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# A nonparametric test for stationarity in functional time series

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*Abstract:* We propose a new measure for stationarity in functional time series that is based on an explicit representation of the $L^2$-distance between the spectral density operator of a nonstationary process and its best ($L^2$-)approximation by a spectral density operator corresponding to a stationary process. This distance can be estimated by the sum of the Hilbert–Schmidt inner products of the periodogram operators (evaluated at different frequencies). Furthermore, the asymptotic normality of an appropriately standardized version of the estimator can be established for the corresponding estimator under the null and alternative hypotheses. As a result, we obtain a simple asymptotic frequency-domain level $\alpha$-test (using the quantiles of the normal distribution) to test for the hypothesis of stationarity of a functional time series. We also briefly discuss other applications, such as asymptotic confidence intervals for the measure of stationarity, or the construction of tests for "relevant deviations from stationarity". We demonstrate in a small simulation study that the new method has very good finite-sample properties. Moreover, we apply our test to annual temperature curves.

*Key words and phrases:* time series, functional data, spectral analysis, local stationarity, measuring stationarity, relevant hypotheses

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## 1. Introduction

In many applications of functional data analysis (FDA), data are recorded sequentially over time and naturally exhibit dependence. As a result, researchers are increasingly analyzing functional data from time series; we refer to the monographs of Bosq (2000) and Horváth and Kokoszka (2012), among others. An important assumption in most of the literature is that of stationarity, which allows us to develop a unified statistical theory. For example, stationary processes with a linear representation have been investigated by, among others, Mas (2000), Bosq (2002), and Dehling and Sharipov (2005). Prediction methods (e.g., Antoniadis and Sapatinas, 2003; Aue et al., 2015; Bosq, 2000) and violations of the independent and identically distributed (i.i.d.) assumption in the context of change point detection are also relatively well documented in the literature (e.g., Aue et al., 2009; Berkes et al., 2009; Horváth et al., 2010). Hörmann and Kokoszka (2010) provide a general framework within which to examine temporal dependence between the functional observations of stationary processes. Frequency domain analysis of stationary functional time series are considered by Panaretos and Tavakoli (2013) under the assumption of functional generalizations of cumulant-mixing conditions.

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In practice, however, it is not clear that the temporal dependence structure is constant and, hence, that stationarity is satisfied. It is therefore desirable to have tests for second-order stationarity or measures for deviations from stationarity for data analyses of functional time series. In the context of Euclidean data (univariate and multivariate), there exists a considerable amount of literature on this problem. Early work can be found in Priestley and Subba Rao (1969), who proposed testing the "homogeneity" of a set of evolutionary spectra. Von Sachs and Neumann (2000) used coefficients with respect to a Haar wavelet series expansion of time-varying periodograms for this purpose; see also Nason (2013), who provided an important extension of their approach and, Cardinali and Nason (2010) and Taylor et al. (2014) for further applications of wavelets to the problem of testing for stationarity. Paparoditis (2009, 2010) proposed rejecting the null hypothesis of second-order stationarity if there is a large $L^2$-distance between a local spectral density estimate and an estimate derived under the assumption of stationarity. Dette et al. (2011) suggested estimating this distance directly using sums of periodograms evaluated at the Fourier frequencies in order to avoid the problem of choosing additional bandwidths (see also Preuß et al. (2013), for an empirical process approach). An alternative method for investigating second-order stationarity can be found in Dwivedi and Subba Rao (2011) and Jentsch and

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Subba Rao (2015), who use the fact that the discrete Fourier transform (DFT) is asymptotically uncorrelated at the canonical frequencies if and only if the time series is second-order stationary. Recently, Jin et al. (2015) proposed a double-order selection test for checking the second-order stationarity of a univariate time series. Furthermore, Das and Nason (2016) investigated an experimental empirical measure of nonstationarity based on the mathematical roughness of the time evolution of the fitted parameters of a dynamic linear model.

