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Power laws distributions in objective priors

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Abstract: Using objective priors in Bayesian applications has become a common way of analyzing data without using subjective information. Formal rules are usually used to obtain these prior distributions, and the data provide the dominant information in the posterior distribution. However, these priors are typically improper, and may lead to an improper posterior. Here, for a general family of distributions, we show that the objective priors obtained for the parameters either follow a power law distribution, or exhibit asymptotic power law behavior. As a result, we observe that the exponents of the model are between 0.5 and 1. Understanding this behavior allows us to use the exponent of the power law directly to verify whether such priors lead to proper or improper posteriors. The general family of distributions we consider includes essential models such as the exponential, gamma, Weibull, Nakagami-m, half-normal, Rayleigh, Erlang, and Maxwell Boltzmann distributions, among others. In summary, we show that understanding the mechanisms that describe the shape of a prior provides essential information that can be used to understand the properties of posterior distributions.

Key words and phrases: Bayesian inference; objective prior; power-law.

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1 Introduction

Bayesian methods have become popular statistical procedures in areas from medicine to engineering (Lloyd-Jones et al., 2019; Wang and Matthies, 2019). In a Bayesian approach, the parameters in a statistical model are assumed to be random variables (Bernardo, 2005), in contrast to a frequentist approach, in which these parameters are constant. Moreover, a subjective ingredient can be included in the model to represent the knowledge of a specialist (see O'Hagan et al. (2006)). On the other hand, in many situations, we are interested in obtaining a prior distribution, that guarantees that the information provided by the data will not be overshadowed by subjective information. In this case, an objective analysis is recommended using non-informative priors derived from formal rules (Consonni et al., 2018; Kass and Wasserman, 1996). Although several studies have assumed weakly informative priors (flat priors) as non-informative priors, Bernardo (2005) argues that using simple proper priors, which are supposed to be non-informative, may hide significant unwarranted assumptions, which may dominate or even invalidate the statistical analysis.

Objective priors are constructed using formal rules (Kass and Wasserman, 1996) and are usually improper, that is, they do not correspond to a proper probability distribution and may lead to improper posteriors, which is undesirable. See Leisen et al. (2019) for a recent discussion about the limitations of objective priors. According to Northrop and Attalides (2016), there are no simple conditions that can be used to prove that an improper prior yields a proper posterior for a particular distribution. Therefore, a case-by-case investigation is needed to check the propriety of the posterior distribution. For the Stacy (1962) general family of distributions, we solve this problem by proving that if an objective prior asymptotically follows a power law model with the exponent in some particular regions, then the posterior distribution can be proper or improper. As a result, one can easily check whether the obtained posterior is proper or improper, by examining the behavior of the improper prior as a power law model.

Understanding when the data follow a power law distribution can indicate the mechanisms that describe the natural phenomenon in question. Power law distributions appear in many physical, biological, and man-made phenomena, for instance, they can be used to describe biological networks (Pržulj, 2007), infectious diseases (Geilhufe et al., 2014), the sizes of craters on the moon (Newman, 2005), the intensity function in repairable systems (Louzada et al., 2019), and energy dissipation in cyclones (Corral et al., 2010) (see also Goldstein et al. (2004); Barrat et al. (2008); Newman (2018)). The probability density function (PDF) of a power law distribution is represented as

$$\pi(\theta) = c\,\theta^{-\lambda}, \tag{1.1}$$

where c is a normalized constant and λ is the exponent parameter. When applying Bayesian methods, the normalizing constant is usually omitted, and thus the prior can be represented by $\pi(\theta) \propto \theta^{-\lambda}$.

Here, we analyze the behavior of different objective priors related to the parameters of various distributions. We show that the asymptotic behavior follows a power law model with an exponent between 0.5 and 1. This may lead to a proper or improper posterior,

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depending on the exponent values of the priors. Power law distributions with an exponent smaller than one have been observed by Goldstein et al. (2004), Deluca and Corral (2013) and Hanel et al. (2017), with the objective priors obtained using Jeffreys' rule (Kass and Wasserman, 1996), Jeffreys' prior (Jeffreys, 1946), and reference priors (Bernardo, 1979, 2005; Berger et al., 2015), respectively. Although the posterior distribution may be proper, the posterior moments can be infinite. Therefore, we also provide sufficient conditions to verify whether the posterior moments are finite. These results are important, because the power law behavior of a prior distribution related to a particular distribution can help us to understand the shape of the prior to use when additional complexity (e.g., random censoring, long-term survival, etc.) is present, or when it is difficult or not possible to obtain priors from formal rules.

The remainder of this paper is organized as follows. Section 2 presents the theorems that provide the necessary and sufficient conditions for the posterior distributions to be proper, depending on the asymptotic behavior of the prior as a power law model. Here, we also present sufficient conditions to check whether the posterior moments are finite. In Section 3, we examine the behavior of the objective priors. Section 4 provides an application to a real data set. Finally, Section 5 concludes the paper.

2 A general model

The Stacy family of distributions plays an important role in statistics, and has proven to be very flexible in modeling data in areas such as climatology, meteorology, medicine,

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reliability, and image processing, among others (Stacy, 1962). A random variable X follows a Stacy distribution if its PDF is given by

$$f(x|\boldsymbol{\theta}) = \alpha\mu^{\alpha\phi}x^{\alpha\phi-1}\exp\left(-(\mu x)^{\alpha}\right)/\Gamma(\phi), \quad x > 0, \tag{2.1}$$

where $\Gamma(\phi) = \int_0^{\infty} e^{-x}x^{\phi-1}dx$ is the gamma function, $\boldsymbol{\theta} = (\phi, \mu, \alpha)$, $\alpha > 0$ and $\phi > 0$ are the shape parameters and $\mu > 0$ is a scale parameter. The Stacy distribution unifies many important distributions, as shown in Table 1, and is sometimes referred to the as generalized gamma (GG) distribution.

Inference procedures related to the parameters are conducted using the joint posterior distribution for $\boldsymbol{\theta}$ given by the product of the likelihood function and the prior distribution $\pi(\boldsymbol{\theta})$, divided by a normalizing constant $d(\boldsymbol{x})$, resulting in

$$p(\boldsymbol{\theta}|\boldsymbol{x}) = \frac{\pi(\boldsymbol{\theta})}{d(\boldsymbol{x})}\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\}, \tag{2.2}$$

where

$$d(\boldsymbol{x}) = \int\limits_{\mathcal{A}} \pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\}d\boldsymbol{\theta}, \tag{2.3}$$

and $\mathcal{A} = \{(0,\infty)\times(0,\infty)\times(0,\infty)\}$ is the parameter space of $\boldsymbol{\theta}$. Considering any prior of the form $\pi(\boldsymbol{\theta}) \propto \pi(\mu)\pi(\alpha)\pi(\phi)$, our main aim is to analyze the asymptotic behavior of the priors that leads to a power law distribution, thus determining the necessary and sufficient conditions for the posterior to be proper, that is, $d(\boldsymbol{x}) < \infty$.

To study such asymptotic behavior, we require the following definitions and propositions to prove the results related to the posterior distribution. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denote the *extended real number line* with the usual order ($\geq$), $\mathbb{R}^+$ denote the positive

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Table 1: Distributions included in the Stacy family of distributions (see equation 2.1).

Distribution	μ	ϕ	α
Exponential	$\cdot$	1	1
Rayleigh	$\cdot$	1	2
Haf-Normal	$\cdot$	0.5	2
Maxwell Boltzmann	$\cdot$	$\frac{3}{2}$	2
scaled chi-square	$\cdot$	0.5n	1
chi-square	2	0.5n	1
Weibull	$\cdot$	1	$\cdot$
Generalized Haf-Normal	$\cdot$	2	$\cdot$
Gamma	$\cdot$	$\cdot$	1
Erlang	$\cdot$	n	$\cdot$
Nakagami	$\cdot$	$\cdot$	2
Wilson-Hilferty	$\cdot$	$\cdot$	3
Lognormal	$\cdot$	$\phi \to \infty$	$\cdot$

$n \in \mathbb{N}$

real numbers, and $\mathbb{R}_0^+$ denote the positive real numbers including zero, and denote $\overline{\mathbb{R}}^+$ and $\overline{\mathbb{R}}_0^+$ analogously. Moreover, if $M \in \mathbb{R}^+$ and $a \in \overline{\mathbb{R}}^+$, define $M \cdot a$ as the usual product if $a \in \mathbb{R}$, and $M \cdot a = \infty$ if $a = \infty$.

Definition 1. Let $a \in \overline{\mathbb{R}}_0^+$ and $b \in \overline{\mathbb{R}}_0^+$. We say that $a \lesssim b$ if there exist $M \in \mathbb{R}^+$ such

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that $a \leq M \cdot b$. If $a \lesssim b$ and $b \lesssim a$, then we say that $a \propto b$.

In other words, by Definition 1, we have $a \lesssim b$ if either $a < \infty$ or $b = \infty$, and we have $a \propto b$ if either $a < \infty$ and $b < \infty$, or $a = b = \infty$.

