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Distributed Sparse Composite Quantile Regression in Ultrahigh Dimensions

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Abstract: We examine distributed estimation and support recovery for ultrahigh-dimensional linear regression models under a potentially arbitrary noise distribution. The composite quantile regression is an efficient alternative to the least squares method, and provides robustness against heavy-tailed noise while maintaining reasonable efficiency in the case of light-tailed noise. The highly nonsmooth nature of the composite quantile regression loss poses challenges to both the theoretical and the computational development in an ultrahigh-dimensional distributed estimation setting. Thus, we cast the composite quantile regression into the least squares framework, and propose a distributed algorithm based on an approximate Newton method. This algorithm is efficient in terms of both computation and communication, and requires only gradient information to be communicated between the machines. We show that the resultant distributed estimator attains a near-oracle rate after a constant number of communications, and provide theoretical guarantees for its estimation and support recovery accuracy. Extensive experiments demonstrate the competitive empirical performance of our algorithm.

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Key words and phrases: Composite quantile regression, distributed estimation, efficiency, heavy-tailed noise, support recovery.

1. Introduction

A fundamental task in statistics is to estimate the coefficients of a linear model. The least squares (LS) regression is routinely used for this task, and has well-established theory (Monahan, 2008). However, in the era of big data, rapid advances in information technology have raised several new challenges. The first lies in the sizes of the data sets, often measured in TBs or even PBs, making them difficult to process on a single machine. Traditional in-memory algorithms are losing power because of communication, storage, and computation restrictions (Lan et al., 2020). Thus, we need distributed algorithms with theoretical guarantees. The second challenge arises from the potentially arbitrary noise. Under very heavy-tailed noise, where the finite variance condition is violated, the LS and Huber regressions are sub-optimal (Fan et al., 2014a; Zhou et al., 2018; Sun et al., 2020). In such cases, the quantile regression (QR, Koenker, 2005) becomes an attractive alternative, because its asymptotic variance does not depend on the moments of the noise distribution. However, in terms of efficiency, a QR can be arbitrarily less efficient than an LS. For example, under the mixture normal noise $0.5\mathcal{N}(-3,1)+0.5\mathcal{N}(3,1)$, the least absolute deviation estimate is 1272.8

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times less efficient than the LS estimate (Gu and Zou, 2020). To shield the QR against potential efficiency loss, while maintaining its robust property, Zou and Yuan (2008) proposed a composite quantile regression (CQR) that combines quantile information across various quantile levels. The third challenge lies in the ultrahigh dimensionality of modern data. Here, a sparsity assumption is often adopted (Zhao and Yu, 2006; Wainwright, 2009; Hastie et al., 2015). Despite the massive amount of literature on sparse LS under ultrahigh dimensions, few works have examined the ultrahigh-dimensional CQR; see Gu and Zou (2020). In a distributed setting, numerous studies focus on statistical estimation (Lee et al., 2017; Battey et al., 2018; Jordan et al., 2019). However, the *support recovery* for the CQR in a distributed setting remains largely unexplored.

We propose a new estimation procedure for an ultrahigh-dimensional CQR in the distributed setting, with theoretical guarantees on its estimation and support recovery accuracy. Specifically, we consider coefficient estimation and support recovery of the following model:

$$Y = \beta_0 + \sum_{j=1}^{p} X_j \beta_j + \varepsilon, \tag{1.1}$$

where $\mathbf{x} = (X_1, \ldots, X_p)^{\mathrm{T}}$ is a p-dimensional covariate vector with mean zero, and ε is the noise. We assume ε is independent of $\mathbf{x}$ and has a density with respect to the Lebesgue measure (see, e.g., Zou and Yuan, 2008; Fan et al., 2014a; Gu and Zou, 2020). Suppose β_0^* and $\boldsymbol{\beta}^* = (\beta_1^*, \ldots, \beta_p^*)^{\mathrm{T}}$ are the true coefficients

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that generate the N independent and identically distributed (i.i.d.) data $(\mathbf{x}_i, Y_i)_{i=1}^N$, where $\mathbf{x}_i = (X_{i1}, \ldots, X_{ip})^{\mathrm{T}}$. Denote the response vector by $\mathbf{y} = (Y_1, \ldots, Y_N)^{\mathrm{T}}$, and the design matrix by $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)^{\mathrm{T}}$. We assume a sparsity structure on $\boldsymbol{\beta}^*$ in the sense that only $s < p$ elements of $\boldsymbol{\beta}^*$ are nonzero.

We consider the CQR for estimating $\boldsymbol{\beta}^*$ in model (1.1) that is robust to heavy-tailed errors, while maintaining reasonable efficiency under light-tailed errors. Denote $F(\cdot)$ and $f(\cdot)$ as the cumulative distribution and the probability density functions, respectively, of ε. To ensure the identifiability of β_0^*, assume $F(0) = 0.5$. Given an ordered sequence of quantile levels $\tau_1 < \tau_2 < \cdots < \tau_K \in (0, 1)$, let $\alpha_k^* \stackrel{\text{def}}{=} \beta_0^* + F^{-1}(\tau_k)$ and $\boldsymbol{\alpha}^* \stackrel{\text{def}}{=} (\alpha_1^*, \ldots, \alpha_K^*)^{\mathrm{T}} \in \mathbb{R}^K$, where $F^{-1}(\tau_k) = \inf\{x : F(x) \geq \tau_k\}$ denotes the τ_kth quantile of ε, for $k = 1, \ldots, K$. The canonical CQR (Zou and Yuan, 2008) estimates $\boldsymbol{\beta}^*$ by minimizing

$$Q(\alpha, \beta; \mathbf{x}, Y) \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^{K} \rho_{\tau_k}(Y - \alpha_k - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta}) \tag{1.2}$$

jointly over $\alpha = (\alpha_1, \ldots, \alpha_K)^{\mathrm{T}} \in \mathbb{R}^K$ and $\boldsymbol{\beta} \in \mathbb{R}^p$, where $\rho_{\tau_k}(u) \stackrel{\text{def}}{=} \{\tau_k - I(u < 0)\}u$ is the check loss at level τ_k, for $k = 1, \ldots, K$ (Koenker, 2005). It is easy to see that

$$(\boldsymbol{\alpha}^{*\mathrm{T}}, \boldsymbol{\beta}^{*\mathrm{T}})^{\mathrm{T}} = \arg\min_{\boldsymbol{\alpha}, \boldsymbol{\beta}} E\{Q(\alpha, \beta; \mathbf{x}, Y)\}.$$

Typically, we take equally spaced τ_k: $\tau_k = k/(K + 1)$, for $k = 1, \ldots, K$. As $K \to \infty$, Zou and Yuan (2008) show that the asymptotic efficiency of the CQR

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relative to the LS has a universal lower bound of 86.4% (Kai et al., 2010). Even with a relatively small K, such as $K = 9$, the CQR estimator achieves a substantial efficiency gain.

Although using the CQR provides robustness to heavy-tailed noise and safeguards against potential efficiency loss, the nonsmoothness of the CQR loss raises computational challenges, owing to limited computing power and memory when the sample size and dimension are both considerable. Doing so also makes theoretical developments difficult. Recently, Gu and Zou (2020) developed the theory for ultrahigh-dimensional sparse penalized CQR in a single-node setting. In addition, to consider scalability, they proposed using an alternating direction method of multipliers algorithm, rather than the linear programming algorithm considered in Zou and Yuan (2008). For ultrahigh-dimensional data stored on multiple machines, we may not be able to use existing algorithms for the sparse penalized CQR, making distributed algorithms with theoretical guarantees increasingly important. The main goals of this study are to develop a new distributed estimation approach for the ultrahigh-dimensional CQR, and to establish its *estimation* and *support recovery* theory.

There are two main data-partitioning schemes in distributed systems: "horizontal" and "vertical." In a "horizontal" distributed setting, assume N observations are scattered evenly across m local nodes, with n observations at each node.

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Denote by $\mathcal{H}_j$ the set of observational indices at the jth node. The divide-and-conquer strategy is popular for such data, owing to its simplicity (Zaharia et al., 2016); see, for example, Li et al. (2012), Zhao et al. (2016), Battey et al. (2018), Shi et al. (2018), Jiang et al. (2018), Fan et al. (2019a) and Fan et al. (2019b). However, the final estimate, which is an average of the m local estimates, is usually no longer sparse. Moreover, although averaging reduces the variance of the local estimates, it might not remove the bias of these local estimates. Hence, restrictions on the number of nodes, for example, $m = O(N^{1/2})$, are routinely imposed to achieve the minimax convergence rate (Braverman et al., 2016). To remove such restrictions on m, Wang et al. (2017) and Jordan et al. (2019) developed multi-round procedures. However, their methodology and theory require second-order differentiability of the loss function, in general, and cannot be applied directly to the highly *nonsmooth* CQR loss. Chen et al. (2020) studied the distributed high-dimensional QR problem, providing theoretical guarantees on its support recovery. However, their method cannot safeguard against the potential efficiency loss incurred by the QR. In a "vertical" distributed setting, each local node holds a subset of all the covariates of the data. To recover the sparsity pattern, He et al. (2022) proposed decorrelating the covariates before aggregating. Their work improves on previous results (Zhou et al., 2014; Song and Liang, 2015) by requiring constraints on the correlation structure of the covariates that

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are less strict.

