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ASYMPTOTIC INDEPENDENCE OF THE SUM AND MAXIMUM OF DEPENDENT RANDOM VARIABLES WITH APPLICATIONS TO HIGH-DIMENSIONAL TESTS

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Abstract: For a set of dependent random variables, and without using stationary or strong mixing assumptions, we derive the asymptotic independence between their sums and maxima. Then, we apply this result to high-dimensional testing problems. Here, we combine the sum-type and max-type tests, and propose a novel test procedure for the one-sample mean test, two-sample mean test and regression coefficient test in a high-dimensional setting. Based on the asymptotic independence between the sums and maxima, we establish the asymptotic distributions of the test statistics. Simulation studies show that our proposed tests perform well regardless of the sparsity of the data. Examples based on real data are also presented to demonstrate the advantages of our proposed methods.

Key words and phrases: Asymptotic normality; Asymptotic independence; Extreme-value distribution; High-dimensional tests; Large p and small n

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1. Introduction

Statistical independence is a very simple structure and is convenient in statistical inference and applications. In this paper, we study the asymptotic independence between two common statistics: the extreme-value statistic $M_p = \max_{1\leqslant i\leqslant p} X_i$, and the sum $S_p = \sum_{i=1}^{p} X_i$, where $\{X_i\}_{i=1}^{p}$ is a sequence of dependent random variables. Then, we apply these results to three high-dimensional testing problems, with numerical examples.

1.1 Independence Between Sum and Maximum

There is growing academic interest in understanding the asymptotic joint distribution of M_p and S_p. In an early research, Chow and Teugels (1978) established the asymptotic independence between M_p and S_p for independent and identically distributed (i.i.d.) random variables. To overcome the limitation imposed by the required assumptions, Anderson and Turkman (1991), Anderson and Turkman (1993), Anderson and Turkman (1995), and Hsing (1995) generalized the asymptotic result to the case in which $\{X_i\}_{i=1}^{p}$ is strong mixing; for the concept of "strong mixing" and its properties, see, for example, Bradley (2005), and the literature therein. In particular, Hsing (1995) showed that for a stationary sequence, the strong mixing property and the asymptotic normality of S_p are enough to guarantee the

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asymptotic independence of the sum and maximum. However, Davis and Hsing (1995) show that in the case of an infinite variance, M_p and S_p are not asymptotically independent, because the asymptotic behavior of S_p is dominated by that of the extreme order statistic. In addition, Ho and Hsing (1996), Ho and McCormick (1999), McCormick and Qi (2000), and Peng and Nadarajah (2003) considered the joint limit distribution of the maximum and sum of the stationary Gaussian sequence $\{X_i\}_{i=1}^p$, in which $E(X_i) = 0$, $\mathrm{Var}(X_i) = 1$, and $r(p) = E(X_i X_{i+p})$. Under different conditions on $r(p)$, the joint limiting distributions of maxima and sums are different. Specifically, Ho and Hsing (1996) showed that M_p and S_p are asymptotically independent, as long as $\lim_{p\to\infty} r(p)\log p = 0$; the two statistics are not independent provided $\lim_{p\to\infty} r(p)\log p = \rho \in (0,\infty)$. Then, by assuming $\lim_{p\to\infty} \frac{\log p}{p} \sum_{i=1}^p |r(i) - r(p)| = 0$, Ho and McCormick (1999) and McCormick and Qi (2000) obtained the asymptotic independence of $M_p - (S_p/n)$ and S_p.

The aforementioned results are all based on the stationary assumption that the covariance structure among $\{X_i\}_{i=1}^p$ has the property that $E(X_i X_{i+h}) = E(X_1 X_{1+h})$, for each integer h and $i = 1, \cdots, p-h$. Although this is a common assumption in research, it is not easy to check. Even though it can be verified using hypothesis testing, the stationary prop-

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erty still may not hold up to certain statistical errors. In fact, in many scenarios this assumption is not true. For example, for stock data from the US S&P 500 index, in which stock returns are considered as variables, if stocks are ordered alphabetically by name, the two stocks, such as AAPL and MSFT, may have both a far distance and a strong correlation, which does not satisfy the stationary assumption.

In this work, we study the asymptotic independence between $\tilde{S}_p = \sum_{i=1}^p Z_i^2$ and $\tilde{M}_p = \max_{1\leqslant i\leqslant p} Z_i^2$, without the stationary assumption. In our case, each Z_i is marginally $N(0,1)$, and the covariance matrix of Z_i, denoted by $\mathbf{\Sigma}_p = (\sigma_{ij})_{1\leqslant i,j\leqslant p}$, satisfies certain conditions. Specifically, we first establish the asymptotically normality of $\tilde{S}_p$ if $[\mathrm{tr}(\mathbf{\Sigma}_p^{2+\delta})]^2 \cdot [\mathrm{tr}(\mathbf{\Sigma}_p^2)]^{-2-\delta} \to 0$ for some $\delta > 0$. Then, we show that the limit distribution of the maximum $\tilde{M}_p - 2\log p + \log\log p$ is a Gumbel distribution, under conditions on the covariance matrix $\mathbf{\Sigma}_p$. Finally, we prove the asymptotic independence between $\tilde{S}_p$ and $\tilde{M}_p$ under the conditions $\max_{1\leqslant i<j\leqslant p} |\sigma_{ij}| \leqslant \varrho$ and $\max_{1\leqslant i\leqslant p} \sum_{j=1}^p \sigma_{ij}^2 \leqslant (\log p)^C$, together with two additional conditions on the maximum and minimum eigenvalues of $\mathbf{\Sigma}_p$. These theoretical results are novel because they do not require the stationary property. Since these results are universal, they may provide many useful implications. In this paper, we apply the above asymptotic independence results to three high-

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dimensional hypothesis testing problems: the one-sample mean test, two-sample mean test and regression coefficient test.

1.2 High-Dimensional Hypothesis Testing

High-dimensional hypothesis testing is an important research area in modern statistics. It is frequently used in many fields, such as genomics, medical imaging, risk management, and web search. The motivation for studying high-dimensional tests is that traditional tests, such as the Hotelling T-squared test, do not work, in general, when the data dimension is larger than the sample size owing to the singularity of the sample covariance matrix. A natural way of solving this problem is to replace the sample covariance matrix in the Hotelling T-squared test statistic with a nonsingular matrix, such as the identity matrix or the diagonal matrix of the sample covariance matrix. In this way, for example, Srivastava (2009), Park and Ayyala (2013), Wang et al. (2015), Feng et al. (2016), Feng et al. (2015) and Feng et al. (2017) developed tests for the one-sample mean problem, and Bai and Saranadasa (1996), Srivastava and Du (2008), Chen and Qin (2010), and Gregory et al. (2015) developed tests for the two-sample mean problem. In addition, Goeman et al. (2006) and Lan et al. (2014), among others, test the regression coefficients in high-dimensional linear models.

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These tests are all sum-type tests, based on a summation of the parameter estimators. It is well known that sum-type tests perform well, in general, when the data are dense; that is, most of the parameters are nonzero under the local alternative. However, such tests may be inefficient when the data are sparse, where only a few parameters are nonzero under local alternative. To establish high-dimensional tests for sparse data, Cai et al. (2014), Zhong et al. (2013), and Chen et al. (2019) propose max-type tests, which typically perform well on sparse data, but worse when the data become dense.

In practice, it is often difficult to determine whether or not the data are sparse. Thus, much effort has been devoted to developing tests with good and robust performance under both data conditions. For example, Fan et al. (2015) propose a power enhancement procedure that uses a screening technique for high-dimensional tests. They combine the power enhancement component with an asymptotically pivotal statistic to strengthen the power under sparse alternatives. Xu et al. (2016) propose an adaptive test for a high-dimensional two-sample mean test that combines information across a class of sum-of-power tests, including tests based on the sum of the squares of the mean differences and the supremum mean difference. Wu et al. (2019) extend the adaptive test to generalized linear models. He et al.

