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On Construction of Nonregular Two-Level Factorial Designs with Maximum Generalized Resolutions

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Abstract

The generalized resolution was introduced and justified as a criterion for selecting nonregular factorial designs. Although there has been extensive research conducted on other aspects of nonregular designs, few works have investigated the construction of nonregular designs with maximum generalized resolutions, as we do in this study. To date, our knowledge of nonregular designs with maximum generalized resolutions is predominantly computational, except for very few theoretical results. We derive lower bounds on relevant J-characteristics and present the construction results. With the assistance of the lower bounds, many of the constructed designs are shown to have maximum generalized resolutions.

Key words and phrases: good Hadamard matrix; orthogonal array; Paley construction; tensor product.

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1 Introduction

Nonregular factorial designs are not regular and, therefore, cannot be specified by defining relations. The generalized resolution, introduced by Deng and Tang (1999), provides a concise characterization for a two-level nonregular design, just as the resolution does for a two-level regular design. As a design selection criterion, the generalized resolution is justifiable from two points of view, one from the properties of projection designs, and the other from the biases on the estimation of the main effects. An extension of the concept to multi-level designs was examined by Evangelaras, Koukouvinos, Dean, and Dingus (2005) and Grömping and Xu (2014).

Studies on nonregular designs started as early as the 1940s, when Plackett and Burman (1946) introduced a class of main-effect plans for run sizes that are multiples of four. However, such studies were rare until Lin and Draper (1992), Wang and Wu (1995), Cheng (1995) and Box and Tyssedal (1996) investigated the projection properties of nonregular designs. There have been extensive research activities since, with much of the later work centered around the minimum G_2-aberration criterion and the construction of minimum G_2-aberration designs; see, for example, Tang and Deng (1999), Xu and Wu (2001), Butler (2003), Xu (2003), and Butler (2004). Parallel flats designs are a class of nonregular designs; see Srivastava and Li (1996), Liao, Iyer, and Vecchia (1996), and Mee and Peralta (2000) for some representative papers. Nonregular designs from quaternary codes were considered by, for example, Xu and Wong (2007) and Phoa and Xu (2009). A recent work on constructing strength-three orthogonal arrays is that of

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Vazquez, Goos, and Schoen (2019). We refer to Xu, Phoa, and Wong (2009) for a general review of topics on nonregular designs.

However, our knowledge of nonregular designs with maximum generalized resolutions is predominantly computational. Important works along this line include Deng and Tang (2002), Sun, Li, and Ye (2008), Schoen, Eendebak, and Nguyen (2010), Bulutoglu and Ryan (2015), and Schoen, Vo-Thanh, and Goos (2017). There are very few theoretical results available in the literature. Xu and Wong (2007) showed that quaternary code designs can be constructed to have a generalized resolution of 3.5. Shi and Tang (2018) obtained a couple of results that allow the construction of designs with large generalized resolutions from good Hadamard matrices.

We undertake a comprehensive and systematic study on the construction of two-level nonregular designs with maximum generalized resolutions. Section 2 introduces the notation and background, and Section 3 provides some preliminary results on saturated orthogonal arrays. The main results are in Section 4, where we derive a general lower bound on the relevant J-characteristics, and present several constructions of nonregular designs with large generalized resolutions. We show that many of the constructed designs have maximum generalized resolutions. Section 5 presents another construction of designs with maximum generalized resolutions. Section 6 concludes the paper.

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2 Notation and Preparation

A two-level factorial design of n runs for m factors is represented by an $n \times m$ matrix $D = (d_{ij})$, with $d_{ij} = \pm 1$, where each column represents a factor and each row specifies a run. Design D is an orthogonal array of strength t, denoted by $\mathrm{OA}(n, 2^m, t)$, if the 2^t level combinations occur with the same frequency in each of its submatrices of t columns. Corresponding to every subset of columns of D is a J-characteristic, defined as $J_u = \sum_{i=1}^{n} \prod_{j \in u} d_{ij}$, where $u \subseteq \{1, 2, \ldots, m\}$. Let r be the smallest positive integer such that $\max_{|u|=r} |J_u| > 0$, where $|u|$ is the cardinality of u. Then, the generalized resolution of design D, as defined in Deng and Tang (1999), is

$$R(D) = r + 1 - \max_{|u|=r} |J_u|/n.$$

Two-level orthogonal arrays are intimately related to Hadamard matrices. A Hadamard matrix of order n is an $n \times n$ square matrix H with entries ± 1 such that $H^T H = H H^T = n I_n$, where I_n is an identity matrix of order n. Therefore, the columns of a Hadamard matrix are mutually orthogonal; so are its rows. The order n of a Hadamard matrix has to be one, two, or a multiple of four. Two Hadamard matrices are isomorphic if one can be obtained from the other by a sequence of isomorphic operations consisting of row permutation, column permutation, sign-switching a row, and sign-switching a column. A Hadamard matrix is said to be normalized if its first column only contains entry $+1$; any Hadamard matrix can be normalized by sign-switching those rows that have -1 in the first column. Deleting the first column from a normalized Hadamard matrix of order n, we obtain a saturated orthogonal array $\mathrm{OA}(n, 2^{n-1}, 2)$.

