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| **Complete List of Authors** | Long Feng,<br>Binghui Liu and<br>Yanyuan Ma |
| **Corresponding Author** | Binghui Liu |
| **E-mail** | liubh100@nenu.edu.cn |

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# HIGH-DIMENSIONAL ALPHA TEST OF LINEAR FACTOR PRICING MODELS WITH HEAVY-TAILED DISTRIBUTIONS

Binghui Liu[1], Long Feng[2,*] and Yanyuan Ma[3]

[1]*Northeast Normal University,* [2]*Nankai University and* [3]*Pennsylvania State University*

*Abstract:* We consider the problem of testing the presence of alpha in linear factor pricing models. We propose a robust spatial sign-based nonparametric test that simultaneously alleviates two prominent difficulties encountered by most existing methods, namely, those caused by the high dimensionality of the securities and the departure from normality of the distributions. We rigorously show that the proposed test has desired theoretical properties and demonstrate its superior performance using Monte Carlo experiments. These results are established when the number of securities is larger than the time dimension of the return series and the distribution of the securities belongs to the family of elliptically symmetric distributions, which extends the normal distribution to many well-known heavy-tailed distributions. We apply the proposed test to the monthly returns on securities in stock markets, showing that it outperforms existing tests.

*Key words and phrases:* Elliptical symmetric distribution, high dimensionality, Linear Factor Pricing Models, robust test for alpha, S&P's 500 securities.

*Corresponding author.

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## 1. Introduction

This study focuses on testing the presence of alpha in linear factor pricing models (LFPMs) when the number of securities is large relative to the time dimension of the return series.

### 1.1 LFPM

The LFPM is one of the most fundamental tools in finance. Motivated by arbitrage pricing theory (APT) (Ross, 1976), LFPMs explain how security returns are related to market factors. Special forms of the LFPM include the well-known single-factor model, that is, the Sharpe–Lintner capital asset pricing model (CAPM) (Sharpe, 1964; Lintner, 1965), Fama–French three-factor model (Fama and French, 1993), and Fama–French five-factor model (Fama and French, 2015). Typically, each factor in these models has significant economic meaning and pricing ability.

Let $N$ be the number of securities, and $T$ be the time dimension of the return series of each security. Let the $t$th excess return of the $i$th security be $Y_{it}$. The LFPM takes the form

$$Y_{it} = \alpha_i + \boldsymbol{\beta}_i^\top \mathbf{f}_t + \epsilon_{it}, \tag{1.1}$$

for $i = 1, \ldots, N$, $t = 1, \ldots, T$, where $\mathbf{f}_t \equiv (f_{t1}, \ldots, f_{tp})^\top \in \mathcal{R}^p$ contains $p$ economic factors at time $t$, $\alpha_i$ is a scalar representing the security-specific intercept, $\boldsymbol{\beta}_i \equiv (\beta_{i1}, \cdots, \beta_{ip})^\top \in \mathcal{R}^p$ is a vector of multiple regression betas with respect to the $p$ factors, and $\epsilon_{it}$ is the

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corresponding idiosyncratic error term. The LFPM seeks to explain differences in expected security returns in terms of how security betas interact with systematic economic factors. Specifically, it predicts a security-specific linear relationship between the expected security return and the economic factors. This linear framework has great intuitive appeal and important practical advantage of simplifying the modeling of security returns, as well as playing a central role in modern theories of security pricing (Zhou, 1993).

The intercept term $\alpha_i$ in (1.1) captures the excessive return of the $i$th security. That is, other than the return associated with overall market factors, some securities may have systematic positive or negative returns due to characteristics of the individual securities, termed excess returns. Thus, testing $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_N)^\top = \mathbf{0}$ allows us to test whether an excess return of a market portfolio exists. In addition, the test has a special meaning in a CAPM, and is also called the test of the mean-variance efficiency (Gibbons et al., 1989).

### 1.2 Feature and utility of our test

In this work, we devise a novel procedure to test the presence of alpha in an LFPM. The test is applicable when the number of securities $N$ is much larger than the time dimension $T$, which is particularly relevant in modern finance, owing to the large number of securities on the market. The test is also applicable when the idiosyncratic errors of the LFPM follow an elliptically symmetric distribution, or are symmetric and independent

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between securities. These distributions are much more general than the familiar normal form, and include many heavy-tailed distributions. This is an important property, because departures from normality in such data tend to be the norm rather than exceptions (Mandelbrot, 1963). The features of our test are established both theoretically and empirically, and the test is shown to be especially beneficial when the number of securities is much larger than the time dimension and the departure from normality is severe.

Our test is a spatial sign-based procedure that belongs to a nonparametric testing framework that often leads to robustness to departures from normality (Oja, 2010). Numerous high-dimensional nonparametric testing methods have been developed, including the high-dimensional multivariate sign- and rank-based methods (Zou et al., 2014; Wang et al., 2015; Guo and Chen, 2016; Paindaveine and Verdebout, 2016; Feng et al., 2016; Feng and Liu, 2017), designed to perform the simple task of testing whether a very long mean vector is zero. However, the test for the presence of alpha examined here is different and much more complex. Although it can be regarded as an extension of the mean test that includes additive factors, the high-dimensionality of the securities together with the additive factors causes much complexity, especially in the case of departures from normality. To the best of our knowledge, there are currently no high-dimensional nonparametric testing methods for LFPMs.

### 1.3 Literature review

The test of alpha in LFPMs has received much attention in the econometrics literature. Early works on LFPMs routinely assume the normality of security returns (Gibbons et al., 1989; MacKinlay and Richardson, 1991; Zhou, 1993). Unfortunately, there is strong evidence suggesting that the normality assumption is not always appropriate for security returns in practice. In fact, numerous studies, since that of Mandelbrot (1963), have documented the heavy tailness in security return distributions relative to the normal. Such tail thickness is associated with the tendency of the security returns to take values of extremely large magnitude with nonnegligible probability. If the sample nonnormalities are severe, the size and/or power of the tests based on the normality assumption may be seriously mismeasured, owing to the sensitivity of these methods to the normality assumption (John and MacDonald, 2012).

This has prompted the econometrics community to derive additional test procedures of alpha in LFPMs, and to study their limiting distributions without the normality assumption. Indeed, several multivariate tests for LFPMs have been proposed that are robust to departures from normality, including parametric procedures based on postulating a nonnormal distribution (Zhou, 1993), semiparametric asymptotic procedures specific to elliptical distributions (Hodgson et al., 2002), nonnormal Bayesian procedures (Tu and Zhou, 2004), and tests using large-sample generalized method of moments (GMM) or bootstrap techniques (John and MacDonald, 2012).

To model the heavy tail property of security returns, an appropriate distribution family popular in the literatue is the elliptical distribution family. In fact, Chamberlain (1983) showed that a mean-variance analysis of the CAPM is consistent with investors' portfolio decision-making if and only if the returns are elliptically distributed. Moreover, in the case of elliptical returns, the CAPM remains valid theoretically. Hodgson et al. (2002) state that it is important to test security-pricing models for the case when the returns are elliptically distributed. Indeed, the elliptical family contains not only the normal distribution, but also many well-known heavy-tailed distributions, including Student's t-distribution as well as the logistic, contaminated normal, and power exponential distributions, among others. Hence, it offers a more flexible framework for modeling security prices or returns. Hodgson et al. (2002) proposed exact tests for both the case in which the returns are elliptically distributed and the case in which the error terms are elliptically distributed. To assess whether the returns are elliptically distributed, they further provide exact tests based on measures of multivariate skewness and kurtosis, complementing studies on the distributional properties of security returns. Beaulieu et al. (2007) developed exact mean-variance efficiency tests for market portfolios in the context of a CAPM, accommodating a wide class of possibly nonnormal error distributions.

In modern financial markets, thousands of securities are traded every day, which

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naturally raises the issue of testing when $N$ is large or diverges to infinity. In these situations, it is unrealistic to require $T$ to increase with the number of securities, because a large enough $T$ is likely to increase the possibility of structural changes in the factor loadings, which may destroy the identical distribution assumption of the factors over time periods, as is commonly assumed in LFPMs. Owing to these difficulties, none of the aforementioned tests are applicable (Pesaran and Yamagata, 2012). To perform a test for alpha when $N > T$, modern procedures include those of Pesaran and Yamagata (2012), Fan et al. (2015), Gagliardini et al. (2016), and Pesaran and Yamagata (2017). However, a common drawback of these methods is their inability to handle many well-known heavy-tailed distributions, such as the multivariate Student's t-distribution and a mixture of multivariate normal distributions, which are commonly used to model securities. Some of these methods also impose additional assumptions on either the error structure or the loadings in an LFPM. Note that Pesaran and Yamagata (2012, 2017) impose moment conditions only on the distributions of the errors, which enables us to apply their tests to certain nonnormal data. However, as mentioned in Zou et al. (2014), the error vector from a multivariate t-distribution or from a mixture of multivariate normal distributions does not satisfy these moment conditions. In addition, in our numerical experiments, we find that the statistical performance of these tests deteriorates quickly when the nonnormality is severe, especially for heavy-tailed distributions; see Section 4.