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(2021) construct U-statistics of different orders that are asymptotically independent of the max-type test statistics in high-dimensional tests, based upon which, they propose an adaptive testing procedure. However, these results are based on the work of Hsing (1995), and thus require that the data be sampled from stationary and α-mixing random variables. In fact, the α-mixing property is rarely checked in practice, which greatly limits the application of these methods. In this study, we use the novel asymptotic independence analysis between the aforementioned sum and maximum to solve the problem without requiring the stationary assumption or the α-mixing property. In addition, we propose a series of high-dimensional tests, including one-sample mean test, two-sample mean test and regression coefficient test. Numerical results demonstrate the strong robustness of the proposed tests, regardless of the sparsity of the data.

This study makes three main contributions to the literature. First, we establish the asymptotic distribution of the maximum of dependent Gaussian random variables under a general assumption. Second, we prove the asymptotic independence between the sum and maximum of dependent Gaussian random variables, without needing the stationary or the α-mixing property. Third, we propose three high-dimensional combo-type tests based on the aforementioned asymptotic properties, namely, the one-sample mean test, two-sample mean test, and regression coefficient test. Numerical examples on simulated and real-world data demonstrate the strong robustness of our tests on both sparse and dense data sets.

The rest of the paper is organized as follows. In Section 2, we state our theoretical results, including the asymptotic distributions of the sum and maximum statistics, and the asymptotic independence between them. In Section 3, we propose a series of tests for high-dimensional data based on these theoretical results. Then, we compare our simulation results with those of several existing tests in Section 4, followed by an application of the proposed tests in Section 5. Finally, Section 6 concludes the paper. Several extended results and proofs are provided in the Supplementary Material.

2. Asymptotic Independence of Sum and Maximum of Dependent Random Variables

First, we study the asymptotic normality of the sum of dependent random variables. For each $p \geqslant 2$, let $Z_{p1}, \cdots, Z_{pp}$ be $N(0,1)$-distributed random variables with $p \times p$ covariance matrix $\mathbf{\Sigma}_p$. If there is no danger of confusion, we simply write "$Z_1, \cdots, Z_p$" rather than "$Z_{p1}, \cdots, Z_{pp}$" and "$\mathbf{\Sigma}$" rather than "$\mathbf{\Sigma}_p$". The following assumption is needed:

$$\lim_{p\to\infty} \frac{[\mathrm{tr}(\mathbf{\Sigma}^{2+\delta})]^2}{[\mathrm{tr}(\mathbf{\Sigma}^2)]^{2+\delta}} = 0, \quad \text{for some } \delta > 0. \tag{2.1}$$

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Assumption (2.1) with $\delta = 2$ is the same as condition (3.7) in Chen and Qin (2010), and here we make it more general. In practice, the true covariance matrix $\mathbf{\Sigma}$ is usually unknown. However, this condition ensures that our results apply to a wide range of problems. For instance, if all eigenvalues of $\mathbf{\Sigma}$ are bounded above and are bounded below from zero, it is trivial to see that (2.1) holds.

Theorem 1. *Under Assumption* (2.1), $\frac{Z_1^2+\cdots+Z_p^2-p}{\sqrt{2tr(\mathbf{\Sigma}^2)}} \to N(0,1)$ *in distribution as* $p \to \infty$.

Theorem 1 shows that the sum of squares of dependent Gaussian random variables has the asymptotic normality if the covariance matrix satisfies Assumption (2.1).

Next, for the same Gaussian random variables, we consider the asymptotic distribution of $\max_{1\leqslant i\leqslant p} Z_i^2$. Let $|A|$ denote the cardinality of the set A. We require the following assumption:

$$\text{Let } \mathbf{\Sigma} = (\sigma_{ij})_{1\leqslant i,j\leqslant p}. \text{ For some } \varrho \in (0,1), \text{ assume } |\sigma_{ij}| \leqslant \varrho, \text{ for all } 1 \leqslant i < j \leqslant p \text{ and } p \geqslant 2. \text{ Suppose } \{\delta_p;\, p \geqslant 1\} \text{ and } \{\kappa_p;\, p \geqslant 1\} \text{ are positive constants, with } \delta_p = o(1/\log p) \text{ and } \kappa = \kappa_p \to 0 \text{ as } p \to \infty. \text{ For } 1 \leqslant i \leqslant p, \text{ define } B_{p,i} = \big\{1 \leqslant j \leqslant p;\, |\sigma_{ij}| \geqslant \delta_p\big\} \text{ and } C_p = \big\{1 \leqslant i \leqslant p;\, |B_{p,i}| \geqslant p^{\kappa}\big\}. \text{ We assume that } |C_p|/p \to 0 \text{ as } p \to \infty. \tag{2.2}$$

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Theorem 2. *Suppose Assumption* (2.2) *holds. Then,* $\max_{1\leqslant i\leqslant p} Z_i^2 - 2\log p + \log\log p$ *converges to a Gumbel distribution with cdf* $F(x) = \exp\{-\frac{1}{\sqrt{\pi}}e^{-x/2}\}$ *as* $p \to \infty$.

Remark 1. *Cai et al. (2014) obtained the above limiting distribution of* $\max_{1\leqslant i\leqslant p} Z_i^2$ *under the assumption that* $\max_{1\leqslant i\leqslant p}\sum_{j=1}^p \sigma_{ij}^2 \leqslant C_0$, *for each* $p \geqslant 1$, *where* C_0 *is a constant free of* p. *In the following, we show that their result is a special case of Theorem 2. In fact, let* $\delta_p = (\log p)^{-2}$, *for* $p \geqslant e^e$. *Then, for each* $1 \leqslant i \leqslant p$, $\delta_p^2 \cdot |B_{p,i}| \leqslant \sum_{j=1}^p \sigma_{ij}^2 \leqslant C_0$. *Hence,* $|B_{p,i}| \leqslant C_0 \cdot (\log p)^2 < p^\kappa$, *where* $\kappa = \kappa_p := 5(\log\log p)/\log p$ *for large* p. *As a result,* $|C_p| = 0$, *which implies the results of Theorem 2.*

A closely related but not the same result by Fan and Jiang (2019) shows that $\delta_p = o(1/\log p)$ in Assumption (2.2) cannot be relaxed. Their statistic is $\max_{1\leqslant i\leqslant p} Z_i$, in contrast to $\max_{1\leqslant i\leqslant p} |Z_i|$ here. We expect that $\delta_p = o(1/\log p)$ is also the critical threshold for $\max_{1\leqslant i\leqslant p} |Z_i|$.

Theorem 2 is proved by using the spirit of the proof of Lemma 6 from Cai et al. (2014). First, the conditions imposed in our theorem are weaker than those required in Lemma 6 of Cai et al. (2014), as discussed in Remark 1. This allows us to apply this type of result to a more general covariance matrix $\mathbf{\Sigma}$. Second, some of the steps in the proof of Theorem 2 are also used in the proof of Theorem 3, stated next.

