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Consistency of BIC Model Averaging

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Abstract: BIC weighting is frequently applied to high-dimensional linear regressions when model averaging is used to address model selection uncertainty. It also plays a central role in model selection diagnostics. However, little research has been done on its consistency or weak consistency, which are crucial properties of model averaging methods. In addition, previous works on model averaging consistency do not consider categorical covariates. As such, with both continuous covariates and categorical predictors (with possibly diverging numbers of levels) allowed, we establish both the consistency and the weak consistency of BIC weighting.

Key words and phrases: BIC-p weighting, Categorical predictors, Consistency, Weak consistency.

1. Introduction

Model averaging is an alternative approach to mitigating model selection uncertainty by weighting estimators across some models. Various model averaging approaches have been proposed; see Buckland et al. (1997),

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Yang (2001), Hjort and Claeskens (2003), Leung and Barron (2006), Hansen (2007), and Zhang et al. (2020), and the references therein.

However, to the best of our knowledge, these previous results focus on estimation accuracy. Little has been done formally on the consistency of model averaging weighting for general high-dimensional linear modeling. Note that model averaging based on consistent model selection criteria does not necessarily lead to consistent weighting. Lai et al. (2015), as an exception, derived the consistency of the generalized fiducial probabilities for candidate models in the absence of categorical predictors.

Note that the consistency of the weighting plays a central role in some important applications. For instance, it provides a theoretical guarantee for assessing variable selection performance in model selection diagnostics (see Nan and Yang (2014) and Yu et al. (2020)) and measuring variable importance (see Ye et al. (2018)). Thus, it is essential to establish this consistency for successful model selection diagnostics. Given this background, and focusing on a high-dimensional BIC (BIC-p) with a sparsity-oriented prior on the models, we derive the consistency of BIC-p weighting and provide theoretical support for previous work in the literature. Detailed proofs of the theorems are provided in the Supplementary Material.

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2. Main Results

For a linear regression model with both categorical and continuous predictors, we assume, without loss of generality, that among the p predictors $\{X_1, \ldots, X_p\}$, the first q, $\{X_1, \ldots, X_q\}$, are categorical, while the others are continuous. The categorical levels of $\{X_1, \ldots, X_q\}$ are denoted by $\{J_1, \ldots, J_q\}$, respectively. For each categorical variable X_i, we define dummy variables $X_{i,j}$ pertaining to the jth categorical level, for $j = 1, \ldots, J_i - 1$, and put $X_{\mathcal{I}_i} = (X_{i,1}, \ldots, X_{i,J_i-1})^{\mathrm{T}}$ with $\mathcal{I}_i \overset{\mathrm{def}}{=} \{(i,1), \ldots, (i, J_i - 1)\}$ in the regression. In a similar fashion, put $X_{\mathcal{I}_i} = X_i$ with $\mathcal{I}_i \overset{\mathrm{def}}{=} \{i\}$ for each continuous predictor X_i. Given observations $\{y_i, x_i\}_{i=1}^n$ with $x_i = (x_{i,\mathcal{I}_1}^{\mathrm{T}}, \ldots, x_{i,\mathcal{I}_p}^{\mathrm{T}})^{\mathrm{T}}$, where $x_{i,\mathcal{I}_j}$ is the ith sample of $X_{\mathcal{I}_j}$, the linear regression model is written in matrix form as

$$Y = \beta_0 + X\beta + \epsilon, \tag{2.1}$$

where $Y = (y_1, \ldots, y_n)^{\mathrm{T}}$ is an n-dimensional response vector, $X = (x_1, \ldots, x_n)^{\mathrm{T}}$ is a covariate matrix, $\beta = (\beta_{\mathcal{I}_1}^{\mathrm{T}}, \ldots, \beta_{\mathcal{I}_p}^{\mathrm{T}})^{\mathrm{T}}$ is a parameter vector of size $p^* = \sum_{i=1}^q J_i + p - 2q$, $\beta_{\mathcal{I}_i} = (\beta_{i,1}, \ldots, \beta_{i,J_i-1})^{\mathrm{T}}$ for $i = 1, \ldots, q$ and $\beta_{\mathcal{I}_i} = \beta_i$ for $i = q+1, \ldots, p$, and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^{\mathrm{T}} \sim N(0, \sigma^2 I_n)$, where I_n is the $n \times n$ identity matrix.