On the other hand, despite the frequent assumption of second-order stationarity in functional data analysis, much less work has been done investigating the stationarity of functional data. A rigorous mathematical framework for locally stationary functional time series was recently developed by van Delft and Eichler (2018), who extended the concept of local stationarity introduced by Dahlhaus (1996, 1997) from univariate time series to functional data. To the best of our knowledge, Aue and van Delft (2019) is the only work that applies this framework to test for the second-order stationarity of a functional time series against smooth alternatives. These authors follow the approach of Dwivedi and Subba Rao (2011), showing that the functional discrete Fourier transform (fDFT) is asymptotically uncorrelated at distinct Fourier frequencies if and only if the process is functional weakly stationary. This result is then

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used to construct a test statistic based on an empirical covariance operator of the fDFTs, which is subsequently projected onto a finite-dimensional subspace. The asymptotic properties of the resulting quadratic form are shown to follow a chi-square distribution, both under the null and under the alternative of functional local stationarity. Although the authors thereby provide an explicit expression for the degree of departure from weak stationarity, the test requires the specification of the parameter $M$, the number of included lagged fDFTs. This can be viewed as a disadvantage, because it affects the power of the test.

We propose a different test, based on an explicit representation of the $L^2$-distance between the spectral density operator of a nonstationary process and its best ($L^2$-)approximation by a spectral density operator corresponding to a stationary process. This measure vanishes if and only if the time series is second-order stationary. Consequently, a test can be obtained by rejecting the hypothesis of stationarity for large values of a corresponding estimate. The $L^2$-distance is estimated using a functional of sums of integrated periodogram operators for which, after appropriate standardization, asymptotic normality can be established under the null hypothesis and any fixed alternative. The resulting test for the hypothesis of stationarity is extremely simple and, therefore, attractive to practitioners. The test uses the quantiles of the

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standard normal distribution, and does not require choosing a bandwidth in order to estimate the time-varying spectral density operators, or using bootstrap methods to obtain critical values. Therefore, the proposed methodology is very efficient, from a computational point of view.

Although a similar concept has been investigated for univariate time series (see Dette et al., 2011), the mathematical derivation of the asymptotic normality requires several sophisticated and new tools for spectral analysis of locally stationary functional time series. In particular, in contrast to the cited reference, our approach does not require a linear representation of the time series using an independent sequence, and we derive several new properties of the periodogram operator, which are of independent interest. Owing to space constraints, these results, together with the more technical arguments (which are rather complicated), are relegated to the Supplementary Material van Delft et al. (2019b). We specifically mention Theorem S3.1 in Section S3 of the Supplementary Material, which provides a representation of the cumulants of the Hilbert–Schmidt inner products of local periodogram tensors (evaluated at different time points and different frequencies) using the trace of cumulants of simple tensors of the local functional discrete Fourier transforms. The Supplementary Material also contains a brief discussion of several other applications of the asymptotic theory.

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The rest of the paper is organized as follows. In Section 2, we introduce the main concept of locally stationary functional time series, define a measure of stationarity for these processes, and introduce its corresponding estimators. Section 3 is devoted to the asymptotic properties of the proposed estimators. In Section 4, we report a small simulation study to demonstrate that the new test has very good finite-sample properties. In this section, we also apply our test to annual temperature curves recorded at several measuring stations in Australia over the past 135 years.

## 2. A measure of stationarity on the function space

### 2.1 Notation and the functional setup

Suppose $\mathscr{H}$ is a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $\mathscr{L}(\mathscr{H})$ be the space of bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$, and let $\{e_n\}_{n\geq 1}$ be some orthonormal basis of $\mathscr{H}$. An operator $A \in \mathscr{L}(\mathscr{H})$ is Hilbert–Schmidt if $|||A|||_2^2 = \sum_{n\geq 1} \|Ae_n\|^2 < \infty$, in which case, we write $A \in S_2(\mathscr{H})$. The space $S_2(\mathscr{H})$ is a Hilbert space with the inner product given by $\langle A, B\rangle_{\mathrm{HS}} = \sum_{n\geq 1}\langle Ae_n, Be_n\rangle$ for $A, B \in S_2(\mathscr{H})$. An operator $A \in \mathscr{L}(\mathscr{H})$ is a trace-class operator, that is, $A \in S_1(\mathscr{H})$, if $|||A|||_1 = \sum_{n\geq 1}\langle (A^\dagger A)^{1/2} e_n, e_n\rangle < \infty$, where $A^\dagger$ denotes the adjoint of $A$. The trace of $A \in \mathscr{L}(\mathscr{H})$ is defined by $\operatorname{Tr} A = \sum_{n\geq 1}\langle Ae_n, e_n\rangle$, which converges if $A \in S_1(\mathscr{H})$. We commonly use the

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rank-one operator $x \otimes y \in \mathscr{L}(\mathscr{H})$, with $x, y \in \mathscr{H}$, defined by $(x \otimes y)(z) = \langle z, y \rangle x$, for $z \in \mathscr{H}$.