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The simplest way to construct Hadamard matrices is to use a tensor product. Let $H = (h_{ij})$ and G be Hadamard matrices of orders n_1 and n_2, respectively. Then, $H \otimes G = (h_{ij}G)$ is a Hadamard matrix of order $n_1 n_2$. A Hadamard matrix of Sylvester type, which corresponds to a regular design, is simply the k-fold tensor product of a Hadamard matrix of order two, for any integer $k \geq 1$.

Paley (1993) presented two constructions of Hadamard matrices. A Galois field of order s is denoted by $GF(s) = \{\alpha_0, \alpha_1, \ldots, \alpha_{s-1}\}$, where s is a prime power. A nonzero element α in $GF(s)$ is called a quadratic residue if $\alpha = \beta^2$, for some β in $GF(s)$. Now, define $\chi(\alpha) = 0$ if $\alpha = 0$, $\chi(\alpha) = 1$ if α is a quadratic residue, and $\chi(\alpha) = -1$ if α is not a quadratic residue.

Both of Paley's constructions use the matrix $Q = (q_{ij})_{s\times s}$, where $q_{ij} = \chi(\alpha_i - \alpha_j)$, for $i, j = 0, 1, \ldots, s-1$. The first asserts that

$$H = \begin{bmatrix} 1 & -1_s^T \\ 1_s & Q + I_s \end{bmatrix} \tag{2.1}$$

is a Hadamard matrix of order $s+1$, where 1_s is a column vector of all ones. The second claims that

$$H = \begin{bmatrix} 1 & 1_s^T & -1 & 1_s^T \\ 1_s & Q + I_s & 1_s & Q - I_s \\ -1 & 1_s^T & -1 & -1_s^T \\ 1_s & Q - I_s & -1_s & -Q - I_s \end{bmatrix} \tag{2.2}$$

is a Hadamard matrix of order $2(s+1)$ if $s = 4k+1$, for some integer k.

We see that, for any given order n that is a multiple of four, Paley's first construction works if $n-1$ is a prime power, and his second construction works if $n/2 - 1$ is a prime

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power that has the form $4k+1$.

Note that the matrix Q is skew-symmetric in Paley's first construction and symmetric in his second construction. The symmetry of Q in Paley's second construction is needed in Section 5.

3 Preliminary results

Our first set of results concerns saturated orthogonal arrays $\mathrm{OA}(n, 2^{n-1}, 2)$. The next result provides a lower bound on the value of $\max_{|u|=3} |J_u|$ for such designs.

Proposition 1. *Let D be an $OA(n, 2^{n-1}, 2)$. We then have that*

$$\max_{|u|=3} |J_u| \geq n - 8 \left\lfloor (n/8) \left(1 - 1/(n-3)^{1/2}\right) \right\rfloor,$$

where $\lfloor x \rfloor$ is the floor function.

Shi and Tang (2018) effectively proved that

$$\max_{|u|=3,4} |J_u| \geq n - 8 \left\lfloor (n/8) \left(1 - 1/(n-3)^{1/2}\right) \right\rfloor.$$

Although this result does not imply Proposition 1 directly, a tiny modification to its proof is all we need to prove Proposition 1.

Consider the Hadamard matrix from Paley's first construction given in (2.1) of Section 2. If we delete the first column, consisting of all ones, we obtain a saturated orthogonal array $\mathrm{OA}(n, 2^{n-1}, 2)$. This array is called a *Paley design* in the literature.

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Proposition 2. *If an $OA(n, 2^{n-1}, 2)$ is a Paley design, we have that*

$$\max_{|u|=3} |J_u| \leq n - 8 \left\lceil (1/8) \left(n - 4 - 2(n-1)^{1/2} \right) \right\rceil ,$$

where $\lceil x \rceil$ is the ceiling function.

For a Paley design, Shi and Tang (2018) effectively established that

$$\max_{|u|=3,4} |J_u| \leq n - 8 \left\lceil (1/8) \left(n - 4 - 2(n-1)^{1/2} \right) \right\rceil .$$

Proposition 2 is an immediate consequence of this result, because $\max_{|u|=3} |J_u| \leq \max_{|u|=3,4} |J_u|$.