Definition 2. Let $\mathrm{g} : \mathcal{U} \to \overline{\mathbb{R}}_0^+$ and $\mathrm{h} : \mathcal{U} \to \overline{\mathbb{R}}_0^+$, where $\mathcal{U} \subset \mathbb{R}$. We say that $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ if there exist $M \in \mathbb{R}^+$ such that $\mathrm{g}(x) \leq M\,\mathrm{h}(x)$, for every $x \in \mathcal{U}$. If $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ and $\mathrm{h}(x) \lesssim \mathrm{g}(x)$, then we say that $\mathrm{g}(x) \propto \mathrm{h}(x)$.

Definition 3. Let $\mathcal{U} \subset \mathbb{R}$ and $a \in \mathcal{U}' \cup \{\infty\}$, where $\mathcal{U}'$ is the closure of $\mathcal{U}$ in $\mathbb{R}$, and let $\mathrm{g} : \mathcal{U} \to \mathbb{R}^+$ and $\mathrm{h} : \mathcal{U} \to \mathbb{R}^+$. We say that $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ if $\limsup_{x\to a} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} < \infty$. If $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ and $\mathrm{h}(x) \underset{x\to a}{\lesssim} \mathrm{g}(x)$, then we say that $\mathrm{g}(x) \underset{x\to a}{\propto} \mathrm{h}(x)$.

The relations $\mathrm{g}(x) \underset{x\to a^+}{\lesssim} \mathrm{h}(x)$ and $\mathrm{g}(x) \underset{x\to a^-}{\lesssim} \mathrm{h}(x)$, for $a \in \mathbb{R}$, are defined analogously. Note that if for some $d \in \mathbb{R}^+$, we have $\lim_{x\to c} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} = d$, then it follows directly that $\mathrm{g}(x) \underset{x\to c}{\propto} \mathrm{h}(x)$. The following proposition is a direct consequence of the above definition.

Proposition 1. *Let $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$, $c \in [a, b]$ and $r \in \mathbb{R}^+$, and let $f_1(x)$, $f_2(x)$, $g_1(x)$, and $g_2(x)$ be nonnegative continuous functions with domain (a, b), such that $f_1(x) \underset{x\to c}{\lesssim} f_2(x)$ and $g_1(x) \underset{x\to c}{\lesssim} g_2(x)$. Then, the following hold:*

$$f_1(x)g_1(x) \underset{x\to c}{\lesssim} f_2(x)g_2(x) \quad \text{and} \quad f_1(x)^r \underset{x\to c}{\lesssim} f_2(x)^r.$$

The following proposition relates to Definitions 2 and 3.

Proposition 2. *Let $\mathrm{g} : (a, b) \to \mathbb{R}_0^+$ and $\mathrm{h} : (a, b) \to \mathbb{R}^+$ be continuous functions on $(a, b) \subset \mathbb{R}$, where $a \in \mathbb{R}$ and $b \in \overline{\mathbb{R}}$. Then, $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ if and only if $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ and $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$.*

Proof. See Supplementary Material S1.2. □

Note that if $\mathrm{g} : (a, b) \to \mathbb{R}^+$ and $\mathrm{h} : (a, b) \to \mathbb{R}^+$ are continuous functions on $(a, b) \subset \mathbb{R}$, then by continuity, it follows directly that $\lim_{x\to c} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} = \dfrac{\mathrm{g}(c)}{\mathrm{h}(c)} > 0$ and, therefore, $\mathrm{g}(x) \underset{x\to c}{\propto} \mathrm{h}(x)$, for every $c \in (a, b)$. This fact and Proposition 2 directly imply the following.

Proposition 3. *Let* $\mathrm{g} : (a, b) \to \mathbb{R}^+$ *and* $\mathrm{h} : (a, b) \to \mathbb{R}^+$ *be continuous functions in* $(a, b) \subset \mathbb{R}$, *where* $a \in \mathbb{R}$ *and* $b \in \overline{\mathbb{R}}$, *and let* $c \in (a, b)$. *Then, if* $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ *(or* $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$*), we have* $\int_a^c g(t)\, dt \lesssim \int_a^c h(t)\, dt$ *(respectively,* $\int_c^b g(t)\, dt \lesssim \int_c^b h(t)\, dt$ *).*

2.1 Case when α is known

Let $p(\boldsymbol{\theta}|\boldsymbol{x}, \alpha)$ be of the form (2.2), but consider α fixed and $\boldsymbol{\theta} = (\phi, \mu)$. Then, the normalizing constant is given by

$$d(\boldsymbol{x}; \alpha) \propto \int_{\mathcal{A}} \frac{\pi(\boldsymbol{\theta})}{\Gamma(\phi)^n} \left\{ \prod_{i=1}^{n} x_i^{\alpha\phi-1} \right\} \mu^{n\alpha\phi} \exp\left\{ -\mu^{\alpha} \sum_{i=1}^{n} x_i^{\alpha} \right\} d\boldsymbol{\theta}, \tag{2.4}$$

where $\mathcal{A} = \{(0, \infty) \times (0, \infty)\}$ is the parameter space. Here, our objective is reduced to analyzing $\pi(\boldsymbol{\theta}) \propto \pi(\mu)\pi(\phi)$ and finding sufficient and necessary conditions for $d(\boldsymbol{x}; \alpha) < \infty$.

Theorem 1. *Suppose that* $\pi(\mu, \phi) < \infty$, *for all* $(\mu, \phi) \in \mathbb{R}_+^2$, $n \in \mathbb{N}^+$, $\pi(\mu, \phi) = \pi(\mu)\pi(\phi)$, *and that the priors exhibit asymptotic power law behavior, with*

$$\pi(\mu) \lesssim \mu^k, \quad \pi(\phi) \underset{\phi\to 0^+}{\lesssim} \phi^{r_0}, \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\lesssim} \phi^{r_\infty},$$

such that $k = -1$ *with* $n > -r_0$, *or* $k > -1$ *with* $n > -r_0 - 1$. *Then,* $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is proper.*

Proof. See Supplementary Material S1.3. □

Theorem 2. *Suppose that $\pi(\mu,\phi) > 0$, $\forall(\mu,\phi) \in \mathbb{R}_+^2$, $n \in \mathbb{N}^+$, $\pi(\mu,\phi) \gtrsim \pi(\mu)\pi(\phi)$, and the priors exhibit asymptotic power law behavior, where $\pi(\mu) \gtrsim \mu^k$ and one of the following holds:*

i) $k < -1$; *or*

ii) $k > -1$, *where* $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ *with* $n \leq -r_0 - 1$; *or*

iii) $k = -1$, *where* $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ *with* $n \leq -r_0$,

then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is improper.

Proof. See Supplementary Material S1.4. □

Theorem 3. *Let $\pi(\mu,\phi) = \pi(\mu)\pi(\phi)$ and the behavior of $\pi(\mu)$ and $\pi(\phi)$ follow the asymptotic power law distribution given by*

$$\pi(\mu) \propto \mu^k, \quad \pi(\phi) \underset{\mu\to 0^+}{\propto} \phi^{r_0}, \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\propto} \phi^{r_\infty},$$

for $k \in \mathbb{R}$, $r_0 \in \mathbb{R}$, and $r_\infty \in \mathbb{R}$. The posterior related to $\pi(\mu,\phi)$ is proper if and only if $k = -1$ with $n > -r_0$, or $k > -1$ with $n > -r_0 - 1$. In this case, the posterior means of μ and ϕ are finite, as are all moments.

Proof. Because the posterior is proper, by Theorem 1, we have $k = -1$ with $n > -r_0$, or $k > -1$ with $n > -r_0 - 1$.

Let $\pi^*(\mu,\phi) = \phi\pi(\mu,\phi)$. Then, $\pi^*(\mu,\phi) = \pi^*(\mu)\pi^*(\phi)$, where $\pi^*(\mu) = \pi(\mu)$ and $\pi^*(\phi) = \phi\pi(\phi)$, and we have

$$\pi^*(\mu) \propto \mu^k, \quad \pi^*(\phi) \underset{\phi\to 0^+}{\propto} \phi^{r_0+1}, \quad \text{and} \quad \pi^*(\phi) \underset{\phi\to\infty}{\propto} \phi^{r_\infty+1}.$$

Because $k = -1$ with $n > -r_0 > -(r_0+1)$ or $k > -1$ with $n > -(r_0+1) - 1$, it follows from Theorem 1 that the posterior

$$\pi^*(\mu,\phi)\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}$$

related to the prior $\pi^*(\mu,\phi)$ is proper. Therefore,

$$E[\phi|\boldsymbol{x}] = \int_0^\infty\int_0^\infty \phi\pi(\mu,\phi)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\phi < \infty.$$

Analogously, one can prove that

$$E[\mu|\boldsymbol{x}] = \int_0^\infty\int_0^\infty \mu\pi(\mu,\phi)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\phi < \infty.$$

Therefore, we have proved that if a prior $\pi(\mu,\phi)$ satisfying the assumptions of the theorem leads to a proper posterior, then the priors $\phi\pi(\mu,\phi)$ and $\mu\pi(\mu,\phi)$ also lead to proper posteriors. Furthermore, it follows by induction that $\mu^s\phi^r\pi(\mu,\phi)$ leads to proper posteriors, for any r and $s \in \mathbb{N}$, which concludes the proof. □

2.2 Case when ϕ is known

Let $p(\boldsymbol{\theta}|\boldsymbol{x},\phi)$ be of the form (2.2), but consider fixed ϕ and $\boldsymbol{\theta} = (\mu,\alpha)$. Then, the normalizing constant is given by

$$d(\boldsymbol{x};\phi) = \int_{\mathcal{A}} \pi(\boldsymbol{\theta})\,\alpha^n\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\boldsymbol{\theta}, \tag{2.5}$$

where $\mathcal{A} = \{(0,\infty)\times(0,\infty)\}$ is the parameter space. Let $\pi(\boldsymbol{\theta}) \propto \pi(\mu)\pi(\alpha)$. Our goal is to find necessary and sufficient conditions where $d(\boldsymbol{x};\phi) < \infty$.

