Hypothesis Testing in Large-scale Functional Linear Regression

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Supplementary Material

S1 Notations

For clarity, we first recall the notations used throughout the supplementary material. For any vector \( u = (u_1, \ldots, u_s)' \), let \( ||u||_q = (\sum_{l=1}^s |u_l|^q)^{1/q} \) for \( q \geq 1 \), \( ||u||_\infty = \max_{l \leq s} |u_l| \), and \( ||u||_0 \) denotes the cardinality of \( \text{supp}(u) \), where \( \text{supp}(u) = \{ l : u_l \neq 0 \} \). If \( S = \text{supp}(u) \), let \( u_S \) be the vector containing only nonzero elements of \( u \), while \( u_{S^c} \) contains only zero elements. For any matrix \( C = [c_{ij}] \), denote \( ||C||_\infty = \max_{i,j} |c_{ij}| \), and if \( C \) is symmetric, \( \lambda_{\min}(C) \) and \( \lambda_{\max}(C) \) are the minimum and maximum eigenvalues. For real sequences \( a_n, b_n \), \( a_n \sim b_n \) if and only if \( c_1 \leq \lim_{n \to \infty} |a_n/b_n| \leq c_2 \) for some positive constants \( c_1, c_2 \). Recall that \( ||X||_{\phi_1} = \sup_{q \geq 1} q^{-1/2} \{ E(|X|^q) \}^{1/q} \) is the sub-Gaussian norm, and \( ||X||_{\phi_2} = \sup_{q \geq 1} q^{-1} \{ E(|X|^q) \}^{1/q} \) the sub-exponential norm, similar to those adopted in Ning and Liu (2017).
Recall that \( \mathcal{P}_n = \{1, \ldots, p_n\} \) is the index set representing all functional predictors, while \( \mathcal{H}_n \subseteq \mathcal{P}_n \) is any nonempty subset of \( \mathcal{P}_n \), with cardinality \( |\mathcal{H}_n| = h_n \leq p_n \), and denote \( \mathcal{H}_n^c = \mathcal{P}_n \setminus \mathcal{H}_n \). For each \( j \leq p_n \), let \( \Lambda_j = \text{diag}\{\omega_{j1}^{1/2}, \ldots, \omega_{js_n}^{1/2}\} \), \( \Lambda = \Lambda_{\mathcal{P}_n} \) be the block diagonal matrix with \( \{\Lambda_j : j \leq p_n\} \) as diagonal submatrices, and \( \Lambda_{\mathcal{H}_n} \) be the block diagonal matrix with \( \{\Lambda_j : j \in \mathcal{H}_n\} \) as diagonal submatrices. Similarly, define \( \hat{\Lambda}_j = \text{diag}\{\hat{\omega}_{j1}^{1/2}, \ldots, \hat{\omega}_{js_n}^{1/2}\} \), \( \hat{\Lambda} = \hat{\Lambda}_{\mathcal{P}_n} \) and \( \hat{\Lambda}_{\mathcal{H}_n} \) be the block diagonal matrix with \( \{\hat{\Lambda}_j : j \in \mathcal{H}_n\} \) as diagonal submatrices. Furthermore, we denote \( \hat{\eta} = \Lambda \eta = (\hat{\eta}_1', \ldots, \hat{\eta}_{ps_n}')' \) with \( \hat{\eta}_j = \Lambda_j \eta_j = (\eta_{j1} \omega_{j1}^{1/2}, \ldots, \eta_{js_n} \omega_{js_n}^{1/2})' \). Similarly, define \( \hat{\nu} = \eta - \eta^* \) and \( \hat{\nu} = \eta - \eta^* = \Lambda \nu \), where \( \eta^* \) and \( \eta^* \) are true values of \( \eta \) and \( \hat{\eta} \), respectively. For any \( \mathcal{H}_n \subseteq \mathcal{P}_n \), \( \hat{\nu}_{\mathcal{H}_n} \) is constructed by stacking the vectors \( \{\hat{\nu}_j = \Lambda_j (\eta_j - \eta_j^*) : j \in \mathcal{H}_n\} \) in a column.

Without loss of generality, let \( \Theta = (G_1, \ldots, G_n)' = [\Theta_{\mathcal{H}_n}, \Theta_{\mathcal{H}_n}^c], \Theta_{\mathcal{H}_n} = (E_1, \ldots, E_n)', \Theta_{\mathcal{H}_n}^c = (F_1, \ldots, F_n)' \), where \( \Theta_{\mathcal{H}_n} \) is constructed by stacking \( \{\Theta_j : j \in \mathcal{H}_n\} \) in a row. Similarly, denote \( \hat{\Theta} = (\hat{G}_1, \ldots, \hat{G}_n)' = [\hat{\Theta}_{\mathcal{H}_n}, \hat{\Theta}_{\mathcal{H}_n}^c] = \Theta \Lambda^{-1} = [\Theta_{\mathcal{H}_n} \Lambda_{\mathcal{H}_n}^{-1}, \Theta_{\mathcal{H}_n}^c \Lambda_{\mathcal{H}_n}^{-1}], \hat{\Theta}_{\mathcal{H}_n} = (\hat{E}_1, \ldots, \hat{E}_n)', \hat{\Theta}_{\mathcal{H}_n}^c = (\hat{F}_1, \ldots, \hat{F}_n)' \), where \( \hat{\Theta}_{\mathcal{H}_n} \) is constructed by stacking \( \{\hat{\Theta}_j = \Theta_j \Lambda_j^{-1} : j \in \mathcal{H}_n\} \) in a row. Fur-
ther, we denote a series of quantities derived from the information matrix $I$.

$$I = E(G_i G_i^t), \quad I_{\mathcal{H}_n \mathcal{H}_n} = E(E_i E_i^t), \quad I_{\mathcal{H}_n \mathcal{H}_n'} = E(E_i F_i^t), \quad I_{\mathcal{H}_n \mathcal{H}_n'} = E(F_i E_i^t),$$

$$I_{\mathcal{H}_n \mathcal{H}_n'} = E(F_i F_i^t), \quad I_{\mathcal{H}_n | \mathcal{H}_n} = I_{\mathcal{H}_n} - w^t I_{\mathcal{H}_n} w, \quad w = I_{\mathcal{H}_n}^{-1} I_{\mathcal{H}_n}^\prime h_{\mathcal{H}_n}, \quad w = E_{\mathcal{H}_n}((w_1, \ldots, w_{h_{\mathcal{H}_n}}),$$

$$\mathcal{I} = E(\mathcal{G}_i \mathcal{G}_i^t) = \Lambda^{-1} I \Lambda^{-1}, \quad \mathcal{I}_{\mathcal{H}_n} = \Lambda^{-1} I_{\mathcal{H}_n} \Lambda^{-1}, \quad \mathcal{I}_{\mathcal{H}_n}^\prime = \Lambda^{-1} I_{\mathcal{H}_n} \Lambda^{-1},$$

$$\mathcal{I}_{\mathcal{H}_n}^\prime = \Lambda^{-1} I_{\mathcal{H}_n} \Lambda^{-1}, \quad \mathcal{I}_{\mathcal{H}_n}^\prime = \Lambda^{-1} I_{\mathcal{H}_n} \Lambda^{-1}, \quad \mathcal{I}_{\mathcal{H}_n}^\prime = \Lambda^{-1} I_{\mathcal{H}_n} \Lambda^{-1},$$

$$\rho_n = \max_{l \leq h_{\mathcal{H}_n}} ||w_l||_0 = \max_{l \leq h_{\mathcal{H}_n}} \rho_{nl}, \quad \rho_{nl} = ||w_l||_0.$$

Using the $\hat{\eta}$ obtained from (2.5) in the main paper under the conditions of Theorem 1, we further denote

$$\mathcal{S}_i = (\mathcal{S}_{i,1}, \ldots, \mathcal{S}_{i,h_{\mathcal{H}_n}}) = \hat{\Lambda}_{\mathcal{H}_n}^{-1}(\hat{w}^t F_i - E_i)(Y_i - F_i^t \hat{\eta}_{\mathcal{H}_n}), \quad i = 1, \ldots, n,$$

$$\mathcal{S}_i = (\mathcal{S}_{i,1}, \ldots, \mathcal{S}_{i,h_{\mathcal{H}_n}}) = \Lambda_{\mathcal{H}_n}^{-1}(w^t F_i - E_i) e_i, \quad i = 1, \ldots, n,$$

$$\mathcal{T}^* = n^{-1/2} \sum_{i=1}^n \hat{\mathcal{S}}_i, \quad \mathcal{T}^* = n^{-1/2} \sum_{i=1}^n \mathcal{S}_i, \quad \mathcal{T}^*_e = n^{-1/2} \sum_{i=1}^n e_i \hat{\mathcal{S}}_i, \quad \mathcal{T}^*_e = n^{-1/2} \sum_{i=1}^n e_i \mathcal{S}_i,$$

$$c_B(\alpha) = \inf \{ t \in \mathbb{R} : P_e(||\mathcal{T}^*_e||_\infty \leq t) \geq 1 - \alpha \}, \quad \alpha \in (0, 1),$$

$$\phi^*(\alpha) = \inf \{ t \in \mathbb{R} : P_e(||\mathcal{T}^*_e||_\infty \leq t) \geq 1 - \alpha \}, \quad \alpha \in (0, 1),$$

where $e = (e_1, \ldots, e_n)'$ consists of i.i.d. standard normal random variables independent of the data, $P_e(\cdot)$ refers to the probability measure with respect to $e$, $c_B(\alpha)$ denotes the 100$(1 - \alpha)$th percentile of $||\mathcal{T}^*_e||_\infty$, and $\phi^*(\alpha)$ denotes the 100$(1 - \alpha)$th percentile of $||\mathcal{T}^*_e||_\infty$. It is also worth to notice that $\{\mathcal{S}_i : i \leq n\}$ are i.i.d. centered random vectors such that $E(\mathcal{S}_i \mathcal{S}_i^t) = \sigma^2 \hat{I}_{\mathcal{H}_n | \mathcal{H}_n'}$. 

S1. NOTATIONS
We present below a series of auxiliary lemmas, quantifying various error bounds for a number of quantities that are useful for showing the main theorems.

S2 Auxiliary Lemmas and Proofs

Lemma 1. (a) Under conditions (P1)–(P4), \(| \rho_\lambda(t_1) - \rho_\lambda(t_2) | \leq \lambda L |t_1 - t_2|\), for all \(t_1, t_2 \in \mathbb{R}\).

(b) Under conditions (P1)–(P4), \(| \rho'_\lambda(t) | \leq \lambda L\), for all \(t \neq 0\).

(c) Under conditions (P1)–(P5), \(\lambda L |t| \leq \rho_\lambda(t) + 2^{-1} \mu t^2\), for all \(t \in \mathbb{R}\).

Proof. The proof of this lemma can be referred to Lemma 4 in Loh and Wainwright (2015).

Lemma 2. Under conditions (P1)–(P4), if \(P_\lambda(\eta^*) - P_\lambda(\eta) \geq 0\), where \(\eta^*\) is the true version of \(\eta\) and \(P_\lambda(\eta) = \sum_{j=1}^{p_n} \rho_\lambda s_n^{-1/2} (n^{-1/2}||\Theta_j \eta_j||_2)\), then

\[
0 \leq P_\lambda(\eta^*) - P_\lambda(\eta) \leq \lambda_n s_n^{1/2} L \left\{ \sum_{j \in \mathcal{A}_n} n^{-1/2} ||\Theta_j (\eta_j - \eta^*_j)||_2 - \sum_{j \in \mathcal{A}^c_n} n^{-1/2} ||\Theta_j (\eta_j - \eta^*_j)||_2 \right\},
\]

where \(\mathcal{A}_n \subseteq \mathcal{P}_n\) is the index set corresponding to the largest \(q_n\) elements of \(\{n^{-1/2}||\Theta_j \eta_j||_2 : j \leq p_n\}\), and \(\mathcal{A}^c_n = \mathcal{P}_n \setminus \mathcal{A}_n\).

