = \sum_{i=1}^n P_ig(x_i)\left\{P_if_0(x_i) - f(x_i)\right\},\ \mathbf{VI} = \sum_{i=1}^n P_ig(x_i)\varepsilon_i,$$

then $\left(\tilde{L}g, \tilde{L}f_0 - h\right)_Z = \mathbf{IV} + \mathbf{V} - \mathbf{VI}$.

The easy part is for $\mathbf{IV}$, where

$$|\mathbf{IV}| \le \lambda_n\left\|f_0^{(q)}\right\|_2\left\|g^{(q)}\right\|_2 \le \lambda_n^{1/2}\left\|f_0^{(q)}\right\|_2\left\|\tilde{L}g\right\| = O_p(n^{1/(4q+2)})\left\|\tilde{L}g\right\|.$$

Using Cauchy's inequality, we have

$$\mathbf{V}^2 \leq \left\{ \sum_{i=1}^{n} (P_i g)^2(x_i) \right\} \left\{ \sum_{i=1}^{n} (P_i f_0(x_i) - f(x_i))^2 \right\},$$

where $\sum_{i=1}^{n} (P_i g)^2(x_i) \leq \left\| \tilde{L} g \right\|^2$. And

$$\sum_{i=1}^{n} (P_i f_0(x_i) - f_0(x_i))^2 \leq \sum_{i=1}^{n} \|(I - P_i) f_0\|_\infty^2 \leq \sum_{i=1}^{n} \|(I - P_i) f_0\|_{H^1}^2 .$$

By Lemma 2,

$$\|(I - P_i) f_0\|_{H^1}^2 = \inf_{e \in \mathbb{S}_{\kappa_i, p+1}} \|f_0 - e\|_{H^1}^2 \leq \left(\Delta_i^{2q} + \Delta_i^{2q+2} \right) \left\| f_0^{(q+1)} \right\|_2^2 .$$

Because $\Delta_i \geq \Delta_{i+1}$ and $E\Delta_i = O(n^{-\nu})$,

$$\sum_{i=1}^{n} \left(\Delta_i^{2q} + \Delta_i^{2q+2} \right) = O_p(n^{1/(2q+1)})$$

and $\mathbf{V} = O_p(n^{1/(4q+2)}) \left\| \tilde{L} g \right\|$.

Now we need to show that $\mathbf{VI} / \left\| \tilde{L} g \right\| = O_p(n^{1/(4q+2)})$.We have already seen in (S2.4) and (S2.5) that $|\sum_{i=1}^{n} g(x_i)\varepsilon_i| = O_p(n^{1/(4q+2)}) \|g\|_K$, and in (S3.2) and (S3.3) that $\|g\|_K = O_p(1) \left\| \tilde{L} g \right\|$. It suffices to show that

$$\sum_{i=1}^{n} (I - P_i) g(x_i) \varepsilon_i = O_p(n^{1/(4q+2)}) \left\| \tilde{L} g \right\| . \qquad \text{(S3.5)}$$

Let $p_g(x)$ be the polynomial of degree at most q that $\int_0^1 \{p_g(x) - g(x)\} dx = 0$ and $p_g^{(k)}(1) - p_g^{(k)}(0) = g^{(k)}(1) - g^{(k)}(0)$ for $k = 0, \ldots q - 1$.

Suppose $g^{(q)} - p_g^{(q)}$ has the Fourier expansion

$$g^{(q)}(x) - p_g^{(q)}(x) = \sum_{k=1}^{\infty} a_k \cos(2\pi kx) + b_k \sin(2\pi kx),$$
$$g(x) - p_g(x) = \sum_{k=1}^{\infty} (2\pi k)^{-2q}\{a_k \cos(2\pi kx) + b_k \sin(2\pi kx)\}.$$

Notice that $(I - P_i)g = (I - P_i)(g - p_g)$ and $\left\|g^{(q)}\right\|_2^2 \geq \sum_{k=1}^{\infty}(a_k^2 + b_k^2)$, by Cauchy-Schwartz inequality,

$$\left\{\sum_{i=1}^{n}(I - P_i)g(x_i)\varepsilon_i\right\}^2 \left\|g^{(q)}\right\|_2^{-2} \leq$$
$$\sum_{k=1}^{\infty}(2\pi k)^{-2q}\left[\left\{\sum_{i=0}^{n}(I - P_i)\cos(2\pi k\cdot)(x_i)\varepsilon_i\right\}^2 + \left\{\sum_{i=1}^{n}(I - P_i)\sin(2\pi k\cdot)(x_i)\varepsilon_i\right\}^2\right]. \tag{S3.6}$$