Our work focuses on the "horizontal" distributed setting. We propose a new distributed procedure for estimating the coefficients of a high-dimensional linear model, with potentially arbitrary noise, using the CQR. We first show that we can estimate $\boldsymbol{\alpha}^*$ and $\boldsymbol{\beta}^*$ from a pseudo-response $\widetilde{Y}_i$, instead of Y_i. This yields a pooled estimate from solving a lasso problem based on $(\mathbf{x}_i, \widetilde{Y}_i)_{i=1}^N$, without any moment conditions on the noise term. The pooled estimate is computationally much more efficient than the penalized CQR in a single-node setting. We further provide a communication-efficient distributed implementation of this pooled estimate where, at each communication, only the $(p+K)$-dimensional gradient information is communicated, instead of the $(p+K)\times(p+K)$ Hessian matrix, by modifying the approximate Newton method of Shamir et al. (2014). Our results demonstrate the accuracy of our method in terms of both estimation and support recovery. We prove that, after a constant number of communications, our estimate achieves a near-oracle rate of $[s\log\{\max(p,N)\}/N]^{1/2}$ for the estimation of $\boldsymbol{\beta}^*$, and $[Ks\log\{\max(p,N)\}/N]^{1/2}$ for that of $\boldsymbol{\alpha}^*$, in terms of the ℓ_2-error (Theorem 2). These rates coincide with those of the ℓ_1-regularized CQR in a single-node setting (Gu and Zou, 2020). We also derive the "beta-min" condition for exact support recovery, which becomes weaker as the number of communications increases (Theorem 4). After a constant number of communications, our "beta-

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min" condition matches that of the classical setting in which all data are in a single node.

The rest of this paper is organized as follows. We describe the distributed algorithm for the penalized CQR in Section 2, and derive the estimation error bounds and the support recovery results for the distributed estimator in Section 3. Extensive simulations in Section 4 provide empirical evidence for our theoretical findings. Section 5 concludes the paper. All technical proofs are relegated to the online Supplementary Material.

We use the following notation. We denote $C, C_0, C_1, \ldots, c, c_0, c_1, \ldots$ as generic constants that may vary at each appearance. We also use the standard asymptotic notation. Given two sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n = O(b_n)$ if there exists a constant $C < \infty$ such that $a_n \le Cb_n$, and $a_n = o(b_n)$ if $a_n/b_n \to 0$. For two sets of random variables $\{X_n\}$ and $\{Y_n\}$, we write $X_n = O_p(Y_n)$ if for any $\epsilon > 0$, there exists a finite $M > 0$ and a finite $n_0 > 0$ such that $\mathrm{pr}(|X_n/Y_n| > M) < \epsilon$, for any $n > n_0$. For a vector $\mathbf{v} = (v_1, \ldots, v_p)^{\mathrm{T}}$, we denote its support by $\mathrm{supp}(\mathbf{v}) \stackrel{\mathrm{def}}{=} \{j \in \mathbb{N}\colon v_j \neq 0\}$. We further define $|\mathbf{v}|_1 \stackrel{\mathrm{def}}{=} \sum_{i=1}^p |v_i|$, $|\mathbf{v}|_2 \stackrel{\mathrm{def}}{=} \left(\sum_{i=1}^p v_i^2\right)^{1/2}$, and $\mathbf{v}^{\min} \stackrel{\mathrm{def}}{=} \min_{i \in \mathrm{supp}(\mathbf{v})} |v_i|$. For $\mathcal{S} \subseteq \{1, \ldots, p\}$ with length $|\mathcal{S}|$, let $\mathbf{v}_{\mathcal{S}} \stackrel{\mathrm{def}}{=} (v_i, i \in \mathcal{S}) \in \mathbb{R}^{|\mathcal{S}|}$. For a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{p \times q}$, we define $|\mathbf{A}|_\infty \stackrel{\mathrm{def}}{=} \max_{1 \le i \le p, 1 \le j \le q} |a_{ij}|$, $\|\mathbf{A}\|_\infty \stackrel{\mathrm{def}}{=} \max_{1 \le i \le p} \sum_{1 \le j \le q} |a_{ij}|$, and $\|\mathbf{A}\|_{\mathrm{op}} \stackrel{\mathrm{def}}{=} \max_{|\mathbf{v}|_2 = 1} |\mathbf{A}\mathbf{v}|_2$. For two subsets $\mathcal{S}_1 \subseteq \{1, \ldots, p\}$ and $\mathcal{S}_2 \subseteq \{1, \ldots, q\}$, we let $\mathbf{A}_{\mathcal{S}_1 \times \mathcal{S}_2} = (a_{ij}, i \in$

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$\mathcal{S}_1, j \in \mathcal{S}_2$). Finally, denote the largest and smallest singular values of $\mathbf{A}$ by $\Lambda_{\max}(\mathbf{A})$ and $\Lambda_{\min}(\mathbf{A})$, respectively.

2. Distributed Sparse CQR

2.1 The Newton update and a surrogate loss

Motivated by the Newton–Raphson method, we cast the CQR as an LS problem. Let $\boldsymbol{\phi} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}^{\mathrm{T}}, \boldsymbol{\beta}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{p+K}$ and $\boldsymbol{g}(\boldsymbol{\phi}; \mathbf{x}, Y) \stackrel{\text{def}}{=} \{\boldsymbol{g}_{\boldsymbol{\alpha}}(\boldsymbol{\phi}; \mathbf{x}, Y)^{\mathrm{T}}, \boldsymbol{g}_{\boldsymbol{\beta}}(\boldsymbol{\phi}; \mathbf{x}, Y)^{\mathrm{T}}\}^{\mathrm{T}}$, where $\boldsymbol{g}_{\boldsymbol{\alpha}}(\boldsymbol{\phi}; \mathbf{x}, Y) \in \mathbb{R}^K$ and $\boldsymbol{g}_{\boldsymbol{\beta}}(\boldsymbol{\phi}; \mathbf{x}, Y) \in \mathbb{R}^p$ are the subgradients of the loss function $Q(\boldsymbol{\phi}; \mathbf{x}, Y)$ with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. Moreover, let $\boldsymbol{H}(\boldsymbol{\phi}) \stackrel{\text{def}}{=} \partial E\{\boldsymbol{g}(\boldsymbol{\phi}; \mathbf{x}, Y)\}/\partial \boldsymbol{\phi}^{\mathrm{T}} \in \mathbb{R}^{(p+K)\times(p+K)}$ denote the population Hessian matrix of $E\{Q(\boldsymbol{\phi}; \mathbf{x}, Y)\}$. Given any initial solution $\boldsymbol{\phi}_0 \stackrel{\text{def}}{=} (\boldsymbol{\alpha}_0^{\mathrm{T}}, \boldsymbol{\beta}_0^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{p+K}$, the population version of the Newton–Raphson iteration has the form

$$\boldsymbol{\phi}_1 \stackrel{\text{def}}{=} (\boldsymbol{\alpha}_1^{\mathrm{T}}, \boldsymbol{\beta}_1^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{\phi}_0 - \boldsymbol{H}(\boldsymbol{\phi}_0)^{-1} E\{\boldsymbol{g}(\boldsymbol{\phi}_0; \mathbf{x}, Y)\}. \tag{2.1}$$

For the CQR loss $Q(\boldsymbol{\phi}; \mathbf{x}, Y)$ in (1.2), we consider a subgradient of the form $\boldsymbol{g}_{\boldsymbol{\alpha}}(\boldsymbol{\phi}; \mathbf{x}, Y) = \{I(Y - \alpha_1 - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta} \le 0) - \tau_1, \ldots, I(Y - \alpha_K - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta} \le 0) - \tau_K\}^{\mathrm{T}}/K$ and $\boldsymbol{g}_{\boldsymbol{\beta}}(\boldsymbol{\phi}; \mathbf{x}, Y) = \sum_{k=1}^{K} \mathbf{x}\{I(Y - \alpha_k - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta} \le 0) - \tau_k\}/K$. Note that the Hessian matrix takes the form

$$\boldsymbol{H}(\boldsymbol{\phi}) = \begin{pmatrix} \partial E\{\boldsymbol{g}_{\boldsymbol{\alpha}}(\boldsymbol{\phi}; \mathbf{x}, Y)\}/\partial \boldsymbol{\alpha}^{\mathrm{T}} & \partial E\{\boldsymbol{g}_{\boldsymbol{\alpha}}(\boldsymbol{\phi}; \mathbf{x}, Y)\}/\partial \boldsymbol{\beta}^{\mathrm{T}} \\ \partial E\{\boldsymbol{g}_{\boldsymbol{\beta}}(\boldsymbol{\phi}; \mathbf{x}, Y)\}/\partial \boldsymbol{\alpha}^{\mathrm{T}} & \partial E\{\boldsymbol{g}_{\boldsymbol{\beta}}(\boldsymbol{\phi}; \mathbf{x}, Y)\}/\partial \boldsymbol{\beta}^{\mathrm{T}} \end{pmatrix}_{(p+K)\times(p+K)},$$

