TESTING CONSTANCY OF CONDITIONAL VARIANCE IN HIGH DIMENSION

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Supplementary Material

The supplementary material includes all the proofs of Theorems 1–2 and Propositions 1–4, as well as a necessary lemma. Some additional simulation results are also presented.

We first give a useful lemma as follows.

Lemma 1. Under Condition (C4), given $Y$, let $A(Y) = \Gamma^T(Y)\Gamma(Y)$, we have

(i) for any positive integer $k$,

$$E\{(Z^T(Y)A^k(Y)Z(Y))^2 \mid Y\} = \text{tr}^2(\Sigma^k_0)+2\text{tr}(\Sigma^{2k}_0)+\Delta(Y)\text{tr}(A^k(Y)\circ A^k(Y)),$$

here we define $F \circ G = (f_{kl}g_{kl})$, where $F = (f_{kl})$ and $G = (g_{kl})$;

(ii) for independent variables $Z_1(Y)$ and $Z_2(Y)$,

$$E\{(Z^T_1(Y)A(Y)Z_2(Y))^4 \mid Y\} = 3\text{tr}^2(\Sigma^2_0)+6\text{tr}(\Sigma^4_0)+6\Delta(Y)\text{tr}(A^2(Y) \circ A^2(Y)).$$
\[ + \Delta^2(Y) \sum_{k,l=1}^{m} (A_{kl}(Y))^4, \]

(iii) \[ \sum_{k,l=1}^{m} (A_{kl}(Y))^4 \leq \text{tr}(\Sigma_0^4) \text{ and } \text{tr}(A^k(Y) \circ A^k(Y)) \leq \text{tr}(\Sigma_0^{2k}). \]

(iv) for any \( m \times m \) positive definite matrix \( F \),

\[ E\{(Z^T(Y)FZ(Y) - \text{tr}(F))^4\} \leq C\text{tr}^2(F^2). \]

where \( C \) is a constant which doesn’t depend on \( Y \).

This lemma is similar to Proposition A.1 in Chen, Zhang and Zhong (2010), replacing all the expectations by the conditional expectations.

Proof of Proposition 1

Proof. For notational convenience, let \( A_i = \text{tr}(\Sigma_i^2) \) and \( C_{ij} = \text{tr}(\Sigma_i\Sigma_j) \);

Denote \( \phi_i(\{s, t\}) \) and \( \phi_i(\{s\}) \) as the counterparts of \( \theta_i(\{s, t\}) \) and \( \theta_i(\{s\}) \) after replacing every \( X_{ik} \) by \( \varepsilon_{ik} \), respectively. Similarly, let \( \omega_i(\{s, t\}) \) and \( \omega_i(\{s, t\}) \) be the counterparts of \( \theta_i(\{s, t\}) \) and \( \theta_i(\{s\}) \) by replacing \( X_{ik} \) by \( \mu_{ik} \), respectively. The test statistic \( T_n \) can be rewritten as

\[
T_n = (H - 1) \sum_{i=1}^{H} A_i - 2 \sum_{i<j} C_{ij} \\
= (H - 1) \sum_{i=1}^{H} \frac{1}{l(l-1)} \sum_{s \neq t} \{(\varepsilon_{is} - \phi_i(\{s, t\}))^T(\varepsilon_{it} - \phi_i(\{s, t\}))\}^2 \\
- 2 \sum_{i<j} \frac{1}{l^2} \sum_{s,t} \{(\varepsilon_{is} - \phi_i(\{s\}))^T(\varepsilon_{jt} - \phi_j(\{t\}))\}^2 + R_1,
\]
where $R_1$ denotes the remaining terms.

Expanding $\phi_i(\{s, t\})$ and $\phi_i(\{s\})$ as $(l-2)^{-1}\sum_{r\neq s, t} e_{ir}$ and $(l-1)^{-1}\sum_{r\neq s} e_{ir}$, respectively, we then have

$$T_n = (H - 1) \sum_{i=1}^{H} \left\{ \frac{1}{l(l - 1)} + \frac{2}{l(l - 1)(l - 2)} + \frac{2(l - 3)}{l(l - 1)(l - 2)^3} \right\} \sum_{s \neq t} (e_{is}^T e_{it})^2$$

$$+ \left\{ \frac{-2}{l(l - 1)(l - 2)} + \frac{-8(l - 3)}{l(l - 1)(l - 2)^3} + \frac{4(l - 3)(l - 4)}{l(l - 1)(l - 2)^4} \right\} \sum_{s, t, r}^* e_{is}^T e_{ir} e_{it} e_{ir}$$

$$+ \frac{l + 4}{l(l - 2)^4} \sum_{s, t, r, q}^* e_{is}^T e_{it} e_{is}^T e_{iq} + \frac{-2l + 8}{l(l - 1)(l - 2)^4} \sum_{s, t, r}^* e_{is}^T e_{it} e_{is}^T e_{ir}$$

$$+ \frac{-4}{l(l - 1)(l - 2)^3} \sum_{s \neq t} e_{is}^T e_{it} e_{is}^T e_{it} + \frac{1}{l(l - 2)^3} \sum_{s}^* (e_{is}^T e_{is})^2$$

$$+ \frac{l - 3}{l(l - 1)(l - 2)^3} \sum_{s \neq t} e_{is}^T e_{is} e_{it} e_{it}$$

$$- 2 \sum_{i < j} \left\{ \frac{1}{(l - 1)^2} \sum_{s, t}^* (e_{is}^T e_{jt})^2 - \frac{1}{(l - 1)^3} \sum_{s} \sum_{t \neq r}^* e_{is}^T e_{is} e_{it} e_{ir} e_{jr} e_{jr} \right\}$$

$$- \frac{1}{(l - 1)^3} \sum_{s} \sum_{t \neq r}^* e_{it} e_{is} e_{is} e_{ir} + \frac{1}{(l - 1)^4} \sum_{s \neq t} \sum_{r \neq q}^* e_{is}^T e_{jr} e_{ir} e_{it} e_{iq} + R_1$$