For all fixed function h,

$$\left\{\sum_{i=1}^{n}(I - P_i)h(x_i)\varepsilon_i\right\}^2 = \sum_{i=1}^{n}(I-P_i)h^2(x_i)\varepsilon_i^2 + 2\sum_{i<j}(I-P_i)h(x_i)(I-P_j)h(x_j)\varepsilon_i\varepsilon_j.$$

if $(\varepsilon_i)_{i=1,2,\ldots}$ are independent of $(\kappa_i)_{i=1,2,\ldots}$ and $(x_i)_{i=1,2,\ldots}$, and pairwise uncorrelated, then for $i < j$,

$$E(I - P_i)h(x_i)(I - P_j)h(x_j)\varepsilon_i\varepsilon_j = E(I - P_i)h(x_i)(I - P_j)h(x_j)E\varepsilon_i\varepsilon_j = 0,$$

otherwise if ε_j is independent of κ_i and x_i for $i \leq j$ and $(\varepsilon_i)_{i=1,2,\ldots}$ are pairwise independent, then for $i < j$,

$$E(I - P_i)h(x_i)(I - P_j)h(x_j)\varepsilon_i\varepsilon_j = E(I - P_i)h(x_i)(I - P_j)h(x_j)\varepsilon_i E\varepsilon_j = 0.$$

In either case ε_i is independent of $(I-P_i)h^2(x_i)$, so

$$E\left\{\sum_{i=1}^n (I-P_i)h(x_i)\varepsilon_i\right\}^2 = E\sum_{i=1}^n (I-P_i)h^2(x_i)\varepsilon_i^2 \leq \left(\sup_{i\geq 1} E\varepsilon_i^2\right) E\sum_{i=1}^n (I-P_i)h^2(x_i).$$

Replace h with $\cos(2\pi k\cdot)$, for some constant C_{11},

$$|(I-P_i)\cos(2\pi k\cdot)(x_i)|^2 \leq \|(I-P_i)\cos(2\pi k\cdot)\|_\infty^2 \leq C_{11}\,\|(I-P_i)\cos(2\pi k\cdot)\|_{H^1}^2 .$$

By (S3.4), for some constant C_{12},

$$\|(I-P_i)\cos(2\pi k\cdot)\|_{H^1}^2 \leq C_{12}\Delta_i^{2s-2}(2\pi k)^s,\ s=1,\ldots,q,$$

and thus the above inequality holds for all $s\in[1,q]$, which means

$$|(I-P_i)\cos(2\pi k\cdot)(x_i)|^2 \leq C_{11}C_{12}\Delta_i^{2s-2}(2\pi k)^s \text{ for all } s\in[1,q].$$

The same argument applies to $\sin(2\pi k\cdot)$, so

$$|(I-P_i)\sin(2\pi k\cdot)(x_i)|^2 \leq C_{11}C_{12}\Delta_i^{2s-2}(2\pi k)^s \text{ for all } s\in[1,q].$$

Insert these and that $\left\|g^{(q)}\right\|_2^2 \leq \|g\|_K^2/\lambda_n$ into (S3.6), we get

$$E\sup_g \|g\|_K^{-2}\left\{\sum_{i=1}^n (I-P_i)g(x_i)\varepsilon_i\right\}^2 \leq C_{11}C_{12}\lambda_n^{-1}\left(\sup_{i\geq 1} E\varepsilon_i^2\right) E\sum_{i=1}^n \Delta_i^{2s-2}\sum_{k=0}^\infty (2\pi k)^{2s-2q}.$$

Put $s = 1+(2q-1)/\{2\nu(2q+1)\}$. Because $\lambda_n \geq D_1 n^{1/(2q+1)}$ for some $D_1\in(0,\infty)$, $\Delta_i \geq \Delta_{i+1}$ and $E\Delta_i = O(n^{-\nu})$,

$$\lambda_n^{-1}\sum_{i=1}^n \Delta_i^{2s-2} = O_p(n^{1/(2q+1)}).$$

Since $\nu > (2q-1)/\{(2q+1)(2q-3)\}$, we have $2s-2q<-1$, so $\sum_{k=0}^\infty (2\pi k)^{2s-2q}$ is some finite constant. And Assumption 4 asserts that $\sup_{i\geq 1} E\varepsilon_i^2$ is finite. Thus we have proved (S3.5), and our proof is complete. □

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