Suppose that $X$ is an $\mathscr{H}$-valued random element. If $\mathbb{E}\|X\| < \infty$, the expected value $\mathbb{E}X$ can be defined as the unique element $\mu \in \mathscr{H}$ that satisfies $\mathbb{E}\langle X, x \rangle = \langle \mu, x \rangle$, for all $x \in \mathscr{H}$. Provided that $\mathbb{E}\|X\|^2 < \infty$, the covariance operator of $X$ is defined as $\mathbb{E}((X-\mu) \otimes (X-\mu))$. An $\mathscr{H}$-valued sequence $\{X_t\}_{t \in \mathbb{Z}}$ is second-order (or weakly) stationary if $\mathbb{E}\|X_t\|^2 < \infty$, $\mathbb{E}X_t = \mu$, and $\mathbb{E}((X_s - \mu) \otimes (X_t - \mu)) = E((X_{s-t} - \mu) \otimes (X_0 - \mu))$, for all $s, t \in \mathbb{Z}$. We say that $\{X_t\}_{t \in \mathbb{Z}}$ is strictly stationary if the joint distribution of $\{X_{t_1}, \ldots, X_{t_n}\}$ and the joint distribution of $\{X_{t_1+h}, \ldots, X_{t_n+h}\}$ coincide, for all $t_1, \ldots, t_n \in \mathbb{Z}$, $n \geq 1$, and $h \geq 1$.

In this paper, we focus on the space $\mathscr{H} := L^2_{\mathbb{C}}([0,1]^k)$, for $k \geq 1$, the space of (the equivalence classes of) square integrable functions $f : [0,1]^k \to \mathbb{C}$, where we denote the norm of $L^2_{\mathbb{C}}([0,1]^k)$ by $\|\cdot\|_2$. The corresponding space of real functions is denoted by $L^2_{\mathbb{R}}([0,1]^k)$, for $k \geq 1$.

### 2.2 Locally stationary functional time series

The second-order dynamics of weakly stationary time series of functional data $\{X_h\}_{h \in \mathbb{Z}}$ can be described completely by the Fourier transform of the sequence of covariance operators, acting on $L^2_{\mathbb{C}}([0,1])$; that is,

$$\mathscr{F}_\omega = \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} \mathbb{E}\big((X_h - \mu) \otimes (X_0 - \mu)\big) e^{-\mathrm{i}\omega h} \quad \omega \in [-\pi, \pi], \tag{2.1}$$

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where $\mu = \mathbb{E}X_0$ denotes the mean function. Following most of the literature on testing for second-order stationarity (see, i.a., Paparoditis, 2009; Dwivedi and Subba Rao, 2011), we assume that our data are centered and, hence, $\mu = 0$. This is without loss of generality, because the mean can be estimated without affecting the properties of our test; see Remark 3.1. If second-order stationarity is violated, we can no longer speak of a frequency distribution over all time and, hence, if it exists, (2.1) must become time-dependent. To allow for a meaningful definition of this object if stationarity is violated, we consider a triangular array $\{X_{t,T} : 1 \leq t \leq T\}_{T\in\mathbb{N}}$ as a doubly indexed functional time series, where $X_{t,T}$ is a random element with values in $L^2_{\mathbb{R}}([0,1])$, for each $1 \leq t \leq T$ and $T \in \mathbb{N}$. The processes $\{X_{t,T} : 1 \leq t \leq T\}$ are extended on $\mathbb{Z}$ by setting $X_{t,T} = X_{1,T}$ for $t < 1$, and $X_{t,T} = X_{T,T}$ for $t > T$. Following van Delft and Eichler (2018), the sequence of stochastic processes $\{X_{t,T} : t \in \mathbb{Z}\}$ indexed by $T \in \mathbb{N}$ is called *locally stationary* if, for all rescaled times $u \in [0,1]$, there exists a $L^2_{\mathbb{R}}([0,1])$-valued strictly stationary process $\{X_t^{(u)} : t \in \mathbb{Z}\}$, such that

$$\left\| X_{t,T} - X_t^{(u)} \right\|_2 \leq \left( \left| \tfrac{t}{T} - u \right| + \tfrac{1}{T} \right) P_{t,T}^{(u)} \qquad a.s., \tag{2.2}$$

for all $1 \leq t \leq T$, where $P_{t,T}^{(u)}$ is a positive real-valued process, such that for some $\rho > 0$ and $C < \infty$, the process satisfies $\mathbb{E}\left(\left|P_{t,T}^{(u)}\right|^{\rho}\right) < C$ for all $t$ and $T$, uniformly in $u \in [0,1]$. If the second-order dynamics are changing gradually over time, the second-order dynamics of the stochastic process $\{X_{t,T} : t \in \mathbb{Z}\}_{T\in\mathbb{N}}$ are