(S.1)

By Lemma 1 in the Supplementary Material, it can be shown that

$$E(T_n - R_1) = \left\{ \frac{2}{l - 2} + \frac{2}{(l - 2)^2} - \frac{2}{l - 1} - \frac{1}{(l - 1)^2} \right\} H(H - 1) \text{tr}(\Sigma_0^2)$$

$$+ \frac{1}{(l - 2)^2} H(H - 1) \text{tr}^2(\Sigma_0) + O\left( \frac{H(H - 1)}{(l - 2)^3} \text{tr}(\Sigma_0^3) \right).$$

To derive the variance of $T_n$, consider its variants $\widetilde{T}_n$

$$\widetilde{T}_n = (H - 1) \sum_{i=1}^{H} \left\{ \frac{1}{(l - 1)^2} - 2 \right\} + 2 \left\{ \frac{2}{l(l - 1)(l - 2)} \right\}^{-1}$$
+ 2(l - 3)\{l(l - 1)(l - 2)^3\}^{-1} \sum_{s \neq t} (\varepsilon_{is}^T \varepsilon_{it})^2 - 2 \sum_{i < j} (l - 1)^{-2} \sum_{s, t} (\varepsilon_{is}^T \varepsilon_{jt})^2.

Tedious algebra yields that

\begin{align*}
\text{var}\left(\sum_{s \neq t} (\varepsilon_{is}^T \varepsilon_{it})^2\right) &= l(l - 1)\{-4l + 6\text{tr}^2(\Sigma^2_0) + 4(l - 2)\delta_2 + 2\delta_3\}, \\
\text{var}\left(\sum_{s, t} (\varepsilon_{is}^T \varepsilon_{jt})^2\right) &= l^2\{-2l + 1\text{tr}^2(\Sigma^2_0) + 2(l - 1)\delta_2 + \delta_3\}, \quad (S.2) \\
\text{cov}\left(\sum_{s \neq t} (\varepsilon_{is}^T \varepsilon_{it})^2, \sum_{s, t} (\varepsilon_{is}^T \varepsilon_{jt})^2\right) &= 2l^2(l - 1)\{-\text{tr}^2(\Sigma^2_0) + \delta_2\}, \\
\text{cov}\left(\sum_{s, t} (\varepsilon_{is}^T \varepsilon_{jt})^2, \sum_{s, t} (\varepsilon_{is}^T \varepsilon_{jt})^2\right) &= l^3\{-\text{tr}^2(\Sigma^2_0) + \delta_2\},
\end{align*}

where \(\delta_2\) and \(\delta_3\) are \(E\{(\varepsilon_{i1}^T \varepsilon_{i2})^2(\varepsilon_{i1}^T \varepsilon_{i3})^2\}\) and \(E\{(\varepsilon_{i1}^T \varepsilon_{i2})^4\}\), respectively. By Lemma 1, we have \(\delta_2 = \text{tr}^2(\Sigma^2_0) + O(\text{tr}\Sigma^4_0), \delta_3 = 3\text{tr}^2(\Sigma^2_0) + O(\text{tr}\Sigma^4_0)\).

By Condition (C3), we have

\begin{equation*}
\text{var}(\hat{T}_n) = 4l^{-2}H^2(H - 1)\text{tr}^2(\Sigma^2_0)(1 + o(1)).
\end{equation*}

By the first part of Condition (C2), the last term in \(E(T_n - R_1)\) is \(o(\sqrt{\text{var}(\hat{T}_n)})\).

By similar arguments, we also have that

\begin{align*}
\text{var}\left(\sum_{s, t, r}^* (\varepsilon_{is}^T \varepsilon_{ir}^T \varepsilon_{it})^2\right) &= 2l(l - 1)(l - 2)\{\delta_2 + (l - 3)\text{tr}(\Sigma^4_0)\}, \\
\text{var}\left(\sum_{s, t, r, q}^* (\varepsilon_{is}^T \varepsilon_{it}^T \varepsilon_{iq}^T \varepsilon_{i}^T)^2\right) &= 8l(l - 1)(l - 2)(l - 3)\{\text{tr}^2(\Sigma^2_0) + 2\text{tr}(\Sigma^4_0)\}, \\
\text{var}\left(\sum_{s \neq t}^* (\varepsilon_{is}^T \varepsilon_{is}^T \varepsilon_{is}^T \varepsilon_{it}^T)^2\right) &= P_t^4\text{tr}^4(\Sigma_0) + 4P_t^3\{\text{tr}^2(\Sigma_0) + O(\text{tr}(\Sigma^4_0))\}\text{tr}^2(\Sigma_0) \\
&+ 2P_t^2\{\text{tr}^2(\Sigma_0) + O(\text{tr}(\Sigma^4_0))\}^2 - (P_t^2)^2\text{tr}^4(\Sigma_0).
\end{align*}
Now it suffices to verify that $R_{φ} = \left(1 + \sum_{t} - \sum_{i} (\epsilon_{it}^T \epsilon_{it})/\epsilon_{is}^T \epsilon_{is} \right)$.

$$\text{var}\left(\sum_{s,t,r}^{*} \epsilon_{is}^T \epsilon_{it} \epsilon_{ir} \epsilon_{ir} \right) = l^4 O\{\text{tr}(\Sigma_0^2)\text{tr}^2(\Sigma_0)\},$$

$$\text{var}\left(\sum_{s \neq t} \epsilon_{is}^T \epsilon_{it} \epsilon_{it} \epsilon_{it} \right) = l^2 O\{\text{tr}(\Sigma_0^2)\text{tr}^2(\Sigma_0)\},$$

$$\text{var}\left(\sum_{s} (\epsilon_{is}^T \epsilon_{is})^2 \right) = lO\{\text{tr}(\Sigma_0^2)\text{tr}^2(\Sigma_0)\}.$$  

Under the condition $p = o(l^3)$, each term in $T_n - \tilde{T}_n - R_1$ is of $o_p(\sqrt{\text{var}(\tilde{T}_n)})$.