To proceed, we need some additional notation and one further assumption. For two sequences of numbers $\{a_p \geqslant 0;\, p \geqslant 1\}$ and $\{b_p > 0;\, p \geqslant 1\}$, we write $a_p \ll b_p$ if $\lim_{p\to\infty} \frac{a_p}{b_p} = 0$. We assume the following:

$$\begin{aligned}&\text{There exist } C > 0 \text{ and } \varrho \in (0,1) \text{ such that } \max_{1\leqslant i<j\leqslant p} |\sigma_{ij}| \leqslant \varrho \text{ and } \max_{1\leqslant i\leqslant p} \sum_{j=1}^{p} \sigma_{ij}^2 \leqslant (\log p)^C,\\ &\text{for all } p \geqslant 3;\ p^{-1/2}(\log p)^C \ll \lambda_{min}(\mathbf{\Sigma}) \leqslant \lambda_{max}(\mathbf{\Sigma}) \ll \sqrt{p}(\log p)^{-1} \text{ and}\\ &\lambda_{max}(\mathbf{\Sigma})/\lambda_{min}(\mathbf{\Sigma}) = O(p^{\tau})\ , \text{ for some } \tau \in (0, 1/4).\end{aligned} \tag{2.3}$$

Assumption (2.3) is actually stronger than both (2.1) and (2.2). To see this, assume (2.3) holds now. To derive (2.1), observe that $\mathrm{tr}(\mathbf{\Sigma}^{2+\delta}) \leqslant p \cdot \lambda_{max}(\mathbf{\Sigma})^{2+\delta}$ and $\mathrm{tr}(\mathbf{\Sigma}^2) \geqslant p \cdot \lambda_{min}(\mathbf{\Sigma})^2$. Then, $\frac{[\mathrm{tr}(\mathbf{\Sigma}^{2+\delta})]^2}{[\mathrm{tr}(\mathbf{\Sigma}^2)]^{2+\delta}} \leqslant \frac{1}{p^\delta} \cdot \left(\frac{\lambda_{max}(\mathbf{\Sigma})}{\lambda_{min}(\mathbf{\Sigma})}\right)^{4+2\delta} = O\left(\frac{1}{p^{\delta-(4+2\delta)\tau}}\right) \to 0$ by choosing $\delta = 2$ and using the assumption $\tau \in (0, 1/4)$, stated in (2.3). We then get (2.1) with $\delta = 2$. To deduce (2.2), we replace "C_0" in Remark 1 with "$(\log p)^C$". By the same argument as that in Remark 1 and choosing $\delta_p = (\log p)^{-2}$, we have $|B_{p,i}| \leqslant C_0 \cdot (\log p)^{C+2} < p^{\kappa}$, where $\kappa = \kappa_p := (C+3)(\log\log p)/\log p$ for $p \geqslant e^e$. Hence, $|C_p| = 0$ and Assumption (2.2) holds.

Theorem 3. *Under Assumption* (2.3), $\frac{Z_1^2+\cdots+Z_p^2-p}{\sqrt{2tr(\mathbf{\Sigma}^2)}}$ *and* $\max_{1\leqslant i\leqslant p} Z_i^2 - 2\log p + \log\log p$ *are asymptotically independent as* $p \to \infty$.

Importantly, note that the above asymptotic independence result holds without the stationary assumption or the α-mixing condition. For the as-

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sumption on the spectrum, in the literature of high-dimensional statistics, it is common to assume $[\lambda_{min}(\mathbf{\Sigma}), \lambda_{max}(\mathbf{\Sigma})] \subset [a, b]$, with $0 < a < b < \infty$. Note that this is stronger than our assumption on the eigenvalues of $\mathbf{\Sigma}$ in (2.3). In fact, Assumption (2.3) allows that the largest eigenvalue goes to infinity and the smallest eigenvalue goes to zero. Thus, Theorem 3 provides more a general result and more freedom and practicality.

3. Application: High-Dimensional Testing Problems

In this section, we apply the theoretical results derived in Section 2 to three high-dimensional testing problems: the one-sample mean test, two-sample mean test, and regression coefficient test. The first test is presented in the following subsections; the two-sample mean test and regression coefficient test are presented in the Supplementary Material.

3.1 One-Sample Mean Test

Assume $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ are i.i.d. p-dimensional random vectors from $N(\boldsymbol{\mu}, \mathbf{\Sigma})$. The classical one-sample mean testing problem considers

$$H_0 : \boldsymbol{\mu} = \mathbf{0} \quad \text{versus} \quad H_1 : \boldsymbol{\mu} \neq \mathbf{0}. \tag{3.1}$$

In the traditional setting, where p is fixed, this topic is covered in classic textbooks on multivariate analysis, such as in Anderson (2003), Eaton

(1983), and Muirhead (1982). Recent research has begun developing tests for the high-dimensional setting, where both n and p go to infinity. In the following, we highlight parts of works before examining our problem of interest: the test (3.1) when $n \leqslant p$. This is a typical problem of interest in high-dimensional statistics with small n and large p.

Let $\bar{\boldsymbol{X}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i$ and $\hat{\mathbf{S}} = \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})^T$ be the sample mean and the sample covariance matrix of $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$, respectively. The Hotelling T^2-statistic is defined as $n\bar{\boldsymbol{X}}^T\hat{\mathbf{S}}^{-1}\bar{\boldsymbol{X}}$; see Hotelling (1931). For $n > p$, Bai and Saranadasa (1996) study the Hotelling statistic. However, when $n \leqslant p$, the matrix $\hat{\mathbf{S}}$ is no longer invertible, which motivates the design of new statistics. By replacing $\hat{\mathbf{S}}$ with its diagonal matrix in the Hotelling T^2-statistic, Srivastava and Du (2008) and Srivastava (2009) propose a scale-invariant test for (3.1), defined as

$$T_{sum}^{(1)} = \frac{n\bar{\boldsymbol{X}}^T\hat{\mathbf{D}}^{-1}\bar{\boldsymbol{X}} - (n-1)p/(n-3)}{\sqrt{2[\mathrm{tr}(\hat{\mathbf{R}}^2) - p^2/(n-1)]}}, \tag{3.2}$$

where $\hat{\mathbf{D}}$ is the diagonal matrix of the sample covariance matrix $\hat{\mathbf{S}}$, and $\hat{\mathbf{R}} = \hat{\mathbf{D}}^{-1/2}\hat{\mathbf{S}}\hat{\mathbf{D}}^{-1/2}$ is the sample correlation matrix. The major ingredient of $T_{sum}^{(1)}$ can be written as a sum of random variables, so we sometimes call it a "sum-type" statistic. In general, sum-type statistics do not perform well in sparse cases, in which only a few entries in $\boldsymbol{\mu}$ in the sum are nonzero;

see Cai et al. (2014) for more a detailed discussion. Zhong et al. (2013) propose two alternative tests by first thresholding two statistics based on the sample means, and then maximizing over a range of thresholding levels. Denote $\bar{\boldsymbol{X}} = (\bar{\boldsymbol{X}}_1, \cdots, \bar{\boldsymbol{X}}_p)^T$. The L_2-version of the thresholding statistic is

$$T_{HC2} = \max_{s \in \mathcal{S}} \frac{T_{2n}(s) - \hat{\mu}(s)}{\hat{\sigma}(s)}, \tag{3.3}$$

where $\mathcal{S}$ is a subset of the interval $(0, 1)$,

$$T_{2n}(s) = \sum_{j=1}^{p} n \left(\bar{\boldsymbol{X}}_j/\sigma_j\right)^2 I\left(|\bar{\boldsymbol{X}}_j| \geqslant \sigma_j \sqrt{\lambda_s/n}\right),$$

$$\hat{\mu}(s) = p\left\{2\lambda_p^{1/2}(s)\phi(\lambda_p^{1/2}(s)) + 2\bar{\Phi}(\lambda_p^{1/2}(s))\right\},$$

$$\hat{\sigma}^2(s) = p\left\{2\left[\lambda_p^{3/2}(s) + 3\lambda_p^{1/2}(s)\right]\phi(\lambda_p^{1/2}(s)) + 6\bar{\Phi}(\lambda_p^{1/2}(s))\right\}.$$