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completely described by the *time-varying spectral density operator* given by

$$\mathscr{F}_{u,\omega} = \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} \mathbb{E}\big(X_{t+h}^{(u)} \otimes X_t^{(u)}\big) e^{-\mathrm{i}\omega h}, \tag{2.3}$$

for each $u \in [0,1]$ and $\{X_t^{(u)} : t \in \mathbb{Z}\}$. Under the technical assumptions stated in Section 3, this object is a Hilbert–Schmidt operator. Note that if the process is, in fact, second-order stationary, then (2.3) reduces to the form (2.1). This framework thus lends itself in a natural way to testing for changing dynamics in the second-order structure.

### 2.3 Minimum distance and its estimation

In this study, we are interested in testing the hypothesis

$$H_0 : \mathscr{F}_{u,\omega} \equiv \mathscr{F}_\omega \quad \text{a.e. on } [-\pi,\pi] \times [0,1] \tag{2.4}$$

versus

$$H_a : \mathscr{F}_{u,\omega} \neq \mathscr{F}_\omega, \text{on a subset of } [-\pi,\pi] \times [0,1] \text{ of positive Lebesgue measure,} \tag{2.5}$$

where $\mathscr{F}_\omega$ is an unknown nonnegative definite Hilbert–Schmidt operator for each $\omega \in [-\pi,\pi]$ that does not depend on rescaled time $u \in [0,1]$. We measure deviations from stationarity using the minimum distance principle. To explain the main idea of this approach, consider a square integrable function

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$g : [0,1] \to \mathbb{R}$. Note that for any constant $a \in \mathbb{R}$, we have

$$d^2(a) = \int_0^1 |g(u) - a|^2 du = \int_0^1 |g(u) - \bar{g}|^2 du + \int_0^1 |\bar{g} - a|^2 du,$$

where $\bar{g} = \int_0^1 g(u) du$. Therefore, minimizing $d^2(a)$ with respect to $a \in \mathbb{R}$ gives the best approximation of the function $g$ by a constant function. The minimum is attained for the choice $a = \bar{g}$ and $d^2(\bar{g}) = \int_0^1 |g(u)|^2 du - \left(\int_0^1 g(u) du\right)^2$. In particular, the function $g$ is constant if and only if $\min_{a\in\mathbb{R}} d^2(a)$ vanishes.

We now transfer this idea to the setting of functional times series, and define a measure for the deviation from second-order stationarity by

$$m^2 = \min_{\mathscr{G}} \int_{-\pi}^{\pi} \int_0^1 |||\mathscr{F}_{u,\omega} - \mathscr{G}_\omega|||_2^2 du d\omega, \tag{2.6}$$

where the minimum is taken over all mappings $\mathscr{G} : [-\pi, \pi] \to S_2(L^2_{\mathbb{C}}([0,1]))$. Note that the hypotheses in (2.4) and (2.5) can be rewritten as

$$H_0 : m^2 = 0 \qquad \text{versus} \qquad H_a : m^2 > 0, \tag{2.7}$$

and a statistical test can be obtained by rejecting the null hypothesis $H_0$ for large values of an appropriate estimator of $m^2$. In order to construct such an estimator, we first derive an alternative representation of the minimum distance $m^2$.

**Lemma 2.1.** *The minimum distance $m^2$, defined in* (2.6)*, can be expressed as*

$$m^2 = \int_{-\pi}^{\pi} \int_0^1 |||\mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega|||_2^2 du d\omega, \tag{2.8}$$

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*where the operators* $\widetilde{\mathscr{F}}_\omega$ *are defined by*

$$\widetilde{\mathscr{F}}_\omega := \int_0^1 \mathscr{F}_{u,\omega} du, \tag{2.9}$$

*for each* $\omega \in [-\pi, \pi]$. *We refer to this operator* $\widetilde{\mathscr{F}}_\omega$ *as the time-integrated local spectral density operator, because it acts on* $L^2_{\mathbb{C}}([0,1])$, *such that* $\widetilde{\mathscr{F}}_\omega$ *no longer depends on* $u \in [0,1]$, *for each* $\omega \in [-\pi, \pi]$.