Now it suffices to verify that $R_1 = o_p(\sqrt{\text{var}(\tilde{T}_n)})$, which has the form

$$R_1 = (H - 1) \sum_{i} \frac{1}{l(l-1)} \sum_{s \neq t} \left\{ 2(\epsilon_{is} - \phi_i(\{s,t\}))^T(\epsilon_{it} - \phi_i(\{s,t\})) \\
+ (\epsilon_{is} - \phi_i(\{s,t\}))^T(\mu_{it} - \omega_i(\{s,t\})) + (\epsilon_{it} - \phi_i(\{s,t\}))^T(\mu_{is} - \omega_i(\{s,t\})) \\
+ (\mu_{is} - \omega_i(\{s,t\}))^T(\mu_{it} - \omega_i(\{s,t\})) \right\} \\
- 2 \sum_{i \leq j} \frac{1}{l^2} \sum_{s,t} \left\{ 2(\epsilon_{is} - \phi_i(\{s\}))^T(\epsilon_{jt} - \phi_j(\{t\})) \\
+ (\epsilon_{is} - \phi_i(\{s\}))^T(\mu_{jt} - \omega_j(\{t\})) + (\epsilon_{jt} - \phi_j(\{t\}))^T(\mu_{is} - \omega_i(\{s\})) \\
+ (\mu_{is} - \omega_i(\{s\}))^T(\mu_{jt} - \omega_j(\{t\})) \right\} \\
+ (\epsilon_{jt} - \phi_j(\{t\}))^T(\mu_{is} - \omega_i(\{s\})) + (\mu_{is} - \omega_i(\{s\}))^T(\mu_{jt} - \omega_j(\{t\})) \right\}$$

$$= R_{1,1} + R_{1,2}.$$  

Expanding $\phi_i$ and $\omega_i$ as in (S.1), by Conditions (C1) and (C2), it can be
seen that in $R_{1,1}$

$$(H - 1) \sum_{i} \frac{1}{l(l-1)} \sum_{s \neq t} \left\{ (\mu_{is} - \omega_i(s,t))^{T}(\mu_{it} - \omega_i(s,t)) \right\}^2$$

$$\leq H \sum_{i} \max_{s, t} \| \mu_{is} - \mu_{it} \|^4 \leq HM^4p^2 \sum_{i} \max_{s, t} \| Y_{is} - Y_{it} \|^4$$

$$\leq HM^4p^2 \sum_{i} r_{1i}^{4a} = \sqrt{\text{var}(\tilde{T}_n)} O_p(H^{-1/2} p^l \sum_{i} r_{1i}^{4a}) = o_p(\sqrt{\text{var}(\tilde{T}_n)}).$$

Based on the specific form of $R_1$, the above result also holds for corresponding term in $R_{1,2}$. Now we consider the next term in $R_{1,1}$. 

$$(H - 1) \sum_{i} \frac{1}{l(l-1)} \sum_{s \neq t} (\varepsilon_{is} - \phi_i(s,t))^{T}(\varepsilon_{it} - \phi_i(s,t))$$

$$= (H - 1) \sum_{i} \frac{1}{l(l-1)} \sum_{s \neq t} \left\{ (\varepsilon_{is} - \phi_i(s,t))^{T}(\varepsilon_{it} - \phi_i(s,t)) - (l-2)^{-2} \text{tr}(\Sigma_0) \right\}$$

$$+ (H - 1) \sum_{i} \frac{\text{tr}(\Sigma_0)}{(l-2)^2} \sum_{s \neq t} \left( \frac{1}{l} \sum_{k} \mu_{ik}^{T} \mu_{ik} - \frac{1}{l(l-1)} \sum_{s \neq t} \mu_{is}^{T} \mu_{it} \right)$$

$$= o_p(\sqrt{\text{var}(\tilde{T}_n)}) + O_p(H^2 p^{-3} \sum_{i} r_{2i}) = o_p(\sqrt{\text{var}(\tilde{T}_n)}).$$

Note that the expectation of the first term in the last equality is exact 0, and with Conditions (C1) and (C2), similar calculations as in (S.2) imply its variance is a higher-order term than var$(\tilde{T}_n)$, so we need only consider the latter term. Denote $\Lambda_i = l^{-1} \sum_{k} \mu_{ik}^{T} \mu_{ik} - \frac{(l(l-1))^{-1}}{l(l-1)} \sum_{s \neq t} \mu_{is}^{T} \mu_{it}$. Let $\mu_i^T = (\mu_{i1}^T, \ldots, \mu_{il}^T)^T$, $\tilde{\mu}_i^T = (\mu_{i1}^T - \mu_{i2}^T, \ldots, \mu_{i,l-1}^T - \mu_{il}^T, \mu_{il}^T)^T = \mu_{i}^{T} S^{-1}$, then $\Lambda_i$ can
be written in quadratic form: $\Lambda_i = l^{-1}\mu_i^T D \mu_i = l^{-1}\tilde{\mu}_i^T SDS^T \tilde{\mu}_i$, where $D$ is the matrix whose diagonal elements are 1 and off-diagonal elements are $-(l - 1)^{-1}$. Note that the last row and last column of $SDS^T$ are all 0, therefore $\Lambda_i \leq C' l^{-1} \sum_{k=1}^{l-1} (\mu_{ik} - \mu_{i,k+1})^2 (\mu_{ik} - \mu_{i,k+1})$ for some constant $C'$. By Conditions (C1) and (C2), $\Lambda_i \leq C' l^{-1} M^2 p \sum_{k=1}^{l-1} \|Y_{ik} - Y_{i,k+1}\|^2 = O_p(l^{-1}pr^2)$. Using similar arguments, we can show the results above hold for all remaining terms in $R_1$, from which our assertion holds.