If the upper bound in Proposition 2 meets the lower bound in Proposition 1, then the Paley design minimizes $\max_{|u|=3} |J_u|$, and thus maximizes the generalized resolution. We can check that this happens for $n = 12, 20, 24, 28, 32, 44, 60, 72$, and 80. Therefore, Paley designs for these orders have maximum generalized resolutions. The values of $\max_{|u|=3|} |J_u|$ for these Paley designs are displayed below:

order n	12	20	24	28	32	44	60	72	80
$\max_{\lvert u\rvert=3\rvert} \lvert J_u\rvert$	4	12	8	12	8	12	12	16	16

Whether or not Paley designs have maximum generalized resolutions for other orders remains to be settled. The answer may well be affirmative, because we have not found a counterexample. A sharper lower bound on the $\max_{|u|=3} |J_u|$ value would help. Nevertheless, Paley designs provide an attractive class of nonregular designs, because they have a generalized resolution bounded below by $3 + (8/n) \left\lceil (1/8) \left(n - 4 - 2(n-1)^{1/2} \right) \right\rceil$, which converges to four as n goes to infinity.

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As mentioned immediately after Proposition 2, in addition to having small $\max_{|u|=3} |J_u|$ values, Paley designs also have small $\max_{|u|=4} |J_u|$ values. This makes these designs even more attractive, because $\max_{|u|=4} |J_u|/n$ represents the largest correlation of two-factor interactions with each other. For example, among all 7570 $\mathrm{OA}(28, 2^{27}, 2)$ (Schoen, Eendebak, and Nguyen, 2010), twelve arrays have $\max_{|u|=3} |J_u| = 12$, and all twelve have the maximum generalized resolution. Only one of these twelve arrays has $\max_{|u|=4} |J_u| = 12$, and the other eleven all have $\max_{|u|=4} |J_u| = 20$. Not surprisingly, the best one is precisely the Paley design.

4 Main Results

We examine the general case in this section and consider an $\mathrm{OA}(n, 2^m, 2)$, for $n/2 \leq m \leq n-1$. The next result generalizes Proposition 1.

Theorem 1. *Let D be an $OA(n, 2^m, 2)$, where $n/2 \leq m \leq n-1$. We then have that*

$$\max_{|u|=3} |J_u| \geq L(n, m) = n - 8 \left\lfloor (n/8)(1 - q^{1/2}) \right\rfloor,$$

where $q = (2m-n)/((m-1)(m-2))$.

Proof. Write $D = (d_1, \ldots, d_m)$. Then, there exist real vectors $e_1, \ldots, e_p$, with $p = n - m - 1$, such that $1_n/n^{1/2}, d_1/n^{1/2}, \ldots, d_m/n^{1/2}, e_1, \ldots, e_p$ form an orthonormal basis in the n-dimensional Euclidean space. Because $d_1 d_2, \ldots, d_1 d_m$ are mutually orthogonal, for any given $k = 1, \ldots, p$, we have $\sum_{j=2}^{m} |\langle d_1 d_j/n^{1/2}, e_k \rangle|^2 \leq 1$, where $\langle x, y \rangle$ denotes the inner product of x and y. Noting that $d_1 d_1 = 1_n$ and $\langle 1_n, e_k \rangle = 0$, we

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obtain $\sum_{j=1}^{m} |\langle d_1 d_j, e_k\rangle|^2 \leq n$. In general, we have $\sum_{j=1}^{m} |\langle d_i d_j, e_k\rangle|^2 \leq n$, for every $i = 1, \ldots, m$. Summing over i and removing redundancy gives $\sum_{1\leq i<j\leq m} |\langle d_i d_j, e_k\rangle|^2 \leq nm/2$. Summing over k and rearranging, we obtain $\sum_{1\leq i<j\leq m} \sum_{k=1}^{p} |\langle d_i d_j, e_k\rangle|^2 \leq pnm/2$. This shows that $\min_{i<j} \sum_{k=1}^{p} |\langle d_i d_j, e_k\rangle|^2 \leq pn/(m-1)$, which allows us to take i^* and j^* so that $\sum_{k=1}^{p} |\langle d_{i^*} d_{j^*}, e_k\rangle|^2 \leq pn/(m-1)$. Now, consider vector $d_{i^*} d_{j^*}$ under the orthonormal basis $1_n/n^{1/2}, d_1/n^{1/2}, \ldots, d_m/n^{1/2}, e_1, \ldots, e_p$. We then have that $n = \sum_{j\neq i^*,j^*} |\langle d_{i^*} d_{j^*}, d_j/n^{1/2}\rangle|^2 + \sum_{k=1}^{p} |\langle d_{i^*} d_{j^*}, e_k\rangle|^2$, owing to the orthogonality of $d_{i^*} d_{j^*}$ to $1_n, d_{i^*}, d_{j^*}$. Using $|\langle d_{i^*} d_{j^*}, d_j\rangle| \leq \max_{|u|=3} |J_u|$ and the just established bound on $\sum_{k=1}^{p} |\langle d_{i^*} d_{j^*}, e_k\rangle|^2$, we obtain that $n^2 \leq (m-2) \max_{|u|=3} |J_u|^2 + pn^2/(m-1)$. Solving for $\max_{|u|=3} |J_u|$ and using $p = n-1-m$, we obtain that $\max_{|u|=3} |J_u| \geq nq^{1/2}$, where $q = (2m-n)/((m-1)(m-2))$. Theorem 1 then follows by noting that $n - \max_{|u|=3} |J_u|$ is a multiple of eight. □