The proof is given in Section S2 of the Supplementary Material. Using the definition of the Hilbert–Schmidt norm, we can rewrite expression (2.8) as

$$m^2 = \int_{-\pi}^{\pi} \int_0^1 |||\mathscr{F}_{u,\omega}|||_2^2 du d\omega - \int_{-\pi}^{\pi} |||\widetilde{\mathscr{F}}_\omega|||_2^2 d\omega, \tag{2.10}$$

where $\widetilde{\mathscr{F}}_\omega$ is given by (2.9). The two terms in (2.10) can now be easily estimated from the available data $\{X_{t,T} : 1 \le t \le T\}$ using sums of periodogram operators.

In order to estimate the two integrals in (2.10), we split the sample into $M$ blocks, with $N$ elements inside each block such that $T = MN = M(T)N(T)$ for each $T \in \mathbb{N}$, where $M, N \in \mathbb{N}$, and $N$ is an even number. Here, $M$ and $N$ correspond to the number of terms used in a Riemann sum that approximates the integrals in (2.10) with respect to $du$ and $d\omega$ and, therefore, they have to be reasonably large. The number of elements in the blocks must grow faster than the number of blocks, but more slowly than the cube number of blocks. The choice of the number of blocks is discussed in Subsection 3.1, and an

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empirical investigation can be found in Section 4. Throughout this paper, we make the following assumption for the asymptotic analysis.

**Assumption 2.1.** *$M \to \infty$, $N \to \infty$ as $T \to \infty$, such that*

$$N/M \to \infty \quad \textit{and} \quad N/M^3 \to 0.$$

For $u \in [0,1]$, $\omega \in [-\pi, \pi]$, and $N \geq 1$, the fDFT evaluated around time $u$ is defined as a random function with values in $L^2_{\mathbb{C}}([0,1])$ given by

$$D_N^{u,\omega} := \frac{1}{\sqrt{2\pi N}} \sum_{s=0}^{N-1} X_{\lfloor uT \rfloor - N/2 + s + 1, T} e^{-\mathrm{i}\omega s}. \tag{2.11}$$

The periodogram tensor is then defined by

$$I_N^{u,\omega} := D_N^{u,\omega} \otimes D_N^{u,\omega}. \tag{2.12}$$

Let $\omega_k = 2\pi k/N$, for $k = 1, \ldots, N$, and $u_j = (N(j-1) + N/2)/T$ for $j = 1, 2, \ldots, M$, be the midpoint of each block. Observe that only the $j$th block of the sample determines the value of $I_N^{u_j,\omega_k}$, for each $k = 1, \ldots, N$. We estimate the two terms in (2.10) by

$$\hat{F}_{1,T} := \frac{1}{T} \sum_{k=1}^{\lfloor N/2 \rfloor} \sum_{j=1}^{M} \langle I_N^{u_j,\omega_k}, I_N^{u_j,\omega_{k-1}} \rangle_{HS}, \tag{2.13}$$

(note that $\hat{F}_{1,T}$ is real-valued for each $T \in \mathbb{N}$, because $\langle I_N^{u,\lambda}, I_N^{u,\omega} \rangle_{HS} = |\langle D_N^{u,\lambda}, D_N^{u,\omega} \rangle|^2$) and

$$\hat{F}_{2,T} := \frac{1}{N} \sum_{k=1}^{\lfloor N/2 \rfloor} \left\| \frac{1}{M} \sum_{j=1}^{M} I_N^{u_j,\omega_k} \right\|_2^2, \tag{2.14}$$

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respectively. We show that the estimation of $\int_{-\pi}^{\pi} |||\widetilde{\mathscr{F}}_{\omega}|||_2^2 d\omega$ using (2.14) introduces a bias term

$$B_{N,T} = \frac{N}{T} \int_{-\pi}^{\pi} \int_0^1 [\mathrm{Tr}(\mathscr{F}_{u,\omega})]^2 du d\omega. \tag{2.15}$$

Because this term is nonvanishing in a $\sqrt{T}$-consistent estimator under Assumption 2.1, it has to be taken into account. We therefore define the estimator of the minimum distance $m^2$ in (2.10) as

$$\widehat{m}_T = 4\pi(\hat{F}_{1,T} - \hat{F}_{2,T} + \hat{B}_{N,T}) \,, \tag{2.16}$$

where

$$\hat{B}_{N,T} = \frac{1}{T} \sum_{k=1}^{\lfloor N/2 \rfloor} \sum_{j=1}^{M} \mathrm{Tr}(I^{u_j,\omega_k}) \mathrm{Tr}(I^{u_j,\omega_{k-1}}) = \frac{1}{T} \sum_{k=1}^{\lfloor N/2 \rfloor} \sum_{j=1}^{M} \|D_N^{u_j,\omega_k}\|_2^2 \|D_N^{u_j,\omega_{k-1}}\|_2^2. \tag{2.17}$$