Proof of Theorem 1

Proof. From the proof of Proposition 2, we need only to show the asymptotic normality of $\tilde{T}_n'$

$$\tilde{T}_n' = (H - 1) \sum_{i=1}^{H} \{l(l - 1)\}^{-1} \sum_{s \neq t} (\epsilon_{is}^T \epsilon_{it})^2 - 2 \sum_{i < j} (l - 1)^{-2} \sum_{s,t} (\epsilon_{is}^T \epsilon_{jt})^2.$$

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_k = \sigma\{\epsilon(1), \ldots, \epsilon(k)\}$ with $k = 1, 2, \ldots, n$, and $E_k(.)$ denote the conditional expectation given $\mathcal{F}_k$, $E_0(.) = E(.)$, then $\{D_k, k = 1, \ldots, n\}$ is a martingale difference sequence with respect to the $\sigma$-fields $\{\mathcal{F}_k, k = 1, \ldots, n\}$, where $\tilde{T}_n' = \sum_{k=1}^{n} D_k$, $D_k = (E_k - E_{k-1})\tilde{T}_n'$. By noting the fact the $\epsilon(i)$'s are conditional independent given $Y(i)$'s, $D_k$ has the exact
form
\[
D_k = \frac{2(H - 1)}{l(l - 1)} \left\{ \varepsilon^T_k Q_{k-1} \varepsilon(k) - \text{tr}(Q_{k-1} \Sigma_0) \right\} - \frac{2}{l^2} \left\{ \varepsilon^T_k W_{k-1} \varepsilon(k) - \text{tr}(W_{k-1} \Sigma_0) \right\},
\]
where \( Q_{k-1} = \sum_{s=(i-1)l+1}^{k-1} (\varepsilon(s) \varepsilon^T(s) - \Sigma_0) \), \( W_{k-1} = \sum_{s=1}^{(i-1)l} (\varepsilon(s) \varepsilon^T(s) - \Sigma_0) \) and \( k = (i-1)l + j \).

To apply martingale central limit theorem (Hall and Hype, 1980) to establish the limiting distribution of \( \tilde{T}'_n \), we would further verify the following two conditions:
\[
\frac{\sum_{k=1}^{n} \sigma^2_k}{\text{var}(\tilde{T}'_n)} \xrightarrow{p} 1, \quad \text{and} \quad \frac{\sum_{k=1}^{n} E(D^4_k)}{\text{var}^2(\tilde{T}'_n)} \xrightarrow{p} 0,
\]
where \( \sigma^2_k = E_{k-1}(D^2_k) \). As it is true that \( E(\sum_{k=1}^{n} \sigma^2_k) = \text{var}(\tilde{T}'_n) \), it suffices to show \( \text{var}(\sum_{k=1}^{n} \sigma^2_k) = o(\text{var}^2(\tilde{T}'_n)) \). By Lemma 1, we express \( \sum_{k=1}^{n} \sigma^2_k \) in the following form:
\[
\sum_{k=1}^{n} \sigma^2_k = \frac{4(H - 1)^2}{l^2(l - 1)^2} \sum_{k=1}^{n} \left\{ 2\text{tr}\{(Q_{k-1} \Sigma_0)^2\} \\
+ E_{k-1}\{\Delta(Y^{(k)}) \text{tr}(\Gamma^T(Y^{(k)}) Q_{k-1} \Gamma(Y^{(k)}) \circ \Gamma^T(Y^{(k)}) Q_{k-1} \Gamma(Y^{(k)}))\} \right\} \\
+ \frac{4}{l^4} \sum_{k=1}^{n} \left\{ 2\text{tr}\{(W_{k-1} \Sigma_0)^2\} \\
+ E_{k-1}\{\Delta(Y^{(k)}) \text{tr}(\Gamma^T(Y^{(k)}) W_{k-1} \Gamma(Y^{(k)}) \circ \Gamma^T(Y^{(k)}) W_{k-1} \Gamma(Y^{(k)}))\} \right\} \\
- \frac{8(H - 1)}{l^3(l - 1)} \sum_{k=1}^{n} \left\{ 2\text{tr}(Q_{k-1} \Sigma_0) \text{tr}(W_{k-1} \Sigma_0) \right\}.
\]
\[
\begin{align*}
+ E_{k-1} \{ & \Delta(Y(\ell)) \text{tr}(\Gamma(Y(\ell)) Q_{\ell-1} \Gamma(Y(\ell)) \circ \Gamma(Y(\ell)) W_{\ell-1} \Gamma(Y(\ell))) \} \\
\equiv & D_{1,1} + D_{1,2} + D_{2,1} + D_{2,2} + D_{3,1} + D_{3,2}.
\end{align*}
\]

Now we calculate the variance of \( D_{i,j} \) \((i = 1, 2, 3; j = 1, 2)\). Consider firstly \( D_{1,1} \). Using Lemma 1, for any positive integer \( k \leq r \), with \( k = (i-1)l + j \), we have

\[
\text{cov}(\text{tr}\{(Q_{\ell-1} \Sigma_0)^2\}, \text{tr}\{(Q_{r-1} \Sigma_0)^2\})
= \text{cov}(\text{tr}\{(Q_{\ell-1} \Sigma_0)^2\}, \text{tr}\{(Q_{k-1} \Sigma_0)^2\})
= 2j(j-1)\text{var}\{\text{tr}\{(\varepsilon_1 \varepsilon_1^T - \Sigma_0) \Sigma_0 (\varepsilon_2 \varepsilon_2^T - \Sigma_0) \Sigma_0\}\}
+ j\text{var}\{\text{tr}\{(\varepsilon_1 \varepsilon_1^T - \Sigma_0) \Sigma_0\}^2\}
= O(j^2)\text{tr}^2(\Sigma_0^2)\text{tr}(\Sigma_0^4). \quad (S.5)
\]