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The rest of the paper is organized as follows. Section 2 describes the proposed nonparametric testing procedure, and explains the formulation of the test and its advantages. The theoretical results of the test, including the limiting null distribution, power performance under the local alternative, and asymptotic relative efficiency, are established in Section 3. Monte Carlo experiment results are presented in Section 4 to evaluate the finite-sample performance of the proposed testing method compared with that of its main competitors. Finally, we conclude the paper in Section 5. Extensive simulation results and all technical proofs are relegated to the Supplementary Material, where we also showcase the superior performance of the test by analyzing stock return data from the Standard & Poor's 500 and CSI 300 indices.

### 1.4 Notation

Let $\mathbf{Y}_{i\cdot} = (Y_{i1}, \cdots, Y_{iT})^\top \in \mathcal{R}^T$ and $\boldsymbol{\epsilon}_{i\cdot} = (\epsilon_{i1}, \cdots, \epsilon_{iT})^\top \in \mathcal{R}^T$, for each $i = 1, \cdots, N$, $\mathbf{Y}_t = (Y_{1t}, \cdots, Y_{Nt})^\top \in \mathcal{R}^N$ and $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \cdots, \epsilon_{Nt})^\top \in \mathcal{R}^N$, for each $t = 1, \cdots, T$, and $\mathbf{Y} = (\mathbf{Y}_{1\cdot}, \cdots, \mathbf{Y}_{N\cdot}) \in \mathcal{R}^{T\times N}$, $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_{1\cdot}, \cdots, \boldsymbol{\epsilon}_{N\cdot}) \in \mathcal{R}^{T\times N}$, $\mathbf{B} = (\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_N)^\top \in \mathcal{R}^{N\times p}$, $\mathbf{F} = (\mathbf{f}_1, \cdots, \mathbf{f}_T)^\top \in \mathcal{R}^{T\times p}$, and $\mathbf{1}_T = (1, \cdots, 1)^\top \in \mathcal{R}^T$. $\mathbf{I}_T$ is the $T \times T$ identity matrix. Let $\mathbf{M}_\mathbf{F} = \mathbf{I}_T - \mathbf{F}(\mathbf{F}^\top\mathbf{F})^{-1}\mathbf{F}^\top$ and $\mathbf{h} = \mathbf{M}_\mathbf{F}\mathbf{1}_T$. For any vector $\mathbf{x}$, let $\|\mathbf{x}\|$ be the $l_2$ norm of $\mathbf{x}$, and let $U(\mathbf{x})$ be the spatial sign function defined as $U(\mathbf{x}) \equiv I(\mathbf{x} \neq \mathbf{0})\mathbf{x}/\|\mathbf{x}\|$. In fact, the spatial sign is a multivariate extension of the sign function. The function projects the

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vector onto a unit sphere and is related to global contrast normalization. Here, "spatial" reflects that the argument of the original "sign" function is one dimensional, whereas the argument of the "spatial sign" function is multidimensional.

Throughout this paper, we assume that $\boldsymbol{\epsilon}_t$ are independently and identically distributed (i.i.d.) from an $N$-variate mean-zero elliptical distribution with probability density function

$$\det(\boldsymbol{\Sigma})^{-1/2} g(\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}\|),\ \boldsymbol{\epsilon} \in \mathcal{R}^N, \tag{1.2}$$

where $\boldsymbol{\Sigma} \in \mathcal{R}^{N\times N}$ is a symmetric positive-definite scatter matrix and $g$ is a generator function. The elliptical distribution is characterized by the scatter matrix and the generator function.

## 2. Methodology

### 2.1 The proposed test

The proposed spatial sign test was motivated by the need to handle a large number of securities ($N > T$) and the need to accommodate distributions of the error $\boldsymbol{\epsilon}_t$ beyond normality.

A natural treatment of the nonnormal distribution is to resort to nonparametric testing procedures. To this end, following the sign-based test in Section 13.3 of Oja (2010), we first form $\widetilde{\mathbf{U}}_t = U\{\widetilde{\boldsymbol{\Sigma}}^{-1/2}(\mathbf{Y}_t - \widetilde{\mathbf{B}}\mathbf{f}_t)\}$, for $t = 1, \ldots, T$, and then construct a

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test statistic

$$Q_0 \equiv N\mathbf{h}^\top(\widetilde{\mathbf{U}}_1,\dots\widetilde{\mathbf{U}}_T)^\top(\widetilde{\mathbf{U}}_1,\dots\widetilde{\mathbf{U}}_T)\mathbf{h}/(\mathbf{h}^\top\mathbf{h}).$$

Note that $\mathbf{h}$ is defined in Section 1.4 and represents the residual of the projection of $\mathbf{F}$ to $\mathbf{1}_N$, and $\mathbf{h}(\mathbf{h}^\top\mathbf{h})^{-1}\mathbf{h}$ is the projection matrix onto the column space of $\mathbf{h}$. Furthermore, $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{\Sigma}}$ are the estimators of $\mathbf{B}$ and the scatter matrix of $\boldsymbol{\epsilon}_t$, $\mathbf{\Sigma}$, under $H_0$, respectively, and are thus required to satisfy

$$\sum_{t=1}^T \widetilde{\mathbf{\Sigma}}^{-1/2}U\{\widetilde{\mathbf{\Sigma}}^{-1/2}(\mathbf{Y}_t-\widetilde{\mathbf{B}}\mathbf{f}_t)\}\mathbf{f}_t^\top = \mathbf{0}, \tag{2.1}$$

$$\frac{1}{T}\sum_{t=1}^T U\{\widetilde{\mathbf{\Sigma}}^{-1/2}(\mathbf{Y}_t-\widetilde{\mathbf{B}}\mathbf{f}_t)\}U\{\widetilde{\mathbf{\Sigma}}^{-1/2}(\mathbf{Y}_t-\widetilde{\mathbf{B}}\mathbf{f}_t)\}^\top = \frac{1}{N}\mathbf{I}_N. \tag{2.2}$$

Note that (2.1) means that given $\widetilde{\mathbf{\Sigma}}$, $\widetilde{\mathbf{B}}$ should minimize the objective function $Q(\boldsymbol{B}) = \sum_{t=1}^T ||\widetilde{\mathbf{\Sigma}}^{-1/2}(\mathbf{Y}_t - \boldsymbol{B}\mathbf{f}_t)||$, where $||\cdot||$ denotes the $l_2$ norm. Here, (2.2) means that the sample spatial sign covariance matrix of $\widetilde{\boldsymbol{\epsilon}}_t = \widetilde{\mathbf{\Sigma}}^{-1/2}(\mathbf{Y}_t-\widetilde{\mathbf{B}}\mathbf{f}_t)$ should be $\frac{1}{N}\mathbf{I}_N$. These two equations are similar to the Hettmansperger–Randles estimators of the location vector and the scatter matrix (Hettmansperger and Randles, 2002).