The lower bound $L(n, m)$ in Theorem 1 reduces to that in Proposition 1 when $m = n-1$ as $q = 1/(n-3)$. For $m = n/2$, Theorem 1 also gives sensible results, because it is easily seen that $L(n, n/2) = 0$ if n is a multiple of eight and $L(n, n/2) = 4$ otherwise. The case of $m = n/2$ is discussed further in Section 5.

Let P_n denote a Paley design of order n, which is a saturated design of n runs for $n-1$ factors. Further, let $P_{n,m}$ be any design that selects m columns from P_n and, for convenience, we still call $P_{n,m}$ a Paley design. In Section 3, we proved that P_n has the maximum generalized resolution for $n = 12, 20, 24, 28, 32, 44, 60, 72$, and 80. Using the bound in Theorem 1, we can establish that for each of these orders, design $P_{n,m}$ also has

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the maximum generalized resolution for a range of m values. For $n = 36$, a Paley design is unavailable because $n - 1 = 35$ is not a prime power. Using the results of Shi and Tang (2018) on Hadamard matrices of order 36, an $\mathrm{OA}(36, 2^{35}, 2)$ with $\max_{|u|=3} |J_u| = 12$ can be obtained. This design and many of its subdesigns can be shown by Theorem 1 to have the maximum generalized resolution. We present in Table 1 the above results for $n = 20, 24, 28, 32, 36, 44, 60, 72$, and 80. The case of $n = 12$ is excluded from Table 1 because, trivially, $P_{12,m}$ has the maximum generalized resolution for any $m \geq 3$.

Table 1 also includes other designs with maximum generalized resolutions, obtained later in this paper.

Table 1. Nonregular designs with maximum generalized resolutions

n	$\max_{\lvert u\rvert=3} \lvert J_u\rvert$	m	source
20	12	$13, 14, \ldots, 19$	Theorem 1
24	8	$13, 14, \ldots, 23$	Theorem 1
28	12	$17, 18, \ldots, 27$	Theorem 1
32	8	$17, 18, \ldots, 31$	Theorem 1
36	12	$21, 22, \ldots, 35$	Theorem 1
44	12	$25, 26, \ldots, 43$	Theorem 1
60	12	$33, 34, \ldots, 59$	Theorem 1
72	16	$52, 53, \ldots, 71$	Theorem 1
80	16	$54, 55, \ldots, 79$	Theorem 1
48	8	$25, 26, \ldots, 44$	Theorem 4, Example 3
64	16	$52, 53, \ldots, 62$	Theorem 3, Example 4
96	16	$60, 61, \ldots, 92$	Theorem 4
128	16	$75, 76, \ldots, 124$	Theorem 4, Example 4
144	16	$83, 84, \ldots, 121$	Theorem 2, Example 1
192	16	$106, 107, \ldots, 176$	Theorem 4, Example 3
768	32	$511, 512, \ldots, 704$	Theorem 4, Example 3
20	4	$3, 4, \ldots, 10$	Theorem 5
28	4	$3, 4, \ldots, 14$	Theorem 5
36	4	$3, 4, \ldots, 18$	Theorem 5
52	4	$3, 4, \ldots, 26$	Theorem 5
60	4	$3, 4, \ldots, 30$	Theorem 5
76	4	$3, 4, \ldots, 38$	Theorem 5
84	4	$3, 4, \ldots, 42$	Theorem 5
100	4	$3, 4, \ldots, 50$	Theorem 5
108	4	$3, 4, \ldots, 54$	Theorem 5
124	4	$3, 4, \ldots, 62$	Theorem 5

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The next major task of this section is to present constructions of nonregular designs with large generalized resolutions. Using Theorem 1, many of the resulting designs are shown to have the maximum generalized resolutions.