Here, $\lambda_s(p) = 2s\log p$ and $\phi(\cdot)$, $\bar{\Phi}(\cdot)$ are the density and survival functions of the standard normal distribution, respectively. Fan et al. (2015) propose a novel procedure by adding a power enhancement component that is asymptotically zero under the null, and diverges under some specific regions of alternatives. Their test statistic is

$$J = J_0 + J_1, \tag{3.4}$$

where the power enhancement component J_0 is $J_0 = \sqrt{p}\sum_{j=1}^{p} \bar{\boldsymbol{X}}_j^2 \hat{\sigma}_j^{-2} I(|\bar{\boldsymbol{X}}_j| > \hat{\sigma}_j \delta_{p,n})$, and J_1 is the standard Wald statistic $J_1 = \frac{\bar{\boldsymbol{X}}^T \widehat{\mathrm{var}}^{-1}(\hat{\boldsymbol{X}})\bar{\boldsymbol{X}} - p}{2\sqrt{p}}$. Here, $\hat{\sigma}_j^2$

is the sample variance of the jth coordinate of the population vector, $\delta_{p,n}$ is a thresholding parameter, and $\widehat{\text{var}}^{-1}(\hat{\boldsymbol{X}})$ is a consistent estimator of the asymptotic inverse covariance matrix of $\bar{\boldsymbol{X}}$. However, the power enhancement component is negligible if the signal is not very strong. As noted earlier, Cai et al. (2014) show that extreme-value statistics are particularly powerful against sparse alternatives, and possess certain optimal properties. Hence, we propose a statistic that combines the sum-type statistic from (3.2) and an extreme-value statistic based on our results in Section 2, and compare it with various baselines numerically in Section 4.1. As shown later, our method performs very well, regardless of the sparsity of the alternative hypothesis.

We now formally introduce our approach. Define

$$T_{max}^{(1)} = n \cdot \max_{1 \leqslant i \leqslant p} \frac{\bar{\boldsymbol{X}}_i^2}{\hat{\sigma}_{ii}^2}, \tag{3.5}$$

where $\bar{\boldsymbol{X}}_i$ is the ith coordinate of $\bar{\boldsymbol{X}} = \frac{1}{n}(\mathbf{X}_1 + \cdots \mathbf{X}_n) \in \mathbb{R}^p$, and $\hat{\sigma}_{ii}^2$ is the sample variance of the ith coordinate of the population vector; that is, if we write $\mathbf{X}_j = (x_{1j}, \cdots, x_{pj})^T$ for each $1 \leqslant j \leqslant n$, then $\hat{\sigma}_{ii}^2$ is the sample variance of the i.i.d. random variables $x_{i1}, x_{i2}, \cdots, x_{in}$. First, we present the asymptotic distribution of $T_{max}^{(1)}$, along with some additional notation. Let $\mathbf{R} = \mathbf{D}^{-1/2}\boldsymbol{\Sigma}\mathbf{D}^{-1/2} = (\rho_{ij})_{1 \leqslant i,j \leqslant p}$ denote the population correlation matrix,

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where $\mathbf{D}$ is the diagonal matrix of $\mathbf{\Sigma}$. We impose the following assumption:

$$\text{There exist } \epsilon \in \left(\frac{1}{2}, 1\right] \text{ and } K > 1, \text{ such that } K^{-1}p^{\epsilon} \leqslant n \leqslant Kp^{\epsilon} \text{ and}$$
$$\sup_{p\geqslant 2} \frac{1}{p}\mathrm{tr}(\mathbf{R}^i) < \infty, \text{ for } i = 2, 3, 4. \tag{3.6}$$

Note that (3.6) is the same as assumptions (3.1) and (3.2) of Srivastava (2009). If the eigenvalues of the correlation matrix $\mathbf{R}$ are bounded, the second condition of (3.6) holds automatically. For rigor of mathematics, we assume n depends on p, and sometimes write n_p when there is possible confusion.

Theorem 4. *Under the null hypothesis in* (3.1)*, the following hold as* $p \to \infty$*:*

(i) If (3.6) *holds, then* $T_{sum}^{(1)} \to N(0, 1)$ *in distribution.*

(ii) If (2.2) *holds, with "*$\mathbf{\Sigma}$*" replaced with "*$\mathbf{R}$*" and* $\log p = o(n^{1/3})$*, then* $T_{max}^{(1)} - 2\log p + \log\log p$ *converges weakly to a Gumbel distribution with cdf* $F(x) = \exp\{-\frac{1}{\sqrt{\pi}}\exp(-x/2)\}$*.*

(iii) Assume (3.6) *is true. If* (2.3) *holds, with "*$\mathbf{\Sigma}$*" replaced with "*$\mathbf{R}$*", then* $T_{sum}^{(1)}$ *and* $T_{max}^{(1)} - 2\log p + \log\log p$ *are asymptotically independent.*

Part (i) of the above theorem is from Srivastava (2009), and is also a corollary of the recent work by Jiang and Li (2021). For the sum-type test, we perform a level-α test, where we reject H_0 when $T_{sum}^{(1)}$ is larger than the $(1-\alpha)$-quantile $z_\alpha = \Phi^{-1}(1-\alpha)$ where $\Phi(y)$ is the cdf of $N(0,1)$. For the max-type test, we then perform a level-α test where we reject H_0 when $T_{max}^{(1)} - 2\log p + \log\log p$ is larger than the $(1-\alpha)$-quantile $q_\alpha = -\log\pi - 2\log\log(1-\alpha)^{-1}$ of the Gumbel distribution $F(x)$.

Based on Theorem 4, we propose a combo-type test statistic by combining the max-type and the sum-type tests. It is defined by

$$T_{com}^{(1)} = \min\{P_S^{(1)}, P_M^{(1)}\}, \tag{3.7}$$

where $P_S^{(1)} = 1 - \Phi\left\{T_{sum}^{(1)}\right\}$ and $P_M^{(1)} = 1 - F(T_{max}^{(1)} - 2\log p + \log\log p)$. Note that $P_S^{(1)}$ and $P_M^{(1)}$ are the p-values for the tests using the statistics $T_{sum}^{(1)}$ and $T_{max}^{(1)}$, respectively, and $T_{com}^{(1)}$ is defined as the smaller one of the two, with an asymptotic distribution characterized by the minimum of two standard uniform random variables.

Corollary 1. *Assume the conditions in Theorem 4(iii) hold. Then,* $T_{com}^{(1)}$ *from* (3.7) *converges weakly to a distribution with density* $G(w) = 2(1-w)I(0 \leqslant w \leqslant 1)$ *as* $p \to \infty$.

According to Corollary 1, the proposed combo-type test allows us to perform a level-α test by rejecting the null hypothesis when $T_{com}^{(1)} < 1 - \sqrt{1-\alpha} \approx \frac{\alpha}{2}$ as α is small. We now discuss the power functions. First, the