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where

$$\frac{\partial E\{\boldsymbol{g}_\alpha(\boldsymbol{\phi};\mathbf{x},Y)\}}{\partial \alpha^{\mathrm{T}}} = \frac{1}{K}\mathrm{diag}(f\{\mathbf{x}^{\mathrm{T}}(\boldsymbol{\beta}-\boldsymbol{\beta}^*)+\alpha_1\},\ldots,f\{\mathbf{x}^{\mathrm{T}}(\boldsymbol{\beta}-\boldsymbol{\beta}^*)+\alpha_K\}),$$

$$\frac{\partial E\{\boldsymbol{g}_\beta(\boldsymbol{\phi};\mathbf{x},Y)\}}{\partial \beta^{\mathrm{T}}} = \frac{1}{K}\sum_{k=1}^{K} E[\mathbf{x}\mathbf{x}^{\mathrm{T}} f\{\mathbf{x}^{\mathrm{T}}(\boldsymbol{\beta}-\boldsymbol{\beta}^*)+\alpha_k\}], \quad \text{and}$$

$$\frac{\partial E\{\boldsymbol{g}_\alpha(\boldsymbol{\phi};\mathbf{x},Y)\}}{\partial \beta^{\mathrm{T}}} = \frac{1}{K}\{E(\mathbf{x} f\{\mathbf{x}^{\mathrm{T}}(\boldsymbol{\beta}-\boldsymbol{\beta}^*)+\alpha_1\}),\ldots,E(\mathbf{x} f\{\mathbf{x}^{\mathrm{T}}(\boldsymbol{\beta}-\boldsymbol{\beta}^*)+\alpha_K\})\}^{\mathrm{T}}.$$

When the initial estimate $\boldsymbol{\phi}_0$ is close to the true parameter $\boldsymbol{\phi}^* \overset{\text{def}}{=} (\boldsymbol{\alpha}^{*\mathrm{T}}, \boldsymbol{\beta}^{*\mathrm{T}})^{\mathrm{T}}$, $\boldsymbol{H}(\boldsymbol{\phi}_0)$ is approximately

$$\boldsymbol{H}(\boldsymbol{\phi}^*) = \frac{1}{K}\left(\begin{array}{ccc|c} f(\alpha_1^*) & & & \\ & \ddots & & \\ & & f(\alpha_K^*) & \\ \hline & & & \sum_{k=1}^{K} f(\alpha_k^*)\Sigma \end{array}\right),$$

where $\Sigma = E(\mathbf{x}\mathbf{x}^{\mathrm{T}})$, and the zero entries are left blank. Replacing $\boldsymbol{H}(\boldsymbol{\phi}_0)$ with $\boldsymbol{H}(\boldsymbol{\phi}^*)$ in (2.1) results in the following iteration:

$$\boldsymbol{\phi}_1 = \boldsymbol{\phi}_0 - \boldsymbol{H}(\boldsymbol{\phi}^*)^{-1} E\{\boldsymbol{g}(\boldsymbol{\phi}_0;\mathbf{x},Y)\}. \tag{2.2}$$

This, together with the Taylor expansion of $E\{g(\boldsymbol{\phi}_0;\mathbf{x},Y)\}$ around $\boldsymbol{\phi}^*$,

$$E\{g(\boldsymbol{\phi}_0;\mathbf{x},Y)\} = \boldsymbol{H}(\boldsymbol{\phi}^*)(\boldsymbol{\phi}_0-\boldsymbol{\phi}^*) + O(|\boldsymbol{\phi}_0-\boldsymbol{\phi}^*|_2^2),$$

ensures an improved convergence rate of $\boldsymbol{\phi}_1$ in the ℓ_2-norm; that is,

$$\begin{aligned} |\boldsymbol{\phi}_1-\boldsymbol{\phi}^*|_2 &= |\boldsymbol{\phi}_0 - \boldsymbol{H}(\boldsymbol{\phi}^*)^{-1}\{\boldsymbol{H}(\boldsymbol{\phi}^*)(\boldsymbol{\phi}_0-\boldsymbol{\phi}^*) + O(|\boldsymbol{\phi}_0-\boldsymbol{\phi}^*|_2^2)\} - \boldsymbol{\phi}^*|_2 \\ &= O(|\boldsymbol{\phi}_0-\boldsymbol{\phi}^*|_2^2). \end{aligned}$$

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Therefore, by refining a consistent estimate $\boldsymbol{\phi}_0$ using the Newton–Raphson iteration (2.2), we obtain an improved estimate $\boldsymbol{\phi}_1$.

Next, we demonstrate how to cast the Newton–Raphson iteration of the CQR problem as an LS problem. Let $f(\boldsymbol{\alpha}^*) \stackrel{\text{def}}{=} \sum_{k=1}^K f(\alpha_k^*)/K$. Because $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are decoupled in the Newton–Raphson iteration (2.2), $\boldsymbol{\alpha}_1 = (\alpha_{1,1}, \ldots, \alpha_{1,K})^{\mathrm{T}}$ and $\boldsymbol{\beta}_1$ admit the explicit forms

$$\alpha_{1,k} = \alpha_{0,k} - f^{-1}(\alpha_k^*) E\{I(Y - \alpha_{0,k} - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta}_0 \le 0) - \tau_k\}, \quad k = 1, \ldots, K, \tag{2.3}$$

and

$$\begin{aligned} \boldsymbol{\beta}_1 &= \boldsymbol{\beta}_0 - \Sigma^{-1} f^{-1}(\boldsymbol{\alpha}^*) E\{\boldsymbol{g}_{\boldsymbol{\beta}}(\boldsymbol{\phi}_0; \mathbf{x}, Y)\} \\ &= \Sigma^{-1} E\left\{\mathbf{x}\left(\mathbf{x}^{\mathrm{T}}\boldsymbol{\beta}_0 - f^{-1}(\boldsymbol{\alpha}^*)\left[\frac{1}{K}\sum_{k=1}^K \{I(Y - \alpha_{0,k} - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta}_0 \le 0) - \tau_k\}\right]\right)\right\}. \end{aligned}$$

Define a pseudo response

$$\widetilde{Y} = \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta}_0 - f^{-1}(\boldsymbol{\alpha}^*)\left[\frac{1}{K}\sum_{k=1}^K \{I(Y - \alpha_{0,k} - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta}_0 \le 0) - \tau_k\}\right].$$

Then, $\boldsymbol{\beta}_1 = \Sigma^{-1} E(\mathbf{x}\widetilde{Y}) = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^p} E(\widetilde{Y} - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta})^2$ is the LS regression coefficient of $\widetilde{Y}$ on $\mathbf{x}$. To encourage sparsity in the coefficient vector, we consider the following penalized LS problem:

$$\boldsymbol{\beta}_{1,\,\mathrm{pen}} = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \frac{1}{2} E(\widetilde{Y} - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta})^2 + P(\boldsymbol{\beta}, \lambda), \tag{2.4}$$

where $P(\cdot, \cdot)$ is a sparsity-inducing penalty. Examples of penalties include the ℓ_1-norm penalty (LASSO, Tibshirani, 1996), smoothly clipped absolute deviation

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penalty (SCAD, Fan and Li, 2001), and minimax concave penalty (MCP, Zhang, 2010); see Hastie et al. (2015) and the references therein for comprehensive reviews of recent developments. In the present context, we adopt the ℓ_1-norm penalty $P(\boldsymbol{\beta}, \lambda) \stackrel{\text{def}}{=} \lambda|\boldsymbol{\beta}|_1$, for ease of exposition. Then, (2.4) becomes

$$\boldsymbol{\beta}_{1,\ell_1} = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \frac{1}{2} E(\widetilde{Y} - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta})^2 + \lambda|\boldsymbol{\beta}|_1. \tag{2.5}$$

At the population level, if we have a consistent estimate $(\boldsymbol{\alpha}_0^{\mathrm{T}}, \boldsymbol{\beta}_0^{\mathrm{T}})^{\mathrm{T}}$ of $(\boldsymbol{\alpha}^{*\mathrm{T}}, \boldsymbol{\beta}^{*\mathrm{T}})^{\mathrm{T}}$, then we can estimate $\boldsymbol{\alpha}^*$ and the ultrahigh-dimensional sparse $\boldsymbol{\beta}^*$ by solving a simple iteration (2.3) and a penalized LS problem (2.5), rather than solving the original penalized CQR.

Now, we define the empirical version of $\boldsymbol{\beta}_{1,\ell_1}$ in a single-node setting. Let $\widehat{\boldsymbol{\alpha}}^{(0)}$ and $\widehat{\boldsymbol{\beta}}^{(0)}$ be the initial estimates of $\boldsymbol{\alpha}^*$ and $\boldsymbol{\beta}^*$, respectively, and let $\widehat{f}(\boldsymbol{\alpha}^*)$ be an estimate of $f(\boldsymbol{\alpha}^*)$. For $i = 1, \ldots, N$, define the pseudo responses

$$\widetilde{Y}_i = \mathbf{x}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}^{(0)} - \widehat{f}^{-1}(\boldsymbol{\alpha}^*)\left[\frac{1}{K}\sum_{k=1}^{K}\{I(Y_i - \widehat{\alpha}_k^{(0)} - \mathbf{x}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}^{(0)} \le 0) - \tau_k\}\right].$$

We estimate $\boldsymbol{\alpha}^*$ and $\boldsymbol{\beta}^*$ using the empirical versions of (2.3) and (2.5), respectively:

$$\widehat{\alpha}_k^{(1)} = \widehat{\alpha}_k^{(0)} - \widehat{f}^{-1}(\alpha_k^*) \cdot \frac{1}{N}\sum_{i=1}^{N}\left\{I(Y_i - \widehat{\alpha}_k^{(0)} - \mathbf{x}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}^{(0)} \le 0) - \tau_k\right\}, \tag{2.6}$$

and

$$\widehat{\boldsymbol{\beta}}_{\ell_1}^{(1)} = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \frac{1}{2N}\sum_{i=1}^{N}(\widetilde{Y}_i - \mathbf{x}_i^{\mathrm{T}}\boldsymbol{\beta})^2 + \lambda_N|\boldsymbol{\beta}|_1. \tag{2.7}$$

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In a single-node setting, problems (2.6) and (2.7) correspond to a simple LS problem and an LS lasso problem, respectively, which are computationally much easier than an ℓ_1-regularized CQR problem.