Theorem 4. *Suppose that $\pi(\mu,\alpha) < \infty$, for all $(\mu,\alpha) \in \mathbb{R}^2_+$, $n \in \mathbb{N}^+$, $\pi(\mu,\alpha) = \pi(\alpha)\pi(\mu)$, and that the priors exhibit asymptotic power law behavior, with*

$$\pi(\mu) \lesssim \mu^k, \quad \pi(\alpha) \underset{\alpha\to 0^+}{\lesssim} \alpha^{q_0}, \quad \pi(\alpha) \underset{\alpha\to\infty}{\lesssim} \alpha^{q_\infty},$$

such that $k=-1$, $n > -q_0$, and $q_\infty \in \mathbb{R}$. Then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is proper.

Proof. See Supplementary Material S1.5. □

Theorem 5. *Suppose that $\pi(\mu,\alpha) > 0$ $\forall(\mu,\alpha) \in \mathbb{R}^2_+$, $n \in \mathbb{N}^+$, $\pi(\mu,\alpha) \gtrsim \pi(\mu)\pi(\alpha)$, and that the priors exhibit asymptotic power law behavior, where $\pi(\mu) \gtrsim \mu^k$ and one of the following holds:*

i) $k < -1$;

ii) $k > -1$, such that $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$ with $q_0 \in \mathbb{R}$; or

iii) $k = -1$, such that $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$ with $n \leq -q_0$.

Then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is improper.

Proof. See Supplementary Material S1.6. □

Theorem 6. *Let $\pi(\mu,\alpha) = \pi(\mu)\pi(\alpha)$, and suppose the behavior of $\pi(\mu)$ and $\pi(\alpha)$ follows an asymptotic power law distribution given by*

$$\pi(\mu) \propto \mu^k, \quad \pi(\alpha) \underset{\mu\to 0^+}{\propto} \alpha^{q_0}, \quad \text{and} \quad \pi(\alpha) \underset{\alpha\to\infty}{\propto} \alpha^{q_\infty},$$

for $k \in \mathbb{R}$, $q_0 \in \mathbb{R}$, and $q_\infty \in \mathbb{R}$. The posterior related to $\pi(\mu, \alpha)$ is proper if and only if $k = -1$, with $n > -q_0$. In this case, the posterior mean of α is finite for this prior, as are all moments relative to α, and the posterior mean of μ is not finite.

Proof. Because the posterior is proper, by Theorem 5, we have $k = -1$ and $n > -q_0$.

Let $\pi^*(\mu, \alpha) = \alpha\pi(\mu, \alpha)$. Then, $\pi^*(\mu, \alpha) = \pi^*(\mu)\pi^*(\alpha)$, where $\pi^*(\alpha) = \alpha\pi(\alpha)$ and $\pi^*(\mu) = \pi(\mu)$, and we have

$$\pi^*(\mu) \propto \mu^{-1}, \quad \pi^*(\alpha) \underset{\mu\to 0^+}{\propto} \alpha^{q_0+1}, \quad \text{and} \quad \pi^*(\alpha) \underset{\alpha\to\infty}{\propto} \alpha^{q_\infty+1}.$$

However, because $n > -q_0 > -(q_0 + 1)$, it follows from Theorem 4 that the posterior

$$\pi^*(\mu, \alpha)\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}$$

relative to the prior $\pi^*(\mu, \alpha)$ is proper. Therefore,

$$E[\alpha|\boldsymbol{x}] = \int_0^\infty\int_0^\infty \alpha\pi(\mu, \alpha)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\alpha < \infty.$$

Analogously, one can prove using item ii) of the Theorem 5 that

$$E[\mu|\boldsymbol{x}] = \int_0^\infty\int_0^\infty \mu\pi(\mu, \alpha)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\alpha = \infty,$$

because in this case, $\mu\pi(\mu) \propto \mu^0$.

Therefore, we have proved that if a prior $\pi(\mu, \alpha)$ satisfying the assumptions of the theorem leads to a proper posterior, then the prior $\alpha\pi(\mu, \alpha)$ also leads to a proper posterior. In addition, it follows by induction that $\alpha^r\pi(\mu, \alpha)$ leads to proper posteriors, for any r in $\mathbb{N}$, which concludes the proof. □

2.3 General case when ϕ, α, and μ are unknown

Theorem 7. *Suppose that $\pi(\mu,\alpha,\phi) < \infty$, for all $(\mu,\alpha,\phi) \in \mathbb{R}_+^3$, $n \in \mathbb{N}^+$, $\pi(\mu,\alpha,\phi) = \pi(\mu)\pi(\alpha)\pi(\mu)$, and the priors exhibit asymptotic power law behavior, with*

$$\pi(\mu) \lesssim \mu^k, \quad \pi(\alpha) \underset{\alpha\to 0^+}{\lesssim} \alpha^{q_0}, \quad \pi(\alpha) \underset{\alpha\to\infty}{\lesssim} \alpha^{q_\infty},$$

$$\pi(\phi) \underset{\phi\to 0^+}{\lesssim} \phi^{r_0}, \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\lesssim} \phi^{r_\infty},$$

such that $k=-1$, $q_\infty < r_0$, $2r_\infty + 1 < q_0$, $n > -q_0$, and $n > -r_0$. Then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is proper.

Proof. See Supplementary Material S1.7. □

Theorem 8. *Suppose that $\pi(\mu,\alpha,\phi) > 0$, $\forall(\mu,\alpha,\phi) \in \mathbb{R}_+^3$, and $n \in \mathbb{N}^+$. Then, the following items are valid:*

i) If $\pi(\mu,\alpha,\phi) \gtrsim \pi(\mu)\pi(\alpha)\pi(\phi)$, for all $\phi \in [b_0, b_1]$, where $0 \le b_0 < b_1$, such that $\pi(\mu) \gtrsim \mu^k$, and one of the following holds:

- *$k < -1$;*
- *$k > -1$, where $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$ with $q_0 \in \mathbb{R}$, or*
- *$k > -1$, where $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$, with $n < -r_0 - 1$ and $b_0 = 0$;*

then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is improper.

ii) If $\pi(\mu,\alpha,\beta) \gtrsim \pi(\mu)\pi(\alpha)\pi(\beta)$, such that $\pi(\mu) \gtrsim \mu^{-1}$, and one of the following holds:

- *$\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ and $\pi(\alpha) \underset{\alpha\to\infty}{\gtrsim} \alpha^{q_\infty}$, where either $q_\infty \ge r_0$ or $n \le -r_0$; or*

- $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$ *and* $\pi(\phi) \underset{\phi\to\infty}{\gtrsim} \phi^{r_\infty}$, *where either* $2r_\infty + 1 \geq q_0$ *or* $n \leq -q_0$;

then $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is improper.*

Proof. See Supplementary Material S1.8. □

Theorem 9. *Suppose that* $0 < \pi(\mu, \alpha, \phi) < \infty$, *for all* $(\mu, \alpha, \phi) \in \mathbb{R}^3_+$, *and that* $\pi(\mu, \alpha, \phi) = \pi(\mu)\pi(\alpha)\pi(\phi)$, *where the priors exhibit asymptotic power law behavior, with*

$$\pi(\mu) \propto \mu^k, \quad \pi(\alpha) \underset{\alpha\to 0^+}{\propto} \alpha^{q_0}, \quad \pi(\alpha) \underset{\alpha\to\infty}{\propto} \alpha^{q_\infty},$$

$$\pi(\phi) \underset{\phi\to 0^+}{\propto} \phi^{r_0}, \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\propto} \phi^{r_\infty}.$$

Then, the posterior is proper if and only if $k = -1$, $q_\infty < r_0$, $2r_\infty + 1 < q_0$, $n > -q_0$, *and* $n > -r_0$. *Moreover, if the posterior is proper, then* $\mu^j\alpha^q\phi^r\pi(\mu, \alpha, \phi)$ *leads to a proper posterior if and only if* $j = 0$ *and* $2(r + r_\infty) + 1 - q_0 < q < r + r_0 - q_\infty$.