Theorem 2. *Let A be an $OA(n_1, 2^{m_1}, 2)$ with $R(A) = 4 - \epsilon_1$, and B be an $OA(n_2, 2^{m_2}, 2)$ with $R(B) = 4 - \epsilon_2$. Then, $D = A \otimes B$ is an $OA(n_1 n_2, 2^{m_1 m_2}, 2)$ with $R(D) = 4 - \epsilon_1 \epsilon_2$.*

Proof. Write $A = (a_1, a_2, \ldots, a_{m_1})$ and $B = (b_1, b_2, \ldots, b_{m_2})$. Then, the columns of $D = A \otimes B$ have the form of $a_i \otimes b_j$. By a result of Tang (2006), we have

$$J(a_{i_1} \otimes b_{j_1}, a_{i_2} \otimes b_{j_2}, a_{i_3} \otimes b_{j_3}) = J(a_{i_1}, a_{i_2}, a_{i_3}) J(b_{j_1}, b_{j_2}, b_{j_3}),$$

where $J(x, y, z)$ is the J-characteristic of three column vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and $z = (z_1, \ldots, z_n)$, defined as $J(x, y, z) = \sum_{i=1}^{n} x_i y_i z_i$. It is obvious that $J(a_{i_1}, a_{i_2}, a_{i_3}) = 0$ if $a_{i_1} = a_{i_2}$ because A is an orthogonal array. We therefore have that $\max_{|u|=3} |J_u(D)| = \max_{|u|=3} |J_u(A)| \max_{|u|=3} |J_u(B)|$. Theorem 2 follows. □

Example 1. *Let A and B both be P_{12}, an $OA(12, 2^{11}, 2)$, that has a generalized resolution $R = 4 - 1/3$. Then, Theorem 2 states that design $D = A \otimes B$, an $OA(144, 2^{121}, 2)$, has a generalized resolution $R(D) = 4 - 1/9$. In terms of J-characteristics, we have that $\max_{|u|=3} |J_u(D)| = 16$. Because $L(144, 121) = 16$, by Theorem 1, design D has the maximum generalized resolution. Deleting some columns from D gives a design with fewer factors. Because $L(144, m) = 16$ for all m values in the range of $83 \leq m \leq 121$, we therefore obtain an $OA(144, 2^m, 2)$ with the maximum generalized resolution for every $m = 83, 84, \ldots, 121$.*

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Example 2. *Let A be P_{12} and B be P_{32}, so $R(A) = 4 - 1/3$ and $R(B) = 4 - 1/4$. Using Theorem 2, we obtain a design $D = A \otimes B$, an $OA(384, 2^{341}, 2)$, that has a generalized resolution $R = 4 - 1/12$. This design does not achieve the lower bound in Theorem 1. Although we do not know if it has the maximum generalized resolution, we do know that it has a large generalized resolution, close to four.*

The next result has a flavor similar to that of Theorem 2, but offers a different perspective on the tensor product construction. Given a Hadamard matrix H, we define

$$\gamma(H) = \max_{|u|=1,3} |J_u(H)|. \tag{4.1}$$

Theorem 3. *Let H be a Hadamard matrix of order n_1 with $\gamma = \gamma(H)$, and A be an $OA(n_2, 2^m, 2)$ with $R(A) = 4 - \epsilon$. Then, design $D = H \otimes A$ is an $OA(n_1 n_2, 2^{n_1 m}, 2)$ with $R(D) = 4 - \epsilon(\gamma/n_1)$.*

Proof. It is obvious that $D = H \otimes A$ is an $\mathrm{OA}(n_1 n_2, 2^{n_1 m}, 2)$. To prove $R(D) = 4 - \epsilon(\gamma/n_1)$, again using Lemma 2 of Tang (2006), we obtain

$$\max_{|u|=1,3} |J_u(D)| = \max_{|u|=1,3} |J_u(H)| \max_{|u|=1,3} |J_u(A)|.$$

Because A and D are orthogonal arrays of strength two, we have that $J_u(D) = J_u(A) = 0$, for all u, with $|u| = 1$. This shows that $\max_{|u|=3} |J_u(D)| = \max_{|u|=1,3} |J_u(H)| \max_{|u|=3} |J_u(A)|$, which entails that $R(D) = 4 - \epsilon(\gamma/n_1)$. □

If H in Theorem 3 has a column of all ones, then $\gamma(H) = n_1$, and thus $R(D) = 4 - \epsilon$. This special case was established in Shi and Tang (2018).

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One can easily check that the Hadamard matrix

$$H = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \tag{4.2}$$

has that $\gamma(H) = 2$.

Theorem 4. *Consider the above Hadamard matrix H of order four, and let A be an $OA(n, 2^m, 2)$ with $R(A) = 4 - \epsilon$. Define $D_0 = A$ and $D_k = H \otimes D_{k-1}$, for $k \geq 1$. Then, D_k, an $OA(n4^k, 2^{m4^k}, 2)$, has that $R(D_k) = 4 - \epsilon/2^k$, for any integer $k \geq 0$.*

Theorem 4 is immediate from Theorem 3.