Then we can rewrite \( \sum_{k=1}^{n} \text{tr}\{(Q_{k-1} \Sigma_0)^2\} \) as \( \sum_{i=1}^{H} \sum_{j=1}^{l} \text{tr}\{(Q_{(i-1)l+j-1} \Sigma_0)^2\} \) based on the specific form of \( Q_{k-1} \). Thus, \( \text{var}(D_{1,1}) = O(H^5l^{-4})\text{tr}^2(\Sigma_0^2)\text{tr}(\Sigma_0^4) \) and by Condition (C3),

\[
\text{var}(D_{1,1})/\text{var}^2(\tilde{T}_\alpha) = O(\{H\text{tr}^2(\Sigma_0^2)\}^{-1}\text{tr}(\Sigma_0^4)) \to 0.
\]

Next, consider the part \( D_{1,2} \). Since \( \Delta(Y) \) is uniformly bounded by some constant \( \Delta_0 \) and as well as \( \Gamma(Y) \), we have \( D_{1,2} \leq D_{1,3} \) where

\[
D_{1,3} = \frac{4(H-1)^2}{l^2(l-1)^2} \sum_{k=1}^{n} \Delta_0 \text{tr}(\Gamma^\tau Q_{k-1} \Gamma \circ \Gamma^\tau Q_{k-1} \Gamma).
\]
By similar argument, we can verify that the result for $D_{1,1}$ also holds for $D_{1,3}$ and for all $D_{i,j}$ with $i = 1, 2, 3; j = 1, 2$. Hence we complete the first part of (S.4).

Now we show the second part of (S.4) holds. Note that $E_{k-1}\{\varepsilon_{(k)}^T Q_{k-1} \varepsilon_{(k)}\} = \text{tr}(Q_{k-1} \Sigma_0)$ and $E_{k-1}\{\varepsilon_{(k)}^T W_{k-1} \varepsilon_{(k)}\} = \text{tr}(W_{k-1} \Sigma_0)$. By the part (iv) of Lemma 1, we have
\[
\sum_{k=1}^{n} E(D_k^4) \leq 8 \left[ \frac{16(H-1)^4}{l^4(l-1)^4} \sum_{k=1}^{n} E\{(\varepsilon_{(k)}^T Q_{k-1} \varepsilon_{(k)} - \text{tr}(Q_{k-1} \Sigma_0))^4\} \right. \\
+ \left. \frac{16}{l^8} \sum_{k=1}^{n} E\{(\varepsilon_{(k)}^T W_{k-1} \varepsilon_{(k)} - \text{tr}(W_{k-1} \Sigma_0))^4\} \right] \\
\leq M(H^5 l^{-5} \text{tr}^2(\Sigma_0^4) + H^3 l^{-5} \text{tr}^2(\Sigma_0^4)) \\
= o(\text{var}^2(T'_n)),
\]
where the last two inequalities come from (S.5) and Condition (C3). The second part of (S.4) is proved and thus the proof is completed. \qed

**Proof of Proposition 4**

*Proof.* Similar to the proof of Proposition 2, we can rewrite $T'_n$ as
\[
T'_n = T^*_n + \{(J_n - U_n) + (U_n - \widehat{W}) + (\widehat{W} - W)\}.
\]

By (S.1), $T^*_n$ can be represented as
\[
T^*_n = (H - 1) \sum_{i=1}^{H} \left\{ \frac{1}{l(l-1)} + \frac{2}{l(l-1)(l-2)} + \frac{2(l-3)}{l(l-1)(l-2)^3} \right\} \sum_{s \neq t} (\varepsilon_{is}^T \varepsilon_{it})^2
\]
Proposition 1, under 

\[ \text{where } \delta \text{ var}(\tilde{H}) \]

Now we elaborately study the variance of \( \tilde{H} \). By Lemma 1 again and similar arguments for (S.2) in Proposition 1, under \( H_0 \),

\[
\begin{align*}
\text{var}(\tilde{T}_n^*) &= 4H^2(H-1) \left\{ \frac{l}{(l-1)(l-2)} + \frac{2}{l(l-1)(l-2)} \right\} \text{tr}^2(\Sigma_0^2) \\
&
\quad + O(l^{-4}) \text{tr}^2(\Sigma_0^2) + O(l^{-3}) \text{tr}(\Sigma_0^4) + O(l^{-2}H^{-1}) \text{tr}(\Sigma_0^4) \\
&
\quad + O(l^{-2}) (\delta_2 - \text{tr}^2(\Sigma_0^2)) + O(l^{-2})(\delta_3 - 3\text{tr}^2(\Sigma_0^2)) + O(l^{-3}) \delta_4 \\
&= 4H^2(H-1) \left\{ \frac{l}{(l-1)(l-2)} + \frac{2}{l(l-1)(l-2)} \right\} \text{tr}^2(\Sigma_0^2) \\
&
\quad + o(l^{-3}) \text{tr}^2(\Sigma_0^2) + O(l^{-2}) \text{tr}(\Sigma_0^4) \\
&= 4H^2(H-1) \left\{ \frac{l}{(l-1)(l-2)} + \frac{2}{l(l-1)(l-2)} \right\} \text{tr}^2(\Sigma_0^2) (1 + o(1)),
\end{align*}
\]

where \( \delta_4 = E\{ \text{tr}(\Sigma_0^2)^2 \} \). The second equation holds because
\[ \delta_2 = \text{tr}^2(\Sigma_0^2) + O(\text{tr}(\Sigma_0)) \], \[ \delta_3 = 3\text{tr}^2(\Sigma_0^2) + O(\text{tr}(\Sigma_0^4)) \], and
\[ \delta_4 \leq \text{var}(\varepsilon_{i1}^T \varepsilon_{i2})^2 \text{var}(\varepsilon_{i1}^T \varepsilon_{i3}^T \varepsilon_{i2}) \]^{1/2} = \text{tr}(\Sigma_0^2) \text{tr}^{1/2}(\Sigma_0^4) = o(\text{tr}^2(\Sigma_0^2)). \]

The last equation is due to the condition \( \text{tr}(\Sigma_0^4) = o(l^{-1})\text{tr}^2(\Sigma_0^2) \).