We take $\widehat{f}(\boldsymbol{\alpha}^*) = \sum_{k=1}^{K} \widehat{f}(\alpha_k^*)/K$ as the average of the K kernel density estimates

$$\widehat{f}(\alpha_k^*) = (Nh)^{-1} \sum_{i=1}^{N} \mathcal{K}\{(Y_i - \widehat{\alpha}_k^{(0)} - \mathbf{x}_i^{\mathrm{T}} \widehat{\boldsymbol{\beta}}^{(0)})/h\}, \quad k = 1, \ldots, K. \tag{2.8}$$

Here, $\mathcal{K}(\cdot)$ is a kernel function fulfilling Condition (C3) of Section 3, and $h > 0$ is the bandwidth. In Sections 3 and 4, we discuss the selection of the bandwidth.

To minimize problems (2.6) and (2.7), we may first pool all the data into a single central node, which we then optimize. However, this may require substantial memory and storage for large amounts of data. Distributed systems require computationally efficient algorithms with very low communication costs (Jordan et al., 2019; Fan et al., 2019b; Lan et al., 2020). In this paper, we introduce a distributed algorithm that robustly and efficiently estimates $\boldsymbol{\alpha}^*$ and $\boldsymbol{\beta}^*$ at near-oracle rates.

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2.2 Distributed estimation

Here, we develop a distributed communication-efficient algorithm to compute $\widehat{\boldsymbol{\alpha}}^{(1)}$ and $\widehat{\boldsymbol{\beta}}^{(1)}_{\ell_1}$ in (2.6) and (2.7) under the "horizontal" distributed setting. Because

$$\widehat{\alpha}_k^{(1)} = \widehat{\alpha}_k^{(0)} - \widehat{f}^{-1}(\alpha_k^*)\frac{1}{N}\sum_{j=1}^{m}\sum_{i\in\mathcal{H}_j}\{I(Y_i - \widehat{\alpha}_k^{(0)} - \mathbf{x}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}^{(0)} \le 0) - \tau_k\}, \quad k = 1, \ldots, K,$$

the communication cost to obtain $\widehat{\boldsymbol{\alpha}}^{(1)}$ is $O(mK)$, which is communication-efficient. For $\widehat{\boldsymbol{\beta}}^{(1)}_{\ell_1}$, we solve problem (2.7) using an approximate Newton method (Wang et al., 2017; Jordan et al., 2019; Fan et al., 2019a), that has a communication cost $O(mp)$. For ease of notation, let

$$\mathbf{z}_{n,j} \overset{\text{def}}{=} \frac{1}{n}\sum_{i\in\mathcal{H}_j}\mathbf{x}_i\widetilde{Y}_i,\ \mathbf{z}_N \overset{\text{def}}{=} \frac{1}{m}\sum_{j=1}^{m}\mathbf{z}_{n,j},\ \widehat{\Sigma}_j \overset{\text{def}}{=} \frac{1}{n}\sum_{i\in\mathcal{H}_j}\mathbf{x}_i\mathbf{x}_i^{\mathrm{T}},\ \text{and}\ \widehat{\Sigma} \overset{\text{def}}{=} \frac{1}{m}\sum_{j=1}^{m}\widehat{\Sigma}_j.$$

We further define the pseudo local and global loss functions, respectively, as

$$\mathcal{L}_j(\boldsymbol{\beta}) \overset{\text{def}}{=} \frac{1}{2n}\sum_{i\in\mathcal{H}_j}(\widetilde{Y}_i - \mathbf{x}_i^{\mathrm{T}}\boldsymbol{\beta})^2 \ \text{and}\ \mathcal{L}_N(\boldsymbol{\beta}) \overset{\text{def}}{=} \frac{1}{m}\sum_{j=1}^{m}\mathcal{L}_j(\boldsymbol{\beta}).$$

Denote the gradient of $\mathcal{L}_N(\boldsymbol{\beta})$ by $\nabla\mathcal{L}_N(\boldsymbol{\beta})$, which is $\widehat{\Sigma}\boldsymbol{\beta} - \mathbf{z}_N$. Given an initial estimate $\widehat{\boldsymbol{\beta}}^{(0)}$, we have

$$\begin{aligned}\mathcal{L}_N(\boldsymbol{\beta}) = \mathcal{L}_N(\widehat{\boldsymbol{\beta}}^{(0)}) + \{\nabla\mathcal{L}_N(\widehat{\boldsymbol{\beta}}^{(0)})\}^{\mathrm{T}}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}^{(0)}) \\ + \frac{1}{2}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}^{(0)})^{\mathrm{T}}\widehat{\Sigma}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}^{(0)}).\end{aligned} \tag{2.9}$$

Recall that in a "horizontal" distributed system, the data are scattered across m nodes. Transmitting the Hessian matrix $\widehat{\Sigma}$ requires a communication cost

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$O(p^2)$, which is typically expensive in a high-dimensional setting. To reduce the communication cost, we approximate the Hessian matrix $\widehat{\Sigma}$ by $\widehat{\Sigma}_1$, which leads to the surrogate loss

$$\widetilde{\mathcal{L}}^*(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \mathcal{L}_N(\widehat{\boldsymbol{\beta}}^{(0)}) + \{\nabla \mathcal{L}_N(\widehat{\boldsymbol{\beta}}^{(0)})\}^{\mathrm{T}}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}^{(0)}) + \frac{1}{2}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}^{(0)})^{\mathrm{T}}\widehat{\Sigma}_1(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}^{(0)}). \tag{2.10}$$

Comparing (2.10) with (2.9), we obtain the approximation error of the surrogate loss

$$\begin{aligned}\widetilde{\mathcal{L}}^*(\boldsymbol{\beta}) - \mathcal{L}_N(\boldsymbol{\beta}) &= (\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}^{(0)})^{\mathrm{T}}(\widehat{\Sigma} - \widehat{\Sigma}_1)(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}^{(0)}) \\ &= O_p\big\{\|\widehat{\Sigma} - \widehat{\Sigma}_1\|_{\mathrm{op}} \cdot |\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}^{(0)}|_2^2\big\},\end{aligned}$$

where the last equality follows from the Cauchy–Schwarz inequality. The approximation error is negligible if either $\|\widehat{\Sigma} - \widehat{\Sigma}_1\|_{\mathrm{op}}$ or $|\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}^{(0)}|_2$ is $o_p(1)$, which is possible if p is much smaller than n or if a sparsity structure exists in the coefficient vector when p is much greater than n.

Ignoring the additive terms in (2.10) irrelevant to $\boldsymbol{\beta}$, the surrogate loss can be simplified to

$$\widetilde{\mathcal{L}}(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \frac{1}{2n}\sum_{i \in \mathcal{H}_1}(\mathbf{x}_i^{\mathrm{T}}\boldsymbol{\beta})^2 - \boldsymbol{\beta}^{\mathrm{T}}\{\mathbf{z}_N + (\widehat{\Sigma}_1 - \widehat{\Sigma})\widehat{\boldsymbol{\beta}}^{(0)}\}.$$

Here, rather than working with the pseudo global loss $\mathcal{L}_N$ in (2.7), we work with $\widetilde{\mathcal{L}}(\boldsymbol{\beta})$ to reduce the communication cost. Specifically, we define

$$\widehat{\boldsymbol{\beta}}^{(1)} \stackrel{\text{def}}{=} \arg\min_{\boldsymbol{\beta} \in \mathbb{R}^p} \widetilde{\mathcal{L}}(\boldsymbol{\beta}) + \lambda_N |\boldsymbol{\beta}|_1. \tag{2.11}$$

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Note that we can calculate $\widehat{\Sigma}\widehat{\boldsymbol{\beta}}^{(0)}$ and $\mathbf{z}_N$ very efficiently in a distributed manner with a communication cost $O(mp)$, because $\widehat{\Sigma}_j\widehat{\boldsymbol{\beta}}^{(0)}$ and $\mathbf{z}_{n,j}$, which form $\widehat{\Sigma}\widehat{\boldsymbol{\beta}}^{(0)}$ and $\mathbf{z}_N$, respectively, are both p-dimensional vectors (see Algorithm 1). There is no need to communicate the $p \times p$ covariance matrix $\widehat{\Sigma}_j$.