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power function of our combo-type test is

$$\begin{aligned}\beta_C^{(1)}(\boldsymbol{\mu},\alpha) &= P\left(T_{com}^{(1)} < 1-\sqrt{1-\alpha}\right) = P\left(P_M^{(1)} < 1-\sqrt{1-\alpha} \text{ or } P_S^{(1)} < 1-\sqrt{1-\alpha}\right) \\ &\geqslant \max\left\{P\left(P_S^{(1)} < 1-\sqrt{1-\alpha}\right), P\left(P_M^{(1)} < 1-\sqrt{1-\alpha}\right)\right\} \\ &\approx \max\left\{\beta_S^{(1)}(\boldsymbol{\mu},\alpha/2), \beta_M^{(1)}(\boldsymbol{\mu},\alpha/2)\right\} \qquad (3.8)\end{aligned}$$

when α is small, where $\beta_M^{(1)}(\boldsymbol{\mu},\alpha)$ and $\beta_S^{(1)}(\boldsymbol{\mu},\alpha)$ are the power functions of $T_{max}^{(1)}$ and $T_{sum}^{(1)}$, respectively, with significance level α. From Srivastava (2009), the power function of $T_{sum}^{(1)}$ is

$$\beta_S^{(1)}(\boldsymbol{\mu},\alpha) = \lim_{p\to\infty} \Phi\left(-z_\alpha + \frac{n\boldsymbol{\mu}^T\mathbf{D}^{-1}\boldsymbol{\mu}}{\sqrt{2\text{tr}(\mathbf{R}^2)}}\right), \qquad (3.9)$$

where $z_\alpha = \Phi^{-1}(1-\alpha)$ is the $(1-\alpha)$-quantile of $N(0,1)$. From (3.8), we have $\beta_C^{(1)}(\boldsymbol{\mu},\alpha) \geqslant \lim_{p\to\infty}\Phi\left(-z_{\alpha/2} + \frac{n\boldsymbol{\mu}^T\mathbf{D}^{-1}\boldsymbol{\mu}}{\sqrt{2\text{tr}(\mathbf{R}^2)}}\right)$. Denote $\mathbf{D} = \text{diag}(\sigma_{11}^2,\cdots,\sigma_{pp}^2)$. By the same argument as that in Theorem 2 of Cai et al. (2014), the asymptotic power of $T_{max}^{(1)}$ converges to one if $\max_{1\leqslant i\leqslant p}|\mu_i/\sigma_{ii}| \geqslant c\sqrt{\log p/n}$, for a certain constant c, and also the nonzero μ_i are randomly uniformly sampled with sparsity level $\gamma < 1/4$, that is, the number of nonzero μ_i is less than $p^\gamma, \gamma < 1/4$. Thus, according to (3.8), the power function of our proposed test $T_{com}^{(1)}$ also converges to one in this case. Similarly, according to Theorem 3 in Cai et al. (2014), the condition $\max_{1\leqslant i\leqslant p}|\mu_i/\sigma_{ii}| \geqslant c\sqrt{\log p/n}$ is minimax rate optimal for testing against sparse alternatives. If c is sufficiently small, then any α-level test is unable to reject the null hypothesis with probability tending to one. Cai et al. (2014) show that $T_{max}^{(1)}$ enjoys a certain optimality against sparse alternatives. By (3.8), our test $T_{com}^{(1)}$ also has this optimality.

For a rough asymptotic power comparison between $T_{sum}^{(1)}$, $T_{max}^{(1)}$ and $T_{com}^{(1)}$, we simply assume that $\boldsymbol{\Sigma} = \mathbf{I}_p$. There are m nonzero μ_i, and all are equal to $\delta \neq 0$. Equation (3.9) gives $\beta_S^{(1)}(\boldsymbol{\mu}, \alpha) = \lim_{p\to\infty} \Phi\left(-z_\alpha + \frac{nm\delta^2}{\sqrt{2p}}\right)$.

We consider two special cases:

(1) *Dense case*: $\delta = O(n^{-\xi})$ and $m = O(p^{1/2}n^{2\xi-1})$, with $\xi \in (1/2, 5/6]$. We also assume $\log p = o(n^{\xi-\frac{1}{2}})$; hence, $\log p = o(n^{1/3})$. As a result, the requirement on p versus n imposed in Theorem 4(ii) is fulfilled. Obviously, the number of nonzero μ_i goes to infinity. The power function for $T_{max}^{(1)}$ is given by $\beta_M^{(1)}(\boldsymbol{\mu}, \alpha) = P\left(T_{max}^{(1)} - 2\log p + 2\log\log p > q_\alpha\right)$. In this case, we will show in Section S3.4 of the Supplementary Material that $\beta_M^{(1)}(\boldsymbol{\mu}, \alpha) \approx \alpha$, which means that $T_{max}^{(1)}$ is not effective or useful. Consequently, we have $\beta_C^{(1)}(\boldsymbol{\mu}, \alpha) \approx \beta_S^{(1)}(\boldsymbol{\mu}, \alpha/2)$. When the significance level α is small, the difference between $\beta_S^{(1)}(\boldsymbol{\mu}, \alpha)$ and $\beta_S^{(1)}(\boldsymbol{\mu}, \alpha/2)$ is negligible. Thus, our proposed test $T_{com}^{(1)}$ has similar performance as $T_{sum}^{(1)}$ in this dense case.

(2) *Sparse case*: $\delta = c\sqrt{\log p/n}$ for a sufficiently large constant c and $m = o((\log p)^{-1}p^{1/2})$. Here, the value of m is much smaller than that

in (1), confirming the notion of "sparse". In this case, $\frac{nm\delta^2}{\sqrt{2p}} \to 0$; thus, $\beta_S^{(1)}(\boldsymbol{\mu}, \alpha) \approx \alpha$ and $T_{sum}^{(1)}$ is not effective or useful. However, $\beta_M^{(1)}(\boldsymbol{\mu}, \alpha) \to 1$, by an argument similar to that of Theorem 2 of Cai et al. (2014) as discussed above, which also leads to $\beta_C^{(1)}(\boldsymbol{\mu}, \alpha) \to 1$ in this sparse case. In addition, in Fan et al. (2015), the quantity $\delta_{p,n}$ is chosen to as $\log\log n\sqrt{\log p/n}$, which implies that the screening set $\{i : \sqrt{n}|\bar{\boldsymbol{X}}_i| > \log\log n\sqrt{\log p}\}$ is empty, with probability tending to one. Thus, the power enhancement component of Fan et al. (2015) is negligible in this case, which makes the standardized Wald test statistic the same as $T_{sum}^{(1)}$ because $\boldsymbol{\Sigma} = \mathbf{I}_p$. That is, their test is also ineffective in this sparse case.

The above theoretical results and analysis, together with the simulation in the next section, indicate that our proposed test $T_{com}^{(1)}$ performs very well, regardless of the sparsity of the alternative hypothesis, and is more convenient to use in various practical scenarios.

To conserve space, we present the combo-type two-sample mean test and regression coefficient test, as well as their simulation results, in the supplementary material.

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4. Simulation Results

In this section, we carry out a series of simulation studies on the testing problems discussed in the previous section to compare different test statistics and validate the advantage of the proposed combo-type tests.

4.1 One-Sample Test Problem

First, we present numerical examples of the one-sample test problem. We compare our combo-type test T_{com} in (3.7) (abbreviated as COM) with the sum-type test $T_{sum}^{(1)}$ in (3.2) by Srivastava (2009) (abbreviated as SUM), the max-type test $T_{max}^{(1)}$ in (3.5) (abbreviated as MAX), the higher criticism test T_{HC2} from (3.3) by Zhong et al. (2013) (abbreviated as HC2), and the power enhancement test J from (3.4) by Fan et al. (2015) (abbreviated as FLY). The data set is simulated as follows.

EXAMPLE 1. We consider $\boldsymbol{X}_i = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_i$, for $i = 1\cdots, n$, and each component of $\boldsymbol{z}_i$ is generated independently from three distributions: (1) the normal distribution $N(0,1)$; (2) the t-distribution $t(3)/\sqrt{3}$; and (3) the mixture normal random variable $V/\sqrt{1.8}$, where V has density function $0.1f_1(x)+0.9f_2(x)$, with $f_1(x)$ and $f_2(x)$ being the densities of $N(0,9)$ and $N(0,1)$, respectively. We have two sample sizes $n = 100, 200$, and three dimensions $p = 200, 400, 600$. Under the null hypothesis, we set $\boldsymbol{\mu} = \mathbf{0}$ and the significance level $\alpha = 0.05$. We consider the following three scenarios of covariance matrices:

(I) AR(1) model: $\mathbf{\Sigma} = (0.5^{|i-j|})_{1\leqslant i,j\leqslant p}$.