Proof. First, we define \(f_n(t) = t/\rho_\lambda s_n^{-1/2}(t)\) for \(t > 0\), and \(f_n(t) = (\lambda_n s_n^{1/2} L)^{-1}\) for \(t = 0\). Under conditions (P1)–(P4), we have that \(f_n(t)\) is nondecreasing in \(t\).
for $t \geq 0$. Then, it follows that

$$
\sum_{j \in \mathcal{A}_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2) = \sum_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2 \left\{ f_n(n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2) \right\}^{-1} \\
\geq \left\{ f_n\left(\max_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2\right) \right\}^{-1} \sum_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2. \quad (S2.1)
$$

Moreover, we also have

$$
\sum_{j \in \mathcal{A}_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2) = \sum_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2 \left\{ f_n(n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2) \right\}^{-1} \\
\leq \left\{ f_n\left(\max_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2\right) \right\}^{-1} \sum_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2. \quad (S2.2)
$$

By combining (S2.1), (S2.2) with $P_{\lambda_n}(\eta^*) - P_{\lambda_n}(\eta) \geq 0$, we have

$$
0 \leq P_{\lambda_n}(\eta^*) - P_{\lambda_n}(\eta) = \sum_{j=1}^{p_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j\eta_j^*||_2) - \sum_{j=1}^{p_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j\eta_j||_2) \\
= \sum_{j=1}^{q_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j\eta_j^*||_2) - \sum_{j=1}^{p_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j\eta_j||_2) \\
\leq \sum_{j=1}^{q_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2) + \sum_{j=1}^{q_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j\eta_j||_2) - \sum_{j=1}^{p_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j\eta_j||_2) \\
= \sum_{j=1}^{q_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2) - \sum_{j=q_n+1}^{p_n} \rho_{\lambda_n s_n^{1/2}}(n^{-1/2}||\Theta_j\eta_j||_2) \\
\leq \left\{ f_n\left(\max_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2\right) \right\}^{-1} \left\{ \sum_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2 - \sum_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2 \right\} \\
\leq \lambda_n s_n^{1/2} L \left\{ \sum_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2 - \sum_{j \in \mathcal{A}_n} n^{-1/2}||\Theta_j(\eta_j - \eta_j^*)||_2 \right\},
$$

which completes the proof. \qed
Lemma 3. Under conditions (A1), (A2), (A5), (B1), we have with probability tending to one, for some constants $c_1$ and $c_2$, where $I$ here denotes the $p_n s_n \times p_n s_n$ identity matrix,

$$\text{det}(\hat{\Lambda}) \neq 0, \quad ||\hat{\Lambda}^{-1} - I||_\infty \leq c_1 n^{\beta/2 - 1/2}, \quad ||\Lambda^{-1} - I||_\infty \leq c_2 n^{\beta/2 - 1/2}.$$

**Proof.** First, under condition (A2) and by Bernstein inequality, we have that for any $t > 0$,

$$P\{n^{-1} \sum_{i=1}^{n} (\omega_{jk}^{-1} \theta_{ijk}^2 - 1) \geq t\} \leq 2 \exp \{-c_0 \min(c_1^{-2} t^2, c_1^{-1} t)n\},$$

for some universal constants $c_0, c_1 > 0$, uniformly in $j = 1, \ldots, p_n$, $k = 1, \ldots, s_n$. It then follows from union bound inequality that

$$P\{\max_{j \leq p_n} \max_{k \leq s_n} |n^{-1} \sum_{i=1}^{n} (\omega_{jk}^{-1} \theta_{ijk}^2 - 1)| \geq t\} \leq 2 p_n s_n \exp \{-c_0 \min(c_1^{-2} t^2, c_1^{-1} t)n\},$$

by choosing $t = c_2 \{\log(p_n s_n)/n\}^{1/2}$, for some sufficiently large $c_2 > 0$, we have that

$$\max_{j \leq p_n} \max_{k \leq s_n} |n^{-1} \sum_{i=1}^{n} (\omega_{jk}^{-1} \theta_{ijk}^2 - 1)| \leq c_2 \{\log(p_n s_n)/n\}^{1/2},$$

with probability tending to one. Accordingly, we have that

$$||\hat{\Lambda}^{-1} - I||_\infty = \max_{j \leq p_n} \max_{k \leq s_n} |(n^{-1} \sum_{i=1}^{n} \omega_{jk}^{-1} \theta_{ijk}^2)^{1/2} - 1|$$

$$\leq \max_{j \leq p_n} \max_{k \leq s_n} |n^{-1} \sum_{i=1}^{n} (\omega_{jk}^{-1} \theta_{ijk}^2 - 1)| \leq c_2 \{\log(p_n s_n)/n\}^{1/2},$$
with probability tending to one. Under (A5) and (B1), we have that \( \|\hat{\Lambda}\Lambda^{-1} - I\|_\infty \leq c_3 n^{\beta/2-1/2} \), with probability tending to one, for some constant \( c_3 > 0 \).

Together with (A1) and (B1), it can be deduced that

\[
\lambda_{\min}(\hat{\Lambda}) \geq \lambda_{\min}(\Lambda) - |\lambda_{\min}(\hat{\Lambda}) - \lambda_{\min}(\Lambda)| \geq \lambda_{\min}(\Lambda) - \|\hat{\Lambda} - \Lambda\|_\infty
\]

\[
\geq \lambda_{\min}(\Lambda) - \lambda_{\max}(\Lambda)\|\hat{\Lambda}\Lambda^{-1} - I\|_\infty \geq c_4 s^{\alpha/2} - c_5 n^{\beta/2-1/2}
\]

\[
\geq c_6 s^{\alpha/2},
\]

with probability tending to one, for some constants \( c_4, c_5, c_6 > 0 \). It follows that \( \det(\hat{\Lambda}) \neq 0 \), with probability tending to one. Moreover, we have

\[
\|\hat{\Lambda}\Lambda^{-1} - I\|_\infty = \max_{j \leq p_n} \max_{k \leq s_n} |\omega_{jk}^{1/2} (n^{-1} \sum_{i=1}^{n} \theta_{ijk}^2)^{-1/2} - 1|
\]

\[
\leq \left\{ \max_{j \leq p_n} \max_{k \leq s_n} \omega_{jk}^{1/2} (n^{-1} \sum_{i=1}^{n} \theta_{ijk}^2)^{-1/2} \right\} \|\hat{\Lambda}\Lambda^{-1} - I\|_\infty
\]

\[
\leq (\|\hat{\Lambda}\Lambda^{-1} - I\|_\infty + 1)\|\hat{\Lambda}\Lambda^{-1} - I\|_\infty,
\]

which further implies that

\[
\|\hat{\Lambda}\Lambda^{-1} - I\|_\infty \leq \|\hat{\Lambda}\Lambda^{-1} - I\|_\infty / (1 - \|\hat{\Lambda}\Lambda^{-1} - I\|_\infty)
\]

\[
\leq c_3 n^{\beta/2-1/2} / (1 - c_3 n^{\beta/2-1/2})
\]

\[
\leq 2c_3 n^{\beta/2-1/2},
\]

with probability tending to one, which completes the proof. \( \square \)

**Lemma 4.** Under conditions (A1), (A2), (A5), (B1), we have the following with
probability tending to one, for some positive constants $c_0, c_1$ and $c_3$, where $\epsilon$ denotes the $n \times 1$ random error vector.

1) $\|n^{-1}\Theta' - E(G_iG_i')\|_{\infty} \leq c_0 n^{\beta/2-1/2}$, $\|n^{-1}\tilde{\Theta}' - E(\tilde{G}_i\tilde{G}_i')\|_{\infty} \leq c_1 n^{\beta/2-1/2}$,

2) $\|n^{-1}\Theta'\epsilon\|_{\infty} \leq c_2 n^{\beta/2-1/2}$, $\|n^{-1}\tilde{\Theta}'\epsilon\|_{\infty} \leq c_3 n^{\beta/2-1/2}$,

3) $\max_{l \leq h_n s_n} \|n^{-1} \sum_{i=1}^{n} (w_i'F_i - Eil)F_i'\|_{\infty} \leq c_4 n^{\beta/2-1/2}$,

4) $\max_{l \leq h_n s_n} \|n^{-1} \sum_{i=1}^{n} (w_i'F_i - Eil)\{E(E_{il}^2)\}^{-1/2}F_i'\|_{\infty} \leq c_5 n^{\beta/2-1/2}$,

5) $\max_{l \leq h_n s_n} \|n^{-1} \sum_{i=1}^{n} (w_i'F_i - Eil)\epsilon_i\| \leq c_6 n^{\beta/2-1/2}$,

6) $\max_{l \leq h_n s_n} \|n^{-1} \sum_{i=1}^{n} (w_i'F_i - Eil)\{E(E_{il}^2)\}^{-1/2}\epsilon_i\| \leq c_7 n^{\beta/2-1/2}$.

Proof. By using Bernstein inequality and union bound inequality repeatedly, the Lemma holds trivially. \qed

Lemma 5. Under conditions (A1), (A2), (A5)-(A6), (B1) and $H_0 : \|\beta\|_{L^2} = 0$ for all $j \in H_n$, we have the following with probability tending to one, for some positive constants $c_0$ and $c_1$.

1) $\|n^{-1}\tilde{\Theta}'_{H_n^c}(Y - \Theta_{H_n^c}^\eta_{H_n^c})\|_{\infty} \leq c_0 (n^{\beta/2-1/2} + q_n s_n^{-\delta})$,

2) $\|n^{-1}\tilde{\Theta}'_{H_n^c}(Y - \Theta_{H_n^c}^\eta_{H_n^c})\|_{\infty} \leq c_1 (n^{\beta/2-1/2} + q_n s_n^{-\delta})$. 
Proof. First, under $H_0: \| \beta_j \|_{L^2} = 0$ for all $j \in \mathcal{H}_n$, we have that

$$
\| n^{-1} \hat{\beta}_n \|_{\mathcal{H}_n} (Y - \Theta \eta_{\mathcal{H}_n}) \|_{\infty} = \| n^{-1} \sum_{i=1}^{n} \hat{F}_i (Y_i - F_i') \|_{\infty}
$$

$$
= \| n^{-1} \sum_{i=1}^{n} \hat{F}_i \epsilon_i + n^{-1} \sum_{i=1}^{n} \sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} \{ \hat{F}_i \theta_{ijk} - E(\hat{F}_i \theta_{ijk}) \} \eta_{jk} + n^{-1} \sum_{i=1}^{n} \sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} E(\hat{F}_i \theta_{ijk}) \eta_{jk} \|_{\infty}
$$

$$
\leq \| n^{-1} \sum_{i=1}^{n} \hat{F}_i \epsilon_i + n^{-1} \sum_{i=1}^{n} \sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} \{ \hat{F}_i \theta_{ijk} - E(\hat{F}_i \theta_{ijk}) \} \eta_{jk} \|_{\infty}
$$

$$
\| n^{-1} \sum_{i=1}^{n} \sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} E(\hat{F}_i \theta_{ijk}) \eta_{jk} \|_{\infty}. \quad (S2.3)
$$

In addition, for each $l = 1, \ldots, (p_n - h_n) s_n$, we have

$$
E \{ (n^{-1} \sum_{i=1}^{n} \hat{F}_i \epsilon_i)^2 \} = n^{-1} \sigma^2 \sim n^{-1}. \quad (S2.4)
$$