For $N > 1$, let $\mathcal{M} \overset{\text{def}}{=} \{M_i, i = 1, \dots, N\}$ be a candidate model set, where $M_i = \bigcup_{j \in \mathcal{A}_i} \mathcal{I}_j$, $\mathcal{A}_i \subset \{1, \dots, p\}$. Let $\| \cdot \|_2$ be the l_2-norm and denote by $|\cdot|$ the number of elements of a set. For the linear regression model (2.1), the index set of true variables is defined as $M_0 \overset{\text{def}}{=} \bigcup_{i=1}^{p} \mathcal{I}_i^0$ with $\mathcal{I}_i^0 = \mathcal{I}_i$ for $\beta_{\mathcal{I}_i} \neq 0$, and $\mathcal{I}_i^0 = \emptyset$ otherwise. Note that under the sparsity assumption, that is, $|M_0| \ll n$, the number of continuous variables in the true model can increase to infinity with n. This also applies to the numbers of categorical variables and the number of levels of each categorical variable.

Throughout, we assume that $|M_0| \log p^* = o(n)$ and $p^* \to \infty$. Define $\mathcal{M} \overset{\text{def}}{=} \{M_i : |M_i| \leq (p^*)^\alpha \wedge (Cn/\log p^*) \text{ and } i \in \{1, \dots, N\}\}$, for some constants $C > 0$ and $0 < \alpha < 1$, such that $|M_0| = o((p^*)^\alpha)$, where $a \wedge b \overset{\text{def}}{=} \min\{a, b\}$. Note that the condition $|M| \leq (p^*)^\alpha \wedge (Cn/\log p^*)$ for $M \in \mathcal{M}$ is much weaker than the condition $|M| \leq k|M_0|$, for some $k > 1$, which was assumed by Chen and Chen (2008), Luo and Chen (2013), and Lai et al. (2015), among others. To ensure that any finitely many categorical variables can be included in our candidate model set, we assume $\max\{J_i : 1 \leq i \leq q \text{ and } \mathcal{I}_i \not\subset M_0\} = o((p^*)^\alpha \wedge (n/\log p^*))$.

For each element M in $\mathcal{M}$, we calculate the corresponding weight w_M using the BIC-p weighting method. Let $RSS_M \overset{\text{def}}{=} \|Y - \hat{\beta}_0 - X_M \hat{\beta}_M\|_2^2$ be the residual sum of squares of the model M, where X_M denotes an $n \times |M|$

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submatrix of the design matrix X, and $\hat{\beta}_0$ and $\hat{\beta}_M$ are the corresponding least squares estimators. Let $I_M \overset{\text{def}}{=} n\log(RSS_M) + |M|\log n - n\log n$. Following Nan and Yang (2014), the BIC-p weight w_M is defined as

$$w_M \overset{\text{def}}{=} \exp\Big(-\frac{I_M}{2} - \psi C_M\Big) / \sum_{M' \in \mathcal{M}} \exp\Big(-\frac{I_{M'}}{2} - \psi C_{M'}\Big), \tag{2.2}$$

where $C_M = |M|\log(e \cdot p^*/|M|) + 2\log(|M|+2)$ and $\psi > 0$ is a constant.

For ease of notation, let $w_i \triangleq w_{M_i}$, for $M_i \in \mathcal{M}$. Given the candidate models $\mathcal{M}$ and a weighting $w = \{w_i, i = 1, ..., N\}$, we define *weight concentration index* (WCI) as $WCI(w) = \sum_{i=1}^N w_i |M_i \nabla M_0|$, where ∇ denotes the symmetric difference of two sets. Clearly, when WCI is close to zero, the weights of the candidate models are concentrated well around the true model. Based on $WCI(w)$, we define consistency and weak consistency as follows.

Definition 1. The weighting w is consistent if

$$WCI(w) \xrightarrow{P} 0, \quad \text{as } n \to \infty,$$

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and the weighting is weakly consistent if

$$\frac{WCI(w)}{|M_0|} \xrightarrow{P} 0, \qquad \text{as } n \to \infty.$$

For the theorems below, $\mathcal{M}$ is assumed to contain the true model, and can be up to the collection of all subset models. Conditions 1−3 are required for consistency.

Condition 1. All levels of each categorical variable are observed and the ratio of the most frequent levels to the least frequent levels is bounded by some constant.

Condition 2. $\min_{i\in\{1,\ldots,p\}}\{\|\beta_{\mathcal{I}_i^0}\|_2^2 : \mathcal{I}_i^0 \neq \emptyset\} \geq c_1\left(|M_0|\log\left(p^*\right)/n\right)^{\kappa}$, where $c_1 > 0$, $\kappa = 1-\varepsilon$ and ε is an arbitrarily small positive constant.