Operationally, $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{\Sigma}}$ are obtained using the following iterative algorithm:

(i) $\widetilde{\boldsymbol{\epsilon}}_t \leftarrow \widetilde{\mathbf{\Sigma}}^{-1/2}(\mathbf{Y}_t-\widetilde{\mathbf{B}}\mathbf{f}_t),\ \ t=1,\cdots,T;$

(ii) $\widetilde{\mathbf{B}} \leftarrow \widetilde{\mathbf{B}} + \widetilde{\mathbf{\Sigma}}^{1/2}\left\{\sum_{t=1}^T U(\widetilde{\boldsymbol{\epsilon}}_t)\mathbf{f}_t^\top\right\}\left(\sum_{t=1}^T \mathbf{f}_t\mathbf{f}_t^\top/\|\widetilde{\boldsymbol{\epsilon}}_t\|\right)^{-1};$

(iii) $\widetilde{\mathbf{\Sigma}} \leftarrow N\widetilde{\mathbf{\Sigma}}^{1/2}\left\{T^{-1}\sum_{t=1}^T U(\widetilde{\boldsymbol{\epsilon}}_t)U(\widetilde{\boldsymbol{\epsilon}}_t)^\top\right\}\widetilde{\mathbf{\Sigma}}^{1/2}.$

Note that the operation $\widetilde{\boldsymbol{\Sigma}}^{-1/2}(\mathbf{Y}_t - \widetilde{\mathbf{B}}\mathbf{f}_t)$ is a centering and standardization step, which is performed inside the spatial-sign function $U(\cdot)$ to achieve test invariance under affine transformations, leading to the notion of "inner centering and standardization" frequently used to describe the test statistic. The main purpose of the spatial sign function $U(\cdot)$ is to weaken the normality condition required by other tests, such as Hotelling's test (Oja, 2010). After these standardization steps, each row of $(\widetilde{\mathbf{U}}_1, \ldots, \widetilde{\mathbf{U}}_T)$ is projected onto the linear space spanned by the residual of projecting $\mathbf{1}_p$ onto the space spanned by $\mathbf{f}_1, \ldots \mathbf{f}_T$. Obviously, this projection is what cannot be explained by the factors, and needs to be picked by an intercept. Hence, it may indicate that we need to include the intercept term. In the spatial sign test described above, the average norm squares of the residuals is used as a summary statistic to quantify the model's goodness-of-fit under $H_0$.

Unfortunately, despite its robustness against the normality of the error distribution, the above test statistic is not feasible when $N > T$, because the sample scatter matrix $\widetilde{\boldsymbol{\Sigma}}$ is not invertible. The singularity of $\widetilde{\boldsymbol{\Sigma}}$ is a direct consequence of the singularity of $T^{-1}\sum_{t=1}^{T} U(\widetilde{\boldsymbol{\epsilon}}_t)U(\widetilde{\boldsymbol{\epsilon}}_t)^\top$ in step (iii) above. To circumvent this problem, we replace $\widetilde{\mathbf{B}}$ with the least-square estimator $\widehat{\mathbf{B}} = (\widehat{\boldsymbol{\beta}}_1, \cdots, \widehat{\boldsymbol{\beta}}_N)^\top$, where, for each $i = 1, \ldots, N$, $\widehat{\boldsymbol{\beta}}_i \equiv (\mathbf{F}^\top\mathbf{F})^{-1}\mathbf{F}^\top\mathbf{Y}_{i\cdot}$ is the restricted OLS estimator of the slope coefficient $\boldsymbol{\beta}_i$ by letting $\alpha_i = 0$. Then, we replace the residual scatter matrix $\boldsymbol{\Sigma}$ with its diagonal matrix $\mathbf{D}$, and weaken

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the requirement on the estimator $\widehat{\mathbf{D}}$ of $\mathbf{D}$ to

$$\sum_{t=1}^{T} U\{\widehat{\mathbf{D}}^{-1/2}(\mathbf{Y}_t - \widehat{\mathbf{B}}\mathbf{f}_t)\}\mathbf{f}_t^\top = \mathbf{0},$$

$$\frac{1}{T}\sum_{t=1}^{T} \mathrm{diag}[U\{\widehat{\mathbf{D}}^{-1/2}(\mathbf{Y}_t - \widehat{\mathbf{B}}\mathbf{f}_t)\}U\{\widehat{\mathbf{D}}^{-1/2}(\mathbf{Y}_t - \widehat{\mathbf{B}}\mathbf{f}_t)\}^\top] = \frac{1}{N}\mathbf{I}_N.$$

Thus, the estimator $\widehat{\mathbf{D}}$ can be obtained using the following iterative algorithm:

(i) $\boldsymbol{\xi}_t \leftarrow \widehat{\mathbf{D}}^{-1/2}\widehat{\boldsymbol{\epsilon}}_t,\ \ t = 1, \cdots, T;$

(ii) $\widehat{\mathbf{D}} \leftarrow N\widehat{\mathbf{D}}^{1/2}\mathrm{diag}\{T^{-1}\sum_{t=1}^{T} U(\boldsymbol{\xi}_t)U(\boldsymbol{\xi}_t)^\top\}\widehat{\mathbf{D}}^{1/2}.$

Note that $\widetilde{\mathbf{B}}$ is an estimator of $\mathbf{B}$ obtained by assuming that $\boldsymbol{\epsilon}_t$ follows an elliptical distribution. In contrast, the least-square estimator $\widehat{\mathbf{B}}$ assumes that $\boldsymbol{\epsilon}_t$ follows a multivariate normal distribution. The most significant difference between the two is that in the high-dimensional case, $\widehat{\mathbf{B}}$ is available, but $\widetilde{\mathbf{B}}$ is not. Furthermore, even when $\boldsymbol{\Sigma}$ is known, and hence $\widetilde{\mathbf{B}}$ becomes available, there is still a difference. Specifically, $\widehat{\mathbf{B}}$ is an unbiased estimator of $\mathbf{B}$ under $H_0$, whereas $\widetilde{\mathbf{B}}$ obtained from (2.1) and (2.2) is not unbiased, as mentioned in Section 13.3 of Oja (2010). In particular, when the dimension is very large, there is a nonnegligible bias term in the proposed test statistic if we replace $\widehat{\mathbf{B}}$ with $\widetilde{\mathbf{B}}$.

After obtaining $\widehat{\mathbf{B}}$ and $\widehat{\mathbf{D}}$, we can form $\widehat{\mathbf{U}}_t = U\{\widehat{\mathbf{D}}^{-1/2}(\mathbf{Y}_t - \widehat{\mathbf{B}}\mathbf{f}_t)\}$ and construct the test statistic similarly as before, yielding

$$Q_1 \equiv N\mathbf{h}^\top(\widehat{\mathbf{U}}_1, \ldots \widehat{\mathbf{U}}_T)^\top(\widehat{\mathbf{U}}_1, \ldots \widehat{\mathbf{U}}_T)\mathbf{h}/(\mathbf{h}^\top\mathbf{h})$$

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$$
\begin{aligned}
&= N(\mathbf{h}^\top \mathbf{h})^{-1} \sum_{t_1,t_2=1,t_1\neq t_2}^{T} h_{t_1} h_{t_2} \widehat{\mathbf{U}}_{t_1}^\top \widehat{\mathbf{U}}_{t_2} + N \\
&= Q + N.
\end{aligned}
$$

What we find is that although the above test statistic can be applied to handle the case of $N > T$, its performance gradually deteriorates as $N$ increases. For example, when $N > T^2$, the performance is not satisfactory, owing to the hidden bias in $Q$. The bias has two main sources. First, when constructing $\widehat{\mathbf{U}}_t = U\{\widehat{\mathbf{D}}^{-1/2}(\mathbf{Y}_t - \widehat{\mathbf{B}}\mathbf{f}_t)\}$, we used the diagonal matrix $\widehat{\mathbf{D}}$ to replace $\widehat{\boldsymbol{\Sigma}}$. This may incur an $O_p(\sqrt{N}/T)$ bias, which may be nonignorable in practice. This phenomenon is observed in other testing problems as well; see, for example, Srivastava et al. (2013) and Feng et al. (2015). Second, compared with the problem without factors, an additional operation is needed to estimate $\mathbf{B}$, and replacing $\mathbf{B}$ with $\widehat{\mathbf{B}}$ contributes another source of bias, because both $\widehat{\mathbf{D}}^{-1/2}$ and $\mathbf{Y}_t - \widehat{\mathbf{B}}\mathbf{f}_t$ contain $\widehat{\mathbf{B}}$. The two sources of bias cannot be extracted and analyzed separately.