We prove in Section S3 (Corollary S3.1) of the Supplementary Material that, under the conditions of Theorem 3.1,

$$\sqrt{T}\big(\hat{B}_{N,T} - B_{N,T}\big) \xrightarrow{p} 0 \quad \text{as} \quad T \to \infty.$$

Therefore, the bias correction does not affect the asymptotic distribution of the test statistic.

As is the case with the periodogram of a real-valued time series, the periodogram tensor defined by (2.12) is not a consistent estimator. However, the estimators $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$ are consistent for the quantities appearing in the

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measure of stationarity defined in (2.10), because they are obtained by averaging periodogram tensors with respect to different Fourier frequencies. These heuristic arguments will be made more precise in the following section, where we state our main asymptotic results.

## 3. Asymptotic normality and statistical applications

In this section, we establish the asymptotic normality of an appropriately standardized version of the statistic $\widehat{m}_T$ defined in (2.16) and, as a by-product, its consistency for estimating the measure of stationarity $m^2$. We denote the joint cumulant of $X_1, \ldots, X_k$ by $\mathrm{Cum}(X_1, \ldots, X_k)$, where $X_1, \ldots, X_k$ are $\mathscr{H}$-valued random elements, such that $\mathbb{E}\|X_t\|^k < \infty$, for each $t = 1, \ldots, k$. The definition of the joint cumulant of $\mathscr{H}$-valued random variables is intricate and, hence, postponed to Section S1 of the Supplementary Material. The functional process $\{X_{t,T} \colon t \in \mathbb{Z}\}_{T\in\mathbb{N}}$ is assumed to satisfy the following set of conditions.

**Assumption 3.1.** Assume that $\{X_{t,T} \colon t \in \mathbb{Z}\}_{T\in\mathbb{N}}$ is a locally stationary zero-mean stochastic process, as introduced in Section 2, and, for even $k \in \mathbb{N}$, let $\kappa_{k;t_1,\ldots,t_{k-1}} : L^2([0,1]^{k/2}) \to L^2([0,1]^{k/2})$ be a positive operator independent of $T$, such that, for all $j = 1, \ldots, k-1$ and some $\ell \in \mathbb{N}$,

$$\sum_{t_1,\ldots,t_{k-1}\in\mathbb{Z}} (1 + |t_j|^{\ell}) |||\kappa_{k;t_1,\ldots,t_{k-1}}|||_1 < \infty. \tag{3.1}$$

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Denote

$$Y_t^{(T)} = X_{t,T} - X_t^{(t/T)} \qquad \text{and} \qquad Y_t^{(u,v)} = \frac{X_t^{(u)} - X_t^{(v)}}{(u-v)}, \tag{3.2}$$

for $T \in \mathbb{N}$, $1 \leq t \leq T$, and $u, v \in [0,1]$, such that $u \neq v$. Suppose, furthermore, that the $k$th order joint cumulants satisfy

(i) $|||\mathrm{Cum}(X_{t_1,T}, \ldots, X_{t_{k-1},T}, Y_{t_k}^{(T)})|||_1 \leq \frac{1}{T}|||\kappa_{k;t_1-t_k,\ldots,t_{k-1}-t_k}|||_1$,

(ii) $|||\mathrm{Cum}(X_{t_1}^{(u_1)}, \ldots, X_{t_{k-1}}^{(u_{k-1})}, Y_{t_k}^{(u_k,v)})|||_1 \leq |||\kappa_{k;t_1-t_k,\ldots,t_{k-1}-t_k}|||_1$,

(iii) $\sup_u |||\mathrm{Cum}(X_{t_1}^{(u)}, \ldots, X_{t_{k-1}}^{(u)}, X_{t_k}^{(u)})|||_1 \leq |||\kappa_{k;t_1-t_k,\ldots,t_{k-1}-t_k}|||_1$,

(iv) $\sup_u |||\frac{\partial^\ell}{\partial u^\ell}\mathrm{Cum}(X_{t_1}^{(u)}, \ldots, X_{t_{k-1}}^{(u)}, X_{t_k}^{(u)})|||_1 \leq |||\kappa_{k;t_1-t_k,\ldots,t_{k-1}-t_k}|||_1$.