(II) $\mathbf{\Sigma} = \mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2}$, with $\mathbf{D} = \text{diag}(\sigma_1^2, \cdots, \sigma_p^2)$ and $\mathbf{R} = \mathbf{I}_p + \boldsymbol{b}\boldsymbol{b}^T - \check{\mathbf{B}}$, where σ_i^2 are generated independently from $Uniform(1, 2)$, $\boldsymbol{b} = (b_1, \cdots, b_p)^T$, and $\check{\mathbf{B}} = \text{diag}(b_1^2, \cdots, b_p^2)$. The first $[p^{0.3}]$ entries of $\boldsymbol{b}$ are independently sampled from $Uniform(0.7, 0.9)$, and the remaining entries are set to zero, where $[\cdot]$ denotes taking the integer part.

(III) $\mathbf{\Sigma} = \boldsymbol{\gamma}\boldsymbol{\gamma}^T + (\mathbf{I}_p - \rho_\epsilon \boldsymbol{W})^{-1}(\mathbf{I}_p - \rho_\epsilon \boldsymbol{W}^T)^{-1}$, where $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_{[p^{\delta_\gamma}]}, 0, 0, \cdots, 0)^T$. Here, γ_i, with $i = 1, \cdots, [p^{\delta_\gamma}]$, are generated independently from $Uniform(0.7, 0.9)$. Let $\rho_\epsilon = 0.5$ and $\delta_\gamma = 0.3$. Let $\boldsymbol{W} = (w_{i_1 i_2})_{1\leqslant i_1,i_2\leqslant p}$ have a so-called rook form, that is, all elements of $\boldsymbol{W}$ are zero, except that $w_{i_1+1,i_1} = w_{i_2-1,i_2} = 0.5$ for $i_1 = 1, \cdots, p-2$ and $i_2 = 3, \cdots, p$, and $w_{1,2} = w_{p,p-1} = 1$.

Tables 1, 2, and 3 report the empirical sizes of the five tests, showing that SUM, MAX, and COM can control the empirical size very well, in most cases. However, the empirical sizes of HC2 and FLY can be much smaller than the nominal level in some cases.

Next, we examine the power of each test. Our simulation shows that

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Table 1: Sizes of tests for Example 1 with Scenario (I), $\alpha = 0.05$.

	Distribution	(1)			(2)			(3)		
	p	200	400	600	200	400	600	200	400	600
$n = 100$	MAX	0.053	0.062	0.082	0.026	0.052	0.045	0.044	0.039	0.061
	SUM	0.064	0.064	0.060	0.052	0.050	0.059	0.063	0.058	0.064
	COM	0.063	0.069	0.059	0.040	0.059	0.055	0.056	0.047	0.061
	HC2	0.028	0.044	0.034	0.033	0.029	0.032	0.038	0.025	0.044
	FLY	0.014	0.009	0.004	0.003	0.003	0.002	0.025	0.018	0.014
$n = 200$	MAX	0.046	0.060	0.049	0.045	0.041	0.045	0.042	0.045	0.032
	SUM	0.065	0.068	0.058	0.053	0.057	0.062	0.056	0.054	0.056
	COM	0.056	0.068	0.048	0.042	0.047	0.052	0.043	0.050	0.039
	HC2	0.019	0.027	0.030	0.031	0.024	0.023	0.029	0.020	0.029
	FLY	0.005	0.000	0.000	0.003	0.000	0.000	0.017	0.012	0.005

the power comparisons are similar for any combination of (n, p), with $n = 100, 200$ and $p = 200, 400, 600$. Hence, we present the case $n = 100$ and $p = 200$ for conciseness. Define $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)^T$. For different numbers of nonzero-mean variables $m = 1, \cdots, 20$, we consider $\mu_j = \delta$ for $0 < j \leqslant m$, and $\mu_j = 0$ for $j > m$. The parameter δ is chosen as $||\boldsymbol{\mu}||^2 = m\delta^2 = 0.5$. Figure 1 reports the power of the five tests. The power of MAX decreases as the number of nonzero-mean variables increases, which is as expected, because, in general, the max-type test is more powerful in the

Table 2: Sizes of tests for Example 1 with Scenario (II), $\alpha = 0.05$.

	Distribution	(1)			(2)			(3)		
	p	200	400	600	200	400	600	200	400	600
$n = 100$	MAX	0.058	0.070	0.065	0.044	0.037	0.039	0.048	0.042	0.047
	SUM	0.053	0.067	0.056	0.054	0.052	0.048	0.054	0.055	0.045
	COM	0.055	0.057	0.061	0.054	0.044	0.040	0.043	0.047	0.047
	HC2	0.022	0.011	0.013	0.005	0.015	0.005	0.011	0.011	0.006
	FLY	0.022	0.011	0.011	0.013	0.010	0.006	0.024	0.015	0.007
$n = 200$	MAX	0.053	0.054	0.076	0.025	0.042	0.025	0.044	0.040	0.041
	SUM	0.053	0.057	0.060	0.053	0.051	0.052	0.055	0.065	0.060
	COM	0.058	0.061	0.066	0.037	0.045	0.044	0.043	0.053	0.055
	HC2	0.003	0.011	0.006	0.010	0.006	0.003	0.004	0.005	0.008
	FLY	0.037	0.033	0.025	0.030	0.022	0.011	0.032	0.026	0.015

sparse case and less powerful in the nonsparse case. The power of SUM increases slightly with m, and is higher than the power of HC2 and FLY in all cases. The proposed COM is as powerful as MAX when the number of variables with nonzero means is small (sparse case), and has almost the same power as SUM when the number of variables with nonzero means grows. In general, COM possesses the advantages of both MAX (in the sparse case) and SUM (in the nonsparse case), and outperforms HC2 and FLY in all scenarios. Note that all tests, other than COM, favor either

Table 3: Sizes of tests for Example 1 with Scenario (III), $\alpha = 0.05$.

	Distribution	(1)			(2)			(3)		
	p	200	400	600	200	400	600	200	400	600
$n = 100$	MAX	0.054	0.066	0.059	0.053	0.040	0.033	0.049	0.039	0.043
	SUM	0.052	0.055	0.059	0.053	0.048	0.060	0.059	0.064	0.061
	COM	0.053	0.066	0.059	0.053	0.050	0.040	0.062	0.046	0.051
	HC2	0.034	0.038	0.035	0.032	0.030	0.025	0.036	0.030	0.030
	FLY	0.013	0.003	0.005	0.013	0.001	0.000	0.020	0.013	0.010
$n = 200$	MAX	0.053	0.058	0.063	0.034	0.027	0.038	0.049	0.039	0.050
	SUM	0.061	0.065	0.062	0.044	0.058	0.068	0.063	0.058	0.057
	COM	0.065	0.075	0.069	0.033	0.048	0.047	0.059	0.051	0.053
	HC2	0.035	0.032	0.032	0.019	0.029	0.019	0.029	0.023	0.024
	FLY	0.001	0.001	0.000	0.004	0.001	0.000	0.016	0.011	0.002

the sparse or the nonsparse case. In practice, it is often difficult to justify whether or not the true underlying model is sparse. Hence, our proposed COM test, with its strong robustness, should be a more favorable choice over competing approaches.