Example 3. *If we take P_{12} as A, then D_k is an $OA(12 \cdot 4^k, 2^{11 \cdot 4^k}, 2)$ that has $R(D_k) = 4 - (1/3)/2^k$, for $k \geq 1$. We also have that $\max_{|u|=3} |J_u(D_k)| = 4 \cdot 2^k = 2^{k+2}$. Now, consider D_1, D_2, and D_3. Design D_1 is an $OA(48, 2^{44}, 2)$ with $\max_{|u|=3} |J_u(D_1)| = 8$. Because $L(48, m) = 8$ for $25 \leq m \leq 44$, any design by selecting m columns from D_1 has the maximum generalized resolution of $4 - 1/6$. Similarly, we obtain from D_2 an $OA(192, 2^m, 2)$ with the maximum generalized resolution for any m, with $106 \leq m \leq 176$, and from D_3 an $OA(768, 2^m, 2)$ with the maximum generalized resolution for any m, with $511 \leq m \leq 704$.*

Example 4. *If we take P_{32} as A, then D_k is an $OA(32 \cdot 4^k, 2^{31 \cdot 4^k}, 2)$, which has that $R(D_k) = 4 - (1/4)/2^k = 4 - 1/2^{k+2}$ and that $\max_{|u|=3} |J_u(D_k)| = 8 \cdot 2^k = 2^{k+3}$. Again,*

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using the lower bound in Theorem 1, we obtain from D_1 *an* $OA(128, 2^m, 2)$ *that has the maximum generalized resolution for any* m*, with* $75 \leq m \leq 124$*.*

Now, let $C_k = ((1,1)^T, (1,-1)^T)^T \otimes D_k$*, for* $k \geq 0$*. Then,* C_k *is* $OA(64 \cdot 4^k, 2^{31 \cdot 2^{2k+1}}, 2)$ *also with* $R(C_k) = 4 - 1/2^{k+2}$*, by Theorem 3. Together, designs* D_k *and* C_k*, for* $k \geq 0$*, cover all run sizes* $n \geq 32$ *that are powers of two. In this regard, the quaternary code designs, constructed by Xu and Wong (2007) for run sizes that are powers of two, are not competitive, because they have a generalized resolution of 3.5.*

An application of Theorem 3 requires a Hadamard matrix H with a small value of $\gamma(H)$, defined in (4.1). A full investigation of this problem is beyond the scope of this study. Notwithstanding, we provide some results from a preliminary study.

Proposition 3. *(i) The value of* $\gamma(H)$ *must be even.*

(ii) If a Hadamard matrix of order $n \geq 4$ *exists, we can find one such that* $\gamma(H) \leq n-2$*.*

(iii) We have that $\gamma(H_1 \otimes H_2) = \gamma(H_1)\gamma(H_2)$ *for any two Hadamard matrices* H_1 *and* H_2*.*

Proof. Parts (i) and (iii) are obvious. To prove part (ii), let H be a Hadamard matrix of order n that contains a column of all ones. Then, $\gamma(H) = n$. Now, we switch the signs of one row of H and obtain another Hadamard matrix, say H'. Because $J_u(H)$ is a multiple of four, for all u with $|u| = 1, 3$ and $J_u(H')$ differs from $J_u(H)$ by ± 2, we have that $J_u(H')$ is even, but not a multiple of four; thus, $\gamma(H') \leq n-2$. □

Let $\gamma_n = \min_H \gamma(H)$, where the minimization is over all Hadamard matrices of order n. Obviously, we have that $\gamma_2 = \gamma_4 = 2$. By a computer search, we obtain that $\gamma_8 = 4$ and $\gamma_{12} = 8$. A useful lower bound on γ_n is provided here.

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Proposition 4. *We have that* $\gamma_n \geq n^{1/2}$.

Proof. Let $H = (h_1, \ldots, h_n)$ be a Hadamard matrix of order n. Considering vector 1_n under the orthonormal basis $\{h_1/n^{1/2}, \ldots, h_n/n^{1/2}\}$, we obtain that $n^2 = \sum_{j=1}^{n} |J(h_j)|^2$. Because $|J(h_j)| \leq \gamma(H)$, we obtain $\gamma(H) \geq n^{1/2}$. Therefore $\gamma_n \geq n^{1/2}$. □

Using the bound in Proposition 4, in conjunction with the fact that γ_n is even, we have that $\gamma_{16} \geq 4$ and $\gamma_n \geq 6$ for $n = 20, 24, 28, 32$, and 36. For $n = 4^k$, we have that $\gamma_{4^k} \geq 2^k$. Using the Hadamard matrix of order four given in (4.2) and Part (iii) of Proposition 3, we can construct a Hadamard matrix H of order 4^k with $\gamma(H) = 2^k$. This establishes that $\gamma_{4^k} = 2^k$, which means that D_k in Theorem 4 is the best if A is also the best.