In practice, when feasible, we recommend using the ℓ_1-penalized CQR estimate (Gu and Zou, 2020; Pietrosanu et al., 2020), fitted from the data collected only at the first node as the initial estimate:

$$\{\widehat{\boldsymbol{\alpha}}^{(0)}, \widehat{\boldsymbol{\beta}}^{(0)}\} \stackrel{\text{def}}{=} \underset{\boldsymbol{\alpha}\in\mathbb{R}^K, \boldsymbol{\beta}\in\mathbb{R}^p}{\arg\min} \frac{1}{2nK} \sum_{i\in\mathcal{H}_1} \sum_{k=1}^{K} \rho_{\tau_k}(Y_i - \alpha_k - \mathbf{x}_i^{\mathrm{T}}\boldsymbol{\beta}) + \lambda_n|\boldsymbol{\beta}|_1. \quad (2.12)$$

Our distributed estimation procedure then proceeds iteratively from the initial estimate. Specifically, for any $t \geq 1$, let $\widehat{\boldsymbol{\beta}}^{(t-1)}$ be the distributed estimate in the $(t-1)$th communication, and let

$$\widehat{f}^{(t)}(\boldsymbol{\alpha}^*) \stackrel{\text{def}}{=} (NKh^{(t)})^{-1} \sum_{k=1}^{K} \sum_{i=1}^{N} \mathcal{K}\{(Y_i - \widehat{\alpha}_k^{(t-1)} - \mathbf{x}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}^{(t-1)})/h^{(t)}\}$$

be the estimate of $f(\boldsymbol{\alpha}^*)$ and $h^{(t)}$ be the associated bandwidth (specified in Theorem 2) in the tth communication. Define

$$\widetilde{Y}_i^{(t)} = \mathbf{x}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}^{(t-1)} - \widehat{f}^{-1}(\boldsymbol{\alpha}^*)\left[\frac{1}{K}\sum_{k=1}^{K}\{I(Y_i - \widehat{\alpha}_k^{(t-1)} - \mathbf{x}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}^{(t-1)} \leq 0) - \tau_k\}\right],$$

for $i = 1, \ldots, N$, and $\mathbf{z}_N^{(t)} = N^{-1}\sum_{i=1}^{N} \mathbf{x}_i\widetilde{Y}_i^{(t)}$. The distributed estimate in the tth communication takes the form

$$\widehat{\alpha}_k^{(t)} \stackrel{\text{def}}{=} \widehat{\alpha}_k^{(t-1)} - \widehat{f}^{-1}(\alpha_k^*)\frac{1}{N}\sum_{i=1}^{N}\{I(Y_i - \widehat{\alpha}_k^{(t-1)} - \mathbf{x}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}^{(t-1)} \leq 0) - \tau_k\} \quad (2.13)$$

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and

$$\widehat{\boldsymbol{\beta}}^{(t)} \stackrel{\text{def}}{=} \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \frac{1}{2n}\sum_{i\in\mathcal{H}_1}(\mathbf{x}_i^{\mathrm{T}}\boldsymbol{\beta})^2 - \boldsymbol{\beta}^{\mathrm{T}}\{\mathbf{z}_N^{(t)} + (\widehat{\Sigma}_1 - \widehat{\Sigma})\widehat{\boldsymbol{\beta}}^{(t-1)}\} + \lambda_{N,t}|\boldsymbol{\beta}|_1. \quad (2.14)$$

Problem (2.14) is an ℓ_1-regularized quadratic program, which can be solved using a first-order method (Combettes and Pesquet, 2011; Bach et al., 2012; Tropp and Wright, 2010), a Newton-type algorithm (Fountoulakis et al., 2014; Dassios et al., 2015), or the coordinate descent algorithm (Friedman et al., 2010). In our implementation, we use the primal dual active set (PDAS, Fan et al., 2014b) method, which is essentially a generalized Newton-type method. It converges after one iteration if the initial value is good enough. To select the regularization parameter, because $\widehat{\boldsymbol{\beta}}^{(t)}$ is a piecewise linear function of $\lambda_{N,t}$ (Osborne et al., 2000), we use a continuation procedure in order to fully exploit the fast convergence of the PDAS method. Specifically, we use the solution from the previous step as the initial value for the current step. When the continuation procedure completes, we have a solution path for (2.14), from which we choose the best regularization parameter based on maximum voting. In particular, we set $\lambda_1 = |\mathbf{z}_N^{(t)} + (\widehat{\Sigma}_1 - \widehat{\Sigma})\widehat{\boldsymbol{\beta}}^{(t-1)}|_\infty$, which has a solution to (2.14) that is exactly zero, by the Karush–Kuhn–Tucker conditions. Let $\lambda_\ell = \lambda_1\rho^{\ell-1}$, with $\rho \in (0,1)$, for $\ell \geq 1$. For some pre-fixed threshold $s_0 \in \mathbb{N}_+$, we apply the PDAS method to compute a solution path $\{\widehat{\boldsymbol{\beta}}^{(t),\lambda_1}, \ldots, \widehat{\boldsymbol{\beta}}^{(t),\lambda_L}\}$ until $|\widehat{\boldsymbol{\beta}}^{(t),\lambda_L}|_0 > s_0$ for a smallest possible L. Let $\mathcal{S}_v = \{\lambda_\ell : |\widehat{\boldsymbol{\beta}}^{(t),\lambda_\ell}|_0 = v, \ell = 1, \ldots, L\}$ be the set of regularization parameters

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at which the solution to (2.14) has v nonzero elements, where $v = 1, \ldots, s_0$. We determine $\lambda_{N,t}$ by maximum voting, that is,

$$\lambda_{N,t} = \max\{\mathcal{S}_{\overline{v}}\} \text{ and } \overline{v} = \text{argmax}_v |\mathcal{S}_v|,$$

where $|\mathcal{S}_v|$ is the cardinality of the set $\mathcal{S}_v$. Our parameter selection rule is seamlessly integrated with the continuation procedure without incurring extra communication and computation costs. Classical cross-validation approaches can be used in the distributed setting (Yu et al., 2021) as well. We summarize our distributed algorithm in Algorithm 1.

3. Theoretical Results

In this section, we show the estimation and support recovery accuracy of the distributed CQR estimate. Denote $\mathcal{S} \overset{\text{def}}{=} \text{supp}(\boldsymbol{\beta}^*)$ as the support of $\boldsymbol{\beta}^*$, and let $s \overset{\text{def}}{=} |\mathcal{S}|$. Following Wainwright (2019), we say a random vector $\mathbf{x} \in \mathbb{R}^p$ is sub-Gaussian if it satisfies $\sup_{|\alpha|_2=1} E \exp\{t(\alpha^{\mathrm{T}}\mathbf{x})^2\} \leq C$, for some $t > 0$ and $C > 0$. We assume the following conditions:

(C1) The density f is bounded and Lipschitz continuous, that is, $|f(x) - f(y)| \leq C_L|x-y|$, for any $x, y \in \mathbb{R}$ and some constant $C_L > 0$. Moreover, we assume $f(\alpha_k^*) \geq \underline{f} > 0$, for all $k = 1, \ldots, K$.

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Algorithm 1 Distributed algorithm for sparse CQR

Input: Data $\{(\mathbf{x}_i, Y_i)_{i\in\mathcal{H}_j}\}$, for $j = 1, \ldots, m$, the number of iterations t, the quantile levels τ_k, for $k = 1, \ldots, K$, a sequence of bandwidths $h^{(g)}$, for $g = 1, \ldots, t$, the regularization parameters λ_n and $\lambda_N^{(g)}$, for $g = 1, \ldots, t$.

1: Compute the initial estimates $\widehat{\boldsymbol{\alpha}}^{(0)}$ and $\widehat{\boldsymbol{\beta}}^{(0)}$ using (2.12), based on $\{(\mathbf{x}_i, Y_i)_{i\in\mathcal{H}_1}\}$.

2: **for** $g = 1, \ldots, t$ **do**

3: Transmit $\widehat{\boldsymbol{\alpha}}^{(g-1)}$ and $\widehat{\boldsymbol{\beta}}^{(g-1)}$ from the first node to the local ones labeled with $2, \ldots, m$.

4: **for** $j = 1, \ldots, m$ **do**

5: Calculate

$$\widehat{f}^{(g,j)}(\boldsymbol{\alpha}^*) = (nKh^{(g)})^{-1} \sum_{k=1}^{K} \sum_{i\in\mathcal{H}_j} \mathcal{K}\{(Y_i - \widehat{\alpha}_k^{(g-1)} - \mathbf{x}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}^{(g-1)})/h^{(g)}\}$$

at the jth node and send it back to the first node.

6: **end for**

7: The first node computes $\widehat{f}^{(g)}(\boldsymbol{\alpha}^*) = m^{-1}\sum_{j=1}^{m}\widehat{f}^{(g,j)}(\boldsymbol{\alpha}^*)$ and transmits it to the local nodes labeled with $2, \ldots, m$.

8: **for** $j = 1, \ldots, m$ **do**

9: Calculate $\widehat{\Sigma}_j\widehat{\boldsymbol{\beta}}^{(g-1)}$ and $\mathbf{z}_{n,j}^{(g)} = n^{-1}\sum_{i\in\mathcal{H}_j}\mathbf{x}_i\widetilde{Y}_i^{(g)}$ at the jth node and send them back to the first node.

10: **end for**

11: Calculate $\widehat{\boldsymbol{\alpha}}^{(g)}$ and $\widehat{\boldsymbol{\beta}}^{(g)}$ on the first node, based on (2.13) and (2.14).

12: **end for**

Output: The final estimates $\widehat{\boldsymbol{\alpha}}^{(t)}$ and $\widehat{\boldsymbol{\beta}}^{(t)}$ obtained from the first node.

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(C2) There exists a constant $c_0 > 0$ such that $c_0^{-1} \leq \Lambda_{\min}(\Sigma) \leq \Lambda_{\max}(\Sigma) \leq c_0$. Furthermore, we assume $\|\Sigma_{\mathcal{S}^c \times \mathcal{S}} \Sigma_{\mathcal{S} \times \mathcal{S}}^{-1}\|_\infty \leq 1 - \alpha$, for some $0 < \alpha < 1$.