5. Real-Data Application

In this section, we apply the results and test statistics obtained in Section 3 to two real data sets: US stock data (dense model), and search engine data

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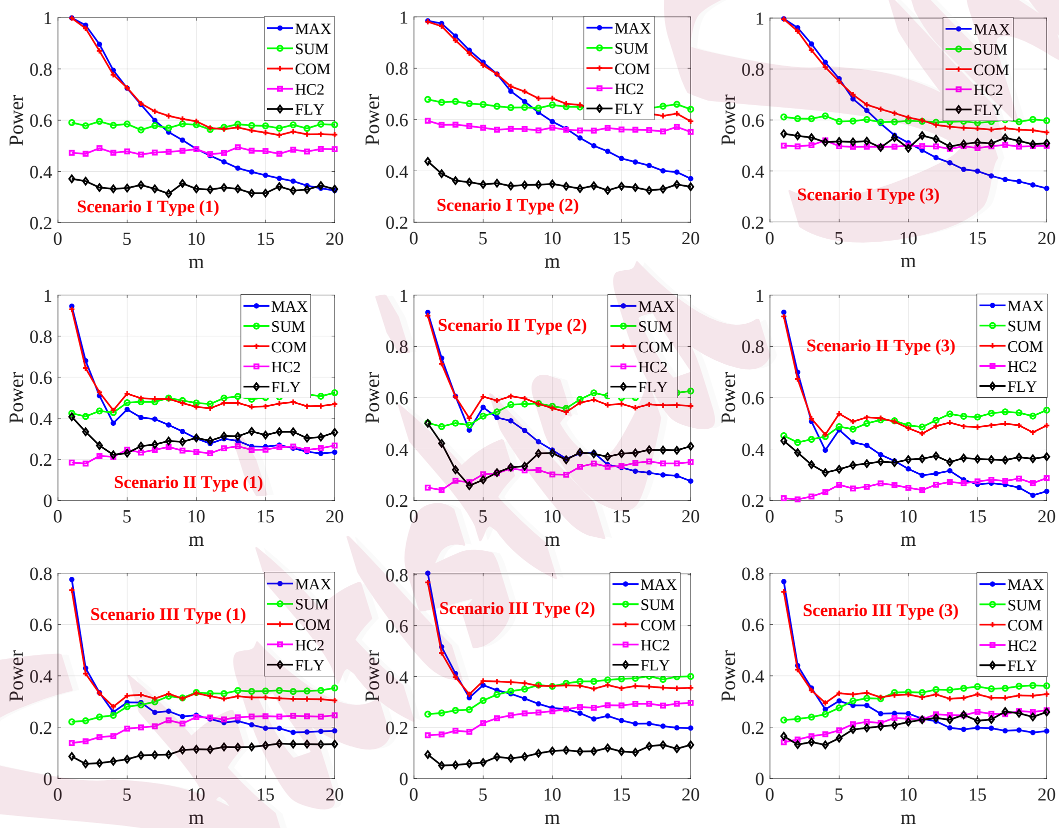
Figure 1: Power versus number of variables with nonzero means for Example 4.1. The x-axis m denotes the number of variables with nonzero means; the y-axis is the empirical power.



(sparse model), given in the supplementary material. As shown here, the proposed combo-type test, COM, performs well on both data sets. Thus, it serves as a "universal" test in practice, regardless of whether or not the true model is sparse.

5.1 US Stock Data

We apply the methods developed for the one-sample mean test in Section 3.1 to a pricing problem in finance. Specifically, we investigate how financial returns of assets are related to their risk-free returns. Let $X_{ij} = R_{ij} - \mathrm{rf}_i$ denote the excess return of the jth asset at time i, for $i = 1, \cdots, n$ and $j = 1, \cdots, p$, where R_{ij} is the return on asset j during period i, and rf_i is the risk-free return rate of all assets during period i. We study the following pricing model:

$$X_{ij} = \mu_j + \xi_{ij}, \tag{5.1}$$

for $i = 1, \cdots, n$ and $j = 1, \cdots, p$, or, in vector form, $\boldsymbol{X}_i = \boldsymbol{\mu} + \boldsymbol{\xi}_i$, where $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{ip})^T$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)^T$, and $\boldsymbol{\xi}_i = (\xi_{i1}, \ldots, \xi_{ip})^T$ is the zero-mean error vector. The pricing model in (5.1) is the zero-factor model in arbitrage pricing theory (Ross, 1976), where "zero-factor" means that no additional factor is used to model the price. A common null hypothesis considered under pricing model (5.1) is $H_0 : \boldsymbol{\mu} = \mathbf{0}$, which means that the excess return of any asset is zero, on average; that is, the return rate of any asset R_{ij} is equal to the risk-free return rate rf_i on average.

We consider the monthly return rates of the stocks that constitute the

S&P 500 index for the period January 2005 to November 2018. Because the stocks in the index change over time, and some stocks were created during this period, we consider only 374 stocks that were included in the index for the full period under review. Figure 2 shows the sample mean of each stock in this period. We observe that most average returns are positive. In fact, as we increase the time range (thus increasing the sample size n), the p-values of MAX, SUM, and COM are eventually smaller than 0.05. These results suggest that we reject the null that asset returns do not only come from the risk-free rates (on average), which is consistent with the views of many economists (Fama and French, 1993, 2015).

Figure 2: Histogram of sample means of stock monthly return rates in the S&P 500.

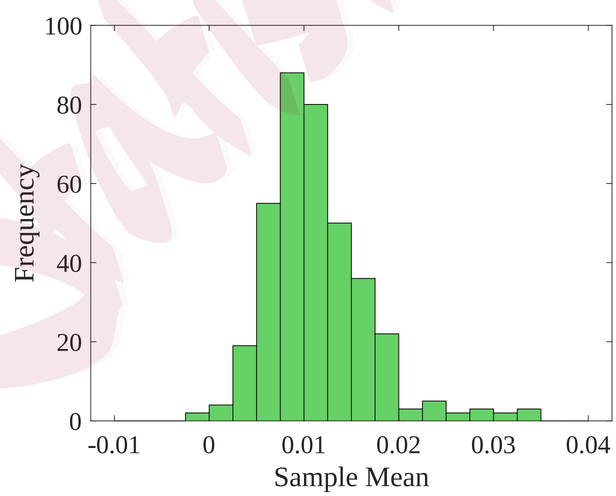


Table 4: Rejection rates of each test in the US stock data.

	MAX	SUM	COM
$n = 30$	0.35	0.39	0.40
$n = 50$	0.40	0.51	0.51
$n = 70$	0.44	0.67	0.62
$n = 100$	0.52	0.86	0.83

We further evaluate the tests using a random sampling procedure. Specifically, we randomly choose n samples from the whole data set and apply MAX, SUM, and COM on this new sample. For each n, we repeat this experiment 1000 times. Table 4 reports the rejection rates for each method with different n. From Table 4, we observe that SUM outperforms MAX in all cases by providing higher rejection rates. This is not surprising, because for these data, the number of variables with nonzero means (assets with non-zero expected excess return) might be large, which is when sum-type tests typically outperform than max-type tests. On the other hand, the combo-type test COM performs similarly to SUM, overall. Therefore, COM does not lose efficiency in this problem.

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6. Conclusion

We have proved the asymptotic independence between the sum and maximum of dependent random variables, without requiring stationary assumptions or strong mixing conditions. We applied our results to high-dimensional testing problems. Our proposed combo-type tests perform well, regardless of the sparsity of the data. Note the following:

1. The normal assumption is essential in the proof of asymptotic independence. Hence, we make a Gaussian assumption in the high-dimensional test problems. Recent studies, such as Liang et al. (2019) and Chen and Xia (2021), have developed high-dimensional normality tests to check whether a p-dimensional random vector with large p is a Gaussian vector. In the literature, we may not need the Gaussian assumption to analyze the asymptotic distribution of the sum-type and max-type test statistics; see, for example Cai et al. (2014) and Chen and Qin (2010). To prove the asymptotic independence between the sum and the maximum of nonnormal dependent random variables deserves further investigation.