Proof. Note that under our hypothesis, Theorems 7 and 8 are complementary, and thus the first part of the theorem is proved. Analogously, by Theorems 7 and 8, the prior $\mu^j\alpha^q\phi^r\pi(\mu, \alpha, \phi)$ leads to a proper posterior if and only if $j = 0$, $q + q_\infty < r + r_0$, $2(r + r_\infty) + 1 < q + q_0$, $n > -q_0 - q$, and $n > -r_0 - r$. The last two proportionals are already satisfied, because $n > -q_0$ and $n > -r_0$. Combining the other inequalities, the proof is complete. □

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3 Objective priors with power law asymptotic behavior

3.1 Some common priors

Jeffreys suggested a common approach that considers different procedures for constructing objective priors. For $\theta \in (0, \infty)$ (see Kass and Wasserman (1996)), Jeffreys suggests using the prior $\pi(\theta) = \theta^{-1}$, that is, a power law distribution with exponent one. The main justification for this choice is its invariance under power transformations of the parameters. Because the parameters of the Stacy family of distributions are contained in the interval $(0, \infty)$, the prior using Jeffreys' first rule is $\pi_1(\mu, \alpha, \phi) \propto (\mu\alpha\phi)^{-1}$.

First, we consider the case when α is known. Hence, this result is valid for the gamma, Nakagami, and Wilson-Hilferty distributions, among others. Jeffreys' first rule when α is known follows a power-law distribution with $\pi(\phi) \propto \phi^{-1}$ and $\pi(\mu) \propto \mu^{-1}$. Hence, the obtained posterior distribution is proper for all $n > 1$, as are its higher moments. This can be proved easily by noting that because $\pi_1(\mu, \phi) \propto \mu^{-1}\phi^{-1}$, we can apply Theorem 6 with $k = r_0 = r_\infty = -1$, and it follows that the posterior and its moments are proper for $n > -r_0 = 1$.

On the other hand, under the general model, in which all parameters are unknown, we have that the posterior distribution (2.2) obtained using Jeffreys' first rule is improper for all $n \in \mathbb{N}^+$. Because $\pi(\phi) \propto \phi^{-1}$, $\pi(\alpha) \propto \alpha^{-1}$, and $\pi(\mu) \propto \mu^{-1}$, that is, power laws with exponent one, we can apply Theorem 8 ii) with $k = q_\infty = r_0 = -1$, where $q_\infty \geq r_0$.

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Therefore, we have $\pi_2(\mu, \alpha, \phi) \propto \phi^{-1}\alpha^{-1}\mu^{-1}$, which leads to an improper posterior for all $n \in \mathbb{N}^+$.

3.2 Priors based on the Fisher information matrix

Here, we consider the cases in which $\pi(\mu) \propto \mu^{-1}$ and the $\pi(\phi)$ have different forms, expressed as

$$\pi_j\left(\boldsymbol{\theta}\right) \propto \frac{\pi_j(\phi)}{\mu}, \tag{3.1}$$

where j is the index related to a particular prior. Therefore, our main focus is to study the behavior of the priors $\pi_j(\phi)$.

One crucial objective prior is based on Jeffreys' general rule (Jeffreys, 1946), and is known as Jeffreys' prior. This prior is obtained as the square root of the determinant of the Fisher information matrix, and is widely used because of its invariance property under one-to-one transformations. The Fisher information matrix for the Stacy family of distributions was derived by Hager and Bain (1970), and its elements are given by

$$I_{\alpha,\alpha}(\boldsymbol{\theta}) = \frac{1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2}{\alpha^2},\ I_{\alpha,\mu}(\boldsymbol{\theta}) = -\frac{\psi(\phi)}{\alpha},\ I_{\mu,\phi}(\boldsymbol{\theta}) = \frac{\alpha}{\mu},$$

$$I_{\alpha,\phi}(\boldsymbol{\theta}) = -\frac{1 + \phi\psi(\phi)}{\mu},\quad I_{\mu,\mu}(\boldsymbol{\theta}) = \frac{\phi\alpha^2}{\mu^2},\quad \text{and}\quad I_{\phi,\phi}(\boldsymbol{\theta}) = \psi'(\phi),$$

where $\psi'(k) = \frac{\partial}{\partial k}\psi(k)$ is the trigamma function.

Van Noortwijk (2001) provided the Jeffreys' prior for the general model, which can be expressed using (3.1) as

$$\pi_3\left(\phi\right) \propto \sqrt{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}. \tag{3.2}$$

Corollary 1. *The prior $\pi_3(\phi)$ has asymptotic behavior given by*

$$\pi_3(\phi) \underset{\phi\to 0^+}{\propto} \phi^0 \quad \textit{and} \quad \pi_3(\phi) \underset{\phi\to\infty}{\propto} \phi^{-1}.$$

As a result, the obtained posterior distribution is improper for all $n \in \mathbb{N}^+$.

Proof. Ramos et al. (2017) proved that

$$\sqrt{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1} \underset{\phi\to 0^+}{\propto} 1 \ \text{ and } \ \sqrt{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1} \underset{\phi\to\infty}{\propto} \frac{1}{\phi}. \tag{3.3}$$

Because $\pi_3(\phi) \underset{\phi\to 0^+}{\propto} 1$, the hypotheses of Theorem 8, ii) hold with $k = -1$ and $r_0 = q_\infty = 0$, where $q_\infty \geq r_0$, and therefore $\pi_3(\boldsymbol{\theta})$ leads to an improper posterior for all $n \in \mathbb{N}^+$. □

Let α be known. Then, Jeffreys' prior has the form (3.1), where $\pi(\phi)$ is given by

$$\pi_4(\phi) \propto \sqrt{\phi\psi'(\phi) - 1}. \tag{3.4}$$

Corollary 2. *The prior $\pi_4(\phi)$ exhibits asymptotic power law behavior given by*

$$\pi_4(\phi) \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}} \quad \textit{and} \quad \pi_4(\phi) \underset{\phi\to\infty}{\propto} \phi^{-\frac{1}{2}}.$$

Then, the obtained posterior and its higher moments are proper for $n \geq 1$.

Proof. Here, we have $\pi(\beta) = \beta^{-1}$, that is, a power law distribution. Following Abramowitz and Stegun (1972), we have $\lim_{z\to 0^+} \dfrac{\psi'(z)}{z^{-2}} = 1$. Then, $\lim_{\phi\to 0^+} \dfrac{\phi\psi'(\phi) - 1}{\phi^{-1}} = \lim_{\phi\to 0^+} \dfrac{\psi'(\phi)}{\phi^{-2}} - \phi = 1$, and thus

$$\phi\psi'(\phi) - 1 \underset{\phi\to 0^+}{\propto} \phi^{-1}, \tag{3.5}$$

which implies $\sqrt{\phi\psi'(\phi)-1} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}$. Moreover, from Abramowitz and Stegun (1972), we have $\psi'(z) = \frac{1}{z} + \frac{1}{2z^2} + o\left(\frac{1}{z^3}\right)$, and thus

$$\frac{\phi\psi'(\phi)-1}{\phi^{-1}} = \frac{1}{2} + o\left(\frac{1}{\phi}\right) \Rightarrow \lim_{\phi\to\infty} \frac{\sqrt{\phi\psi'(\phi)-1}}{\phi^{-\frac{1}{2}}} = \frac{1}{\sqrt{2}},$$

which implies $\sqrt{\phi\psi'(\phi)-1} \underset{\phi\to\infty}{\propto} \phi^{-\frac{1}{2}}$.

Therefore, we can apply Theorem 3 with $k=-1$ and $r_0 = r_\infty = -\frac{1}{2}$, resulting in a proper posterior and finite posterior moments for all $n > -r_0 = \frac{1}{2}$. □

Fonseca et al. (2008) derived objective priors for Student's t-distribution, and showed that the standard Jeffreys prior returned an improper posterior. On the other hand, assuming that one of the parameters is independent, the obtained independent Jeffreys prior returned a proper posterior. The proposed Jeffreys prior with an independent structure has the form $\pi_{J2}(\boldsymbol{\theta}) \propto \sqrt{|\operatorname{diag} I(\boldsymbol{\theta})|}$, where $\operatorname{diag} I(\cdot)$ is the diagonal matrix of $I(\cdot)$. For the general distribution, the prior is given by (3.1), with

$$\pi_5(\phi) \propto \sqrt{\phi\psi'(\phi)\left(1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2\right)}. \tag{3.6}$$

Note that for (3.6), it is only necessary to know the behavior $\pi_5(\phi)$ when $\phi \to 0^+$, which provides enough information to verify that the posterior is improper.