Moreover, for each $l = 1, \ldots, (p_n - h_n) s_n$, we have

$$
E \left[ n^{-1} \sum_{i=1}^{n} \sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} \{ \hat{F}_i \theta_{ijk} - E(\hat{F}_i \theta_{ijk}) \} \eta_{jk} \right]^2
$$

$$
\leq n^{-2} \sum_{i=1}^{n} E \left[ \sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} \{ \hat{F}_i \theta_{ijk} - E(\hat{F}_i \theta_{ijk}) \} \eta_{jk} \right]^2
$$

$$
\leq n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{q_n} E \left[ \sum_{k=s_n+1}^{\infty} \{ \hat{F}_i \theta_{ijk} - E(\hat{F}_i \theta_{ijk}) \} \eta_{jk} \right]^2
$$

$$
\leq n^{-2} q_n \sum_{i=1}^{n} \left( \sum_{j=1}^{q_n} \sum_{k=s_n+1}^{\infty} \eta_{jk}^2 k^{2\delta} \right) \left( \sum_{k_1=s_n+1}^{\infty} E(\hat{F}_i \theta_{ijk_1}) k_1^{-2\delta} \right)
$$

$$
= n^{-2} q_n \sum_{i=1}^{n} \left( \sum_{j=1}^{q_n} \sum_{k=s_n+1}^{\infty} \eta_{jk}^2 k^{2\delta} \right) \left( \sum_{k_1=s_n+1}^{\infty} E(\hat{F}_i \theta_{ijk_1}) \omega_{jk_1} k_1^{-2\delta} \right)
$$

$$
\leq cn^{-1} q_n^2 s_n^{-2\delta} = o(n^{-1}), \quad (S2.5)
$$
where the last inequality is by (A1), (A2), (A6) and (B1). Therefore, by combining (S2.4) with (S2.5), we have that for any \(l = 1, \ldots, (p_n - h_n)s_n\), there exists a universal constant \(c_0 > 0\) such that

\[
\lim_{n \to \infty} P\left[|n^{-1} \sum_{i=1}^{n} \sum_{j \in H_n} \sum_{k=s_n+1}^{\infty} \{\tilde{F}_{il}\theta_{ijk} - E(\tilde{F}_{il}\theta_{ijk})\} \eta_{jk}| \leq c_0|n^{-1} \sum_{i=1}^{n} \tilde{F}_{il}\epsilon_i|\right] = 1,
\]

which implies that for any \(t > 0\),

\[
\lim_{n \to \infty} P\left[|n^{-1} \sum_{i=1}^{n} \tilde{F}_{il}\epsilon_i + n^{-1} \sum_{i=1}^{n} \sum_{j \in H_n} \sum_{k=s_n+1}^{\infty} \{\tilde{F}_{il}\theta_{ijk} - E(\tilde{F}_{il}\theta_{ijk})\} \eta_{jk}| \geq t\right]
\leq \lim_{n \to \infty} P\left(|n^{-1} \sum_{i=1}^{n} \tilde{F}_{il}\epsilon_i| \geq t\right) + \lim_{n \to \infty} P\left(|n^{-1} \sum_{i=1}^{n} \sum_{j \in H_n} \sum_{k=s_n+1}^{\infty} \{\tilde{F}_{il}\theta_{ijk} - E(\tilde{F}_{il}\theta_{ijk})\} \eta_{jk}| \geq t\right)
\leq \lim_{n \to \infty} P\left(|n^{-1} \sum_{i=1}^{n} \tilde{F}_{il}\epsilon_i| \geq t\right) + \lim_{n \to \infty} P\left(|n^{-1} \sum_{i=1}^{n} \tilde{F}_{il}\epsilon_i| \geq c_0^{-1}t\right)
\leq 2 \lim_{n \to \infty} P\left(|n^{-1} \sum_{i=1}^{n} \tilde{F}_{il}\epsilon_i| \geq c_1 t\right),
\]

(S2.6)

where \(c_1 = \min(1, c_0^{-1}) > 0\). Moreover, under (A2) and by Bernstein inequality, we have that for any \(l = 1, \ldots, (p_n - h_n)s_n\), and any \(t > 0\),

\[
\lim_{n \to \infty} P\left(|n^{-1} \sum_{i=1}^{n} \tilde{F}_{il}\epsilon_i| \geq c_1 t\right) \leq \lim_{n \to \infty} 2 \exp \left\{ -c_2 \min(c_3^{-2}t^2, c_3^{-1}t)n \right\},
\]

(S2.7)

where \(c_2\) and \(c_3\) are some universal positive constants. By combining (S2.6) with (S2.7), we have that for any \(l = 1, \ldots, (p_n - h_n)s_n\), and any \(t > 0\),

\[
\lim_{n \to \infty} P\left[|n^{-1} \sum_{i=1}^{n} \tilde{F}_{il}\epsilon_i + n^{-1} \sum_{i=1}^{n} \sum_{j \in H_n} \sum_{k=s_n+1}^{\infty} \{\tilde{F}_{il}\theta_{ijk} - E(\tilde{F}_{il}\theta_{ijk})\} \eta_{jk}| \geq t\right]
\leq 2 \lim_{n \to \infty} P\left(|n^{-1} \sum_{i=1}^{n} \tilde{F}_{il}\epsilon_i| \geq c_1 t\right) \leq \lim_{n \to \infty} 4 \exp \left\{ -c_2 \min(c_3^{-2}t^2, c_3^{-1}t)n \right\},
\]
invoking union bound inequality, we have that for any $t > 0$,

$$
\lim_{n \to \infty} P\left[ ||n^{-1} \sum_{i=1}^{n} \tilde{F}_i \epsilon_i + n^{-1} \sum_{i=1}^{n} \sum_{j \in \mathcal{H}_n^c} \sum_{k=s_n+1}^\infty \{ \tilde{F}_i \theta_{ij} - E(\tilde{F}_i \theta_{ij}) \} \eta_{jk} ||_\infty \geq t \right]
\leq \lim_{n \to \infty} 4p_n s_n \exp \left\{ - c_2 \min(c_3^{-2} t^2, c_3^{-1} t) n \right\},
$$

by choosing $t = c_4 \{ \log(p_n s_n)/n \}^{1/2}$, for some sufficiently large $c_4 > 0$, we have that

$$
||n^{-1} \sum_{i=1}^{n} \tilde{F}_i \epsilon_i + n^{-1} \sum_{i=1}^{n} \sum_{j \in \mathcal{H}_n^c} \sum_{k=s_n+1}^\infty \{ \tilde{F}_i \theta_{ij} - E(\tilde{F}_i \theta_{ij}) \} \eta_{jk} ||_\infty
\leq c_4 \{ \log(p_n s_n)/n \}^{1/2} \leq c_5 n^{\beta/2 - 1/2}
$$

(S2.8)

with probability tending to one, for some $c_5 > 0$. Furthermore, we have

$$
||n^{-1} \sum_{i=1}^{n} \sum_{j \in \mathcal{H}_n^c} \sum_{k=s_n+1}^\infty E(\tilde{F}_i \theta_{ij}) \eta_{jk} ||_\infty = \max_{l \leq (p_n - h_n) s_n} \left| \sum_{i=1}^{n} \sum_{j \in \mathcal{H}_n^c} \sum_{k=s_n+1}^\infty E(\tilde{F}_i \theta_{ij}) \eta_{jk} \right|
\leq \max_{l \leq (p_n - h_n) s_n} n^{-1} \sum_{i=1}^{n} \sum_{j \in \mathcal{H}_n^c} \sum_{k=s_n+1}^\infty E(\tilde{F}_i \theta_{ij}) \eta_{jk}
\leq \max_{l \leq (p_n - h_n) s_n} n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{q_n} \sum_{k=s_n+1}^\infty E(\tilde{F}_i \theta_{ij}) \eta_{jk}
\leq \max_{l \leq (p_n - h_n) s_n} n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{q_n} \sum_{k=s_n+1}^\infty \left( \sum_{k_1=s_n+1}^\infty \eta_{jk_1 k_1} 2^{2\delta} \right)^{1/2} \left( \sum_{k_1=s_n+1}^\infty \eta_{jk_1 k_1} k_1 \right)^{1/2}
\leq n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{q_n} \sum_{k=s_n+1}^\infty \omega_{jk} k^{-2\delta} \left( \sum_{k_1=s_n+1}^\infty \eta_{jk_1 k_1} 2^{2\delta} \right)^{1/2} \left( \sum_{k_1=s_n+1}^\infty \eta_{jk_1 k_1} k_1 \right)^{1/2}
\leq c_6 q_n s_n^{-\delta},
$$

(S2.9)

for some constant $c_6 > 0$, where the last inequality is by (A1) and (A6). By combining (S2.1), (S2.8) with (S2.9), we conclude that

$$
||n^{-1} \tilde{\Theta}_{\mathcal{H}_n} (Y - \tilde{\Theta}_{\mathcal{H}_n} \eta_{\mathcal{H}_n^c}) ||_\infty \leq c_7 (n^{\beta/2 - 1/2} + q_n s_n^{-\delta}),
$$

with probability tending to one, for some constant $c_7 > 0$,
which completes the proof of part 1). For part 2), we have that

\[
\left\| n^{-1} \Theta_{\mathcal{H}_n}' (Y - \Theta_{\mathcal{H}_n} \eta_{\mathcal{H}_n}) \right\|_{\infty} = \left\| n^{-1} \Lambda_{\mathcal{H}_n} \hat{\Theta}'_{\mathcal{H}_n} (Y - \Theta_{\mathcal{H}_n} \eta_{\mathcal{H}_n}) \right\|_{\infty} \\
\leq \lambda_{\text{max}}(\Lambda_{\mathcal{H}_n}) \left\| n^{-1} \hat{\Theta}'_{\mathcal{H}_n} (Y - \Theta_{\mathcal{H}_n} \eta_{\mathcal{H}_n}) \right\|_{\infty} \\
\leq c_8 (n^\beta/2 - 1/2 + q_n s_n^{-1}),
\]

with probability tending to one, for some constant \( c_8 > 0 \), which completes the proof. \( \square \)

**Lemma 6.** Under conditions (A1)–(A3), (A5), (B1) and (P1)–(P5), we have the following with probability tending to one, for some positive constants \( c_0 \) and \( c_1 \), where \( \tilde{\nu} = \Lambda \nu = \tilde{\eta} - \tilde{\eta}^* = \Lambda (\eta - \eta^*) \),

1) \( n^{-1} \sum_{j=1}^{p_n} \left\| \Theta_j (\eta_j - \eta_j^*) \right\|_2^2 \leq m_1 \{1 + o(1)\} \|	ilde{\nu}\|_2^2 \), where the constant \( m_1 \) is defined in (A3).

2) \( ||\tilde{\nu}||_1 \leq c_0 s_n^{1/2} \sum_{j=1}^{p_n} n^{-1/2} ||\Theta_j (\eta_j - \eta_j^*)||_2, \quad \lambda_n ||\tilde{\nu}||_1 \leq c_1 \{ P_{\lambda_n}(\eta^*) + P_{\lambda_n}(\eta) + ||\tilde{\nu}||_2^2 \}. \)
Proof. First, we have that
\[
\begin{align*}
 n^{-1} \sum_{j=1}^{p_n} \| \Theta_j(\eta_j - \eta_j^*) \|_2^2 &= n^{-1} \sum_{j=1}^{p_n} \| \hat{\Theta}_j(\hat{\eta}_j - \hat{\eta}_j^*) \|_2^2 = \sum_{j=1}^{p_n} (\hat{\eta}_j - \hat{\eta}_j^*)(n^{-1}\hat{\Theta}'_j\hat{\Theta}_j)(\hat{\eta}_j - \hat{\eta}_j^*) \\
 \leq& \lambda_{\text{max}}(\hat{I})\| \hat{\nu} \|_2^2 + \sum_{j=1}^{p_n} (\hat{\eta}_j - \hat{\eta}_j^*)'(n^{-1}\hat{\Theta}'_j\hat{\Theta}_j) - E(n^{-1}\hat{\Theta}'_j\hat{\Theta}_j)(\hat{\eta}_j - \hat{\eta}_j^*) \\
 \leq& \lambda_{\text{max}}(\hat{I})\| \hat{\nu} \|_2^2 + \| n^{-1}\hat{\Theta}'\hat{\Theta} - E(n^{-1}\hat{\Theta}'\hat{\Theta}) \|_\infty \sum_{j=1}^{p_n} \| \hat{\eta}_j - \hat{\eta}_j^* \|_1^2 \\
 \leq& \{ \lambda_{\text{max}}(\hat{I}) + s_n\| n^{-1}\hat{\Theta}'\hat{\Theta} - E(n^{-1}\hat{\Theta}'\hat{\Theta}) \|_\infty \} \| \hat{\nu} \|_2^2 \\
 \leq& (m_1 + c_2 s_n n^{\beta/2 - 1/2})\| \hat{\nu} \|_2^2,
\end{align*}
\]
for some constant $c_2 > 0$, where the last inequality is by (A3) and Lemma 4.