Condition 3. Let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalues, respectively. Then, for all M such that $|M| \leq k|M_0|$,

$$0 < c_{\min} \leq \lambda_{\min}\left(\frac{1}{n}X_M^{\mathrm{T}}X_M\right) \leq \lambda_{\max}\left(\frac{1}{n}X_M^{\mathrm{T}}X_M\right) \leq c_{\max} < \infty,$$

for some fixed $k > 1$.

Condition 1 excludes the case of an extremely unbalanced design matrix. Condition 2 requires that the minimum of the l_2-norms of the coeffi-

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cients of both the grouped dummy variables and the continuous variables in the true model are not too small. Note that we impose a restriction on the ith group effect of $\beta_{\mathcal{I}_i^0}$ rather than the individual contribution of $\beta_{i,j}$, for $j = 1, \ldots, J_i - 1$. Therefore, for the true grouped dummy variables, some or even most individual effects can be very small. Condition 3 is the sparse Riesz condition, which is a commonly used regularity condition for $p \gg n$ (see Zhang and Huang (2008); Lai et al. (2015)).

Theorem 1. *Under Conditions 1–3,* $\log(|M_0|)/\log p^* \leq \delta < \alpha$ *and* $\log n$ $/\log p^* \leq \eta$*, for some positive constants* δ, α, *and* η, *if* $\psi > (2C(k-1)((\alpha\wedge\eta)-1))^{-1}k\log(1-4C(1+(\alpha\wedge\eta))) + (k/(k-1) - (\alpha\wedge\eta)/2)/(1-(\alpha\wedge\eta))$, *for some* $C \in (0, 1/(4(1+(\alpha\wedge\eta))))$, *then we have*

$$\max_{M\in\mathcal{M}, M\neq M_0} \frac{w_M}{w_{M_0}} \xrightarrow{P} 0, \qquad \textit{as } n \to \infty. \tag{2.3}$$

Furthermore, the weighting is consistent; that is,

$$WCI(w) \xrightarrow{P} 0, \qquad \textit{as } n \to \infty. \tag{2.4}$$

Note that the lower bound on ψ in Theorem 1 is always a positive constant, because $\alpha \in (0,1)$ and $C \in (0, 1/(4(1+(\alpha\wedge\eta))))$. Theorem 1

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states that the weight w_{M_0} of the true model tends to one as $n \to \infty$.

Typically, there may be some relatively small coefficients in the true (or best) model that violate Condition 2. For $i = 1, \ldots, p$ and a given arbitrary constant $c_2 > 0$, we define

$$\mathcal{I}_i^S \overset{\text{def}}{=} \begin{cases} \mathcal{I}_i^0 & \text{if } \mathcal{I}_i^0 \neq \emptyset \text{ and } \|\beta_{\mathcal{I}_i^0}\|_2^2/|\mathcal{I}_i^0| < c_2|M_0|\log{(p^*)}/n, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $M_0^S \overset{\text{def}}{=} \bigcup_{i=1}^p \mathcal{I}_i^S$ denote the set with indices of smaller coefficients. Note that we allow the l_2-norms of the coefficients of the variables in the set M_0^S to be arbitrarily small, but the number of these variables should be limited. Thus, a condition required for the weak consistency of BIC-p weighting is stated as follows.

Condition 4. $|M_0^S|/|M_0| \leq \xi_n$, where $\{\xi_n\}$ is a nonnegative sequence converging to zero as $n \to \infty$.

Theorem 2. *Under Conditions 1 and 3–4,* $\log{(|M_0|)}/\log p^* \leq \delta < \alpha$*, and* $\log n/\log p^* \leq \eta$*, for some positive constants* δ*,* α*, and* η*, if* $\psi > (2C(k-1)((\alpha\wedge\eta)-1))^{-1}k\log(1-4C(1+(\alpha\wedge\eta)))+(k/(k-1)-(\alpha\wedge\eta)/2)/(1-(\alpha\wedge\eta))$,

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for some $C \in (0, 1/(4(1 + (\alpha \wedge \eta))))$*, then* w *is weakly consistent; that is,*

$$\frac{WCI(w)}{|M_0|} \xrightarrow{P} 0, \qquad \textit{as } n \to \infty. \tag{2.5}$$

Not surprisingly, the weak consistency requires milder conditions that are much more realistic practice.

Supplementary Material

The proofs of Theorems 1 and Theorem 2 are provided in the online Supplementary Material.

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