(C3) Assume the kernel function $\mathcal{K}(\cdot)$ is differentiable with a bounded derivative $\mathcal{K}'(\cdot)$. Moreover, $\mathcal{K}(\cdot)$ is integrable, with $\int_{-\infty}^{\infty} \mathcal{K}(u)\mathrm{d}u = 1$ and $\mathcal{K}(u) = 0$, for $|u| \geq 1$.

(C4) The covariate vector $\mathbf{x}$ is sub-Gaussian. The dimension p satisfies $p = O(N^\nu)$, for some $\nu > 0$. The sample size n at each local node satisfies $n \geq N^\omega$, for some $0 < \omega < 1$, the sparsity level s satisfies $s = O(n^r)$, for some $0 \leq r < 1/3$, and the number of quantile levels K satisfies $K = O(n^r)$, for some $0 \leq r < 1/3$.

(C5) The initial estimates $\widehat{\boldsymbol{\alpha}}^{(0)}$ and $\widehat{\boldsymbol{\beta}}^{(0)}$ satisfy $\mathrm{pr}(\mathrm{supp}(\widehat{\boldsymbol{\beta}}^{(0)}) \subseteq \mathcal{S}) \to 1$ and $|\widehat{\boldsymbol{\beta}}^{(0)} - \boldsymbol{\beta}^*|_2 = O_p(a_n)$ and $|\widehat{\boldsymbol{\alpha}}^{(0)} - \boldsymbol{\alpha}^*|_2 = O_p(K^{1/2} a_n)$, where $a_n = (s \log N/n)^{1/2}$.

Condition (C1) is standard on the smoothness of the noise density (Gu and Zou, 2020; Chen et al., 2020). The irrepresentable condition (C2) is widely used in the high-dimensional statistics literature to establish support recovery; see, for example, Zhao and Yu (2006),Wainwright (2009), Hastie et al. (2015), and Wainwright (2019). Condition (C3) imposes regular conditions on the kernel function, and is mild and satisfied by many common kernel functions.

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Condition (C4) is commonly assumed in the distributed estimation literature; see, for example, Chen et al. (2020), Wang et al. (2017), and Jordan et al. (2019). In Algorithm 1, the initial estimator $\widehat{\boldsymbol{\beta}}^{(0)}$ is obtained using the data scattered at the central node. Such an initial estimate satisfies (C5) under Conditions (C1), (C2), and (C4) (see Theorem 1 of Gu and Zou, 2020). We use $\log N$ throughout for simplicity, because we have $\log\{\max(N, p)\} = C_1 \log N$, for some constant $C_1 > 0$, by Condition (C4).

We first present the convergence rates of the distributed estimates $\widehat{\boldsymbol{\alpha}}^{(1)}$ and $\widehat{\boldsymbol{\beta}}^{(1)}$ from the first communication.

Theorem 1. *Set* $\lambda_N = C_0\{(\log N/N)^{1/2} + (s \log N/n)^{1/2} a_n\}$ *and the bandwidth* $h \asymp a_n$, *where* $C_0 > 0$ *is a sufficiently large constant. Under Conditions (C1)–(C5), we have*

$$|\widehat{\boldsymbol{\alpha}}^{(1)} - \boldsymbol{\alpha}^*|_2 = O_p\{(Ks \log N/N)^{1/2} + (Ks^2 \log N/n)^{1/2} a_n\}$$

and

$$|\widehat{\boldsymbol{\beta}}^{(1)} - \boldsymbol{\beta}^*|_2 = O_p\{(s \log N/N)^{1/2} + (s^2 \log N/n)^{1/2} a_n\}.$$

Let $a_N^{(g)} = (s \log N/N)^{1/2} + s^{(2g+1)/2} (\log[N]/n)^{(g+1)/2}$, for $g = 1, \ldots, t$. Applying Theorem 1 leads to the convergence rates of the distributed estimates $\widehat{\boldsymbol{\alpha}}^{(t)}$ and $\widehat{\boldsymbol{\beta}}^{(t)}$ from the tth communication.

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Theorem 2. *Set* $\lambda_N^{(g)} = C_0\{(\log N/N)^{1/2} + (s\log N/n)^{1/2}a_N^{(g-1)}\}$ *and the bandwidth* $h^{(g)} \asymp a_N^{(g-1)}$, *for* $g = 1, \ldots, t$, *where* $C_0 > 0$ *is a sufficiently large constant. Under Conditions (C1)–(C5), we have*

$$|\widehat{\boldsymbol{\alpha}}^{(t)} - \boldsymbol{\alpha}^*|_2 = O_p\{(Ks\log N/N)^{1/2} + (K)^{1/2}s^{(2t+1)/2}(\log N/n)^{(t+1)/2}\}$$

and

$$|\widehat{\boldsymbol{\beta}}^{(t)} - \boldsymbol{\beta}^*|_2 = O_p\{(s\log N/N)^{1/2} + s^{(2t+1)/2}(\log N/n)^{(t+1)/2}\}.$$

When the number of communications t is large enough, that is,

$$t \geq \log(N/n)/\log\{c_0 n/(s^2\log N)\}, \quad \text{for some} \quad c_0 > 0, \tag{3.1}$$

we have $s^{(2t+1)/2}(\log N/n)^{(t+1)/2} = O\{(s\log N/N)^{1/2}\}$. Therefore, $|\widehat{\boldsymbol{\beta}}^{(t)} - \boldsymbol{\beta}^*|_2 = O_p\{(s\log N/N)^{1/2}\}$, and the distributed estimate $\widehat{\boldsymbol{\beta}}^{(t)}$ attains the minimax optimal rate $O\{(s\log N/N)^{1/2}\}$. This is also the optimal rate when the data are pooled at a single central node (Gu and Zou, 2020). In view of Condition (C4), the right-hand side of (3.1) is bounded by a constant. To achieve the oracle rate, our distributed algorithm requires the number of communications, t, to increase logarithmically with the number of nodes, m. In contrast, existing distributed first-order algorithms require the number of communications to increase polynomially with m; see Table 1 of Zhang and Xiao (2015) for details.

We present the support recovery of our distributed method in the following two theorems.

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Theorem 3. *Under the conditions of Theorem 1, we have* $\mathrm{supp}(\widehat{\boldsymbol{\beta}}^{(1)}) \subseteq \mathcal{S}$, *with probability approaching one. Suppose, in addition, for a sufficiently large positive constant* C,

$$\boldsymbol{\beta}^{*\min} \geq C\|\Sigma^{-1}_{\mathcal{S}\times\mathcal{S}}\|_\infty \left\{(\log N/N)^{1/2} + |\widehat{\boldsymbol{\beta}}^{(0)} - \boldsymbol{\beta}^*|_2 (s\log N/n)^{1/2}\right\}.$$

Then, we have $\mathrm{supp}(\widehat{\boldsymbol{\beta}}^{(1)}) = \mathcal{S}$, *with probability approaching one.*

Theorem 4. *Under the conditions of Theorem 2, we have* $\mathrm{supp}(\widehat{\boldsymbol{\beta}}^{(t)}) \subseteq \mathcal{S}$, *with probability approaching one. Suppose, for a sufficiently large positive constant* C,

$$\boldsymbol{\beta}^{*\min} \geq C\|\Sigma^{-1}_{\mathcal{S}\times\mathcal{S}}\|_\infty \left\{(\log N/N)^{1/2} + s^t (\log N/n)^{(t+1)/2}\right\}.$$

Then, we have $\mathrm{supp}(\widehat{\boldsymbol{\beta}}^{(t)}) = \mathcal{S}$, *with probability approaching one.*

The "beta-min" condition, which is commonly assumed in the literature on high-dimensional statistics, weakens as t increases and matches the oracle rate for the "beta-min" condition, that is, $\boldsymbol{\beta}^{*\min} \geq C\|\Sigma^{-1}_{\mathcal{S}\times\mathcal{S}}\|_\infty (\log N/N)^{1/2}$, after a constant number of communications (Wainwright, 2009).

We have assumed evenly scattered data across the nodes, for ease of demonstration. In fact, the number of data points, n, is just the "working" sample size at the first node, or the central node, as it is known as in distributed computing. Once it is specified, our approach does not depend on the partition of the data.

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Several works in the distributed computing literature examine heavy-tailed noise. Chen et al. (2020) consider the distributed QR estimation and similarly cast the nonsmooth QR problem as an LS problem. Note that our study of the CQR is motivated by the potential loss of efficiency of the QR under certain noise distributions, and our work is technically more challenging than the distributed QR, because our loss function consists of quantile check losses at multiple levels. Furthermore, in contrast to the validation method of Chen et al. (2020), we suggest a new tuning procedure in Section 2.2 that does not incur extra communication and computation costs. Luo et al. (2022) consider the distributed adaptive Huber regression, which can also handle certain cases of heavy-tailed noise. Their theoretical analysis does not assume independence between the noise and the covariates, but does require that the covariates to be bounded. Moreover, the Huber regression does not work under very heavy-tailed noise such as the Cauchy, whereas the CQR does. Battey et al. (2021) suggest a convoluted smoothing of the check loss to handle the nonsmoothness of the QR. This alternative smoothing procedure can be applied to our CQR loss, especially when the construction of the pseudo response is not stable in small samples. However, it may not be as computationally efficient as our method, owing to our simple LS formulation. In addition, the aforementioned works focus mainly on the estimation error bounds of their respective estimates, whereas we establish the support recovery theory

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in addition to the estimation error. Support recovery is an important topic in high-dimensional distributed settings but is usually very challenging (Neykov et al., 2016).