Since $s_n n^{\beta/2 - 1/2} = o(1)$, it follows that $n^{-1} \sum_{j=1}^{p_n} \| \Theta_j(\eta_j - \eta_j^*) \|_2^2 \leq m_1 \{ 1 + o(1) \} \| \hat{\nu} \|_2^2$, with probability tending to one, which completes the proof of part 1).

Moreover, we have
\[
\begin{align*}
\| \hat{\nu} \|_1 &= \sum_{j=1}^{p_n} s_n \sum_{k=1}^{s_n} | \hat{\eta}_{jk} - \hat{\eta}_{jk}^* | \\
 &\leq m_0^{-1/2} s_n^{1/2} \sum_{j=1}^{p_n} \{ \lambda_{\text{min}}(\hat{I}) \}^{1/2} \| \hat{\eta}_j - \hat{\eta}_j^* \|_2 \\
 &\leq m_0^{-1/2} s_n^{1/2} \sum_{j=1}^{p_n} \{ n^{-1}\| \Theta_j(\eta_j - \eta_j^*) \|_2^2 + \| E(n^{-1}\hat{\Theta}'\hat{\Theta}) - n^{-1}\hat{\Theta}'\hat{\Theta} \|_\infty \| \hat{\eta}_j - \hat{\eta}_j^* \|_1^2 \}^{1/2} \\
 &\leq m_0^{-1/2} s_n^{1/2} \sum_{j=1}^{p_n} \{ n^{-1/2}\| \Theta_j(\eta_j - \eta_j^*) \|_2 + \| E(n^{-1}\hat{\Theta}'\hat{\Theta}) - n^{-1}\hat{\Theta}'\hat{\Theta} \|_\infty \| \hat{\eta}_j - \hat{\eta}_j^* \|_1 \} \\
 &\leq \{ m_0^{-1/2} s_n^{1/2} \sum_{j=1}^{p_n} n^{-1/2}\| \Theta_j(\eta_j - \eta_j^*) \|_2 \} + c_3 s_n^{1/2} n^{\beta/4 - 1/4} \| \hat{\nu} \|_1,
\end{align*}
\]
for some constant $c_3 > 0$, where the last inequality is by Lemma 4. Invoking
(B1), we have that \(|\tilde{\nu}|_1 \leq c_4 s_n^{1/2} \sum_{j=1}^{p_n} n^{-1/2} ||\Theta_j(\eta_j - \eta_j^*)||_2\), with probability tending to one, for some constant \(c_4 > 0\), which completes the proof of the first inequality of 2). For the second inequality of 2), by combining the first inequality of 2) with Lemma 1(c), we have

\[
\lambda_n ||\tilde{\nu}||_1 \leq c_4 \lambda_n s_n^{1/2} \sum_{j=1}^{p_n} n^{-1/2} ||\Theta_j(\eta_j - \eta_j^*)||_2
\]

\[
= c_4 L^{-1} \sum_{j=1}^{p_n} \lambda_n s_n^{1/2} L n^{-1/2} ||\Theta_j(\eta_j - \eta_j^*)||_2
\]

\[
\leq c_4 L^{-1} \sum_{j=1}^{p_n} \{\rho_{\lambda_n s_n^{1/2}} (n^{-1/2} ||\Theta_j(\eta_j - \eta_j^*)||_2) + 2^{-1} \mu n^{-1} ||\Theta_j(\eta_j - \eta_j^*)||_2^2\}
\]

\[
\leq c_4 L^{-1} \{P_{\lambda_n}(\eta^*) + P_{\lambda_n}(\eta) + 2^{-1} \mu n^{-1} \sum_{j=1}^{p_n} ||\Theta_j(\eta_j - \eta_j^*)||_2^2\}.
\]  \(\text{(S2.10)}\)

Finally, by combining (S2.10) with part 1), we have that \(\lambda_n ||\tilde{\nu}||_1 \leq c_5 \{P_{\lambda_n}(\eta^*) + P_{\lambda_n}(\eta) + ||\tilde{\nu}||_2^2\}\), with probability tending to one, for some constant \(c_5 > 0\), which completes the proof of part 2).

**Lemma 7.** Under conditions (A1)–(A3), (A5), (B1)–(B3), we have the following with probability tending to one, for some positive constant \(c_0\) and \(c_2\), where \(\rho_{nl} = ||w_l||_0 \) and \(\rho_n = \max_{l \leq h_s n} ||w_l||_0 = \max_{l \leq h_s n} \rho_{nl}\):

1. \(\max_{l \leq h_s n} ||n^{-1} \sum_{i=1}^{n} F_i F'_i (\hat{w}_l - w_l)||_\infty \leq c_0 n^{3/2 - 1/2}\),

2. \(\max_{l \leq h_s n} ||\hat{w}_l - w_l||_2 \leq c_1 \rho_n^{1/2} s_n^{\alpha} n^{\beta/2 - 1/2}\), \(\max_{l \leq h_s n} ||\hat{w}_l - w_l||_1 \leq c_2 \rho_n s_n^{\alpha} n^{\beta/2 - 1/2}\).
Proof. First, by Lemma 4, we have

$$\max_{l \leq h_n s_n} ||n^{-1} \sum_{i=1}^{n} (w_l' F_i - E_{il}) F_i'||_\infty \leq c_0 n^{\beta/2 - 1/2},$$

with probability tending to one, for some constant $c_0 > 0$. Moreover, under (A5), (B1) and (B3), we have $\tau_n \sim n^{\beta/2 - 1/2}$. By choosing $\tau_n = c_0 n^{\beta/2 - 1/2}$, we have

$$\max_{l \leq h_n s_n} ||n^{-1} \sum_{i=1}^{n} (w_l' F_i - E_{il}) F_i'||_\infty \leq \tau_n = c_0 n^{\beta/2 - 1/2}. \tag{S2.11}$$

Under (7) of the main paper, we have

$$\max_{l \leq h_n s_n} ||n^{-1} \sum_{i=1}^{n} (\hat{w}_l' F_i - E_{il}) F_i'||_\infty \leq \tau_n = c_0 n^{\beta/2 - 1/2}. \tag{S2.12}$$

By combining (S2.11) with (S2.12), we have

$$\max_{l \leq h_n s_n} ||n^{-1} \sum_{i=1}^{n} (\hat{w}_l' F_i - E_{il}) F_i'||_\infty \leq 2 \tau_n = 2 c_0 n^{\beta/2 - 1/2}, \tag{S2.13}$$

which completes the proof of part 1). In addition, by the definition of the Dan-tizig selector method, we have

$$P \left( \bigcap_{l=1}^{h_n s_n} \{ ||\hat{w}_l||_1 \leq ||w_l||_1 \} \right) \rightarrow 1, \quad \text{as} \quad n \rightarrow \infty. \tag{S2.14}$$

Denote $S_l = \{ j : w_{lj} \neq 0 \}$ as the support set for each vector $w_l$, then (S2.14)
implies that

\[
\max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_1 \leq \max_{l \leq h_n s_n} \|(\hat{w}_l - w_l) s_l\|_1 + \max_{l \leq h_n s_n} \|(\hat{w}_l) s_l\|_1
\]

\[
\leq \max_{l \leq h_n s_n} \|(\hat{w}_l - w_l) s_l\|_1 + \max_{l \leq h_n s_n} \{\|(w_l) s_l\|_1 - \|(\hat{w}_l) s_l\|_1\}
\]

\[
\leq 2\max_{l \leq h_n s_n} \||(\hat{w}_l - w_l) s_l\|_1 \leq 2\max_{l \leq h_n s_n} \rho_n^{1/2}\|(\hat{w}_l - w_l) s_l\|_2
\]

\[
\leq 2\rho_n^{1/2} \max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_2, \\ (S2.15)
\]

with probability tending to one. Furthermore, we have

\[
\max_{l \leq h_n s_n} (\hat{w}_l - w_l)'(n^{-1}\sum_{i=1}^{n} F_i F_i')(\hat{w}_l - w_l)
\]

\[
\leq \max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_1 \|n^{-1}\sum_{i=1}^{n} F_i F_i'(\hat{w}_l - w_l)\|_{\infty}
\]

\[
\leq 2c_0 n^{\beta/2-1/2} \max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_1, \\ (S2.16)
\]

with probability tending to one, where the last inequality is by (S2.13). In addition, we also have

\[
\max_{l \leq h_n s_n} (\hat{w}_l - w_l)'(n^{-1}\sum_{i=1}^{n} F_i F_i')(\hat{w}_l - w_l)
\]

\[
\geq \max_{l \leq h_n s_n} \left\{ \lambda_{\min}(I) \|\hat{w}_l - w_l\|_2^2 - \|\hat{w}_l - w_l\|_1^2 \|E(G_i G_i') - n^{-1}\Theta'\Theta\|_{\infty} \right\}
\]

\[
\geq \lambda_{\min}(I) \left( \max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_2^2 - \|E(G_i G_i') - n^{-1}\Theta'\Theta\|_{\infty} \right) \left( \max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_1^2 \right)
\]

\[
\geq c_1 s_n^{-a} \left( \max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_2^2 \right) - c_2 n^{\beta/2-1/2} \left( \max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_1^2 \right)
\]

\[
\geq (c_1 s_n^{-a} - c_3 \rho_n n^{\beta/2-1/2}) \left( \max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_2^2 \right), \\ (S2.17)
\]
with probability tending to one, where the second last inequality is by (A1), (A3) and Lemma 4, while the last inequality is by (S2.15). Combining (S2.15), (S2.16), (S2.17) with (B2) entails that
\[
\max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_2 \leq c_4 \rho_n^{1/2} s_n^{a_3} n^{\beta/2-1/2},
\]  
(S2.18)
with probability tending to one, for some constant \(c_4 > 0\). Finally, combining (S2.15) with (S2.18) entails that
\[
\max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_1 \leq c_5 \rho_n s_n^{a_3} n^{\beta/2-1/2},
\]  
with probability tending to one, for some constant \(c_5 > 0\), which completes the proof of part 2).