Note that these assumptions allow for a meaningful definition of local cumulant spectral operators of order $k$, from which we can obtain a closed-form expression of the variance of $\widehat{m}_T$. For further detail, we refer to Section S1 and Section S5 of the Supplementary Material. In addition note that, using the Cauchy–Schwarz inequality, we can bound the $2k+1$th joint cumulant and moment tensors in terms of the $2k+2$th joint cumulant and moment tensors.

The following result establishes the asymptotic normality of $\widehat{m}_T$ (appropriately standardized). The proof is given in Section S3 of the Supplementary Material.

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**Theorem 3.1.** *Suppose that Assumption 2.1 and Assumption 3.1 hold. Then,*

$$\sqrt{T}(\hat{m}_T - m^2) \xrightarrow{d} N(0, v^2) \quad as \quad T \to \infty,$$

*where the expression for the asymptotic variance $v^2$ can be found in Section S3 of the Supplementary Material.*

**Remark 3.1.** *The assumption of a zero or constant mean function is common in the context of testing for stationarity in the frequency domain (see, i.a., Paparoditis, 2009; Dwivedi and Subba Rao, 2011; Jentsch and Subba Rao, 2015). Note that Theorem 3.1 remains true if $\mu = 0$ does not hold. For example, to address this problem, we define $\hat{\mu}_T = T^{-1}\sum_{t=1}^{T} X_{t,T}$, replace $D_N^{u,\omega}$ by*

$$\tilde{D}_N^{u,\omega} = (2\pi N)^{-1/2} \sum_{s=0}^{N-1} (X_{\lfloor uT \rfloor - N/2+s+1,T} - \hat{\mu}_T) e^{-i\omega s},$$

*and replace $I_N^{u,\omega}$ with $\tilde{I}_N^{u,\omega} = \tilde{D}_N^{u,\omega} \otimes \tilde{D}_N^{u,\omega}$ in the quantities* (2.13), (2.14), *and* (2.15), *which define the statistic* (2.16). *A proof of our claim can be found in Subsection S5.6 of the Supplementary Material. In the general case, where the mean functions vary smoothly in time, a local-window estimator has to be subtracted (see e.g., Dette et al., 2017, who considered this scenario for one-dimensional locally stationary long-range dependent time series).*

Under the null hypothesis, the statistic has a very succinct form.

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**Corollary 3.1.** *Suppose that Assumption 2.1 and Assumption 3.1 hold. Then, under the null hypothesis $H_0$, we have*

$$\sqrt{T}\hat{m}_T \xrightarrow{d} N(0, v^2_{H_0}) \quad as \quad T \to \infty,$$

*where the asymptotic variance $v^2_{H_0}$ is given by*

$$v^2_{H_0} = 4\pi \int_{-\pi}^{\pi} |||\widetilde{\mathscr{F}}_\omega|||_2^4 d\omega. \tag{3.3}$$

Observing the equivalent representation of the hypotheses in (2.7), it is reasonable to reject the null hypotheses (2.4) of a stationary functional process whenever

$$\widehat{m}_T > \frac{\widehat{v}_{H_0}}{\sqrt{T}} u_{1-\alpha}, \tag{3.4}$$

where $u_{1-\alpha}$ denotes the $(1-\alpha)$-quantile of the standard normal distribution, and $\widehat{v}^2_{H_0}$ is an appropriate estimator of the asymptotic variance under the null hypothesis given in (3.3). The asymptotic variance $v^2_{H_0}$ can be estimated using the statistic

$$\hat{v}^2_{H_0} = \frac{16\pi^2}{N} \sum_{k=1}^{\lfloor N/2 \rfloor} \left[ \frac{1}{M} \sum_{j=1}^{M} \langle I_N^{u_j,\omega_k}, I_N^{u_j,\omega_{k-1}} \rangle_{HS} \right]^2. \tag{3.5}$$

Corollary 3.1 and the following result show that the test defined by (3.4) is an asymptotic level $\alpha$-test. The proof is given in Section S5.5 of the Supplementary Material.