4. Simulation Studies

4.1 Design of the simulations

We simulate the data from model (1.1) with $\beta_0^* = 0$ and $\boldsymbol{\beta}^* = (3, 1.5, 0, 0, 2, \mathbf{0}_{p-5})^{\mathrm{T}}$, and the covariates $\mathbf{x}_i$ are drawn from $\mathcal{N}(0, \Sigma)$, with $\Sigma = (0.5^{|k-l|})_{p\times p}$. We fix $p = 500$ throughout. For the noise distribution, we follow Zou and Yuan (2008) and Gu and Zou (2020), and consider three shapes:

(a) the normal distribution, $\varepsilon \sim \mathcal{N}(0, 1)$,

(b) the Student's t distribution with three degrees of freedom, $\varepsilon \sim t(3)$, and

(c) the Cauchy distribution, $\varepsilon \sim f(\varepsilon) = 1/\{\pi(1 + \varepsilon^2)\}$.

The initial estimator is taken as the ℓ_1-regularized CQR estimator defined in (2.12) using the local data at the first node. The constant C_0 of $\lambda_N^{(g)}$ is chosen using majority voting along the solution paths calculated using the method of Huang et al. (2018). We use the bi-weight kernel function $\mathcal{K}(x) = 105(1 - 3x^2)(1 - x^2)^2 I(|x| \le 1)/64$, and set the bandwidth as $h^{(g)} = ca_N^{(g-1)}$, for some constant $c > 0$ (Theorem 2). We take $c = 1$, for simplicity. A sensitivity analysis

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of the choice of c is provided in Section 4.5. Throughout, we take $K = 19$ and $\tau_k = \frac{k}{K+1}$, for $k = 1, \ldots, K$.

We compare our distributed estimate with its pooled counterpart and the divide-and-conquer estimate. Specifically, the divide-and-conquer method computes the ℓ_1-regularized CQR at each local node and combines the local estimates using simple averaging.

Two criteria are used to evaluate the performance of the methods: the estimation error, $|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*|_2$, and the F_1-score,

$$F_1 \stackrel{\text{def}}{=} \frac{2(\text{TP})}{2(\text{TP}) + \text{FP} + \text{FN}},$$

where TP, FP, and FN denote the numbers of true positives, false positives, and false negatives, respectively. The F_1-score ranges from zero to one, with larger values indicating better performance (see, e.g., Goutte and Gaussier, 2005), and is widely used in the literature to evaluate support recovery accuracy.

We repeat each setting with one hundred independent runs.

4.2 Effect of the number of communications

We investigate how the performance of our distributed estimate varies according to the number of iterations (or communications). We fix the sample size at $N = 5000$, the local sample size at $n = 500$, and the number of nodes at $m = 10$. Shown in Figure 1 is the plot of the mean estimation error (over one

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hundred independent runs) versus the number of iterations, where the error bar corresponds to one standard error. The distributed and pooled estimates exhibit similar performance under all three noise scenarios, and both become stable in just a few iterations. In addition, they both outperform the divide-and-conquer estimate by a large margin.

4.3 Effect of the noise distribution

Here, we demonstrate the robustness of the CQR to heavy-tailed noise and its preservation of efficiency under light-tailed noise by considering the three aforementioned noise distributions. We additionally include the LS lasso estimate in a single-node setting (pooled data) in the comparison. We vary the sample size N, and summarize the estimation errors and F_1-scores of the four estimates in Table 1. In all settings, the performance of our distributed estimate matches that of the pooled estimate, and both outperform the divide-and-conquer and lasso estimates. In particular, the distributed and pooled estimates perform similarly to the lasso estimate under normal noise, but exhibit much better performance under the Cauchy noise. We omit the lasso estimate in subsequent comparisons, owing to its instability under heavy-tailed noises.

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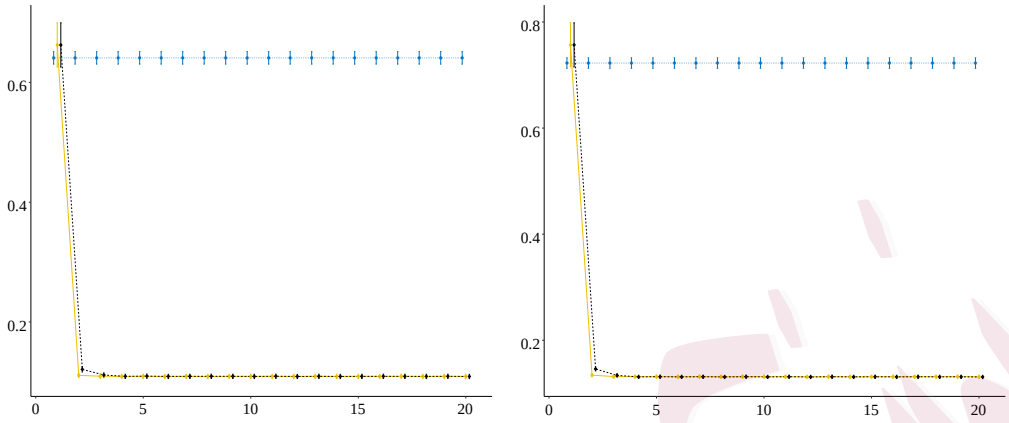
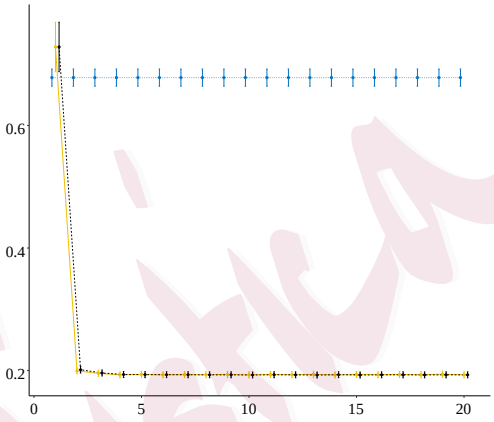


(a) Normal (b) Student's $t(3)$



(c) Cauchy

Figure 1: The horizontal axis shows the number of iterations (communications), and the vertical axis represents the estimation errors of the divide-and-conquer (···✳···), distributed (- ◇ -), and pooled (—⊟—) estimates when the noise comes from the (a) normal, (b) Student's $t(3)$, and (c) Cauchy distributions. The error bar corresponds to one standard error. The horizontal values are jittered to avoid overlapping. The overall sample size, local sample size, and dimension are fixed at $N = 5000$, $n = 500$, and $p = 500$, respectively.

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Table 1: The estimation errors and F_1-scores of the distributed, pooled, and divide-and-conquer estimates, and the least squares lasso estimate fitted using the pooled data under varying sample sizes N when the noise comes from the normal, Student's $t(3)$, and Cauchy distributions. The local sample size n is fixed at 500.

N	distr		pooled		dc		lasso	
	est. error	F_1-score	est. error	F_1-score	est. error	F_1-score	est. error	F_1-score
	Normal noise							
2500	0.0998	1.0000	0.0997	1.0000	0.1770	0.0826	0.0817	0.4636
5000	0.0718	1.0000	0.0717	1.0000	0.1716	0.0409	0.0576	0.4793
10000	0.0515	1.0000	0.0515	1.0000	0.1673	0.0241	0.0398	0.4521
15000	0.0378	1.0000	0.0379	1.0000	0.1639	0.0186	0.0307	0.4748
20000	0.0349	1.0000	0.0348	1.0000	0.1662	0.0158	0.0295	0.5512
25000	0.0308	1.0000	0.0307	1.0000	0.1655	0.0146	0.0288	0.8413
	Student's $t(3)$ noise							
2500	0.1242	1.0000	0.1244	1.0000	0.1974	0.0756	0.1433	0.4199
5000	0.0884	1.0000	0.0885	1.0000	0.1894	0.0394	0.0990	0.4067
10000	0.0596	1.0000	0.0596	1.0000	0.1841	0.0238	0.0695	0.4858
15000	0.0465	1.0000	0.0465	1.0000	0.1826	0.0183	0.0538	0.4761
20000	0.0432	1.0000	0.0432	1.0000	0.1824	0.0159	0.0481	0.4482
25000	0.0383	1.0000	0.0383	1.0000	0.1816	0.0143	0.0441	0.4417
	Cauchy noise							
2500	0.1713	1.0000	0.1713	1.0000	0.1908	0.1042	3.1826	0.2245
5000	0.1254	1.0000	0.1255	1.0000	0.1897	0.0470	3.1247	0.2505
10000	0.0846	1.0000	0.0846	1.0000	0.1878	0.0276	3.0907	0.2454
15000	0.0732	1.0000	0.0732	1.0000	0.1845	0.0211	3.1835	0.2361
20000	0.0619	1.0000	0.0618	1.0000	0.1819	0.0175	3.1540	0.2490
25000	0.0541	1.0000	0.0540	1.0000	0.1837	0.0158	2.9773	0.2418

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4.4 Effect of the overall and local sample sizes

We investigate the performance of the estimates under different combinations of the overall sample size N and the local sample size n. The estimation errors and F_1-scores are reported in Table 2. In terms of the estimation error, both the distributed and the pooled estimates outperform the divide-and-conquer estimate in almost all the settings, except when $N = 5000$ and $n = 1000$ under the Cauchy noise. In this exceptional case, however, they outperform the divide-and-conquer estimate again when the sample size N keeps growing, for example, from $N = 5000$ to $N = 10000$. This demonstrates the sub-optimality of the divide-and-conquer estimate compared with our distributed estimate when the number of nodes m grows. In terms of the support recovery, the F_1-scores of the distributed and pooled estimates are equal to one in all settings, and are much better than that of the divide-and-conquer estimate. This is not surprising, because the divide-and-conquer method usually results in a dense estimate.