**Lemma 8.** Under conditions (A1)–(A3), (A5)–(A6), (B1)–(B3) and (P1)–(P5), we have the following with probability tending to one, for some positive constant \(c_0\) and \(c_1\), where \(\hat{S}_i = (\hat{S}_{i1}, \ldots, \hat{S}_{ih_n s_n}) = \hat{\Lambda}_n^{-1}(\hat{\omega}'F_i - E_i)(Y_i - F_i'\hat{\eta}_{H_n})\) and \(S_i = (S_{i1}, \ldots, S_{i h_n s_n}) = \Lambda_n^{-1}(w'F_i - E_i)\epsilon_i,\)
\[
\max_{l \leq h_n s_n} \left| n^{-1/2} \sum_{i=1}^n (\hat{S}_{il} - S_{il}) \right| \leq c_0 \left( n^{\beta - 1/2} \rho_n s_n^{a_3/2} + \lambda_n n^{\beta/2} q_n s_n^{a_3/2} + n^{\beta/2-1/2} q_n s_n^{-\delta} \log s_n \right) + n^{\beta} \rho_n q_n s_n^{3a_3/2-\delta} \log s_n,
\]
\[
\max_{l \leq h_n s_n} \left\{ n^{-1} \sum_{i=1}^n (\hat{S}_{il} - S_{il})^2 \right\}^{1/2} \leq c_1 \left\{ \lambda_n n^{\beta} q_n s_n^{a_3/2} + n^{\beta/2} q_n s_n^{-\delta} \log s_n + \rho_n s_n^{3a_3/2} n^{\beta-1/2} (\log n)^{1/2} \lambda_n \rho_n q_n s_n^{2a_3/2} n^{3\beta/2-1/2} + n^{\beta-1/2} \rho_n q_n s_n^{3a_3/2-\delta} \log s_n \right\}.
\]

**Proof.** First, for each \(l = 1, \ldots, h_n s_n\), we have
\[
\hat{S}_{il} - S_{il} = \Delta_{1il} + \Delta_{2il} + \Delta_{3il} + \Delta_{4il} + \Delta_{5il} + \Delta_{6il},
\]
where we denote

\[
\Delta_{1il} = \left[ \{ E(E_{il}^2) \} / (n^{-1} \sum_{i=1}^{n} E_{i,il}^2) \right]^{1/2} - 1 \left[ \{ E(E_{il}^2) \}^{-1/2} (w_i'F_i - E_{il}) \epsilon_i \right],
\]

\[
\Delta_{2il} = \left[ \{ E(E_{il}^2) \} / (n^{-1} \sum_{i=1}^{n} E_{i,il}^2) \right]^{1/2} \left[ \{ E(E_{il}^2) \}^{-1/2} (w_i'F_i - E_{il}) F'_i (\eta H^\theta - \hat{\eta} H^\theta) \right],
\]

\[
\Delta_{3il} = \left[ \{ E(E_{il}^2) \} / (n^{-1} \sum_{i=1}^{n} E_{i,il}^2) \right]^{1/2} \left[ \{ E(E_{il}^2) \}^{-1/2} (w_i'F_i - E_{il}) (\sum_{j \in H^\theta k = s_n + 1} \theta_{ijk} \eta_{jk}) \right],
\]

\[
\Delta_{4il} = \left[ \{ E(E_{il}^2) \} / (n^{-1} \sum_{i=1}^{n} E_{i,il}^2) \right]^{1/2} \left[ \{ E(E_{il}^2) \}^{-1/2} (\hat{w}_l - w_i)' F_i \epsilon_i \right],
\]

\[
\Delta_{5il} = \left[ \{ E(E_{il}^2) \} / (n^{-1} \sum_{i=1}^{n} E_{i,il}^2) \right]^{1/2} \left[ \{ E(E_{il}^2) \}^{-1/2} (\hat{w}_l - w_i)' F_i F'_i (\eta H^\theta - \hat{\eta} H^\theta) \right],
\]

\[
\Delta_{6il} = \left[ \{ E(E_{il}^2) \} / (n^{-1} \sum_{i=1}^{n} E_{i,il}^2) \right]^{1/2} \left[ \{ E(E_{il}^2) \}^{-1/2} (\hat{w}_l - w_i)' F_i (\sum_{j \in H^\theta k = s_n + 1} \theta_{ijk} \eta_{jk}) \right].
\]

Accordingly, one can show that

\[
\max_{l \leq h_s} n^{-1/2} \sum_{i=1}^{n} (\hat{S}_{il} - S_{il}) \leq \max_{l \leq h_s} n^{-1/2} \sum_{i=1}^{n} \Delta_{1il} + \max_{l \leq h_s} n^{-1/2} \sum_{i=1}^{n} \Delta_{2il}
\]

\[
+ \max_{l \leq h_s} n^{-1/2} \sum_{i=1}^{n} \Delta_{3il} + \max_{l \leq h_s} n^{-1/2} \sum_{i=1}^{n} \Delta_{4il}
\]

\[
+ \max_{l \leq h_s} n^{-1/2} \sum_{i=1}^{n} \Delta_{5il} + \max_{l \leq h_s} n^{-1/2} \sum_{i=1}^{n} \Delta_{6il}, \quad (S2.19)
\]
For \( \max_{l \leq h_n, s_n} n^{-1/2} \sum_{i=1}^{n} \Delta_{1il} \), we have

\[
\begin{align*}
\max_{l \leq h_n, s_n} n^{-1/2} \sum_{i=1}^{n} \Delta_{1il} &= \max_{l \leq h_n, s_n} n^{-1/2} \sum_{i=1}^{n} \left[ \left\{ E(E_{il}^2)/(n^{-1} \sum_{i_1=1}^{n} E_{i_1l}^2) \right\}^{1/2} - 1 \right] \left[ \left\{ E(E_{il}^2) \right\}^{-1/2} (w_i' F_i - E_{il}) \epsilon_i \right] \\
&\leq \|\Lambda \hat{\Lambda}^{-1} - I\|_\infty \max_{l \leq h_n, s_n} n^{-1/2} \sum_{i=1}^{n} (w_i' F_i - E_{il}) \left\{ E(E_{il}^2) \right\}^{-1/2} \epsilon_i \\
&= n^{1/2} \|\Lambda \hat{\Lambda}^{-1} - I\|_\infty \max_{l \leq h_n, s_n} n^{-1} \sum_{i=1}^{n} (w_i' F_i - E_{il}) \left\{ E(E_{il}^2) \right\}^{-1/2} \epsilon_i \\
&\leq c_0 n^{\beta - 1/2},
\end{align*}
\]

with probability tending to one, for some constant \( c_0 > 0 \), where the last inequality is by Lemma 3 and Lemma 4.

For \( \max_{l \leq h_n, s_n} n^{-1/2} \sum_{i=1}^{n} \Delta_{2il} \), we have

\[
\begin{align*}
\max_{l \leq h_n, s_n} n^{-1/2} \sum_{i=1}^{n} \Delta_{2il} &\leq (1 + \|\Lambda \hat{\Lambda}^{-1} - I\|_\infty) \max_{l \leq h_n, s_n} n^{-1/2} \sum_{i=1}^{n} \left[ \left\{ E(E_{il}^2) \right\}^{-1/2} (w_i' F_i - E_{il}) F_i' (\eta H_n - \hat{\eta} H_n) \right] \\
&\leq n^{1/2} (1 + \|\Lambda \hat{\Lambda}^{-1} - I\|_\infty) \|\eta H_n - \hat{\eta} H_n\|_1 \max_{l \leq h_n, s_n} n^{-1} \sum_{i=1}^{n} (w_i' F_i - E_{il}) \left\{ E(E_{il}^2) \right\}^{-1/2} F_i' \|_\infty \\
&\leq n^{1/2} (1 + \|\Lambda \hat{\Lambda}^{-1} - I\|_\infty) \|\hat{\eta} - \eta\|_1 \max_{l \leq h_n, s_n} n^{-1} \sum_{i=1}^{n} (w_i' F_i - E_{il}) \left\{ E(E_{il}^2) \right\}^{-1/2} F_i' \|_\infty \\
&\leq c_1 \lambda_n n^{\beta/2} q_n s_n^{a/2+1},
\end{align*}
\]

with probability tending to one, for some constant \( c_1 > 0 \), where the last inequality is by Lemma 3, Lemma 4 and Theorem 1.
For \( \max_{t \leq h_n s_n} |n^{-1/2} \sum_{i=1}^n \Delta_{3il}| \), we have

\[
\max_{t \leq h_n s_n} |n^{-1/2} \sum_{i=1}^n \Delta_{3il}| \\
\leq (1 + \|\Lambda \tilde{\Lambda}^{-1} - I\|_{\infty}) \max_{t \leq h_n s_n} n^{-1/2} \left\{ E\left( (E_{il}^2) \right)^{-1/2} \right\} \left( \sum_{j \in H_n} \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right) \\
\leq n^{1/2} \left( 1 + \|\Lambda \tilde{\Lambda}^{-1} - I\|_{\infty} \right) \left[ \max_{t \leq h_n s_n} n^{-1/2} \left\{ E\left( (E_{il}^2) \right)^{-1/2} \right\} \right] \\
\cdot \left( n^{-1} \sum_{i=1}^n \left| \sum_{j \in H_n} \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right| \right). \quad (S2.22)
\]

Regarding \( \max_{t \leq h_n s_n} \max_{i \leq n} | \{ E\left( (E_{il}^2) \right)^{-1/2} \} | \), based on (A2), (A5) and (B1), we have

\[
\max_{t \leq h_n s_n} \max_{i \leq n} | \{ E\left( (E_{il}^2) \right)^{-1/2} \} | \\
\leq c_1 \{ \log(np_n s_n) \}^{1/2} \leq c_2 n^{\delta/2}, \quad (S2.23)
\]

with probability tending to one, for some constants \( c_1, c_2 > 0 \).

Regarding \( n^{-1} \sum_{i=1}^n \left| \sum_{j \in H_n} \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right| \), we have

\[
E\left( n^{-1} \sum_{i=1}^n \left| \sum_{j \in H_n} \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right| \right) \leq E\left( n^{-1} \sum_{i=1}^n \sum_{j=1}^{q_n} \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right) \\
\leq E\left\{ n^{-1} \sum_{i=1}^n \sum_{j=1}^{q_n} \left( \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right)^{1/2} \left( \sum_{k_1=s_n+1}^{\infty} \eta_{jk_1}^2 k_1^{2\delta} \right)^{1/2} \right\} \\
\leq n^{-1} \sum_{i=1}^n \sum_{j=1}^{q_n} \left( \sum_{k=s_n+1}^{\infty} \eta_{jk_1}^2 k_1^{2\delta} \right)^{1/2} E\left\{ \left( \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right)^{1/2} \right\} \\
\leq n^{-1} \sum_{i=1}^n \sum_{j=1}^{q_n} \left( \sum_{k_1=s_n+1}^{\infty} \eta_{jk_1}^2 k_1^{2\delta} \right)^{1/2} E\left\{ \left( \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right)^{1/2} \right\} \\
\leq O(q_n s_n^{-\delta}), \quad (S2.24)
\]
where the last inequality is by (A1) and (A6). Hence, by combining (S2.22), (S2.23) with (S2.24), we have

$$\max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{3il}| \leq O_p(n^{\beta/2 + 1/2} q_n s_n^{-\delta}). \quad (S2.25)$$