Hence, we correct the bias using a bootstrap, leading to the following spatial sign-based test statistic:

$$
T_{SS} = \frac{Q - \delta_Q}{\sqrt{2\widehat{\mathrm{tr}(\mathbf{R}^2)}}}, \tag{2.3}
$$

where

$$
Q = N(\mathbf{h}^\top \mathbf{h})^{-1} \sum_{t_1,t_2=1,t_1\neq t_2}^{T} h_{t_1} h_{t_2} \widehat{\mathbf{U}}_{t_1}^\top \widehat{\mathbf{U}}_{t_2}, \tag{2.4}
$$

and for any $t = 1, \ldots, T$, $h_t$ is the $t$th element of $\mathbf{h}$, $\widehat{\mathbf{U}}_t = U\{\widehat{\mathbf{D}}^{-1/2}(\mathbf{Y}_t - \widehat{\mathbf{B}}\mathbf{f}_t)\}$. Here, $\mathbf{R} = \mathbf{D}^{-1/2}\boldsymbol{\Sigma}\mathbf{D}^{-1/2}$ is the correlation matrix, and

$$\widehat{\mathrm{tr}(\mathbf{R}^2)} \equiv \frac{N^2}{\mathbf{h}^\top\mathbf{h}(\mathbf{h}^\top\mathbf{h}-1)} \sum_{t_1,t_2=1,t_1\neq t_2}^{T} h_{t_1}^2 h_{t_2}^2 \left\{U(\widehat{\mathbf{D}}^{-1/2}\widetilde{\boldsymbol{\epsilon}}_{t_1}^{(t_1,t_2)})^\top U(\widehat{\mathbf{D}}^{-1/2}\widetilde{\boldsymbol{\epsilon}}_{t_2}^{(t_1,t_2)})\right\}^2,$$

where $\widetilde{\boldsymbol{\epsilon}}_{t_1}^{(t_1,t_2)} = \mathbf{Y}_{t_1} - \widehat{\mathbf{B}}_{t_1}^{(t_1 t_2)}\mathbf{f}_{t_1}$, $\widetilde{\boldsymbol{\epsilon}}_{t_2}^{(t_1,t_2)} = \mathbf{Y}_{t_2} - \widehat{\mathbf{B}}_{t_2}^{(t_1 t_2)}\mathbf{f}_{t_2}$, and $\widehat{\mathbf{B}}_{t_1}^{(t_1 t_2)}$ and $\widehat{\mathbf{B}}_{t_2}^{(t_1 t_2)}$ are the least-square estimators of $\mathbf{B}$ based on the first half and second half of the sample $\{(\mathbf{Y}_t, \mathbf{f}_t)\}_{t\neq t_1,t_2}$, respectively. Let $\sigma_T = \sqrt{2\mathrm{tr}(\mathbf{R}^2)}$. As presented in the following theoretical analysis, the variance of $Q$ is asymptotically described by $\sigma_T^2$ under some specific conditions. The construction of $\widehat{\mathrm{tr}(\mathbf{R}^2)}$ follows the estimation of the asymptotic variance in Feng et al. (2016).

Here, $T_{SS}$ is a standardized version of $Q$, $Q$ is a condensed version of $Q_1$, and $Q_1$ is a high-dimensional substitute for $Q_0$. Note that $Q_0$ is the traditional spatial sign test statistic for low-dimensional data proposed by Oja (2010). Intuitively, $Q$ is composed of the sum of the weighted inner products of pairs of the spatial sign vectors $(\widehat{\mathbf{U}}_{t_1}, \widehat{\mathbf{U}}_{t_2})$, with weights $h_{t_1}h_{t_2}$, where we use the leave-out strategy, that is, we exclude the diagonal elements $h_t^2\widehat{\mathbf{U}}_t^\top\widehat{\mathbf{U}}_t$. In particular, for each $\widehat{\mathbf{U}}_t$, the diagonal matrix $\widehat{\mathbf{D}}^{-1/2}$ is multiplied to scale $\mathbf{Y}_t - \widehat{\mathbf{B}}\mathbf{f}_t$. Then, $\widehat{\mathbf{U}}_t$ extracts the direction of the resulting vector $\widehat{\mathbf{D}}^{-1/2}(\mathbf{Y}_t - \widehat{\mathbf{B}}\mathbf{f}_t)$.

We have not explained $\delta_Q$ in (2.3). Indeed, $\delta_Q$ is an assessment of $E(Q)$, and can be ignored in theory, because $E(Q) = o\{\sqrt{2\mathrm{tr}(\mathbf{R}^2)}\}$ when $\min(T, N) \to \infty$, as shown in Theorem 1. However, to ensure the test precision for all relative sizes of $T$ and $N$, we suggest setting $\delta_Q = \widehat{E}(Q)$, instead of ignoring it. To this end, a practical approach is to employ a bootstrap procedure. Specifically, let $\breve{\boldsymbol{\epsilon}}_i = \mathbf{Y}_i - \breve{\alpha}_i - \breve{\boldsymbol{\beta}}_i^\top \mathbf{f}_t$, where $(\breve{\alpha}_i, \breve{\boldsymbol{\beta}}_i)$ is the least-square estimator of $(\alpha_i, \boldsymbol{\beta}_i)$ in model (1.1). Similarly to the above restricted OLS estimator $\widehat{\boldsymbol{\beta}}_i$, we use the normal OLS estimator $\breve{\boldsymbol{\beta}}_i$ because it is an unbiased estimator of $\boldsymbol{\beta}_i$. We randomly generate a Rademacher variable $\eta_{it}^*$ and form

$$\mathbf{Y}_{it}^* = \widehat{\boldsymbol{\beta}}_i^\top \mathbf{f}_t + \breve{\boldsymbol{\epsilon}}_{it} \eta_{it}^*.$$

Then, the bootstrap test statistic $Q^*$ described in (2.4) can be computed based on the bootstrap sample $(\mathbf{Y}_{it}^*, \mathbf{f}_t)$, for $i = 1, \cdots, N$ and $t = 1, \cdots, T$. We repeat the sampling procedure $B = 100$ times, and use the average $Q^*$ as $\delta_Q$.

We establish that $T_{SS}$ has a standard normal distribution when $\min(N, T) \to \infty$. Hence, we perform a level-$\alpha$ test by rejecting $H_0$ when $T_{SS}$ is larger than the $(1-\alpha)$ quantile of the standard normal distribution.

## 2.2 Comparison with existing tests and advantages

Our test is the first high-dimensional nonparametric testing method for $\boldsymbol{\alpha} = \mathbf{0}$ in LFPMs. It allows the number of securities $N$ to be larger than the number of observations $T$, which is most relevant in modern finance because of the large number of securities that are priced every day. It also allows the error distribution to be any elliptical distribution, which is arguably the most relevant distribution family in security-pricing problems. For example, Hodgson et al. (2002) states the importance of testing security-pricing models for elliptically distributed returns, and Chamberlain (1983) shows that a mean-variance analysis of a CAPM is consistent with an investor's portfolio decision-making if and only if the returns are elliptically distributed. Moreover, in the case of elliptical returns, the CAPM remains valid theoretically. Hodgson et al. (2002) propose exact tests for elliptically distributed returns/errors. Indeed, the elliptical family is very general, and contains not only the normal distribution, but also many well-known heavy-tailed distributions, including the Student's t, logistic, contaminated normal, and power exponential distributions, among others, as well as offering a more flexible framework for modeling security prices or returns.

In contrast, classical tests of $\boldsymbol{\alpha} = \mathbf{0}$ in LFPMs apply only to a fixed $N$, while requiring that $T$ diverges. These tests rely mainly on multivariate statistical analysis tools or time series treatments; see, for example, Jensen (1968), Douglas (1968), Black et al. (1972), Fama and MacBeth (1973), Gibbons et al. (1989), and Fama and French (2004). Among these classical methods, the GRS test (Gibbons et al., 1989) is considered the most popular. It is an exact multivariate F-test if the error distribution is normal, but deviates from the F-test when the normality assumption is violated. The GRS test has been extended by, among others, MacKinlay and Richardson (1991) and Zhou (1993). However, most of these works require the number of securities $N$ to be smaller than the

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number of observations $T$, and that the errors follow a normal distribution. These two drawbacks lead to invalid testing results when either one of these requirements is violated. Hence, the test is often inapplicable for analyzing modern finance problems, owing to the large number of securities and the frequently observed heavy-tailed behavior of many securities (Mandelbrot, 1963).

To relax the normality assumption, several procedures were proposed to analyze the large sample property of the test statistic, which no longer has an exact F distribution. For example, John and MacDonald (2012) used a large-sample GMM and bootstrap technique, Hodgson et al. (2002) derived a semiparametric asymptotic procedure specific to elliptical distributions, Zhou (1993) prescribed parametric procedures based on postulating a non-normal distribution, and Tu and Zhou (2004) used a non-normal Bayesian procedure. However, these procedures still apply only in the fixed $N$ setting.

Recently, Pesaran and Yamagata (2012), Fan et al. (2015), Gagliardini et al. (2016), and Pesaran and Yamagata (2017) proposed several new methods that allow $N$ to grow with $T$. However, these methods still impose various assumptions on the error distribution that do not contain many well-known heavy-tailed distributions, such as the multivariate Student's t and the mixture of multivariate normal distributions.