Corollary 3. *The prior (3.6) has asymptotic power law behavior given by $\pi_5(\phi) \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}$, and the obtained posterior is improper for all $n \in \mathbb{N}^+$.*

Proof. By Abramowitz and Stegun (1972), we have the recurrence relations

$$\psi(\phi) = -\frac{1}{\phi} + \psi(\phi+1) \quad \text{and} \quad \psi'(\phi) = \frac{1}{\phi^2} + \psi'(\phi+1). \tag{3.7}$$

It follows that

$$2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2 + 1 =$$
$$2\left(-\frac{1}{\phi} + \psi(\phi+1)\right) + \phi\left(\frac{1}{\phi^2} + \psi'(\phi+1)\right) + \phi\left(\frac{1}{\phi^2} - \frac{2}{\phi}\psi(\phi+1) + \psi(\phi+1)^2\right) + 1 =$$
$$1 + \phi\left(\psi(\phi+1)^2 + \psi'(\phi+1)\right).$$

Hence, $2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2 + 1 \underset{\phi\to 0^+}{\propto} 1$, which implies that

$$\pi_5(\phi) \propto \sqrt{\phi\psi'(\phi)\left(1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2\right)} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}, \tag{3.8}$$

that is, a power law distribution with exponent $\frac{1}{2}$. Then, we can apply Theorem 8 ii) with $k = -1$, $r_0 = -\frac{1}{2}$, and $q_\infty = 0$, where $q_\infty \geq r_0$. Therefore, $\pi_5(\boldsymbol{\theta})$ leads to an improper posterior. □

This approach can be further extended by considering that only one parameter is independent. For instance, let (θ_1, θ_2) be dependent parameters, and θ_3 be independent. Then, under the partition, the $((\theta_1, \theta_2), \theta_3)$ Jeffreys prior is given by

$$\pi(\boldsymbol{\theta}) \propto \sqrt{\left(I_{11}(\boldsymbol{\theta})I_{22}(\boldsymbol{\theta}) - I_{12}^2(\boldsymbol{\theta})\right) I_{33}(\boldsymbol{\theta})}. \tag{3.9}$$

For the general model, the partition $((\phi, \mu), \alpha)$ Jeffreys prior is of the form (3.1), with

$$\pi_6(\phi) \propto \sqrt{\left(\phi\psi'(\phi) - 1\right)\left(1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2\right)}. \tag{3.10}$$

Corollary 4. *The prior (3.10) has asymptotic power law behavior given by* $\pi_6(\phi) \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}$*, and the obtained posterior is improper for all* $n \in \mathbb{N}^+$*.*

Proof. From equation (3.5), we have $\phi\psi'(\phi) - 1 \underset{\phi\to 0^+}{\propto} \frac{1}{\phi}$. Together with (3.8), this implies that

$$\pi_6(\phi) \propto \sqrt{\left(\phi\psi'(\phi) - 1\right)\left(1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2\right)} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}, \tag{3.11}$$

that is, a power-law distribution with exponent $\frac{1}{2}$. Then, we can apply Theorem 8, ii) with $k=-1$, $r_0=-\frac{1}{2}$, and $q_\infty=0$, where $q_\infty \geq r_0$. Therefore, $\pi_6(\boldsymbol{\theta})$ leads to an improper posterior. □

The partition $((\alpha,\mu),\phi)$ Jeffreys prior is given by (3.1), where

$$\pi_7(\phi) \propto \sqrt{\psi'(\phi)(\phi^2\psi'(\phi)+\phi-1)}\,. \tag{3.12}$$

This is similar to the two cases above. From the recurrence relations (3.7), we have

$$\phi^2\psi'(\phi)+\phi-1 = \phi\left(1+\phi\psi'(\phi+1)\right) \Rightarrow \phi^2\psi'(\phi)+\phi-1 \underset{\phi\to 0^+}{\propto} \phi. \tag{3.13}$$

Because $\psi'(\phi) \propto \frac{1}{\phi^2}$, it follows that

$$\pi_7(\phi) \propto \sqrt{\psi'(\phi)(\phi^2\psi'(\phi)+\phi-1)} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}},$$

with the same values $k=-1$, $r_0=-\frac{1}{2}$, and $q_\infty=0$, where $q_\infty \geq r_0$. Thus, the prior $\pi_7(\boldsymbol{\theta})$ leads to an improper posterior.

3.3 Reference priors

Reference priors are another important class of objective priors, introduced by Bernardo (1979) and later extended (Berger and Bernardo, 1989, 1992; Berger et al., 1992). Reference priors play an important role in objective Bayesian analysis, and have desirable properties, such as invariance, consistent marginalization, and consistent sampling properties. Bernardo (2005) reviewed different procedures to derive reference priors that consider the ordered parameters of interest. We apply the following proposition to obtain the reference priors for the generalized gamma distribution.

Proposition 4. *[Bernardo (1979), pg 40, Theorem 14] Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$ be a vector of the ordered parameters of interest, and let $p(\boldsymbol{\theta}|\boldsymbol{x})$ be the posterior distribution that has an asymptotically normal distribution with dispersion matrix $V(\hat{\boldsymbol{\theta}}_n)/n$, where $\hat{\boldsymbol{\theta}}_n$ is a consistent estimator of $\boldsymbol{\theta}$ and $H(\boldsymbol{\theta}) = V^{-1}(\boldsymbol{\theta})$. In addition, V_j is the upper $j \times j$ sub-matrix of V, $H_j = V_j$, and $h_{j,j}(\boldsymbol{\theta})$ is the lower-right element of H_j. If the parameter space of θ_j is independent of $\boldsymbol{\theta}_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_m)$, for $j = 1, \ldots, m$, and $h_{j,j}(\boldsymbol{\theta})$ are factorized in the form $h_{j,j}^{\frac{1}{2}}(\boldsymbol{\theta}) = f_j(\theta_j) g_j(\boldsymbol{\theta}_{-j})$, for $j = 1, \ldots, m$, then the reference prior for the ordered parameters $\boldsymbol{\theta}$ is given by*

$$\pi(\boldsymbol{\theta}) = \pi(\theta_j|\theta_1, \ldots, \theta_{j-1}) \times \cdots \times \pi(\theta_2|\theta_1)\pi(\theta_1),$$

where $\pi(\theta_j|\theta_1, \ldots, \theta_{j-1}) = f_j(\theta_j)$, for $j = 1, \ldots, m$. Note that there is no need for compact approximations, even if the conditional priors are not proper.

The reference priors obtained from Proposition 4 belong to the class of improper priors given by

$$\pi(\boldsymbol{\theta}) \propto \pi(\phi)\alpha^{-1}\mu^{-1}. \tag{3.14}$$

Therefore, both $\pi(\mu) \propto \mu^{-1}$ and $\pi(\alpha) \propto \alpha^{-1}$ follow power law distributions with exponent one. We focus on the asymptotic power law behavior of $\pi(\phi)$. Let (α, ϕ, μ) be the ordered parameters of interest. Then, the conditional priors of the (α, ϕ, μ)-reference prior are given by

$$\pi(\alpha) \propto \alpha^{-1}, \quad \pi(\phi|\alpha) \propto \sqrt{\frac{\phi\psi'(\phi) - 1}{\phi}}, \quad \pi(\mu|\alpha, \phi) \propto \mu^{-1}.$$

Therefore, the (α, ϕ, μ)-reference prior is of the form (3.14), with

$$\pi_8(\phi) \propto \sqrt{\frac{\phi\psi'(\phi) - 1}{\phi}} \underset{\phi \to 0^+}{\propto} \phi^{-1},$$

which is also a power law distribution with exponent one. Therefore, item ii) of Theorem 8 can be applied with $k = r_0 = q_\infty = 1$, where $q_\infty \geq r_0$, which implies that $\pi_8(\alpha, \phi, \mu)$ leads to an improper posterior for all $n \in \mathbb{N}^+$.

Assuming that (α, μ, ϕ) are the ordered parameters, then the conditional reference priors are

$$\pi(\alpha) \propto \alpha^{-1}, \quad \pi(\mu|\alpha) \propto \mu^{-1}, \quad \pi(\phi|\alpha, \mu) \propto \sqrt{\psi'(\phi)},$$

and the (α, μ, ϕ)-reference prior is of the form (3.14), with

$$\pi_9(\phi) \propto \sqrt{\psi'(\phi)}.$$

From $\psi'(\phi) \underset{\phi \to 0^+}{\propto} \phi^{-2}$, we have $\sqrt{\psi'(\phi)} \underset{\phi \to 0^+}{\propto} \phi^{-1}$, that is, a PL distribution with exponent one. Similarly to the case of $\pi_8(\mu, \alpha, \phi)$, we have $\pi_9(\mu, \alpha, \phi)$, which leads to an improper posterior for all $n \in \mathbb{N}^+$.

Consider the case in which α is known with $\alpha = 1$, reducing to the gamma distribution. Then, $\pi(\phi, \mu) \propto \mu^{-1}\sqrt{\psi'(\phi)}$ is the (μ, ϕ)-reference prior, and the joint posterior densities when $\alpha = 1$ using the (μ, ϕ)-reference are proper for $n \geq 2$ as are its higher moments.

The above results follow from $\psi'(\phi) \underset{\phi \to 0^+}{\propto} \phi^{-2}$ and $\psi'(\phi) \underset{\phi \to \infty^+}{\propto} \phi^{-1}$, and thus $\pi_9(\phi)$ exhibits asymptotic power law behavior given by

$$\pi_9(\phi) \underset{\phi \to 0^+}{\propto} \phi^{-1} \quad \text{and} \quad \pi_9(\phi) \underset{\phi \to \infty^+}{\propto} \phi^{-\frac{1}{2}}.$$

Therefore, from the above power law distributions and the distribution $\pi(\mu)$ that has a PL with exponent one, we can apply Theorem 3 with $k=-1$, $r_0=-1$, and $r_\infty=-0.5$. It follows that the posterior, as well as all its moments are proper for all $n>-r_0=1$.