Now we consider the remaining terms. Since \( p = o(\{\sum_i r_{i1}^4\}^{-1} n^{1/2} l^{-5/2}) \) and \( p = o(\{\sum_i r_{2i}\}^{-1} n^{1/2} l^{1/2}) \), by the same arguments used in Proposition 1 and 2, it can be seen that \( R_1 = o_p(l^{-1} \sqrt{\text{var}(\tilde{T}_n^*)}) \), which still holds for \( U_n - \tilde{W} \) and \( \tilde{W} - W \). Combining these results with the fact \( J_n - U_n = o_p(l^{-1} \sqrt{\text{var}(\tilde{T}_n^*)}) \) when \( p = o(l^5) \), we can complete the proof of our Proposition 4.

**Proof of Proposition 2**

*Proof.* From the proof of Proposition 1, the restriction \( p = o(l^3) \) mainly comes from the term in the \( T_n \)

\[ J_n = (H - 1) \sum_{i=1}^{H} \left\{ \frac{-2l + 8}{l(l - 1)(l - 2)^4} \sum_{s,t,r} \varepsilon_{is}^T \varepsilon_{it} \varepsilon_{ir}^T \varepsilon_{it} + \frac{-4}{l(l - 1)(l - 2)^3} \sum_{s \neq t} \varepsilon_{is}^T \varepsilon_{it} \varepsilon_{it}^T \varepsilon_{it} \right. \\
\left. + \frac{1}{l(l - 2)^3} \sum_s (\varepsilon_{is}^T \varepsilon_{is})^2 + \frac{l - 3}{l(l - 1)(l - 2)^3} \sum_{s \neq t} \varepsilon_{is}^T \varepsilon_{is} \varepsilon_{it}^T \varepsilon_{it} \right\}. \]

It can be shown that \( \text{var}(J_n)/\text{var}(\tilde{T}_n) = O(l^{-3} \text{tr}^{-1}(\Sigma_0^2) \text{tr}^2(\Sigma_0)) \), from which we can observe the requirement on \( l \).

Consider the modified test statistic \( T_n' \). Let \( W = H(H - 1) \{\text{tr}(\Sigma_0)^2\}^2/(l- \).
2), and $T'_n$ can be rewritten as

$$T'_n = (H - 1) \sum_{i=1}^{H} A_i - 2 \sum_{i<j} C_{ij} - W$$

$$= \left\{ \left( (H - 1) \sum_{i=1}^{H} A_i - J_n \right) - 2 \sum_{i<j} C_{ij} \right\} + (J_n - W)$$

$$= T^*_n + (J_n - W).$$

By similar arguments in Proposition 1, it can be verified that without the condition $p = o(l^3)$,

$$E(T^*_n) = \{ 2/(l-2)+2/(l-2)^2 - 2/(l-1)/l(l-1)^{2} \} H(H-1)\text{tr}(\Sigma^2_0)+o(\sqrt{\text{var}(\tilde{T}_n)})$$

and \(\text{var}(T^*_n) = \text{var}(\tilde{T}_n)(1+o(1))\). Then we need to deal with the terms $J_n - W$. Define $U_n = (l-2)^{-2}(H-1) \sum_{i=1}^{H} l^{-1} \sum_{s} \epsilon_{is}^{T} \epsilon_{is} - \{ l(l-1) \}^{-1} \sum_{s \neq t} \epsilon_{is}^{T} \epsilon_{it}$.

Some calculations yield that

$$J_n - U_n = (l-2)^{-2}(H-1) \sum_{i=1}^{H} \left\{ \frac{8}{l^2(l-1)(l-2)^2} \sum_{s,t,r}^{*} \epsilon_{is}^{T} \epsilon_{it} \epsilon_{ir} \right.$$

$$- \frac{8}{l^2(l-1)(l-2)} \sum_{s \neq t} \epsilon_{is}^{T} \epsilon_{it} \epsilon_{ir} + \frac{2}{l^2(l-2)} \sum_{s} \epsilon_{is}^{T} \epsilon_{is}^{2}$$

$$- \frac{2}{l^2(l-1)(l-2)} \sum_{s \neq t} \epsilon_{is}^{T} \epsilon_{it} \epsilon_{ir} - \frac{1}{l^2(l-1)^{2}} \sum_{s \neq t \neq q} \epsilon_{is}^{T} \epsilon_{it} \epsilon_{iq} \left\}.$$

Taking similar procedures as for (A.2) and (A.3), we can show that $E(J_n - U_n) = O(l^{-3}\text{tr}(\Sigma^2_0)) = o(\sqrt{\text{var}(\tilde{T}_n)})$ and $\text{var}(J_n - U_n)/\text{var}(\tilde{T}_n) = O(l^{-7}\text{tr}^{-1}(\Sigma^2_0)\text{tr}^2(\Sigma_0))$.

Consequently, it is only required that $p = o(l^7)$.