For $$\max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{4il}|$$, we have

$$\max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{4il}| \leq (1 + ||\Lambda \hat{\Lambda}^{-1} - I||_{\infty}) \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \{ E(E_{il}^2) \}^{-1/2} (\hat{w}_l - w_l)' F_i \epsilon_i|$$

$$\leq n^{1/2} \left( 1 + ||\Lambda \hat{\Lambda}^{-1} - I||_{\infty} \right) \left[ \max_{l \leq h_n s_n} \{ E(E_{il}^2) \}^{-1/2} \right]$$

$$\cdot \left( \max_{l \leq h_n s_n} ||\hat{w}_l - w_l||_1 \right) (||n^{-1} \Theta' \epsilon||_{\infty})$$

$$\leq n^{1/2} \left( 1 + ||\Lambda \hat{\Lambda}^{-1} - I||_{\infty} \right) \left[ \lambda_{\min}(\Lambda) \right]^{-1}$$

$$\cdot \left( \max_{l \leq h_n s_n} ||\hat{w}_l - w_l||_1 \right) (||n^{-1} \Theta' \epsilon||_{\infty})$$

$$\leq c_3 n^{\beta - 1/2} \rho_n s_n^{3a/2}, \quad (S2.26)$$

with probability tending to one, for some constant $$c_3 > 0$$, where the last inequality is by Lemma 3, Lemma 4, Lemma 7 and (A1).
For \( \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{6il}| \), we have

\[
\max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{5il}|
\leq (1 + \|\Lambda \hat{\Lambda}^{-1} - I\|_{\infty}) \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \left[ \{E(E_{il}^2)\}^{1/2} (\hat{w}_l - w_l)' F_i (\sum_{j \in H^k_n} \sum_{k = s_n + 1}^{\infty} \theta_{ijk} \eta_{jk}) \right] |
\]
\[
\leq n^{1/2} \left(1 + \|\Lambda \hat{\Lambda}^{-1} - I\|_{\infty}\right) \left[ \{\lambda_{\min}(\Lambda)\}^{-1} \right] \left( \max_{l \leq h_n s_n} ||\hat{w}_l - w_l||_1 \right)
\]
\[
\cdot \left\{ \max_{l \leq h_n s_n} \sum_{i=1}^{n} F_i (\sum_{j \in H^k_n} \sum_{k = s_n + 1}^{\infty} \theta_{ijk} \eta_{jk}) \right\} \]
\[
\leq n^{1/2} \left(1 + \|\Lambda \hat{\Lambda}^{-1} - I\|_{\infty}\right) \left[ \{\lambda_{\min}(\Lambda)\}^{-1} \right] \left( \max_{l \leq h_n s_n} ||\hat{w}_l - w_l||_1 \right)
\]
\[
\cdot \left( \max_{l \leq (h_n - h_n) s_n} \max_{i \leq n} |F_{il}| \right) \left( n^{-1} \sum_{i=1}^{n} \sum_{j \in H^k_n} \sum_{k = s_n + 1}^{\infty} \theta_{ijk} \eta_{jk} \right)
\]
\[
\leq O_p(n^\beta \rho_n q_n s_n^{3a/2 - \delta}), \quad \text{(S2.27)}
\]

with probability tending to one, for some constant \( c_4 > 0 \), where the last inequality is by Lemma 3, Lemma 7, Theorem 1 and (A1).

For \( \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{6il}| \), we have

\[
\max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{6il}|
\leq (1 + \|\Lambda \hat{\Lambda}^{-1} - I\|_{\infty}) \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \left[ \{E(E_{il}^2)\}^{1/2} (\hat{w}_l - w_l)' F_i (\sum_{j \in H^k_n} \sum_{k = s_n + 1}^{\infty} \theta_{ijk} \eta_{jk}) \right] |
\]
\[
\leq n^{1/2} \left(1 + \|\Lambda \hat{\Lambda}^{-1} - I\|_{\infty}\right) \left[ \{\lambda_{\min}(\Lambda)\}^{-1} \right] \left( \max_{l \leq h_n s_n} ||\hat{w}_l - w_l||_1 \right)
\]
\[
\cdot \left\{ \max_{l \leq h_n s_n} \sum_{i=1}^{n} F_i (\sum_{j \in H^k_n} \sum_{k = s_n + 1}^{\infty} \theta_{ijk} \eta_{jk}) \right\} \]
\[
\leq n^{1/2} \left(1 + \|\Lambda \hat{\Lambda}^{-1} - I\|_{\infty}\right) \left[ \{\lambda_{\min}(\Lambda)\}^{-1} \right] \left( \max_{l \leq h_n s_n} ||\hat{w}_l - w_l||_1 \right)
\]
\[
\cdot \left( \max_{l \leq (h_n - h_n) s_n} \max_{i \leq n} |F_{il}| \right) \left( n^{-1} \sum_{i=1}^{n} \sum_{j \in H^k_n} \sum_{k = s_n + 1}^{\infty} \theta_{ijk} \eta_{jk} \right)
\]
\[
\leq O_p(n^\beta \rho_n q_n s_n^{3a/2 - \delta}), \quad \text{(S2.28)}
\]

where the last inequality is by Lemma 3, Lemma 7, (A1), (A2) and (S2.24).
In summary, by combining (S2.19), (S2.20), (S2.21), (S2.25), (S2.26), (S2.27) with (S2.28), we have

\[
\max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} (\hat{S}_{il} - S_{il})| \leq \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{1il}| + \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{2il}|
\]
\[
+ \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{3il}| + \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{4il}|
\]
\[
+ \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{5il}| + \max_{l \leq h_n s_n} |n^{-1/2} \sum_{i=1}^{n} \Delta_{6il}|
\]
\[
\leq c_0 n^{\beta - 1/2} + c_1 \lambda n^{\beta/2} q_n s_n^{a+1/2} + O_p(n^{\beta/2+1} q_n s_n^{-\delta})
\]
\[
+ c_3 n^{\beta - 1/2} \rho_n s_n^{3a/2} + c_4 \lambda n^{\beta/2} q_n s_n^{a+1} + O_p(n^{\beta} \rho_n q_n s_n^{3a/2-\delta})
\]
\[
\leq c_5 (n^{\beta - 1/2} \rho_n s_n^{3a/2} + \lambda n^{\beta/2} q_n s_n^{a+1} + n^{\beta/2+1} q_n s_n^{-\delta} \log s_n)
\]
\[
+ n^\beta \rho_n q_n s_n^{3a/2-\delta} \log s_n),
\]

with probability tending to one, for some constant $c_5 > 0$, which completes the proof of part 1).

Next, we start to prove part 2). First, we have

\[
\max_{l \leq h_n s_n} \{n^{-1} \sum_{i=1}^{n} (\hat{S}_{il} - S_{il})^2\}
\]
\[
\leq 100 \left( \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{1il}^2 + \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{2il}^2 + \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{3il}^2 \right)
\]
\[
+ \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{4il}^2 + \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{5il}^2 + \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{6il}^2 \right). \quad (S2.29)
\]
For $\max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{2l}^2$, we have

$$\max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{2l}^2 \leq \left( ||\hat{\Lambda}^{-1} - I||_\infty \right)^2 \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \left( \{E(E_{il}^2)\}^{-1/2} (w_i'F_i - E_{il}) \epsilon_i \right)^2$$

$$\leq \left( ||\hat{\Lambda}^{-1} - I||_\infty \right)^2 \max_{l \leq h_n s_n} \max_{1 \leq n} n^{-1} \sum_{i=1}^{n} \left( \{E(E_{il}^2)\}^{-1/2} (w_i'F_i - E_{il}) \epsilon_i \right)^2$$

$$\leq c_6 n^{2\beta - 1} \log n, \quad (S2.30)$$

with probability tending to one, for some constant $c_6 > 0$, where the last inequality is by Lemma 3 and (A2).

For $\max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{2l}^2$, we have

$$\max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{2l}^2 \leq \left( 1 + ||\hat{\Lambda}^{-1} - I||_\infty \right)^2 \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \left( \{E(E_{il}^2)\}^{-1/2} (w_i'F_i - E_{il}) \eta_{il} - \hat{\eta}_{il} \right)^2$$

$$\leq \left( 1 + ||\hat{\Lambda}^{-1} - I||_\infty \right)^2 \left( ||\hat{\eta} - \eta||_1 \right)^2 \left( \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} ||\{E(E_{il}^2)\}^{-1/2} (w_i'F_i - E_{il}) F_{il} ||_\infty \right)^2$$

$$\leq \left( 1 + ||\hat{\Lambda}^{-1} - I||_\infty \right)^2 \left( ||\hat{\eta} - \eta||_1 \right)^2 \left( \max_{l \leq h_n s_n} \max_{1 \leq n} n^{-1} \sum_{i=1}^{n} \left( \{E(E_{il}^2)\}^{-1/2} (w_i'F_i - E_{il}) F_{il} \right)^2 \right)$$

$$\leq c_7 \lambda_n^2 n^{2\beta} q_n^2 s_n^{a+2}, \quad (S2.31)$$

with probability tending to one, for some constant $c_7 > 0$, where the last inequality is by Lemma 3, Theorem 1 and (A2).
For $\max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{3l}^2$, we have

$$
\max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{3l}^2 \\
\leq \left(1 + ||\hat{\Lambda}^{-1} - I||_\infty\right)^2 \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \left\{ E(E_{il}^2) \right\}^{-1/2} (w'_i F_i - E_{il}) \left( \sum_{j \in H_n} \sum_{k = s_n + 1}^{\infty} \theta_{ijk} \eta_{jk} \right)^2 \\
\leq \left(1 + ||\hat{\Lambda}^{-1} - I||_\infty\right)^2 \left[ \max_{l \leq h_n s_n} \max_{i \leq n} \left\{ E(E_{il}^2) \right\}^{-1/2} (w'_i F_i - E_{il}) \right] \\
\cdot \left\{ n^{-1} \sum_{i=1}^{n} \left( \sum_{j \in H_n} \sum_{k = s_n + 1}^{\infty} \theta_{ijk} \eta_{jk} \right)^2 \right\}. 
$$

(S2.32)

Regarding $\max_{l \leq h_n s_n} \max_{i \leq n} \left| \{ E(E_{il}^2) \}^{-1/2} (w'_i F_i - E_{il}) \right|^2$, we have

$$
\max_{l \leq h_n s_n} \max_{i \leq n} \left| \{ E(E_{il}^2) \}^{-1/2} (w'_i F_i - E_{il}) \right|^2 \\
\leq c_8 \log(n h_n s_n) \leq c_8 \log(n p_n s_n) \\
\leq c_9 n^\beta, 
$$

(S2.33)

with probability tending to one, for some constants $c_8, c_9 > 0$, where the last inequality is by (A2), (A5) and (B1).
Regarding $n^{-1} \sum_{i=1}^{n} (\sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk})^2$, we have

$$E \left\{ n^{-1} \sum_{i=1}^{n} \left( \sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right)^2 \right\} \leq E \left\{ n^{-1} \sum_{i=1}^{n} \left( \sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} |\theta_{ijk} \eta_{jk}| \right)^2 \right\}$$

$$\leq E \left\{ n^{-1} q_n \sum_{i=1}^{n} \sum_{j=k_{s_n+1}}^{\infty} \sum_{k_{1}=s_n+1}^{\infty} \theta_{ijk}^2 k_{2\delta} \left( \sum_{k_{1}=s_n+1}^{\infty} \eta_{jk_{1}k_{2\delta}}^2 \right) \right\}$$

$$= n^{-1} q_n \sum_{i=1}^{n} \sum_{j=k_{s_n+1}}^{\infty} \sum_{k_{1}=s_n+1}^{\infty} \omega_{jk} k_{2\delta} \left( \sum_{k_{1}=s_n+1}^{\infty} \eta_{jk_{1}k_{2\delta}}^2 \right)$$

$$= O(q_n^2 s_n^{2\delta}). \quad (S2.34)$$

Hence, by combining (S2.32), (S2.33), (S2.34) with Lemma 3, one can show that

$$\max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{3il}^2 \leq O_p(n^\beta q_n^2 s_n^{2\delta}). \quad (S2.35)$$

For $\max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{4il}^2$, we have

$$\max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} \Delta_{4il}^2$$

$$\leq \left( 1 + \|\Lambda^{-1} - I\|_\infty \right)^2 \left[ \left\{ \lambda_{\text{min}}(\Lambda) \right\}^{-1} \right]^2 \left( \max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_1 \right)^2 \left( n^{-1} \sum_{i=1}^{n} \|F_i \epsilon_i\|_\infty^2 \right)$$

$$\leq \left( 1 + \|\Lambda^{-1} - I\|_\infty \right)^2 \left[ \left\{ \lambda_{\text{min}}(\Lambda) \right\}^{-1} \right]^2 \left( \max_{l \leq h_n s_n} \|\hat{w}_l - w_l\|_1 \right)^2 \left( \max_{l \leq h_n s_n} \max_{l \leq h_n s_n} |F_{il} \epsilon_i| \right)^2$$

$$\leq c_{10} q_n^2 s_n^{3\beta} n^{2\beta-1} \log n, \quad (S2.36)$$

with probability tending to one, for some constant $c_{10} > 0$, where the last inequality is by Lemma 3, Lemma 7, (A1) and (A2).
For $\max_{l \leq h} n^{-1} \sum_{i=1}^{n} \Delta_{5il}^2$, we have

$$\max_{l \leq h} n^{-1} \sum_{i=1}^{n} \Delta_{5il}^2 \leq \left(1 + ||\Lambda \hat{\Lambda}^{-1} - I||_{\infty}\right)^2 \left(\{\lambda_{\min}(\Lambda)\}^{-1}\right)^2 \left(\max_{l \leq h} ||\hat{\omega}_l - \omega_l||_1\right)^2 \left(\max_{i \leq n} ||F_i||_{\infty}\right)^2$$

$$\leq c_{11} \lambda_n^2 \rho_n^2 q_n^2 s_n^4 a_2 n^{3\beta - 1},$$

(S2.37)

with probability tending to one, for some constant $c_{11} > 0$, where the last inequality is by Lemma 3, Lemma 7, Theorem 1, (A1) and (A2).