Assuming now that ϕ is known, with $\phi=1$, the distribution reduces to the Weibull distribution. In this case, $\pi(\mu,\alpha)\propto\alpha^{-1}\mu^{-1}$ is the (α,μ)-reference prior. Note that each prior follows a power law distribution. The joint posterior density using the (α,μ)-reference is proper for $n\geq 2$, although its higher moments relative to μ are improper. This result is a direct consequence from Theorem 6, considering that $k=-1$ and $q_0=q_\infty=-1$, which leads to a proper posterior.

Returning to the general model, if (μ,ϕ,α) is the vector of ordered parameters, it follows that the conditional priors are

$$\pi(\mu)\propto\mu^{-1},\ \ \pi(\phi|\mu)\propto\sqrt{\psi'(\phi)-\frac{\psi(\phi)^2}{2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi^2)+1}},\ \ \pi(\alpha|\phi,\mu)\propto\alpha^{-1},$$

and the (μ,ϕ,α)-reference prior is of the form (3.14), with

$$\pi_{10}(\phi)\propto\sqrt{\psi'(\phi)-\frac{\psi(\phi)^2}{2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi^2)+1}}.$$

Corollary 5. *The prior $\pi_{10}(\phi)$ exhibits asymptotic power law behavior given by $\pi_{10}(\phi)\underset{\phi\to 0^+}{\propto}\phi^{-1}$, and the obtained posterior is improper for all $n\in\mathbb{N}^+$.*

Proof. From Abramowitz and Stegun (1972), we have

$$\psi(\phi)=\log(\phi)-\frac{1}{2\phi}-\frac{1}{12\phi^2}+o\left(\frac{1}{\phi^2}\right)\quad\text{and}\quad\psi'(\phi)=\frac{1}{\phi}+\frac{1}{2\phi^2}+o\left(\frac{1}{\phi^2}\right).\tag{3.15}$$

Thus, it follows directly that

$$\psi(\phi)^2=\log(\phi)^2-\frac{\log(\phi)}{\phi}+o\left(\frac{1}{\phi}\right).$$

Therefore, $2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1=\phi\log(\phi)^2+\log(\phi)+2+o(1)$ and

$$\begin{aligned}\pi_{10}(\phi) &\propto \sqrt{\psi'(\phi)-\frac{\psi(\phi)^2}{2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1}}\\ &= \sqrt{\frac{\left(\frac{1}{\phi}+\frac{1}{2\phi^2}+o\left(\frac{1}{\phi^2}\right)\right)\left(\phi\log(\phi)^2+\log(\phi)+2+o(1)\right)-\log(\phi)^2+\frac{\log(\phi)}{\phi}+o\left(\frac{1}{\phi}\right)}{\phi\log(\phi)^2+\log(\phi)+2+o(1)}}\\ &= \sqrt{\frac{\frac{1}{\phi}\left(\log(\phi)^2+o(\log(\phi)^2)\right)}{\phi\left(\log(\phi)^2+o(\log(\phi)^2)\right)}}=\frac{1}{\phi}\sqrt{\frac{1+o(1)}{1+o(1)}}.\end{aligned}$$

Thus,

$$\pi_{10}(\phi)\propto\sqrt{\psi'(\phi)-\frac{\psi(\phi)^2}{2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1}}\underset{\phi\to 0^+}{\propto}\phi^{-1},$$

and therefore Theorem 8 ii) can be applied with $k=q_0=r_\infty=-1$, where $2r_\infty+1\geq q_0$. Thus, $\pi_{10}(\boldsymbol{\theta})$ leads to an improper posterior.

□

Finally, let (ϕ,α,μ) be the ordered parameters. Then, the conditional priors are

$$\pi(\phi)\propto\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}},\quad \pi(\alpha|\phi)\propto\alpha^{-1},\quad \pi(\mu|\alpha,\phi)\propto\mu^{-1},$$

and the (ϕ,α,μ)-reference prior is of the form (3.14), with

$$\pi_{11}(\phi)\propto\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}}. \tag{3.16}$$

Note that the (ϕ,μ,α)-reference prior is the same as the (ϕ,α,μ)-reference prior, and that the (μ,α,ϕ)-reference prior has the same form as $\pi_9(\boldsymbol{\theta})$, which completes all possible reference priors obtained from Proposition 4.

Corollary 6. *The prior $\pi_{11}(\phi)$ has the asymptotic power law behavior given by*

$$\pi_{11}(\phi)\underset{\phi\to 0^+}{\propto}\phi^{-\frac{1}{2}}\quad\textit{and}\quad\pi_{11}(\phi)\underset{\phi\to\infty}{\propto}\phi^{-\frac{3}{2}}.$$

Then, the obtained posterior distribution is proper for $n \geq 2$, and its higher moments are improper for all $n \in \mathbb{N}^+$.

Proof. From (3.3) and by the asymptotic relations (3.15), we have

$$\phi^2\psi^{'}(\phi) + \phi - 1 = 2\phi - \frac{1}{2} + o(1) \underset{\phi\to\infty}{\propto} \phi.$$

Together with equation (3.13), this implies that

$$\sqrt{\phi^2\psi'(\phi)+\phi-1} \underset{\phi\to 0^+}{\propto} \sqrt{\phi} \quad \text{and} \quad \sqrt{\phi^2\psi'(\phi)+\phi-1} \underset{\phi\to\infty}{\propto} \sqrt{\phi}.$$

Hence, from the above proportionalities, we have

$$\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}} \quad \text{and} \quad \sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}} \underset{\phi\to\infty}{\propto} \phi^{-\frac{3}{2}}.$$

Therefore, Theorem 7 can be applied with $k = q_0 = q_\infty = -1$, $r_0 = -\frac{1}{2}$, and $r_\infty = -\frac{3}{2}$, where $k = -1$, $q_\infty < r_0$, and $2r_\infty + 1 < q_0$. Therefore, $\pi_{11}(\mu, \alpha, \phi)$ leads to a proper posterior for every $n > -q_0 = 1$.

To prove that the higher moments are improper, suppose $\alpha^q\phi^r\mu^j\pi(\boldsymbol{\theta})$ leads to a proper posterior for $r \in \mathbb{N}$, $q \in \mathbb{N}$, and $k \in \mathbb{N}$. By Theorem 9, we have $j = 0$, $q + q_\infty < r + r_0$, $2(r + r_\infty) \leq q + q_0$, and $n \geq -q_0$, that is, $k = 0$ and $2r - 1 < q < r + \frac{1}{2}$. The inequality $2r - 1 < r + \frac{1}{2}$ leads to $r < \frac{3}{2}$, that, $r = 0$ or $r = 1$. By the previous inequality, $r = 0$ leads to $-1 < q < \frac{1}{2}$, that is, $q = 0$. Now, for $r = 1$, we have the inequality $1 < q < \frac{3}{2}$, which does not have an integer solution. Therefore, the only possible values for which $\alpha^q\phi^r\mu^j\pi(\boldsymbol{\theta})$ is proper are $q = r = j = 0$, that is, the higher moments are improper. □

Zellner (1977, 1984) discussed another procedure for obtaining an objective prior that is based on the information measure known as Shannon entropy. The prior is known as

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the MDI prior, and can be obtained by solving

$$\pi_Z(\boldsymbol{\theta}) \propto \exp\left(\int f(t\,|\,\phi,\mu,\alpha)\log f(t\,|\,\phi,\mu,\alpha)dt\right). \tag{3.17}$$

Ramos et al. (2017) showed that the MDI prior (3.17) for the GG distribution is given by

$$\pi_Z(\boldsymbol{\theta}) \propto \frac{\alpha\mu}{\Gamma(\phi)}\exp\left\{\psi(\phi)\left(\phi-\frac{1}{\alpha}\right)-\phi\right\}. \tag{3.18}$$

Note that this distribution is not a power law distribution, because for any ϕ close enough to zero such that $\psi(\phi)<0$, letting $\beta=\frac{-\psi(\phi)}{\alpha}$, we have

$$\begin{aligned}\lim_{\alpha\to 0^+}\frac{1}{\alpha^k}\frac{\alpha\mu}{\Gamma(\phi)}\exp\left\{\psi(\phi)\left(\phi-\frac{1}{\alpha}\right)-\phi\right\} &= \lim_{\beta\to\infty}\frac{\beta^{k-1}}{(-\psi(\phi))^{k-1}}\frac{\mu}{\Gamma(\phi)}\exp\left\{\psi(\phi)\phi+\beta-\phi\right\}\\ &= \frac{\mu}{\Gamma(\phi)}\frac{\exp\left\{\psi(\phi)\phi-\phi\right\}}{(-\psi(\phi))^{k-1}}\lim_{\beta\to\infty}\beta^{k-1}\exp\left\{\beta\right\}=\infty,\end{aligned} \tag{3.19}$$

and thus $\lim_{\alpha\to 0^+}\frac{\pi_Z(\boldsymbol{\theta})}{\alpha^k}=\infty$, for any $k\in\mathbb{R}$. That is, $\pi_Z(\boldsymbol{\theta})$ is not a power law distribution. This distribution also leads to an improper posterior, as discussed in Ramos et al. (2017).