It remains to verify that $W - U_n = o_p(\sqrt{\text{var}(\tilde{T}_n)})$. Let $\tilde{W} = (l -$
2) \(2H(H-1)(H^{-1} \sum_i B_i)^2\), where \(B_i = l^{-1} \sum_s \varepsilon_{is}^T \varepsilon_{is} - \{l(l-1)\}^{-1} \sum_{s \neq t} \varepsilon_{is}^T \varepsilon_{it}\).

Note that \(E(B_i) = \text{tr}(\Sigma_0)\) and \(\text{var}(B_i) = O(l^{-1} \text{tr}(\Sigma_0^2))\). Observe

\[
\widetilde{W} - U_n = \frac{(H - 1)}{(l - 2)^2} \sum_i \{ (B_i - \text{tr}(\Sigma_0))(B_i - H^{-1} \sum_j B_j) \} \\
= O(Hl^{-5/2} \text{tr}^{1/2}(\Sigma_0^2)) \sum_i \{ B_i - \text{tr}(\Sigma_0) \} \\
= O_p(H^{3/2}l^{-3} \text{tr}(\Sigma_0^2)) = O_p(l^{-2}) \sqrt{\text{var}(\widetilde{T}_n)} \\
= o_p(\sqrt{\text{var}(\widetilde{T}_n)}).
\]

Also,

\[
W - \widetilde{W} = \frac{H(H-1)}{(l-2)^2} \left\{ (H^{-1} \sum_i R_i)^2 + 2H^{-2} \sum_i B_i \sum_i R_i \right\} \\
R_i = 2l^{-1} \sum_s \varepsilon_{is}^n \mu_{is} - 2\{l(l-1)\}^{-1} \sum_{s \neq t} \varepsilon_{is}^n \mu_{it} \\
+ l^{-1} \sum_s \mu_{is}^n \mu_{is} - \{l(l-1)\}^{-1} \sum_{s \neq t} \mu_{is}^n \mu_{it} \\
\equiv R_{i1} + R_{i2}.
\]

As mentioned in Proposition 1, \(R_{i2} = \Lambda_i = O_p(l^{-1}pr_{2i})\), we need only consider the terms \(R_{i1}\). By the similar argument, \(\sum_i R_{i1}\) is a higher-order term than \(\sum_i R_{i2}\). Combine all these results, we have

\[
W - \widetilde{W} = H(H-1)/(l-2)^2 O_p(p^2/n \sum_i r_{2i}) \\
= \sqrt{\text{var}(\widetilde{T}_n)} O_p(H^{-1/2}l^{-2}p \sum_i r_{2i})
\]
\[ = \text{o}_p(\sqrt{\text{var}(\tilde{T}_n)}), \]

from which we complete the proof of Proposition 2.

\[ \Box \]

**Proof of Proposition 3**

\[ \text{Proof.} \text{ Rewrite } \hat{\text{tr}}(\Sigma_0^2) \text{ as} \]

\[ \hat{\text{tr}}(\Sigma_0^2) = \left\{ 1 + \frac{2}{l-2} + \frac{2}{(l-2)^2} \right\}^{-1} \left\{ \left( \frac{1}{H} \sum_{i=1}^{H} A_i - \frac{J_n}{H(H-1)} \right) + \frac{J_n - W}{H(H-1)} \right\} \]

\[ \equiv e_1 + e_2. \]

Similar to the proof of Proposition 1, we can show that

\[ E(e_1) = \text{tr}(\Sigma_0^2) + O(l^{-3}\text{tr}(\Sigma_0^2)), \]

\[ \text{var}(e_1) = H^{-4}\text{var}\{(H-1) \sum_i A_i - J_n\}(1 + o(1)) = O(n^{-2}\text{tr}^2(\Sigma_0^2) + (H^2l)^{-1}\text{tr}(\Sigma_0^4)). \]

Accordingly, \( \{e_1 - \text{tr}(\Sigma_0^2)\}/\text{tr}(\Sigma_0^2) \rightarrow 0. \) By Proposition 2 and taking similar procedures, we have \( J_n - W = \text{o}_p\{H^{3/2}l^{-1}\text{tr}(\Sigma_0^2)\}. \) Thus, \( e_2/\text{tr}(\Sigma_0^2) \rightarrow 0. \)

\[ \Box \]

**Proof of Theorem 2**

\[ \text{Proof.} \text{ For technical convenience, we assume that the } n \text{ data can be exactly divided to } H \text{ equal slices. Denote } E_0 \text{ and } E_1 \text{ represent the expectation under } H_0\text{ and under } H_1 \text{ respectively. If a slice } A_i \text{ falls into the area } \{Y < a\}, \text{ then} \]
\( E_1(A_i) = E_0(A_i) \). On the contrary, if it falls into the area \( \{ Y > a \} \), we have \( E_1(A_i) = (1 + \theta_n)^2 E_0(A_i) \). Also note that the number of slices which partly falls into the former and rest falls into the latter is at most one, there is little need to consider this situation as long as \( H \) is sufficiently large. So we further assume \( H \) is an even number. Similar to the proof of Proposition, by Condition \((C2)\) that \( n/l^5 \to 0 \) and the fact that \( b_l/a_l = 1 + O(l^{-2}) \), we have

\[
E_1(\{ T'_n \}) = (H - 1) \sum_i E_1(A_i) - 2 \sum_{i<j} E_1(C_{ij}) - E_1(W)
\]

\[
= \frac{H(H-1)}{2} \left\{ a_l \text{tr}(\Sigma_0^2) + (l-2)^{-2} \text{tr}^2(\Sigma_0) + a_l(1 + \theta_n)^2 \text{tr}(\Sigma_0^2)
\right. \\
+ (l-2)^{-2}(1 + \theta_n)^2 \text{tr}^2(\Sigma_0) + O((l-2)^{-3} \text{tr}(\Sigma_0^2)) \right\} - \frac{H}{2} - 1 \frac{H}{2} b_l \text{tr}(\Sigma_0^2)
\]