For $\max_{l \leq h} n^{-1} \sum_{i=1}^{n} \Delta_{6il}^2$, we have

$$\max_{l \leq h} n^{-1} \sum_{i=1}^{n} \Delta_{6il}^2 \leq \left(1 + ||\Lambda \hat{\Lambda}^{-1} - I||_{\infty}\right)^2 \left(\{\lambda_{\min}(\Lambda)\}^{-1}\right)^2 \left(\max_{l \leq h} ||\hat{\omega}_l - \omega_l||_1\right)^2$$

$$\cdot \left\{ n^{-1} \sum_{i=1}^{n} \left( \sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right)^2 \right\}$$

$$= \left(1 + ||\Lambda \hat{\Lambda}^{-1} - I||_{\infty}\right)^2 \left(\{\lambda_{\min}(\Lambda)\}^{-1}\right)^2 \left(\max_{l \leq h} ||\hat{\omega}_l - \omega_l||_1\right)^2 \left(\max_{i \leq n} ||F_i||_{\infty}\right)^2$$

$$\cdot \left\{ n^{-1} \sum_{i=1}^{n} \left( \sum_{j \in \mathcal{H}_n} \sum_{k=s_n+1}^{\infty} \theta_{ijk} \eta_{jk} \right)^2 \right\}$$

$$\leq O_p(\rho_n^2 q_n^2 s_n^3 a_2 n^{3\beta - 1}),$$

(S2.38)

where the last inequality is by Lemma 3, Lemma 7, (S2.34), (A1) and (A2).

In summary, by combining (S2.30), (S2.31), (S2.35), (S2.36), (S2.37), (S2.38)
with (S2.29), we observe that

$$\max_{l \leq h_n s_n} \left\{ n^{-1} \sum_{i=1}^{n} (\hat{S}_{il} - S_{il})^2 \right\}$$

$$\leq c_{12} \left\{ \lambda_n^2 n^{2\beta} q_n^2 s_n^{a+2} + n^\beta q_n^2 s_n^{-2\delta} (\log s_n)^2 + \rho_n^2 s_n^{3a} n^{2\beta-1} \log n$$

$$+ \lambda_n^2 \rho_n^2 q_n^2 s_n^{2a+2} n^{3\beta-1} + n^{2\beta-1} \rho_n^2 q_n^2 s_n^{3a-2\delta} (\log s_n)^2 \right\}, \quad (S2.39)$$

with probability tending to one, for some constant $c_{12} > 0$, which further implies that

$$\max_{l \leq h_n s_n} \left\{ n^{-1} \sum_{i=1}^{n} (\hat{S}_{il} - S_{il})^2 \right\}^{1/2}$$

$$\leq c_{13} \left\{ \lambda_n n^\beta q_n s_n^{a/2+1} + n^\beta q_n s_n^{-\delta} \log s_n + \rho_n s_n^{3a/2} n^{\beta-1/2} (\log n)^{1/2}$$

$$+ \lambda_n \rho_n q_n s_n^{2a+1} n^{3\beta/2-1/2} + n^{\beta-1/2} \rho_n q_n s_n^{3a/2-\delta} \log s_n \right\}, \quad (S2.40)$$

with probability tending to one, for some constant $c_{13} > 0$, which completes the proof of part 2).

\[ \Box \]

### S3 Proofs of Main Theorems

**Proof of Theorem 1.** Recall that we denote $\nu$ as $\nu = \eta - \eta^*$, and $\bar{\nu} = \bar{\eta} - \bar{\eta}^* = \Lambda \nu$. By first order necessary condition of the optimization theory, any local minimizer $\hat{\eta}$ of $Q_n(\eta)$ from (5) of the main paper must satisfy $\hat{\eta} \in \{ \eta : \langle \nabla L_n(\eta) + \nabla P_{\lambda_n}(\eta), -\nu \rangle \geq 0, \quad ||\nu||_1 \leq R_n \}$, where $\langle a, b \rangle = a^t b$ for any vectors $a, b$. Hence, in order to prove Theorem 1, it is sufficient to show that for
any \( \eta \in \{ \eta : \langle \nabla L_n(\eta) + \nabla P_{\lambda_n}(\eta), -\nu \rangle \geq 0, \ ||\nu||_1 \leq R_n \} \), parts 1) and 2) of Theorem 1 hold. First, we assume

\[
\eta \in \{ \eta : \langle \nabla L_n(\eta) + \nabla P_{\lambda_n}(\eta), -\nu \rangle \geq 0, \ ||\nu||_1 \leq R_n \}. \tag{S3.41}
\]

Then, it can be deduced that

\[
\langle \nabla L_n(\eta) - \nabla L_n(\eta^*), \nu \rangle = \nu' (n^{-1} \bar{\Theta}' \bar{\Theta}) \nu = \nu' E(n^{-1} \bar{\Theta}' \bar{\Theta}) \nu - \nu' \{ E(n^{-1} \bar{\Theta}' \bar{\Theta}) - n^{-1} \bar{\Theta}' \bar{\Theta} \} \nu
\]

\[
\geq \lambda_{\min}(\bar{I}) ||\nu||^2_2 - ||\nu'||_1 ||\nu||^2_1 \geq m_0 ||\nu||^2_2 - c_0 n^{\beta/2-1/2} ||\nu||_1^2.
\]

(S3.42)

with probability tending to one, for some constant \( c_0 > 0 \), where the last inequality is by (A3) and Lemma 4. In addition, with a little abuse of notation, denote

\[
P_{\lambda_n, \mu}(\eta) = P_{\lambda_n}(\eta) + 2^{-1} \mu n^{-1} \sum_{j=1}^{p_n} ||\Theta_j \eta_j||^2_2 = \sum_{j=1}^{p_n} \{ \rho_{\lambda_n, \mu}^{-1/2} (n^{-1/2} ||\Theta_j \eta_j||_2) + 2^{-1} \mu n^{-1} ||\Theta_j \eta_j||^2_2 \}.
\]

Under (P5), it is not hard to verify that \( P_{\lambda_n, \mu}(\eta) \) is convex in \( \eta \), i.e.,

\[
P_{\lambda_n, \mu}(\eta^*) - P_{\lambda_n, \mu}(\eta) \geq \langle \nabla P_{\lambda_n, \mu}(\eta), -\nu \rangle,
\]

which further implies that

\[
\langle \nabla P_{\lambda_n}(\eta), -\nu \rangle \leq P_{\lambda_n}(\eta^*) - P_{\lambda_n}(\eta) + 2^{-1} \mu n^{-1} \sum_{j=1}^{p_n} ||\Theta_j (\eta_j - \eta_j^*)||^2_2. \tag{S3.43}
\]
By combining (S3.41), (S3.42) with (S3.43), we have

\[ m_0 \| \hat{\nu} \|^2 + c_0 n^{\beta/2 - 1/2} \| \hat{\nu} \|^2 \leq \langle \nabla L_n(\eta) - \nabla L_n(\eta^*), \nu \rangle \]

\[ = \langle \nabla L_n(\eta) + \nabla P_{\lambda_n}(\eta), \nu \rangle + \langle \nabla P_{\lambda_n}(\eta), -\nu \rangle + \langle -\nabla L_n(\eta^*), \nu \rangle \]

\[ \leq P_{\lambda_n}(\eta^*) - P_{\lambda_n}(\eta) + 2^{-1} \mu n^{-1} \sum_{j=1}^{p_n} \| \Theta_j(\eta_j - \eta_j^*) \|^2 + \| n^{-1} \hat{\Theta}'(Y - \Theta \eta^*) \|_\infty \| \hat{\nu} \|_1. \]

\[ (S3.44) \]

By combining (S3.41), (S3.44) with Lemma 5, we have an upper bound for

\[ m_0 \| \hat{\nu} \|^2, \] with probability tending to one for some \( c_1 > 0 \),

\[ P_{\lambda_n}(\eta^*) - P_{\lambda_n}(\eta) + 2^{-1} \mu n^{-1} \sum_{j=1}^{p_n} \| \Theta_j(\eta_j - \eta_j^*) \|^2 + c_1 (n^{\beta/2 - 1/2} R_n + q_n R_n^{-\delta}) \| \hat{\nu} \|_1. \]

\[ (S3.45) \]

By combining (S3.45) with Lemma 6 and (A9), we have \( [m_0 - 2^{-1} m_1 \mu \{ 1 + o(1) \}] \| \hat{\nu} \|^2 \leq \{ 1 + o(1) \} \{ P_{\lambda_n}(\eta^*) - P_{\lambda_n}(\eta) \} \). Together with Lemma 2 and (A3), we have

\[ 0 \leq [m_0 - 2^{-1} m_1 \mu \{ 1 + o(1) \}] \| \hat{\nu} \|^2 \]

\[ \leq c_2 \lambda_n^{-1/2} \left\{ \sum_{j \in A_n} n^{-1/2} \| \Theta_j(\eta_j - \eta_j^*) \|_2 - \sum_{j \in A_n^c} n^{-1/2} \| \Theta_j(\eta_j - \eta_j^*) \|_2 \right\}, \]

\[ (S3.46) \]
with probability tending to one, for some constant $c_2 > 0$. On one hand, (S3.46) implies that

$$||\hat{\nu}||_2^2 \leq c_3 \lambda_n s_n^{1/2} \sum_{j \in A_n} n^{-1/2} ||\Theta_j(\eta_j - \eta_j^*)||_2 \leq c_3 \lambda_n s_n^{1/2} q_n^{1/2} \left\{ \sum_{j \in A_n} n^{-1} ||\Theta_j(\eta_j - \eta_j^*)||_2^2 \right\}^{1/2}$$

$$\leq c_3 \lambda_n s_n^{1/2} q_n^{1/2} \left\{ \sum_{j=1}^{p_n} n^{-1} ||\Theta_j(\eta_j - \eta_j^*)||_2^2 \right\}^{1/2} \leq c_4 \lambda_n s_n^{1/2} q_n^{1/2} ||\hat{\nu}||_2,$$

with probability tending to one, for some constants $c_3, c_4 > 0$, where the last inequality is by Lemma 6. Then, it follows that we have $||\hat{\nu}||_2 \leq c_4 \lambda_n s_n^{1/2} q_n^{1/2}$, which further entails that