2. To obtain the asymptotic distributions of the sum and the maximum of dependent random variables, we assume the correlations between the random variables are not very strong. Recent studies consider high-dimensional testing problems without the weak correlation assumption, such as Wang

and Xu (2021) and Zhang et al. (2020). The analogue of our asymptotic independence result between the sum and the maximum of dependent random variables with arbitrary covariance structures is also an interesting and challenging problem.

3. The asymptotic independence result in Theorem 3 is universal. As such, we believe it can be generalized and applied widely, for example, in change point detection and statistical process controls.

Supplementary Material

In the online Supplementary Material, we propose the combo-type two-sample mean test and regression coefficient test, and present corresponding simulation results to show the advantages of our proposed tests. The Supplementary Material also includes technical proofs of our theoretical results.

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References

Anderson, C. and K. Turkman (1991). The joint limiting distribution of sums and maxima of stationary sequences. *Journal of Applied Probability 28*(1), 33–44.

Anderson, C. and K. Turkman (1993). Limiting joint distributions of sums and maxima in a statistical context. *Theory of Probability & Its Applications 37*(2), 314–316.

Anderson, C. and K. Turkman (1995). Sums and maxima of stationary sequences with heavy tailed distributions. *Sankhyā: The Indian Journal of Statistics, Series A*, 1–10.

Anderson, T. W. (2003). *An Introduction to Multivariate Statistical Analysis*. John Wiley & Sons.

Bai, Z. and H. Saranadasa (1996). Effect of high dimension: by an example of a two sample problem. *Statistica Sinica*, 311–329.

Bradley, R. C. (2005). Basic properties of strong mixing conditions. a survey and some open questions. *Probability surveys 2*, 107–144.

Cai, T. T., W. Liu, and Y. Xia (2014). Two-sample test of high dimensional means under dependence. *Journal of the Royal Statistical Society: Series B: Statistical Methodology*, 349–372.

Chen, H. and Y. Xia (2021). A normality test for high-dimensional data based on the nearest neighbor approach. *Journal of the American Statistical Association*, Online published.

Chen, S. X., J. Li, and P.-S. Zhong (2019). Two-sample and anova tests for high dimensional means. *The Annals of Statistics 47*(3), 1443–1474.

Chen, S. X. and Y.-L. Qin (2010). A two-sample test for high-dimensional data with applications to gene-set testing. *The Annals of Statistics 38*(2), 808–835.

Chow, T. L. and J. L. Teugels (1978). The sum and the maximum of iid random variables. In *Proceedings of the 2nd Prague Symposium on Asymptotic Statistics*, Volume 45, pp. 394–403.

Davis, R. A. and T. Hsing (1995). Point process and partial sum convergence for weakly dependent random variables with infinite variance. *The Annals of Probability*, 879–917.

Eaton, M. L. (1983). Multivariate statistics: a vector space approach. *JOHN WILEY & SONS, INC., 605 THIRD AVE., NEW YORK, NY 10158, USA, 1983, 512*.

Fama, E. F. and K. R. French (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics 33*, 3–56.

Fama, E. F. and K. R. French (2015). A five-factor asset pricing model. *Journal of financial economics 116*(1), 1–22.

Fan, J. and T. Jiang (2019). Largest entries of sample correlation matrices from equi-correlated normal populations. *The Annals of Probability 47*(5), 3321–3374.

Fan, J., Y. Liao, and J. Yao (2015). Power enhancement in high-dimensional cross-sectional tests. *Econometrica 83*(4), 1497–1541.

Feng, L., C. Zou, and Z. Wang (2016). Multivariate-sign-based high-dimensional tests for the two-sample location problem. *Journal of the American Statistical Association 111*(514), 721–735.

Feng, L., C. Zou, Z. Wang, and L. Zhu (2015). Two-sample behrens-fisher problem for high-dimensional data. *Statistica Sinica*, 1297–1312.

Feng, L., C. Zou, Z. Wang, and L. Zhu (2017). Composite t 2 test for high-dimensional data. *Statistica Sinica*, 1419–1436.

Goeman, J. J., S. A. Van De Geer, and H. C. Van Houwelingen (2006). Testing against a high dimensional alternative. *Journal of the Royal Statistical Society: Series B (Statistical Methodology) 68*(3), 477–493.

Gregory, K. B., R. J. Carroll, V. Baladandayuthapani, and S. N. Lahiri (2015). A two-sample test for equality of means in high dimension. *Journal of the American Statistical Association 110*(510), 837–849.

He, Y., G. Xu, C. Wu, and W. Pan (2021). Asymptotically independent u-statistics in high-dimensional testing. *The Annals of Statistics 49*(1), 154–181.

Ho, H.-C. and T. Hsing (1996). On the asymptotic joint distribution of the sum and maximum of stationary normal random variables. *Journal of Applied Probability 33*(1), 138–145.

Ho, H.-C. and W. P. McCormick (1999). Asymptotic distribution of sum and maximum for gaussian processes. *Journal of Applied Probability 36*(4), 1031–1044.

Hotelling, H. (1931). The generalization of student's ratio. *The Annals of Mathematical Statistics 2*(3), 360–378.

Hsing, T. (1995). A note on the asymptotic independence of the sum and maximum of strongly mixing stationary random variables. *The Annals of Probability*, 938–947.

Jiang, T. and P. Li (2021). Mean test with fewer observation than dimension and ratio unbiased estimator for correlation matrix. *arXiv preprint arXiv:2108.06892*.

Lan, W., H. Wang, and C.-L. Tsai (2014). Testing covariates in high-dimensional regression. *Annals of the Institute of Statistical Mathematics 66*(2), 279–301.

Liang, J., M.-L. Tang, and X. Zhao (2019). Testing high-dimensional normality based on classical skewness and kurtosis with a possible small sample size. *Communications in Statistics - Theory and Methods 48*(23), 5719–5732.

McCormick, W. and Y. Qi (2000). Asymptotic distribution for the sum and maximum of gaussian processes. *Journal of Applied Probability 37*(4), 958–971.

Muirhead, R. J. (1982). *Aspects of multivariate statistical theory.* John Wiley & Sons.

Park, J. and D. N. Ayyala (2013). A test for the mean vector in large dimension and small samples. *Journal of Statistical Planning and Inference 143*(5), 929–943.

Peng, Z. and S. Nadarajah (2003). On the joint limiting distribution of sums and maxima of stationary normal sequence. *Theory of Probability & Its Applications 47*(4), 706–709.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory 13*(3), 341–360.

Srivastava, M. S. (2009). A test for the mean vector with fewer observations than the dimension under non-normality. *Journal of Multivariate Analysis 100*(3), 518–532.

Srivastava, M. S. and M. Du (2008). A test for the mean vector with fewer observations than the dimension. *Journal of Multivariate Analysis 99*(3), 386–402.

Wang, L., B. Peng, and R. Li (2015). A high-dimensional nonparametric multivariate test for mean vector. *Journal of the American Statistical Association 110*(512), 1658–1669.

Wang, R. and W. Xu (2021). An approximate randomization test for high-dimensional two-sample behrens-fisher problem under arbitrary covariances. *arXiv preprint arXiv:2108.01860*.

Wu, C., G. Xu, and W. Pan (2019). An adaptive test on high-dimensional parameters in generalized linear models. *Statistica Sinica 29*(4), 2163–2186.

Xu, G., L. Lin, P. Wei, and W. Pan (2016). An adaptive two-sample test for high-dimensional means. *Biometrika 103*(3), 609–624.

Zhang, J.-T., J. Guo, B. Zhou, and M.-Y. Cheng (2020). A simple two-sample test in high dimensions based on l 2-norm. *Journal of the American Statistical Association 115*(530), 1011–1027.

Zhong, P.-S., S. X. Chen, and M. Xu (2013). Tests alternative to higher criticism for high-dimensional means under sparsity and column-wise dependence. *The Annals of Statistics 41*(6), 2820–2851.

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