\[
- \frac{H}{2} - 1 \frac{H}{2} b_l (1 + \theta_n)^2 \text{tr}(\Sigma_0^2) - \frac{H^2}{2} b_l (1 + \theta_n) \text{tr}(\Sigma_0^2)
\]

\[
- H(H-1)(l-2)^{-2} \{1 + \theta_n/2\}^2 \text{tr}^2(\Sigma_0) + o(\sigma_{T_n,0})
\]

\[
= b_l H^2 \theta_n^2 \text{tr}(\Sigma_0^2)/4 + H(H-1)(l-2)^{-2} \theta_n^2 \text{tr}^2(\Sigma_0)/4
\]

\[
+ (a_l - b_l) H(H-1) \{1 + (1 + \theta_n)^2\} \text{tr}(\Sigma_0^2)/2 + o(\sigma_{T_n,0})
\]

\[
\equiv \mu_{T_n,1} + o(\sigma_{T_n,0}).
\]

If a slice \( A_i \) falls into the area \( \{ Y > a \} \), then

\[
\text{var} \left( \sum_{s \neq t} (\epsilon_{is}^2 \epsilon_{it}^2)^2 \right) = l(l-1) \{4(1 + \theta_n)^4 \text{tr}^2(\Sigma_0^2) + O(\text{tr}(\Sigma_0^4))\}.
\]
Tedious calculations yield

$$\sigma^2_{T_n,1} = 2l^{-2}H(H-1)^2\text{tr}^2(\Sigma_0^2)\{1+(1+\theta_n)^4\}(1+o(1)) = \sigma^2_{T_n,0}\{1+(1+\theta_n)^4\}/2.$$ 

Under the alternative $H_1$,

$$\hat{\sigma}_{T_n,0} = 2l^{-1}H^{3/2}\left\{\frac{1+(1+\theta_n)^2}{2}\text{tr}(\Sigma_0^2) + \frac{\theta_n^2\text{tr}^2(\Sigma_0)}{4l^2}\right\}(1+o_p(1)).$$

Similar to the proof of Theorem 1, we can establish the asymptotic normality of $\hat{T}_n'$. Here we omit the details.

**Further simulation results**

An application of our proposed test lies in sufficient dimension reduction (SDR) which tries to reduce the dimension by replacing original predictors with a minimal set of their linear combinations without loss of information in regression. Many SDR methods were developed based on the paradigm of inverse regression (Li, 1991) and they usually rely on the validity of the constant variance condition (Cook and Weisberg, 1991), i.e., $\text{cov}(X | \beta^T X) = \Sigma_0$, where $\beta \in \mathbb{R}^{p \times d}$ is a basis matrix of the central subspace and $X$ stands for the predictor vector. In reality, we can test if $\text{cov}(X | \hat{\beta}^T X)$ is approximately a constant matrix at the value $\hat{\beta}$ that is close to the true $\beta$. In order not to affect the validity of detection, we can divide data into two parts, the one for estimating $\hat{\beta}$, and the other for testing.
We added some numerical results regarding this strategy. To show the performance of empirical size, we use the simple linear model 1

\[
x_{ik} \sim N(0, 0.2) \text{ independently for } k = 1, 2
\]

\[
x_{ik} \sim N(0, 1 + 0.6 \cdot I(x_{i2} > 0)) \text{ for } k = 3, \ldots, p
\]

\[
y_i = 4x_{i1} + \delta \epsilon_i
\]

to generate \( n = 320 \) data points. \( \epsilon_i \) is a standard normal distribution, independent of \( x_i \)'s. Clearly noting that the central subspace contains only one SDR direction, standardized as \( \beta = (1, 0, \ldots, 0)^T \) and \( \text{cov}(X \mid x_k) \) is constant only for \( k = 1 \). This setting greatly diminishes the number of \( \beta \) which makes null hypothesis hold. Similarly, to show the performance of empirical power, we use model 2 followed

\[
x_{ik} \sim N(0, 1) \text{ independently for } k = 1, 3, \ldots, p
\]

\[
x_{i2} = (p - 2)^{-1/2} \sum_{k=3}^{p} (x_{ik}^2 - 1) + e_i
\]

\[
y_i = 4x_{i2} + \delta \epsilon_i
\]

where the SDR direction is \( \beta = (0, 1, 0, \ldots, 0)^T \), \( e_i, \epsilon_i \) are both standard normal distribution, independent of \( x_i \)'s. Similar to model 1, \( \text{cov}(X \mid x_k) \) depends on \( x_k \) only for \( k = 2 \), so that it greatly diminishes the number of \( \beta \) which violates \( H_0 \). Here we use the first \( n_1 = 160 \) data points to estimate \( \beta \) by SIR, then use the rest \( n - n_1 = 160 \) data points to test
$H_{01} : \operatorname{cov}(X \mid \beta^T X) = \Sigma_0$ and $H_{02} : \operatorname{cov}(X \mid \hat{\beta}^T X) = \Sigma_0$, respectively.

All the simulation results are obtained based on 1,000 repetitions and the nominal level is fixed as 0.05.

For the first 160 data, the classical SIR doesn’t work in the ”large $p$, small $n$” cases. Actually, there are many statistical methods that estimating central subspace for high dimensional data, for example the SSIR method (Ni, Cook and Tsai, 2005) and the CISE method (Chen, Zou and Cook, 2010). Here for convenience, we simply consider the relative small dimension $p = 20$, and obtain the estimate by classical SIR procedure. Table 1 shows the simulation result and its performance is reasonably well.

References


Table 1: The performance based on our strategy, both in model 1 and in model 2.

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Table 2: Empirical sizes at 5% significance under the model of $d = 1$ and $y \sim U(2, 4)$.

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Table 3: Empirical sizes at 5% significance under the model of $d = 1$ and $y \sim N(3, 0.2)$.

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