$$||\nu||_2 = ||\Lambda^{-1}\hat{\nu}||_2 \leq \lambda_{\max}(\Lambda^{-1}) ||\hat{\nu}||_2 = \left\{ \lambda_{\min}(\Lambda) \right\}^{-1} ||\hat{\nu}||_2 \leq c_5 \lambda_n s_n^{\alpha/2 + 1/2} q_n^{1/2},$$

with probability tending to one, for some constant $c_5 > 0$, which completes the proof of part 1). On the other hand, (S3.46) also implies that $\sum_{j \in A_n} n^{-1/2} ||\Theta_j(\eta_j - \eta_j^*)||_2 \geq \sum_{j \in A_n} n^{-1/2} ||\Theta_j(\eta_j - \eta_j^*)||_2$. Together with Lemma 6, we have

$$||\hat{\nu}||_1 \leq c_5 s_n^{1/2} \sum_{j=1}^{p_n} n^{-1/2} ||\Theta_j(\eta_j - \eta_j^*)||_2 \leq 2c_5 s_n^{1/2} \sum_{j \in A_n} n^{-1/2} ||\Theta_j(\eta_j - \eta_j^*)||_2$$

$$\leq 2c_5 s_n^{1/2} q_n^{1/2} \left\{ \sum_{j=1}^{p_n} n^{-1} ||\Theta_j(\eta_j - \eta_j^*)||_2^2 \right\}^{1/2} \leq c_6 s_n^{1/2} q_n^{1/2} ||\hat{\nu}||_2 \leq c_7 \lambda_n s_n q_n,$$

with probability tending to one, for some constants $c_5, c_6, c_7 > 0$, which further entails that

$$||\nu||_1 = ||\Lambda^{-1}\hat{\nu}||_1 \leq \lambda_{\max}(\Lambda^{-1}) ||\hat{\nu}||_1 = \left\{ \lambda_{\min}(\Lambda) \right\}^{-1} ||\hat{\nu}||_1 \leq c_8 \lambda_n s_n^{\alpha/2 + 1} q_n,$$

with probability tending to one, for some constant $c_8 > 0$, which completes the proof of part 2).
Proof of Theorem 2. First, we have
\[ \|\hat{T}^*\|_\infty - \|T^*\|_\infty \leq \|\hat{T}^* - T^*\|_\infty = \max_{l \leq h_n s_n} \left| n^{-1/2} \sum_{i=1}^{n} (\hat{S}_{il} - S_{il}) \right| \leq c_0 g(n), \]
(S3.47)
with probability tending to one, for some constant $c_0 > 0$, where $g(n) = n^{\beta - 1/2} \rho_n s_n^{3s/2} + \lambda_n n^{\beta/2} q_n s_n^{a+1} + n^{\beta/2 + 1/2} q_n s_n^{-\delta} \log s_n + n^{\beta} q_n s_n^{3s/2 - \delta} \log s_n$.

Second, we have
\[ \|\hat{T}^e\|_\infty - \|T^e\|_\infty \leq \|\hat{T}^e - T^e\|_\infty = \max_{l \leq h_n s_n} \left| n^{-1/2} \sum_{i=1}^{n} e_i (\hat{S}_{il} - S_{il}) \right|. \]
(S3.48)

Since \( \{e_1, \ldots, e_n\} \) is a set of independent and identically distributed standard normal random variables independent of the data, by the Hoeffding inequality, we have that for any $l = 1, \ldots, h_n s_n$ and $t > 0$,
\[ P_e \{ n^{-1/2} \sum_{i=1}^{n} e_i (\hat{S}_{il} - S_{il}) \geq t \} \leq 2 \exp \left[ -t^2 / \{ 2n^{-1} \sum_{i=1}^{n} (\hat{S}_{il} - S_{il})^2 \} \right], \]
where $P_e(\cdot)$ means the probability with respect to $e$. Then, the union bound inequality yields,
\[ P_e \{ \max_{l \leq h_n s_n} n^{-1/2} \sum_{i=1}^{n} e_i (\hat{S}_{il} - S_{il}) \geq t \} \leq 2 h_n s_n \exp \left[ -t^2 / \{ 2 \max_{l \leq h_n s_n} n^{-1} \sum_{i=1}^{n} (\hat{S}_{il} - S_{il})^2 \} \right]. \]

Together with Lemma 8, it is easy to see that
\[ P_e \{ \max_{l \leq h_n s_n} n^{-1/2} \sum_{i=1}^{n} e_i (\hat{S}_{il} - S_{il}) \leq c_1 f(n) \} \rightarrow 1, \]
with probability tending to one, for some constant $c_1 > 0$, where

$$f(n) = \lambda_n n^{3\beta/2} q_n s_n^{\alpha/2 + 1} + n^{\beta} q_n s_n^{-\delta} \log s_n + \rho_n s_n^{3\alpha/2} n^{3\beta/2 - 1/2} (\log n)^{1/2}$$

$$+ \lambda_n \rho_n q_n s_n^{2\alpha + 1} n^{\beta - 1/2} + n^{3\beta/2 - 1/2} \rho_n q_n s_n^{3\alpha/2 - \delta} \log s_n.$$  

Together with (S3.48), we obtain

$$P_e \{ \|\hat{T}^* - T^*\|_\infty \leq c_1 f(n) \} \to 1. \quad (S3.49)$$

Moreover, we have

$$g(n) \{\log(p_n s_n)\}^{1/2} \sim g(n) n^{\beta/2} = n^{3\beta/2 - 1/2} \rho_n s_n^{3\alpha/2} + \lambda_n n^{\beta} q_n s_n^{\alpha + 1} + n^{\beta + 1/2} q_n s_n^{-\delta} \log s_n$$

$$+ n^{3\beta/2} \rho_n q_n s_n^{3\alpha/2 - \delta} \log s_n = o(1), \quad (S3.50)$$

under (A5), (B1)–(B3). In addition, we also have

$$f(n) \{\log(p_n s_n)\}^{1/2} \sim f(n) n^{\beta/2} = \lambda_n n^{2\beta} q_n s_n^{\alpha/2 + 1} + n^{3\beta/2} q_n s_n^{-\delta} \log s_n + \rho_n s_n^{3\alpha/2} n^{2\beta - 1/2} (\log n)^{1/2}$$

$$+ \lambda_n \rho_n q_n s_n^{2\alpha + 1} n^{\beta/2 - 1/2} + n^{2\beta - 1/2} \rho_n q_n s_n^{3\alpha/2 - \delta} \log s_n = o(1), \quad (S3.51)$$

under (A5), (B1)–(B3). Furthermore, (A4) implies that

$$E(S_{il}^2) \geq c_2, \quad (S3.52)$$

for some universal constant $c_2 > 0$, and (A2) implies that

$$\|S_{il}\|_{\phi_2} \leq c_3, \quad (S3.53)$$
for some universal constant $c_3 > 0$. Hence, by combining (S.47), (S.49), (S.50), (S.51), (S.52), (S.53) with (A5), we have

$$\lim_{n \to \infty} \sup_{\alpha \in (0, 1)} \left| P\{\|\hat{T}^*\|_\infty \leq c_B(\alpha)\} - (1 - \alpha) \right| = 0,$$

by Lemma H.7 in Ning and Liu (2017), which completes the proof.

**Proof of Theorem 3.** First, for the decorrelated Wald test, by definition, we have

$$\hat{W}^* = n^{1/2} \hat{A}_n^{-1}(\hat{I}_{H_n|\hat{H}_n} - \hat{w}' \hat{I}_{\hat{H}_n|\hat{H}_n}) \hat{\eta}_{H_n} = n^{1/2} \hat{A}_n^{-1}(\hat{I}_{H_n|\hat{H}_n} - \hat{w}' \hat{I}_{\hat{H}_n|\hat{H}_n}) \hat{\eta}_{H_n},$$

(S3.54)

where

$$\hat{\eta}_{H_n} = \hat{\eta}_{H_n} - \left\{ \frac{\partial \hat{S}(\hat{\eta}_{H_n}, \hat{\eta}_{H_n})}{\partial \eta_{H_n}} \right\}^{-1} \hat{S}(\hat{\eta}_{H_n}, \hat{\eta}_{H_n})$$

$$= -\left\{ \hat{A}_n^{-1}(\hat{w}' \Theta_{H_n} \Theta_{H_n} - \hat{w}' \Theta_{H_n} \Theta_{H_n}) \right\}^{-1} n^{-1/2} \hat{T}^*.$$  

(S3.55)

Hence, by combining (S3.54) with (S3.55), it is easy to see that

$$\hat{W}^* = -\hat{T}^*.$$  

(S3.56)

Second, for the decorrelated likelihood ratio test, by definition, we have

$$\hat{L}^* = \hat{A}_n^{-2} \text{diag}\{((\Theta_{H_n} \hat{w} - \Theta_{H_n})'(\Theta_{H_n} \hat{w} - \Theta_{H_n})/n) \hat{\Sigma},$$

(S3.57)

where $\hat{\Sigma} = (\hat{\Sigma}_1, \ldots, \hat{\Sigma}_{(j-1)s_n+k}, \ldots, \hat{\Sigma}_{s_n^2})'$ whose $((j-1)s_n + k)$'th element
where \( \Theta_{H_n,(j-1)s_n+k} \) represents the \((j-1)s_n+k\) th column of \( \Theta_{H_n} \). Hence, we have

\[
\hat{Y}_{(j-1)s_n+k} = 2n\left\{ \hat{L}_{jk}(0,0,\hat{\eta}_{H_n^c}) - \hat{L}_{jk}(\hat{\eta}_{jk},0,\hat{\eta}_{H_n^c}) \right\} = \left\| Y - \Theta_{H_n^c} \hat{\eta}_{H_n^c} \right\|^2 - \left\| Y - \Theta_{H_n^c} \hat{\eta}_{H_n^c} - \hat{\eta}_{jk}(\Theta_{H_n,(j-1)s_n+k} - \Theta_{H_n^c} \hat{w}(j-1)s_n+k) \right\|^2
\]

\[
\hat{Y}_{(j-1)s_n+k} = 2\hat{\eta}_{jk}(\Theta_{H_n,(j-1)s_n+k} - \Theta_{H_n^c} \hat{w}(j-1)s_n+k)'(Y - \Theta_{H_n^c} \hat{\eta}_{H_n^c}) - \hat{\eta}_{jk}^2(\Theta_{H_n,(j-1)s_n+k} - \Theta_{H_n^c} \hat{w}(j-1)s_n+k)'(\Theta_{H_n,(j-1)s_n+k} - \Theta_{H_n^c} \hat{w}(j-1)s_n+k),
\]

with \( \hat{\eta}_{jk} = \left\{ (\Theta_{H_n,(j-1)s_n+k} - \Theta_{H_n^c} \hat{w}(j-1)s_n+k)'(\Theta_{H_n,(j-1)s_n+k} - \Theta_{H_n^c} \hat{w}(j-1)s_n+k) \right\}^{-1}. \}

\[
\hat{Y}_{(j-1)s_n+k} = \left\{ (\Theta_{H_n,(j-1)s_n+k} - \Theta_{H_n^c} \hat{w}(j-1)s_n+k)'(\Theta_{H_n,(j-1)s_n+k} - \Theta_{H_n^c} \hat{w}(j-1)s_n+k) \right\}^{-1} \cdot \left\{ (\Theta_{H_n,(j-1)s_n+k} - \Theta_{H_n^c} \hat{w}(j-1)s_n+k)'(Y - \Theta_{H_n^c} \hat{\eta}_{H_n^c}) \right\}^2. \quad (S3.58)
\]

Then, by combining (S3.57) with (S3.58), one has \( \hat{L}_{(j-1)s_n+k} = \hat{T}_{(j-1)s_n+k}^* \). Combining with (S3.56) yields \( ||\hat{T}^*||_\infty = ||\hat{W}^*||_\infty = ||\hat{L}^*||_\infty^{1/2} \), which completes the proof. \( \square \)
Bibliography