3.4 Hierarchical models

Power law behavior occurs in other models as well, such as hierarchical models. For instance, Fonseca et al. (2019) derived different objective priors for such a hierarchical structure. Consider a Student's t model with unknown degrees of freedom, where $y|\theta\sim$ student(θ) is a standard Student's t model with fixed mean and precision, and unknown

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degrees of freedom v. The model may be rewritten in a hierarchical setting as

$$y|w \sim N(0, 1/w)$$

$$w|\theta \sim Gamma(\theta/2, \theta/2).$$

The model has two levels of hierarchy where θ appears in the second level. The Jeffreys prior may be written as

$$\pi_{h_1}(\theta) \propto \sqrt{I_y(\theta)},$$

where $I_y(\theta) = I_w(\theta) - E_y[I_w(\theta \mid y)]$, and where $I_w(\theta)$ and $E_y[I_w(\theta \mid y)]$ are given by

$$I_w(\theta) = \frac{1}{4}\psi^{(2)}\left(\frac{\theta}{2}\right) - \frac{1}{2\theta}$$

and

$$E_y[I_w(\theta \mid y)] = \frac{1}{4}\psi^{(1)}\left(\frac{\theta+1}{2}\right) + \frac{\theta+2}{2\theta(\theta+3)} - \frac{1}{\theta+1},$$

respectively.

Corollary 7. *The prior $\pi_{h_1}(\phi)$ exhibits the asymptotic power law behavior given by*

$$\pi_{h_1}(\theta) \underset{\theta\to 0^+}{\propto} \theta^{-1} \quad \textit{and} \quad \pi_{h_1}(\theta) \underset{\theta\to\infty}{\propto} \theta^{-2},$$

Proof. First, note that

$$I_y(\theta) = \frac{1}{4}\psi^{(1)}\left(\frac{\theta}{2}\right) - \frac{1}{4}\psi^{(1)}\left(\frac{\theta+1}{2}\right) - \frac{\theta+5}{2\theta(\theta+1)(\theta+3)}.$$

Now, following Abramowitz and Stegun (1972), we know that $\lim_{x\to 0^+} x^2\psi^{(1)}(x) = 1$ and $\lim_{x\to\infty} x\psi^{(1)}(x) = 1$. Thus it follows that

$$\lim_{\theta\to\infty} \theta\psi^{(1)}\left(\frac{\theta+1}{2}\right) = \lim_{\theta\to\infty} 2\left(\frac{\theta}{\theta+1}\right)\left[\left(\frac{\theta+1}{2}\right)\psi^{(1)}\left(\frac{\theta+1}{2}\right)\right] = 2\times 1\times 1 = 2.$$

Similarly, we have $\lim_{\theta\to 0^+} \theta^2\psi^{(1)}\left(\frac{\theta}{2}\right) = 4$, $\lim_{\theta\to 0^+} \theta^2\psi^{(1)}\left(\frac{\theta+1}{2}\right) = 0$, and $\lim_{\theta\to\infty} \theta\psi^{(1)}\left(\frac{\theta}{2}\right) = 2$. Combining these items, we have

$$\lim_{\theta\to 0^+} \theta^2 I_y(\theta) = \frac{4}{4} - 0 - 0 = 1 \Rightarrow I_y(\theta) \underset{\theta\to 0^+}{\propto} \frac{1}{\theta^2}.$$

Moreover, from Abramowitz and Stegun (1972), we have the asymptotic relation

$$\psi^{(1)}(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} + o(x^4), \text{ as } x \to \infty.$$

Letting $w(\theta) = \frac{1}{\theta} + \frac{1}{2\theta^2} + \frac{1}{6\theta^3}$, we conclude that

$$\begin{aligned}\lim_{\theta\to\infty} \theta^4 I_y(\theta) &= \lim_{\theta\to\infty} \theta^4\left[\frac{1}{4}w\left(\frac{\theta}{2}\right) - \frac{1}{4}w\left(\frac{\theta+1}{2}\right) - \frac{\theta+5}{2\theta(\theta+1)(\theta+3)}\right] \\ &= \lim_{\theta\to\infty} \frac{21\,\theta^4 + 48\,\theta^3 + 29\,\theta^2 + 6\,\theta}{6(\theta+1)^3(\theta+3)} = \frac{21}{6} \Rightarrow I_y(\theta) \underset{\theta\to\infty}{\propto} \frac{1}{\theta^4}.\end{aligned}$$

Thus, we have proved that $I_y(\theta) \underset{\theta\to 0^+}{\propto} \theta^{-2}$ and $I_y(\theta) \underset{\theta\to\infty}{\propto} \theta^{-4}$. Together with $\pi_{h_1} \propto \sqrt{I_y(\theta)}$, this concludes the proof. $\square$

Hence, the Jeffreys prior under this hierarchical model also follows an asymptotic power law distribution.

In another case, we assume independent Laplace priors for the regression coefficients, as discussed in Fonseca et al. (2019). They assume that

$$y|w_1 \sim N(Xw_1, \sigma^2 I_n)$$

where y is the $n \times 1$ vector of responses, X is an $n \times p$ matrix of covariates, and w_1 is an the $n \times 1$ vector of regression coefficients

In a Bayesian context, the lasso constraint is equivalent to using the independent Laplace prior

$$p(w_{1,j}) = \frac{\gamma}{2}\exp\left\{-\gamma\,|w_{1,j}|\right\}.$$

The lasso prior is obtained as a uniform scale mixture by considering the conditional setting

$$w_{1,j} \mid w_{2,j} \sim \text{Unif}\left(-\sigma w_{2,j}, \sigma w_{2,j}\right)$$

$$w_{2,j} \sim \text{Gamma}(2, \theta).$$

After some algebraic manipulations, the final Jeffreys prior has a closed-form given by $\pi(\theta) \propto \theta^{-1}$, that is, a power law distribution with $\lambda = 1$. The results show that power law behavior also occurs in hierarchical models.

4 A Real Application

Van Noortwijk (2001) analyzed a data set related to the annual maximum discharge of the Rhine River at Lobith, the Netherlands, between 1901 and 1998, where the Dutch river dikes have to withstand water levels and discharges with an average return period of up to 1250 years. Maximum river discharge is usually associated with floods, which cause much damage. The values of m^3/s are provided in Figure 1.

Van Noortwijk (2001) used the GG distribution to predict the exceedance probabilities of the annual maximum discharge. The posterior distribution was constructed using the Jeffreys prior (3.2). However, we proved in Corollary (1) that the obtained posterior is improper for all $n \in \mathbb{N}^+$, and should not be used to compute the posterior estimates. The estimates for the parameters ϕ, $1/\mu$, and α are, 1.380, 4936.0, and 2.310, respectively. Van Noortwijk (2001) did not provide the credibility interval for $1/\mu$. However, the credibility intervals for ϕ and α both were (0.01,6.00). In this case, there is a strong

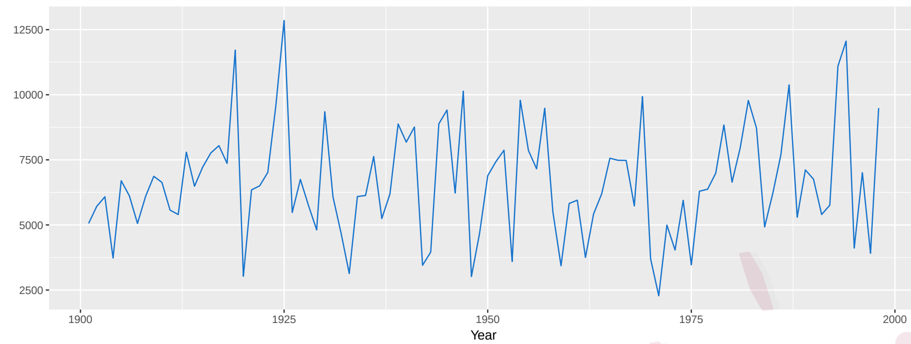

Figure 1: Time series plot for the annual maximum discharge (m^3/s) of the Rhine River at Lobith for the period 1901-1998.

indication that the inference was conducted improperly. The range of the credibility intervals was probably influenced by the use of an improper posterior distribution, and so the results are not reliable.

The posterior distribution using the (ϕ, α, μ)-reference prior (3.16) is proper for $n \geq 2$, and can be used to analyze these data. Owing to the consistent marginalization property of the reference prior, the reference marginal posterior distribution of ϕ and α is

$$p_{12}(\phi, \alpha|\boldsymbol{x}) \propto \alpha^{n-2}\frac{\Gamma(n\phi)}{\Gamma(\phi)^n}\sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi^2\psi'(\phi) + \phi - 1}}\left(\frac{\sqrt[n]{\prod_{i=1}^n t_i^\alpha}}{\sum_{i=1}^n t_i^\alpha}\right)^{n\phi},$$

and the conditional posterior distributions for μ, given ϕ and α, are given by

$$p_{12}(\mu|\phi, \alpha, \boldsymbol{x}) \sim \mathrm{GG}\left(n\phi, \left(\sum_{i=1}^n t_i^\alpha\right)^{\frac{1}{\alpha}}, \alpha\right).$$

The above distributions help to obtain posterior estimates using Markov chain Monte Carlo methods. We conducted a simulation study, reported in Supplementary Material A.9, that shows that the obtained posterior estimates are accurate, especially when compared with simple proper flat priors. Because we have proved that the posterior mean

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for the parameter does not return finite values, the posterior medians for ϕ, μ, and α, are considered to be posterior estimates. Moreover, following Van Noortwijk (2001), we also present the annual maximum river discharge (MRD), in which the probability of exceedance is 1/1250 per year. The posterior summaries are shown in Table 2.