4.5 Sensitivity analysis for the bandwidth

We investigate the sensitivity of the bandwidth selection by varying the sample size N and the constant c in the bandwidth $h^{(g)} = ca_N^{(g-1)}$ from 1 to 20. We summarize the results for the Cauchy noise in Table 3. The results for the other two noise distributions are relegated to the Supplementary Material. We can see

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Table 2: The estimation errors and F_1-scores of the distributed, pooled, and divide-and-conquer estimates under different combinations of the overall sample size N and the local sample size n when the noise comes from the normal, Student's $t(3)$, and Cauchy distributions.

n		200			500			1000		
N		5000	10000	20000	5000	10000	20000	5000	10000	20000
		Normal noise								
distr	est. error	0.0711	0.0506	0.0377	0.0711	0.0491	0.0342	0.0719	0.0478	0.0358
	F_1-score	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
pooled	est. error	0.0710	0.0505	0.0376	0.0710	0.0490	0.0342	0.0718	0.0478	0.0357
	F_1-score	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
dc	est. error	0.2565	0.2544	0.2536	0.1698	0.1661	0.1636	0.1251	0.1185	0.1174
	F_1-score	0.0187	0.0136	0.0121	0.0407	0.0239	0.0156	0.0812	0.0415	0.0254
		Student's $t(3)$ noise								
distr	est. error	0.0870	0.0606	0.0442	0.0860	0.0623	0.0427	0.0873	0.0577	0.0448
	F_1-score	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
pooled	est. error	0.0869	0.0612	0.0436	0.0861	0.0622	0.0425	0.0873	0.0576	0.0447
	F_1-score	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
dc	est. error	0.2953	0.2929	0.2906	0.1868	0.1876	0.1825	0.1394	0.1302	0.1292
	F_1-score	0.0192	0.0139	0.0121	0.0403	0.0239	0.0158	0.0794	0.0424	0.0238
		Cauchy noise								
distr	est. error	0.1236	0.0878	0.0606	0.1326	0.0897	0.0614	0.1215	0.0865	0.0608
	F_1-score	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
pooled	est. error	0.1241	0.0878	0.0611	0.1327	0.0897	0.0615	0.1215	0.0866	0.0610
	F_1-score	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
dc	est. error	0.3674	0.3657	0.3630	0.1900	0.1876	0.1828	0.1083	0.1019	0.0941
	F_1-score	0.0199	0.0144	0.0123	0.0492	0.0278	0.0174	0.1384	0.0797	0.0407

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488 from Table 3 that the distributed and pooled estimates are quite robust to the
489 choice of the bandwidth constant c, and exhibit similar performance under all
490 choices of the constant c.

Table 3: The estimation errors and F_1-scores of the distributed, pooled, and divide-and-conquer estimates under different combinations of the sample size N and the bandwidth constant c when the noise comes from the Cauchy distribution. The local sample size n is fixed at 500.

N	c	distr		pooled		dc	
		est. error	F_1-score	est. error	F_1-score	est. error	F_1-score
5000	1	0.1216	1.0000	0.1216	1.0000	0.1953	0.0483
10000	1	0.0868	1.0000	0.0865	1.0000	0.1870	0.0268
20000	1	0.0610	1.0000	0.0610	1.0000	0.1835	0.0177
5000	5	0.1213	1.0000	0.1211	1.0000	0.1904	0.0503
10000	5	0.0912	1.0000	0.0912	1.0000	0.1889	0.0278
20000	5	0.0606	1.0000	0.0606	1.0000	0.1852	0.0179
5000	10	0.1251	1.0000	0.1252	1.0000	0.1917	0.0494
10000	10	0.0853	1.0000	0.0853	1.0000	0.1859	0.0289
20000	10	0.0586	1.0000	0.0585	1.0000	0.1862	0.0176
5000	20	0.1192	1.0000	0.1193	1.0000	0.1844	0.0486
10000	20	0.0880	1.0000	0.0885	1.0000	0.1886	0.0277
20000	20	0.0615	1.0000	0.0616	1.0000	0.1851	0.0178

491 4.6 The initial estimation method

492 The initial estimates play a key role in determining the performance of our
493 distributed estimate. We investigate the sensitivity of our final estimate to the

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initialization by considering different initial estimates: (1) the LS lasso estimate based on the local sample at the central node; (2) the ℓ_1-regularized CQR estimate (CQR lasso); (3) the ℓ_1-regularized adaptive Huber regression estimate (Huber lasso, Sun et al., 2020; Wang et al., 2021); and (4) the perturbed true parameters with a normal noise, that is, $\widehat{\alpha}_k^{(0)} \sim \mathcal{N}(\alpha_k^*, \sigma^2)$, for $k = 1, \ldots, K$, and $\widehat{\beta}_j^{(0)} \sim I(\beta_j^* \neq 0) \cdot \mathcal{N}(\beta_j^*, \sigma^2)$, for $j = 1, \ldots, p$. We examine two noise levels, $\sigma = 0.05$ and $\sigma = 0.1$, for the last type of initialization. For the first and third types of initialization, we set $\widehat{\boldsymbol{\alpha}}^{(0)}$ as the empirical quantiles of the response at the central node. Because our algorithm may diverge without a carefully chosen initialization, we compare the estimates after only one iteration. We fix the overall sample size at $N = 5000$, the local sample size at $n = 500$, and the dimension at $p = 500$. The results are reported in Table 4.

Comparing the fourth type of initialization under different noise levels, we see that a more precise initial estimate yields a better distributed estimate. Comparing the first three types of initialization, we see that under a heavy-tailed noise, such as the Cauchy, the distributed estimate with the LS lasso initialization has the largest estimation error, which is likely caused by this "bad" initialization. Though the Huber lasso initialization mitigates this issue, its performance is not nearly as good as that of the CQR lasso initialization. Moreover, the latter gives stable estimates under all types of noise. Therefore, we suggest using the CQR lasso

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initialization to handle arbitrary noise.

Table 4: The estimation errors and F_1-scores of the distributed estimates under different types of initialization when the noise comes from the normal, Student's $t(3)$, and Cauchy distributions. The overall sample size is $N = 5000$, the local sample size is $n = 500$, and the dimension is $p = 500$.

Initialization	Normal noise		Student's $t(3)$ noise		Cauchy noise	
	est. error	F_1-score	est. error	F_1-score	est. error	F_1-score
LS lasso	0.1149	1.0000	0.1534	1.0000	1.0561	0.9986
CQR lasso	0.1220	1.0000	0.1472	1.0000	0.1993	1.0000
Huber lasso	0.1145	1.0000	0.1500	1.0000	0.6364	0.9986
$\sigma = 0.05$	0.1179	1.0000	0.1424	1.0000	0.1973	1.0000
$\sigma = 0.1$	0.1318	1.0000	0.1466	1.0000	0.2108	1.0000

4.7 Computational efficiency

We investigate the computational efficiency of our distributed algorithm by comparing the timing with that of competing methods. In addition to the pooled and divide-and-conquer estimates, we include the ℓ_1-regularized CQR estimate based on the pooled data in the comparison. We fix the local sample size at $n = 500$, and vary the overall sample size N. The estimation errors, F_1-scores, and wall times of the four methods are reported in Table 5. It can be seen that our distributed estimate is computationally much more efficient than the single-node ℓ_1-regularized CQR, while exhibiting similar performance.

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Table 5: The estimation errors, F_1-scores, and wall times of the distributed, pooled, divide-and-conquer, and single-node ℓ_1-regularized CQR estimates under varying sample size N and the Cauchy noise. The local sample size is fixed at $n = 500$.

N	distr			pooled		
	est. error	F_1-score	Time	est. error	F_1-score	Time
5000	0.1302	1.0000	7.5889	0.1310	1.0000	7.8213
10000	0.0959	1.0000	9.1686	0.0962	1.0000	15.1164
15000	0.0737	1.0000	9.2264	0.0740	1.0000	19.6047
N	dc			ℓ_1-regularized CQR		
	est. error	F_1-score	Time	est. error	F_1-score	Time
5000	0.1872	0.0519	19.0113	0.0423	0.8109	9.4916
10000	0.1866	0.0280	22.9009	0.0339	0.8609	17.8304
15000	0.1832	0.0213	24.1322	0.0357	0.8387	24.7910

5. Conclusion

We have developed a distributed algorithm for the penalized CQR by transforming the highly nonsmooth CQR problem into an ordinary LS, which facilitates both computational and theoretical developments. We have proposed a communication-efficient distributed implementation of the transformed problem that communicates gradient information only. Note that our distributed algorithm assumes a centralized system, so the local workers are idle when the central node executes the optimization. Future work should consider a decentralized distributed algorithm that uses all of the system's computing power.

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Supplementary Material

The online Supplementary Material contains proofs of all our theoretical results, as well as some additional simulations.

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