In addition to the flexibility of allowing a large $N$ and an elliptical error distribution, our test is also more powerful, sometimes much more powerful, than competing tests also

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designed for large $N$, as we show in our numerical experiments.

## 3. Theoretical properties

We now establish the theoretical level and power properties of our test.

First, we state several required assumptions.

(A1) The $p$-dimensional vector of common factors $\mathbf{f}_t$ is distributed independently of the errors $\boldsymbol{\epsilon}_{it'}$, for all $i = 1, \cdots, N$ and all $t, t' = 1, \cdots, T$. The number of factors $p$ is fixed and $\mathbf{f}_t^\top \mathbf{f}_t \leqslant K < +\infty$, for a constant $K$ and all $t = 1, \cdots, T$. The matrix $T^{-1}(\mathbf{1}_T, \mathbf{F})^\top(\mathbf{1}_T, \mathbf{F})$ is positive definite, and as $T \to \infty$, $T^{-1}\mathbf{1}_T^\top \mathbf{M}_{\mathbf{F}} \mathbf{1}_T > \tau_{\min}$ for some positive constant $\tau_{\min}$.

(A2) The error vectors $\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_T$ are i.i.d. from the $N$-variate mean-zero elliptical distribution with probability density function

$$\det(\boldsymbol{\Sigma})^{-1/2} g(\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}\|), \ \boldsymbol{\epsilon} \in \mathcal{R}^N.$$

Assumption (A1) ensures the independence between the factors and the errors. It also ensures the uniform boundedness of $\mathbf{f}_t^\top \mathbf{f}_t$. These two conditions are basic assumptions that appear in many related studies, such as in Assumption 1 of Pesaran and Yamagata (2017). The remaining requirements on $\mathbf{F}$ and $\mathbf{M}_{\mathbf{F}}$ in Assumption (A1) are similar to those required in Condition (C3) of Lan et al. (2018) and in Assumption 1 of Pesaran and

Yamagata (2012, 2017). Note that, similarly to the corresponding assumption in Pesaran and Yamagata (2017), we do not require mutual independence of $\mathbf{f}_t$, that is, temporal dependence between $\mathbf{f}_t$ can be allowed, and the restriction is imposed using conditions on the matrix $\mathbf{M}_{\mathbf{F}}$. Note too that different forms of restrictions on the temporal dependence of $\mathbf{f}_t$ appear in the literature. For example, Fan et al. (2015) allows the factors $\mathbf{f}_t$ to be weakly correlated across $t$, but satisfy the strong mixing condition, that is, Assumption 4.1 (iii) of Fan et al. (2015).

Assumption (A2) allows for weak or strong error cross-correlations once the effects of the factors are removed from the returns on individual securities, characterized by the scatter matrix $\mathbf{\Sigma}$, under the elliptical distribution framework (1.2). Compared with the mixed weak-factor spatial framework imposed in Pesaran and Yamagata (2017), the family of elliptical distributions is more general and can contain many more types of heavy-tailed distributions, such as the Student's t, logistic, contaminated normal, and power exponential distributions, among others (Zou et al., 2014). It also allows a mixture of these distributions. As mentioned in Pesaran and Yamagata (2017), such residual interdependencies can arise owing to spatial or other network-type spillover effects not captured by the common factors $\mathbf{f}_t$.

Furthermore, we acknowledge that Assumption (A2) still requires that the errors are symmetrically distributed, which is not always satisfied in practice. We assume the sym-

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metry to facilitate the mathematical derivations. Indeed, there are some asymmetric extensions in the literature from the family of elliptical distributions (Fang, 2003; Genton and Loperfido, 2005). However, tests based on spatial sign functions under asymmetric distributions require more in-depth study, because establishing the corresponding asymptotic theory will be challenging.

Additionally, we impose the following assumption on $\mathbf{\Sigma}$.

(A3) a $\mathrm{tr}(\mathbf{R}^4) = o\{\mathrm{tr}^2(\mathbf{R}^2)\}$;

b $T^{-2}N^2/\mathrm{tr}(\mathbf{R}^2) = O(1)$ and $\log N = o(T)$;

c $\mathrm{tr}(\mathbf{R}^2) - N = o(T^{-1}N^2)$.

Here, $\mathbf{R} \equiv \mathbf{D}^{-1/2}\mathbf{\Sigma}\mathbf{D}^{-1/2}$ is the correlation matrix.

Assumption (A3)-a is used in Feng and Sun (2016), and holds automatically if all the eigenvalues of $\mathbf{R}$ are bounded, which is commonly assumed in the literature on estimating high-dimensional covariance matrices (Bickel and Levina, 2008). Conditions (A3)-b and (A3)-c are used in Feng et al. (2016) to restrict the difference between $\boldsymbol{\epsilon}_t^\top\boldsymbol{\epsilon}_t$ and $\boldsymbol{\epsilon}_t^\top\mathbf{R}\boldsymbol{\epsilon}_t$, thus ensuring the consistency of the estimators of $\mathbf{D}$. The three conditions in Assumption (A3) ensure that the error terms are weakly cross-sectionally correlated through the returns on individual securities. If we assume $\mathrm{tr}(\mathbf{R}^2) = O(N)$, Conditions (A3)-b,c reduce to $N = O(T^{\delta})$, where $\delta \in (1, 2)$.

Finally, we impose an assumption on the local alternative intercept vector $\boldsymbol{\alpha}$.

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(A4) $\boldsymbol{\alpha}^T\mathbf{D}^{-1}\boldsymbol{\alpha} = O(c_1^{-2}T^{-1}N^{-1}\sigma_T)$ and $\boldsymbol{\alpha}^T\mathbf{D}^{-1}\boldsymbol{\Sigma}\mathbf{D}^{-1}\boldsymbol{\alpha} = o(c_1^{-2}T^{-1}N^{-1}\sigma_T^2)$, where $c_1 = E(\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}_t\|^{-1})$ and $\sigma_T = \sqrt{2\mathrm{tr}(\mathbf{R}^2)}$.

Assumption (A4) is similar to Condition (C4) of Feng and Sun (2016), which serves as a high-dimensional extension of the classical local alternative hypothesis. In fact, Assumption (A4) restricts the difference between $\boldsymbol{\alpha}$ and $\mathbf{0}$ to be not too large, which enables the variance of $Q$ to be asymptotically described by $\sigma_T^2$.

We now establish the theoretical properties of our testing procedure in terms of its null and alternative distributions. These properties ensure the validity of the test and enable the power calculation in practice. The proof techniques are quite different to those of existing works, such as Feng et al. (2016). For example, we investigate the effect of estimating the residuals, handle the additional weights related to the factors, and analyze the influence of the factors on the unbiasedness of the statistic. These analyses do not appear in the literature and require novel treatments.

**Proposition 1.** *Under Assumptions (A1)–(A3) and $H_0$, when $\min(T, N) \to \infty$, $Q/\sigma_T \xrightarrow{\mathcal{D}} N(0, 1)$. Here, $\sigma_T$ is defined in Assumption (A4).*

The trace $\mathrm{tr}(\mathbf{R}^2)$, which is needed in the construction of the test statistic $T_{SS}$, can be assessed using

$$\widehat{\mathrm{tr}(\mathbf{R}^2)} \equiv \frac{N^2}{\mathbf{h}^\top\mathbf{h}(\mathbf{h}^\top\mathbf{h}-1)} \sum_{t_1,t_2=1,t_1\neq t_2}^{T} h_{t_1}^2 h_{t_2}^2 \left\{U(\widehat{\mathbf{D}}^{-1/2}\widetilde{\boldsymbol{\epsilon}}_1^{(t_1,t_2)})^\top U(\widehat{\mathbf{D}}^{-1/2}\widetilde{\boldsymbol{\epsilon}}_2^{(t_1,t_2)})\right\}^2.$$

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Recall that in $\widetilde{\boldsymbol{\epsilon}}_1^{(t_1,t_2)}$ and $\widetilde{\boldsymbol{\epsilon}}_2^{(t_1,t_2)}$, $\widehat{\mathbf{B}}_1^{(t_1t_2)}$ and $\widehat{\mathbf{B}}_2^{(t_1t_2)}$ are the least-square estimators of $\mathbf{B}$ based on the first-half and second-half, respectively, of the sample $\{(\mathbf{Y}_t, \mathbf{f}_t)\}_{t \neq t_1,t_2}$. The purpose of splitting the samples is to ensure the independence of $\widehat{\mathbf{B}}_1^{(t_1t_2)}$ and $\widehat{\mathbf{B}}_2^{(t_1t_2)}$, thus avoiding additional bias.