Table 2: Posterior median, standard deviations, and 95% credible intervals for ϕ, μ, and α.

$\boldsymbol{\theta}$	Median	SD	$\mathrm{CI}_{95\%}(\boldsymbol{\theta})$
ϕ	3.741	1.857	(1.072; 7.609)
$1/\mu$	3,061.6	2,160.5	(999.08; 7,970)
α	1.620	0.562	(1.070; 3.194)
MRD	14,887	4,055.9	(10,535; 22,591)

The MRD presented by Van Noortwijk is 15,150. Thus, the improper analysis returned an overestimated annual maximum discharge. Therefore, based on our estimates, Dutch river dikes have to withstand water levels and discharges of up to $14,887$ m^3/s.

5 Conclusion

Objective priors play an important role in Bayesian analysis. For several important distributions, we have shown that such objective priors are improper priors, and may lead to an improper posterior. In these cases, we cannot conduct a Bayesian inference, which is un-

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desirable. An exciting aspect of our findings is that such priors either follow a power-law distribution or present an asymptotic behavior to this distribution. Our mathematical formalism is general and covers important distributions widely used in the literature. The exponent of the obtained power law distributions is contained between 0.5 and 1. Hence, they are improper with an infinite mean and variance.

We have provided sufficient and necessary conditions for the posteriors to be proper, depending on the exponent of the power law model. For instance, if ϕ is known, the (α, μ)-reference prior for the Weibull and generalized half-normal distributions, follow power law distributions with exponent one and return proper posteriors. By considering α as fixed, we showed that Jeffreys' first rule and Jeffreys' prior both return proper posterior distributions, and finite higher moments, which are valid for the gamma, Nakagami-m, and Wilson-Hilferty distributions. Moreover, we have provided situations in which the obtained posteriors are improper and should not be used, opening up new opportunities for analyzing real data.

The observed behavior also occurs in classes of distributions. For example, in the Lomax distribution, which is a modified version of the Pareto model, the reference priors for the two parameters of the model follow power law distributions with exponent one (Ferreira et al., 2020). This behavior is also observed in a Gaussian distribution when μ is a known parameter. In this case, the Jeffreys prior for the standard deviation σ follows a power law distribution with exponent one and the obtained posterior is proper. Under the Behrens-Fisher problem, the Jeffreys prior obtained for the parameters exhibits the same behavior with exponent two, and the reference prior has exponent three (Liseo, 1993).

We applied the proposed theoretical results to show that the Bayesian approach was misused to analyze a data set related to the annual maximum discharge of the Rhine river at Lobith, Netherlands. Hence, we computed the correct posterior estimates using a proper posterior distribution. There are several possible extensions of this work. The power law distributions can be considered as objective priors in the presence of censored data or long-term survival. In future work, we would also like to apply our approach to other distributions, such as generalized linear models.

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References

Abramowitz, M. and I. A. Stegun (1972). *Handbook of Mathematical Functions* (10 ed.). Washington, D.C.: NBS.

Barrat, A., M. Barthelemy, and A. Vespignani (2008). *Dynamical processes on complex networks*. Cambridge university press.

Berger, J. O. and J. M. Bernardo (1989). Estimating a product of means: Bayesian analysis with reference priors. *Journal of the American Statistical Association 84*(405), 200–207.

Berger, J. O. and J. M. Bernardo (1992). Ordered group reference priors with application to the multinomial problem. *Biometrika 79*(1), 25–37.

Berger, J. O., J. M. Bernardo, et al. (1992). On the development of reference priors. *Bayesian statistics 4*(4), 35–60.

Berger, J. O., J. M. Bernardo, D. Sun, et al. (2015). Overall objective priors. *Bayesian Analysis 10*(1), 189–221.

Bernardo, J. M. (1979). Reference posterior distributions for bayesian inference. *Journal of the Royal Statistical Society. Series B (Methodological)*, 113–147.

Bernardo, J. M. (2005). Reference analysis. *Handbook of statistics 25*, 17–90.

Consonni, G., D. Fouskakis, B. Liseo, I. Ntzoufras, et al. (2018). Prior distributions for objective bayesian analysis. *Bayesian Analysis 13*(2), 627–679.

Corral, Á., A. Ossó, and J. E. Llebot (2010). Scaling of tropical-cyclone dissipation. *Nature Physics 6*(9), 693–696.

Deluca, A. and Á. Corral (2013). Fitting and goodness-of-fit test of non-truncated and truncated power-law distributions. *Acta Geophysica 61*(6), 1351–1394.

Ferreira, P. H., E. Ramos, P. L. Ramos, J. F. Gonzales, V. L. Tomazella, R. S. Ehlers, E. B. Silva, and F. Louzada (2020). Objective bayesian analysis for the lomax distribution. *Statistics & Probability Letters 159*, 108677.

Fonseca, T. C., M. A. Ferreira, and H. S. Migon (2008). Objective bayesian analysis for the student-t regression model. *Biometrika 95*(2), 325–333.

Fonseca, T. C., H. S. Migon, and H. Mirandola (2019). Reference bayesian analysis for hierarchical models. *arXiv preprint arXiv:1904.11609*.

Geilhufe, M., L. Held, S. O. Skrøvseth, G. S. Simonsen, and F. Godtliebsen (2014). Power law approximations of movement network data for modeling infectious disease spread. *Biometrical Journal 56*(3), 363–382.

Goldstein, M. L., S. A. Morris, and G. G. Yen (2004). Problems with fitting to the power-law distribution. *The European Physical Journal B-Condensed Matter and Complex Systems 41*(2), 255–258.

Hager, H. W. and L. J. Bain (1970). Inferential procedures for the generalized gamma distribution. *Journal of the American Statistical Association 65*(332), 1601–1609.

Hanel, R., B. Corominas-Murtra, B. Liu, and S. Thurner (2017). Fitting power-laws in empirical data with estimators that work for all exponents. *PloS one 12*(2).

Jeffreys, H. (1946). An invariant form for the prior probability in estimation problems. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, Volume 186, pp. 453–461. The Royal Society.

Kass, R. E. and L. Wasserman (1996). The selection of prior distributions by formal rules. *Journal of the American Statistical Association 91*(435), 1343–1370.

Leisen, F., C. Villa, S. G. Walker, et al. (2019). On a class of objective priors from scoring rules. *Bayesian Analysis*.

Liseo, B. (1993). Elimination of nuisance parameters with reference priors. *Biometrika 80*(2), 295–304.

Lloyd-Jones, L. R., J. Zeng, J. Sidorenko, L. Yengo, G. Moser, K. E. Kemper, H. Wang, Z. Zheng, R. Magi, T. Esko, et al. (2019). Improved polygenic prediction by bayesian multiple regression on summary statistics. *Nature communications 10*(1), 1–11.

Louzada, F., J. A. Cuminato, O. M. Rodriguez, V. L. Tomazella, P. H. Ferreira, P. L. Ramos, S. R. Niaki, O. A. Gonzatto, I. C. Perissini, L. F. Alegría, et al. (2019). A repairable system subjected to hierarchical competing risks: Modeling and applications. *IEEE Access 7*, 171707–171723.

Newman, M. (2018). *Networks*. Oxford university press.

Newman, M. E. (2005). Power laws, pareto distributions and zipf's law. *Contemporary physics 46*(5), 323–351.

Northrop, P. and N. Attalides (2016). Posterior propriety in bayesian extreme value analyses using reference priors. *Statistica Sinica 26*(2).

O'Hagan, A., C. E. Buck, A. Daneshkhah, J. R. Eiser, P. H. Garthwaite, D. J. Jenkinson, J. E. Oakley, and T. Rakow (2006). *Uncertain judgements: eliciting experts' probabilities*. John Wiley & Sons.

Pržulj, N. (2007). Biological network comparison using graphlet degree distribution. *Bioinformatics 23*(2), e177–e183.

Ramos, P. L., J. A. Achcar, F. A. Moala, E. Ramos, and F. Louzada (2017). Bayesian analysis of the generalized gamma distribution using non-informative priors. *Statistics 51*(4), 824–843.

Stacy, E. W. (1962). A generalization of the gamma distribution. *The Annals of Mathematical Statistics*, 1187–1192.

Van Noortwijk, J. M. (2001). Bayes estimates of flood quantiles using the generalised gamma distribution. *System and Bayesian Reliability*, 351–374.

Wang, C. and H. G. Matthies (2019). Novel model calibration method via non-probabilistic interval characterization and bayesian theory. *Reliability Engineering & System Safety 183*, 84–92.

Zellner, A. (1977). Maximal data information prior distributions. *In New Methods in the applications of Bayesian Methods*.

Zellner, A. (1984). Maximal data information prior distributions. *Basic Issues in Econometrics*.