5 Further Results

This section provides another construction of nonregular designs with maximum generalized resolutions. The focus here is on designs with $\max_{|u|=3} |J_u| = 4$ when n is not a multiple of eight.

Consider an $\mathrm{OA}(n, 2^m, 2)$ when n is not a multiple of eight. From Deng and Tang (2002), we know that J_u is also not a multiple of eight for all u with $|u| = 3$. This means that $\max_{|u|=3} |J_u| \geq 4$. The best situation is $\max_{|u|=3} |J_u| = 4$, which happens if and only if $|J_u| = 4$, for all $|u| = 3$. An $\mathrm{OA}(n, 2^m, 2)$ with $|J_u| = 4$, for all $|u| = 3$, has a maximum generalized resolution of $4 - 4/n$.

For $n = 12$, design P_{12}, an $\mathrm{OA}(12, 2^{11}, 2)$, has $|J_u| = 4$, for $|u| = 3$. For $n = 20, 28$,

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and 36, computational results from Deng and Tang (2002), Sun, Li, and Ye (2008), and Schoen, Vo-Thanh, and Goos (2017) show that OA$(n, 2^m, 2)$s with $\max_{|u|=3} |J_u| = 4$ can be found for $m \leq n/2$. This is no coincidence, because the following general result can be established.

Let $n = 8k + 4$. Suppose that $s = n/2 - 1 = 4k + 1$ is a prime power. Consider the following two matrices:

$$C = \begin{bmatrix} -1 & 1_s^T \\ 1_s & Q - I_s \\ -1 & -1_s^T \\ -1_s & -Q - I_s \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 1_s^T \\ 1_s & Q - I_s \\ 1 & 1_s^T \\ -1_s & -Q - I_s \end{bmatrix}, \tag{5.1}$$

where Q is given in Section 2. We see that matrix C is an $n \times (n/2)$ submatrix of the Hadamard matrix from Paley's second construction, and thus has orthogonal columns. However, C is not an orthogonal array because all of its columns have more -1s than $+1$s. Matrix D is an OA$(n, 2^{n/2}, 2)$, obtained from C by multiplying -1 to the $(n/2+1)$th row of C, while leaving all other rows untouched.

Theorem 5. *Suppose that $n/2 - 1 = 4k + 1$ is a prime power. Then, the above design D as in* (5.1) *is an OA$(n, 2^{n/2}, 2)$ with $\max_{|u|=3} |J_u| = 4$, and therefore has a maximum generalized resolution $R = 4 - 4/n$.*

Proof. We need to show that $|J_u| = 4$, for all $|u| = 3$. Note that a projection design of D onto three factors has many mirror image runs. When calculating the J value of a projection design onto three factors, these mirror image runs do not contribute, so we only need to focus on those runs that have no mirror images. Looking at the projection

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design onto the first three factors, we see that only the following six runs need to be considered: runs 1, 2, 3 and runs $(n/2+1)$, $(n/2+2)$, and $(n/2+3)$. Using the symmetry of matrix Q, the J value for the three columns of the matrix consisting of these six runs must be ± 4. For other projection designs onto three factors, without loss of generality, we consider the projection design onto columns 2, 3, and 4. For this projection design, we see that we need only consider runs 1, 2, 3, 4 and runs $(n/2+1)$, $(n/2+2)$, $(n/2+3)$, and $(n/2+4)$. Owing to the symmetry of Q, matrix A_1 collecting runs 1, 2, 3, and 4 as rows and matrix A_2 collecting runs $(n/2+1)$, $(n/2+2)$, $(n/2+3)$, and $(n/2+4)$ as rows must have forms

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & q_1 & q_2 \\ q_1 & -1 & q_3 \\ q_2 & q_3 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -q_1 & -q_2 \\ -q_1 & -1 & -q_3 \\ -q_2 & -q_3 & -1 \end{bmatrix}.$$

The J value for the three columns of the 8×3 matrix obtained by combining the rows of A_1 and A_2 is therefore given by $2 - 2(q_1q_2 + q_1q_3 + q_2q_3)$, which is equal to ± 4 for all possible scenarios of $q_1 = \pm 1$, $q_2 = \pm 1$, and $q_3 = \pm 1$. We have therefore shown that $J_u = \pm 4$, for all $|u| = 3$. $\square$

For $n \leq 128$, Theorem 5 allows the construction of an $\mathrm{OA}(n, 2^{n/2}, 2)$ with a generalized resolution $R = 4 - 4/n$, for $n = 20, 28, 36, 52, 60, 76, 84, 100, 108,$ and 124. Obviously, designs from deleting some columns from this $\mathrm{OA}(n, 2^{n/2}, 2)$ have the same generalized resolution.