**Proposition 2.** *Under Assumptions (A1)–(A3),* $\widehat{tr(\mathbf{R}^2)}/tr(\mathbf{R}^2) \xrightarrow{\mathcal{P}} 1$ *when* $\min(T,N) \to \infty$.

The results in Propositions 1 and 2 prompt us to construct the test statistic in (2.3) and obtain the limiting null distribution of the test statistic $T_{SS}$ in Theorem 1.

**<u>Theorem</u> 1.** *Under Assumptions (A1)–(A3) and* $H_0$, *when* $\min(T,N) \to \infty$, $T_{SS} \xrightarrow{\mathcal{D}} N(0,1)$.

Theorem 1 allows us to perform a level-$\alpha$ test by rejecting the null hypothesis when $T_{SS} > z_{1-\alpha}$, where $z_{1-\alpha}$ is the $(1-\alpha)$ quantile of $N(0,1)$.

We further study the asymptotic distribution of $T_{SS}$ under the alternative hypothesis.

**<u>Theorem</u> 2.** *Under Assumptions (A1)–(A4), when* $\min(T,N) \to \infty$, $T_{SS} \xrightarrow{\mathcal{D}} N(\mu_{SS},1)$, *where* $\mu_{SS} = \lim_{\min(T,N)\to\infty} \sigma_T^{-1}\phi^2$, $\phi^2 = \omega T N c_1^2 \boldsymbol{\alpha}^\top \mathbf{D}^{-1}\boldsymbol{\alpha}$ *and* $c_1 = E(\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}_t\|^{-1})$.

Here, $\omega = 1 - E(\mathbf{f}_t)^\top\{E(\mathbf{f}_t\mathbf{f}_t^\top)\}^{-1}E(\mathbf{f}_t) \in (0,1]$. Under Assumption (A1), we have $T^{-1}\mathbf{h}^\top\mathbf{h} \to \omega$ by Lemmas B.1 and E.2 of Fan et al. (2015). Theorem 2 is used to compute the asymptotic power function of $T_{SS}$, which is equal to

$$\beta_{SS}(\boldsymbol{\alpha}) = \lim_{\min(T,N)\to\infty} \Phi\{-z_\alpha + \omega T N c_1^2 \boldsymbol{\alpha}^\top \mathbf{D}^{-1}\boldsymbol{\alpha}/\sigma_T\}.$$

According to Theorem 6 of Pesaran and Yamagata (2017), the asymptotic power function of their proposed test, abbreviated as the PY test, is

$$\beta_{PY}(\boldsymbol{\alpha}) = \lim_{\min(T,N)\to\infty} \Phi(-z_\alpha + \omega T N c_2^{-1} \boldsymbol{\alpha}^\top \mathbf{D}^{-1}\boldsymbol{\alpha}/\sigma_T),$$

where $c_2 = E(\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}_t\|^2)$. Hence, the asymptotic relative efficiency of the proposed SS test with respect to the PY test is

$$\begin{aligned}\mathrm{ARE}(\mathrm{SS},\mathrm{PY}) &= \{E(\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}_t\|^{-1})\}^2 E(\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}_t\|^2)\\ &\geqslant \{E(\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}_t\|^{-1})\}^2 \{E(\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}_t\|)\}^2 \geqslant 1.\end{aligned}$$

The above expression of $\beta_{SS}(\boldsymbol{\alpha})$ indicates that the SS test is consistent (in the sense that its power tends to one) if $\omega T N c_1^2 \boldsymbol{\alpha}^\top \mathbf{D}^{-1}\boldsymbol{\alpha}/\sigma_T \to \infty$. In particular, the power function of the SS test before the limit operation has the same order as that of the PY test. Similarly to the PY test, the SS test has power even if the number of securities with nonzero alphas does not increase with $N$. For example, if $\boldsymbol{\alpha}^\top \mathbf{D}^{-1}\boldsymbol{\alpha} \asymp N^{\delta_\alpha}$, $T \asymp N^d$, and $\mathrm{tr}(\boldsymbol{R}^2) \asymp O(N)$, then $\omega T N c_1^2 \boldsymbol{\alpha}^\top \mathbf{D}^{-1}\boldsymbol{\alpha}/\sigma_T \asymp N^{\delta_\alpha + d - 1/2}$, where the condition $\mathrm{tr}(\boldsymbol{R}^2) \asymp O(N)$ appears in Pesaran and Yamagata (2017). Here, $\eta_1 \asymp \eta_2$ means that $\eta_1$ and $\eta_2$ are of the same order. In such a situation, the SS test is still consistent if $\delta_\alpha + d > 1/2$, which does not require $\delta_\alpha > 0$.

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To illustrate the above asymptotic power functions more intuitively, we consider them in some special cases. According to Appendix 3 in the supplementary material of Zou et al. (2014), when $\boldsymbol{\epsilon}_t \sim t(\mathbf{0}, \mathbf{I}_N, v)$ with $v > 2$, we have $c_1 = \frac{\Gamma\{(v+1)/2\}}{v^{1/2}\Gamma(v/2)} \frac{\Gamma\{(N-1)/2\}}{\Gamma(N/2)}$, $c_2 = \frac{v}{v-2}N$, $\sigma_T = \sqrt{2N}$, and

$$\begin{aligned}
\beta_{SS}(\boldsymbol{\alpha}) =& \Phi\left\{-z_\alpha + \lim_{\min(T,N)\to\infty} \omega T \sqrt{\frac{N}{2}} \frac{\Gamma^2\{(v+1)/2\}}{v\Gamma^2(v/2)} \frac{\Gamma^2\{(N-1)/2\}}{\Gamma^2(N/2)} \boldsymbol{\alpha}^\top \boldsymbol{\alpha}\right\} \\
=& \Phi\left\{-z_\alpha + \lim_{\min(T,N)\to\infty} \frac{\omega T}{\sqrt{2N}} \frac{2\Gamma^2\{(v+1)/2\}}{v\Gamma^2(v/2)} \boldsymbol{\alpha}^\top \boldsymbol{\alpha}\right\}, \\
\beta_{PY}(\boldsymbol{\alpha}) =& \Phi\left\{-z_\alpha + \lim_{\min(T,N)\to\infty} \frac{\omega T}{\sqrt{2N}} \frac{v-2}{v} \boldsymbol{\alpha}^\top \boldsymbol{\alpha}\right\}.
\end{aligned}$$

Hence,

$$\mathrm{ARE}(\mathrm{SS}, \mathrm{PY}) = \frac{2}{v-2} \frac{\Gamma^2\{(v+1)/2\}}{\Gamma^2(v/2)} > 1.$$

Note that because the asymptotic power function of the power enhancement test proposed by Fan et al. (2015), abbreviated as the PE test, is not derived by the authors, we cannot provide the theoretical relative efficiency of the SS test with respect to the PE test. However, we do provide numerical results to show the superiority of our SS test. Finally, although the test procedure in Gagliardini et al. (2016) can also be used to perform the test, as mentioned in Pesaran and Yamagata (2017), it does not retain the test level under the null, theoretically or empirically. For this reason, we do not include it here.

## 4. Monte Carlo experiments

We now conduct Monte Carlo experiments to compare the finite-sample performance of the proposed SS test with that of several existing methods.

### 4.1 Simulation design

**Simulation 1** This simulation is designed to mimic the commonly used Fama–French three-factor model, where the factors $\mathbf{f}_t$ have strong serial correlation and heterogeneous variance. Specifically, we consider a modified version of the example studied in Section 5.1 of Pesaran and Yamagata (2017). The response $Y_{it}$ is generated according to the following LFPM with $p = 3$:

$$Y_{it} = \alpha_i + \sum_{j=1}^{p} \beta_{ij} f_{tj} + \epsilon_{it},$$

where the three factors $f_{t1}$, $f_{t2}$, and $f_{t3}$ are the three Fama–French factors (Market factor, SMB, HML). We generate each factor from an autoregressive conditional heteroskedasticity process and the GARCH(1,1) model. Specifically,

$$\begin{aligned}
f_{t1} =& 0.53 + 0.06 f_{t-1,1} + h_{t1}^{1/2}\zeta_{t1}, \text{ Market factor,}\\
f_{t3} =& 0.19 + 0.19 f_{t-1,2} + h_{t2}^{1/2}\zeta_{t2}, \text{ SMB factor,}\\
f_{t3} =& 0.19 + 0.05 f_{t-1,3} + h_{t3}^{1/2}\zeta_{t3}, \text{ HML factor,}
\end{aligned}$$

where for $j = 1, 2, 3$, $\zeta_{tj}$ are generated independently from a standard normal distribution, and the variance term $h_{tj}$ is generated as follows:

$$h_{t1} = 0.89 + 0.85 h_{t-1,1} + 0.11 \zeta^2_{t-1,1}, \text{ Market factor,}$$

$$h_{t2} = 0.62 + 0.74 h_{t-1,2} + 0.19 \zeta^2_{t-1,2}, \text{ SMB,}$$

$$h_{t3} = 0.80 + 0.76 h_{t-1,3} + 0.15 \zeta^2_{t-1,3}, \text{ HML.}$$

The above process is simulated over the periods $t \in \{-49, \cdots, 0, 1, \cdots, T\}$ with the initial values $f_{-50,j} = 0$ and $h_{-50,j} = 1$, for any $j \in \{1, 2, 3\}$, and the generated data that belong to the periods $\{1, \cdots, T\}$ are extracted as the simulation data.

To verify the robustness of the proposed testing method in terms of nonnormal or heavy-tailed distributions, we generate $\boldsymbol{\epsilon}_t$ from the following three distribution settings:

(I) Multivariate normal distribution: $\boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma})$;

(II) Multivariate t-distribution: $\boldsymbol{\epsilon}_t \sim t_N(\mathbf{0}, \boldsymbol{\Sigma}, 3)$;

(III) Multivariate mixture normal distribution: $\boldsymbol{\epsilon}_t$ are generated from $\gamma N_N(\mathbf{0}, \boldsymbol{\Sigma}) + (1 - \gamma) N_N(\mathbf{0}, 9\boldsymbol{\Sigma})$, denoted by $\mathrm{MN}_{N,\gamma,9}(\mathbf{0}, \boldsymbol{\Sigma})$; $\gamma$ is fixed to be 0.8.

Here, the covariance matrix $\boldsymbol{\Sigma} = \mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2}$, with $\mathbf{D} = \mathrm{diag}\{\sigma_1^2, \cdots, \sigma_N^2\}$ and $\mathbf{R} = \mathbf{I}_N + \boldsymbol{b}\boldsymbol{b}^\top - \check{\mathbf{B}}$, where $\sigma_i^2$ are generated independently from $U(20, 100)$, $\boldsymbol{b} = (b_1, \cdots, b_N)^\top$, and $\check{\mathbf{B}} = \mathrm{diag}\{b_1^2, \cdots, b_N^2\}$. To generate different degrees of error cross-sectional dependence, we generate the first and last $[N^{\delta_\gamma}]$ elements of $\mathbf{b}$ independently from $U(0.7, 0.9)$,

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and set the remaining elements in the middle to be zero, where the exponent $\delta_\gamma = 0.25, 0.5, 0.6$. Here, $\delta_\gamma = 0.25$ corresponds to the case of weak correlation, and $\delta_\gamma = 0.6$ corresponds to that of strong correlation. We conducted the simulations for $T = 50, 100$, $N = 100, 200, 500$, and $p = 1, 3, 5$.

Finally, the three groups of coefficients corresponding to the three factors $\beta_{i1}$, $\beta_{i2}$, and $\beta_{i3}$ are generated independently from $U(0.2, 2)$, $U(-1, 1.5)$, and $U(-1.5, 1.5)$, respectively. Then, we set $\boldsymbol{\alpha} = \mathbf{0}$ under the null hypothesis, and under the alternative hypothesis, we consider two cases: (1) the dense case: we generate $\alpha_i$ independently from $N(0, 1)$ for $i = 1, 2, \cdots, [N^{0.8}]$, keeping the remaining $\alpha_i$ zero; (2) the sparse case: we generate $\alpha_i$ independently from $N(0, 16)$ for $i = 1, 2, \cdots, [N^{0.3}]$, keeping the remaining $\alpha_i$ zero. Here, $[\cdot]$ denotes the integer part of a real number.

Note that to provide a comprehensive picture, we consider additional simulation settings in the Supplementary Material.

## 4.2 Simulation results

We now present the results of SS, PY, and PE under Simulation 1, where we generate each factor with an autoregressive conditional heteroskedasticity process to mimic the commonly used Fama–French three-factor model. All results are based on 1,000 replications. The results of the three testing methods in terms of both empirical size and empirical power are summarized in Tables 1 and 2.

Table 1 summarizes the results of Simulation 1 for $T = 50$. It indicates that if the error of the LFPM has a multivariate normal distribution, all tests in the comparison exhibit similar power performance, and in most cases, the proposed SS test controls the empirical size better than the two competitors do, especially when $N$ is large. Furthermore, if the error follows a heavy-tailed or nonnormal distribution, such as the Student's t or mixture of multivariate normal distributions, regardless of which LFPM is the true mean model, the SS test outperforms the two competitors in terms of both empirical size and empirical power for both the dense and the sparse cases, especially in high-dimensional situations. This indicates that the SS test is much more robust to departures from normality than is PY or PE, as is expected based on their construction. Table 2 summarizes the results of Simulation 1 for $T = 100$. It indicates that when $T$ increases to 100, the results are similar to those in Table 1, and the advantage of the SS test becomes even more prominent.

Note that the effective sample size, that is, the degrees of freedom of the test statistic, is $v = T - p - 1$, a monotonically decreasing function of $p$, which leads to the result shown in Tables 1 and 2 that the power decreases when $p$ increases from one to five for each testing procedure.

In summary, the experiment results all illustrate the robustness of the SS test to departures from normality of the error distribution, and show the test consistency and better power performance than that of the PY and PE tests for high-dimensional securities in LFPMs.

## 5. Discussion

In this paper, we have proposed a robust test, SS, for testing the presence of alpha in LFPMs that aims to simultaneously alleviate the difficulties of high-dimensional securities and a departure from normality of the error distribution. The theoretical properties of the proposed SS test are established for the family of elliptical distributions, which is a much broader distribution family than the commonly studied mixed weak-factor spatial representation because it includes heavy-tailed distributions. These theoretical results are illustrated in a Monte Carlo experiment, the findings of which suggest that the proposed SS test is much more robust to departures from normality than are the two prominent examples of existing methods, PY and PE, in terms of both empirical size and empirical power. The proposed test is used to analyze a data set of monthly returns on securities in the S&P 500 index over the period 2005–2018, where a large number of securities exhibit a nonnormal distribution property. As suggested by the results of the application for the CAPM and the Fama–French three-factor model, the proposed SS test is more inclined to reject the null hypothesis than is PE or PY, benefiting from its superior power. Pesaran and Yamagata (2017) further considered a mixed weak-factor spatial framework that allows unobserved factors. It would be interesting to extend our method to this more general setting as well.

## Supplementary Material

The online Supplementary Material contains additional numerical results, a real-data analysis, and the technical proofs.

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School of Mathematics and Statistics & KLAS, Northeast Normal University, Changchun, China 130024

E-mail: (liubh100@nenu.edu.cn)

School of Statistics and Data Science, LPMC & KLMDASR, Nankai University, Tianjin, China 300071

E-mail: (flnankai@nankai.edu.cn)

Department of Statistics, Pennsylvania State University, University Park, Pennsylvania U.S. 16802

E-mail: (yanyuanma@yahoo.com)

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Table 1: Size and power of the SS, PY, and PE tests in Simulation 1 with $T = 50$.