An inescapable question arising from Theorem 5 is how large m can be if an $\mathrm{OA}(n, 2^m, 2)$ is to have $R = 4-4/n$ when $n = 8k+4$. Theorem 5 states that $m \geq n/2$ if $n/2-1 = 4k+1$

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is a prime power. The next result gives an upper bound on m.

Theorem 6. *If an* $OA(n, 2^m, 2)$ *is to have* $\max_{|u|=3} |J_u| = 4$ *for* $n = 8k + 4 \geq 20$*, then it is necessary that* $m \leq n/2 + 2$.

Proof. We need only prove that $L(n, m) \geq 12$ for $m = n/2 + 3$, where $L(n, m)$ is the lower bound on $\max_{|u|=3} |J_u|$ in Theorem 1. Because $n = 8k + 4$, we have that $L(n, m) = (8k + 4) - 8 \left\lfloor k + 0.5 - (n/8)q^{1/2} \right\rfloor$. Thus, to prove $L(n, m) \geq 12$, we need only prove that $0.5 - (n/8)q^{1/2} < 0$, which is equivalent to $q > (4/n)^2$. Plugging $n = 8k + 4$ and $m = n/2+3 = 4k+5$ into $q > (4/n)^2$, and then doing some elementary algebra, we obtain an equivalent inequality $2k(2k - 1) > 3$. This last inequality holds whenever $k \geq 2$ and, correspondingly, $n = 8k + 4 \geq 20$. □

A weaker version than Theorem 6 can be proved using results from supersaturated designs. Suppose that there exists a design of $n = 8k+4$ runs and m factors with $|J_u| = 4$, for all $|u| = 3$. By half fractioning (Lin, 1993), we can construct a supersaturated design of $n/2$ runs for $m-1$ factors with the property that the inner product of any two columns is ± 2. By Cheng and Tang (2001, Theorem 4), we obtain that $m - 1 \leq n/2 + 2$, and hence that $m \leq n/2 + 3$.

Theorem 6 cannot be improved by just using Theorem 1, because we can show in a way similar to proving Theorem 6 that $L(n, n/2 + 2) = 4$. Thus, the existence of an $\mathrm{OA}(n, 2^{n/2+1}, 2)$ or an $\mathrm{OA}(n, 2^{n/2+2}, 2)$ with $R = 4 - 4/n$ for $n \geq 44$ is still theoretically possible, even though the impossibility has been established for $n = 20, 28$, and 36 by Sun, Li, and Ye (2008) and Schoen, Vo-Thanh, and Goos (2017).

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When n is a multiple of eight, an OA$(n, 2^{n/2}, 3)$, a strength-three array, can be constructed by folding over a Hadamard matrix of order $n/2$. Shi and Tang (2018) showed that if a Hadamard matrix of order $n/2$ has type $b_{\max}$, then its foldover has the maximum generalized resolution given by $R = 4 + 8b_{\max}/(n/2) = 4 + 16b_{\max}/n$.

6 Conclusion

We provide a comprehensive study on the construction of nonregular designs with maximum generalized resolutions. We derive lower bounds on the value of $\max_{|u|=3} |J_u|$, and present several methods for constructing designs with large generalized resolutions. With the help of lower bounds, many designs are shown to have maximum generalized resolutions. The following is a summary of existing results and the new results obtained in this study.

- From the computational results of Sun, Li, and Ye (2008), Schoen, Eendebak, and Nguyen (2010), and Schoen, Vo-Thanh, and Goos (2017), we can deduce that designs with maximum generalized resolutions have been obtained for $n = 12, 16, 20$, and 24 with all $m \leq n - 1$, for $n = 28$ with $m \leq 14$ and $m = 27$, for $n = 32$ with $17 \leq m \leq 31$, and for $n = 36$ with $m \leq 18$. The other designs in Table 1 are new.

- The general result of Xu and Wong (2007) is that the QC designs of n runs they constructed have a generalized resolution of 3.5, where n is a power of two. In contrast, our design D_k, an OA$(32 \cdot 4^k, 2^{31 \cdot 4^k}, 2)$, given in Example 4, has $R(D_k) =$

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$4 - 1/2^{k+2}$; our design C_k, an OA$(64 \cdot 4^k, 2^{31 \cdot 2^{2k+1}}, 2)$, also given in Example 4, has $R(C_k) = 4 - 1/2^{k+2}$. Together, designs D_k and C_k, for $k \geq 0$, cover all run sizes $n \geq 32$ that are powers of two.

Future work could examine the minimum G_2-aberration properties of the constructed designs, and more specifically, select those designs, using the G_2-aberration criterion, from among the constructed designs that have large or maximum generalized resolutions. Hadamard matrices of type $b_{\max}$ can be used to construct saturated strength-three orthogonal arrays with maximum generalized resolutions, but the general problem of constructing strength-three arrays with maximum generalized resolutions is yet to be solved. This is an important topic for further investigation.

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