| Scenarios | | $N(\mathbf{0}, \mathbf{\Sigma})$ | | | $t_N(\mathbf{0}, \mathbf{\Sigma}, 3)$ | | | $\mathrm{MN}_{p,\gamma,9}(\mathbf{0}, \mathbf{\Sigma})$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\delta_\gamma$ | | | $\delta_\gamma$ | | | $\delta_\gamma$ | | |
| $N$ | Method | 0.25 | 0.5 | 0.6 | 0.25 | 0.5 | 0.6 | 0.25 | 0.5 | 0.6 |
| Sizes | | | | | | | | | | |
| 100 | SS | 0.049 | 0.048 | 0.044 | 0.043 | 0.045 | 0.056 | 0.054 | 0.042 | 0.055 |
| | PY | 0.049 | 0.055 | 0.061 | 0.045 | 0.047 | 0.062 | 0.050 | 0.056 | 0.066 |
| | PE | 0.064 | 0.052 | 0.036 | 0.040 | 0.028 | 0.024 | 0.055 | 0.033 | 0.029 |
| 200 | SS | 0.051 | 0.052 | 0.053 | 0.048 | 0.044 | 0.052 | 0.043 | 0.047 | 0.057 |
| | PY | 0.050 | 0.049 | 0.070 | 0.073 | 0.057 | 0.051 | 0.086 | 0.057 | 0.070 |
| | PE | 0.055 | 0.033 | 0.021 | 0.037 | 0.033 | 0.021 | 0.085 | 0.050 | 0.034 |
| 500 | SS | 0.050 | 0.046 | 0.060 | 0.045 | 0.046 | 0.044 | 0.042 | 0.048 | 0.051 |
| | PY | 0.062 | 0.054 | 0.066 | 0.088 | 0.092 | 0.069 | 0.103 | 0.115 | 0.101 |
| | PE | 0.043 | 0.035 | 0.016 | 0.061 | 0.054 | 0.034 | 0.091 | 0.071 | 0.045 |
| dense | | | | | | | | | | |
| 100 | SS | 0.60 | 0.54 | 0.49 | 0.43 | 0.40 | 0.33 | 0.45 | 0.39 | 0.34 |
| | PY | 0.64 | 0.55 | 0.48 | 0.17 | 0.17 | 0.12 | 0.18 | 0.17 | 0.17 |
| | PE | 0.66 | 0.56 | 0.47 | 0.17 | 0.12 | 0.11 | 0.17 | 0.14 | 0.10 |
| 200 | SS | 0.75 | 0.69 | 0.58 | 0.52 | 0.44 | 0.38 | 0.52 | 0.46 | 0.36 |
| | PY | 0.80 | 0.70 | 0.58 | 0.20 | 0.19 | 0.16 | 0.24 | 0.22 | 0.16 |
| | PE | 0.80 | 0.70 | 0.55 | 0.18 | 0.15 | 0.10 | 0.22 | 0.19 | 0.11 |
| 500 | SS | 0.93 | 0.87 | 0.74 | 0.54 | 0.47 | 0.42 | 0.55 | 0.51 | 0.40 |
| | PY | 0.94 | 0.91 | 0.76 | 0.26 | 0.22 | 0.21 | 0.31 | 0.28 | 0.25 |
| | PE | 0.94 | 0.89 | 0.79 | 0.21 | 0.19 | 0.14 | 0.27 | 0.22 | 0.15 |
| sparse | | | | | | | | | | |
| 100 | SS | 0.57 | 0.52 | 0.48 | 0.42 | 0.40 | 0.34 | 0.50 | 0.47 | 0.41 |
| | PY | 0.59 | 0.54 | 0.48 | 0.24 | 0.20 | 0.18 | 0.35 | 0.32 | 0.30 |
| | PE | 0.56 | 0.51 | 0.44 | 0.20 | 0.18 | 0.14 | 0.25 | 0.18 | 0.18 |
| 100 | SS | 0.58 | 0.53 | 0.44 | 0.39 | 0.38 | 0.32 | 0.50 | 0.45 | 0.37 |
| | PY | 0.60 | 0.54 | 0.46 | 0.20 | 0.19 | 0.14 | 0.36 | 0.31 | 0.27 |
| | PE | 0.58 | 0.50 | 0.40 | 0.18 | 0.16 | 0.11 | 0.24 | 0.20 | 0.13 |
| 100 | SS | 0.60 | 0.54 | 0.40 | 0.33 | 0.31 | 0.26 | 0.46 | 0.41 | 0.32 |
| | PY | 0.64 | 0.57 | 0.45 | 0.18 | 0.18 | 0.14 | 0.29 | 0.29 | 0.23 |
| | PE | 0.61 | 0.54 | 0.36 | 0.15 | 0.12 | 0.07 | 0.20 | 0.16 | 0.10 |

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Table 2: Size and power of the SS, PY, and PE tests in Simulation 1 with $T = 100$.

| Scenarios | | $N(\mathbf{0}, \mathbf{\Sigma})$ | | | $t_N(\mathbf{0}, \mathbf{\Sigma}, 3)$ | | | $\mathrm{MN}_{p,\gamma,9}(\mathbf{0}, \mathbf{\Sigma})$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\delta_\gamma$ | | | $\delta_\gamma$ | | | $\delta_\gamma$ | | |
| $N$ | Method | 0.25 | 0.5 | 0.6 | 0.25 | 0.5 | 0.6 | 0.25 | 0.5 | 0.6 |
| Sizes | | | | | | | | | | |
| 100 | SS | 0.046 | 0.054 | 0.051 | 0.040 | 0.052 | 0.052 | 0.053 | 0.051 | 0.054 |
| | PY | 0.076 | 0.059 | 0.068 | 0.034 | 0.032 | 0.038 | 0.048 | 0.051 | 0.058 |
| | PE | 0.036 | 0.027 | 0.008 | 0.034 | 0.014 | 0.008 | 0.056 | 0.025 | 0.012 |
| 200 | SS | 0.057 | 0.049 | 0.056 | 0.043 | 0.054 | 0.058 | 0.053 | 0.052 | 0.046 |
| | PY | 0.051 | 0.042 | 0.061 | 0.038 | 0.027 | 0.051 | 0.063 | 0.044 | 0.057 |
| | PE | 0.052 | 0.021 | 0.005 | 0.027 | 0.010 | 0.007 | 0.053 | 0.027 | 0.006 |
| 500 | SS | 0.047 | 0.049 | 0.048 | 0.046 | 0.043 | 0.052 | 0.042 | 0.052 | 0.057 |
| | PY | 0.050 | 0.050 | 0.056 | 0.040 | 0.033 | 0.040 | 0.077 | 0.062 | 0.065 |
| | PE | 0.045 | 0.015 | 0.002 | 0.023 | 0.020 | 0.018 | 0.049 | 0.022 | 0.011 |
| dense | | | | | | | | | | |
| 100 | SS | 0.97 | 0.95 | 0.94 | 0.89 | 0.85 | 0.81 | 0.86 | 0.84 | 0.79 |
| | PY | 0.96 | 0.94 | 0.90 | 0.33 | 0.30 | 0.29 | 0.45 | 0.42 | 0.35 |
| | PE | 0.97 | 0.93 | 0.90 | 0.35 | 0.26 | 0.21 | 0.48 | 0.37 | 0.27 |
| 200 | SS | 0.99 | 0.99 | 0.98 | 0.94 | 0.94 | 0.91 | 0.96 | 0.92 | 0.91 |
| | PY | 1.00 | 0.99 | 0.97 | 0.40 | 0.38 | 0.29 | 0.54 | 0.47 | 0.40 |
| | PE | 0.99 | 0.99 | 0.99 | 0.35 | 0.30 | 0.27 | 0.56 | 0.47 | 0.35 |
| 500 | SS | 1.00 | 1.00 | 1.00 | 0.97 | 0.98 | 0.96 | 0.99 | 0.99 | 0.97 |
| | PY | 1.00 | 1.00 | 1.00 | 0.43 | 0.40 | 0.31 | 0.65 | 0.62 | 0.46 |
| | PE | 1.00 | 1.00 | 1.00 | 0.36 | 0.34 | 0.29 | 0.63 | 0.57 | 0.47 |
| sparse | | | | | | | | | | |
| 100 | SS | 0.77 | 0.75 | 0.74 | 0.71 | 0.70 | 0.67 | 0.74 | 0.70 | 0.66 |
| | PY | 0.80 | 0.75 | 0.73 | 0.39 | 0.35 | 0.30 | 0.58 | 0.54 | 0.49 |
| | PE | 0.78 | 0.73 | 0.70 | 0.34 | 0.32 | 0.27 | 0.49 | 0.37 | 0.32 |
| 200 | SS | 0.80 | 0.79 | 0.72 | 0.72 | 0.71 | 0.63 | 0.76 | 0.75 | 0.65 |
| | PY | 0.81 | 0.79 | 0.71 | 0.34 | 0.30 | 0.28 | 0.56 | 0.53 | 0.48 |
| | PE | 0.82 | 0.78 | 0.69 | 0.31 | 0.27 | 0.21 | 0.39 | 0.38 | 0.28 |
| 500 | SS | 0.85 | 0.81 | 0.72 | 0.76 | 0.71 | 0.65 | 0.79 | 0.76 | 0.66 |
| | PY | 0.87 | 0.81 | 0.74 | 0.27 | 0.22 | 0.22 | 0.55 | 0.49 | 0.42 |
| | PE | 0.87 | 0.83 | 0.72 | 0.20 | 0.15 | 0.14 | 0.34 | 0.31 | 0.20 |