**Lemma 3.1.** *Under the assumptions of Theorem 3.1, the estimator defined in* (3.5) *is consistent; that is, $\hat{v}^2_{H_0} \to v^2_{H_0}$ in probability as $T \to \infty$.*

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### 3.1 The choice of $M$ and $N$

Here, we provide heuristic arguments on how to choose the number of blocks $M$ and the number of elements in the blocks $N$. Because we assume that $T = MN$, the choice of $M$ determines the value of $N$, and vice versa. Our test is based on the estimator of the distance $m^2$ defined by (2.6). One way to choose the values of $M$ and $N$ is to choose those values that minimize the leading terms in the asymptotic expansion of the mean squared error (MSE) of the estimator of $m^2$. We have that

$$\mathrm{MSE}(\hat{m}_T) = \mathrm{Var}\,\hat{m}_T + |\mathbb{E}\,\hat{m}_T - m^2|^2, \tag{3.6}$$

with $\mathrm{Var}\,\hat{m}_T = (4\pi)^2\{\mathrm{Var}\,\hat{F}_{1,T} + \mathrm{Var}\,\hat{F}_{2,T} - 2\,\mathrm{Cov}[\hat{F}_{1,T}, \hat{F}_{2,T}]\}$ and $\mathbb{E}\,\hat{m}_T = 4\pi(\mathbb{E}\,\hat{F}_{1,T} - \mathbb{E}\,\hat{F}_{2,T})$. Note that we ignore the estimator $\hat{B}_{N,T}$ of the bias defined in (2.17) because it is of lower order in (3.6). The asymptotic expressions of $\mathbb{E}\,\hat{F}_{1,T}$, $\mathbb{E}\,\hat{F}_{2,T}$, $\mathrm{Var}\,\hat{F}_{1,T}$, $\mathrm{Var}\,\hat{F}_{2,T}$, and $\mathrm{Cov}[\hat{F}_{1,T}, \hat{F}_{2,T}]$ are given in Section S5 of the Supplementary Material. Here, we assume Gaussianity in order to avoid dealing with the fourth-order terms. The leading terms of the asymptotic expression of the MSE are double Riemann sums. In addition, $M$ and $N$ determine the error that we make by approximating double integrals using double Riemann sums. Suppose that $g : [0,1] \times [0,\pi] \to \mathbb{R}$ is a Riemann integrable function, twice differentiable in its first argument, and once in its second. Using the

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error bounds for the midpoint and the right endpoint approximations of the integrals, we obtain

$$\Big|\frac{1}{N}\sum_{k=1}^{\lfloor N/2\rfloor}\frac{1}{M}\sum_{j=1}^{M}g(u_j,\omega_k)-\frac{1}{2\pi}\int_0^{\pi}\int_0^1 g(u,\omega)\,du\,d\omega\Big|$$
$$\leq \frac{1}{24M^2}\cdot\frac{1}{N}\sum_{k=1}^{\lfloor N/2\rfloor}\max_{u\in[0,1]}|g''_u(u,\omega_k)|+\frac{\pi^2}{2N}\int_0^1\max_{\omega\in[0,\pi]}|g'_\omega(u,\omega)|\,du, \quad (3.7)$$

where $u_j = (N(j-1)+N/2)/T$ for $1\leq j\leq M$, and $\omega_k = 2\pi k/N$ for $1\leq k\leq \lfloor N/2\rfloor$. Rather than give the complete bound of the MSE, we only explain the idea behind the bound. One of the terms in the expression of $T\operatorname{Var}\hat{F}_{1,T}$ is given by

$$R_{NM}=\frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\langle\mathscr{F}_{u_j,\omega_k},\mathscr{F}_{u_j,\omega_{k-1}}\rangle_{HS}\langle\mathscr{F}_{u_j,-\omega_k},\mathscr{F}_{u_j,-\omega_{k-1}}\rangle_{HS}.$$

We have that

$$R_{NM}\leq\Big|R_{NM}-\frac{1}{2\pi}\int_0^{\pi}\int_0^1|||\mathscr{F}_{u,\omega}|||_2^4\,du\,d\omega\Big|+\frac{1}{2\pi}\int_0^{\pi}\int_0^1|||\mathscr{F}_{u,\omega}|||_2^4\,du\,d\omega. \quad (3.8)$$

The second term in (3.8) does not depend on the choice of $M$ and $N$. We use the inequality (3.7) to bound the first term in (3.8). Provided that the integral is finite and the Riemann sum in (3.7) converges, similar arguments applied to the other terms in the MSE show that we need to minimize the expression $C_1/M^2+C_2/N$ over all possible values of $M$ and $N$, where $C_1$ and $C_2$ are two positive constants that are unknown, because these depend on the

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time-varying spectral density operator. The right-hand side of (3.7) is minimized for

$$M = (2C_1/C_2)^{1/3} \cdot T^{1/3} \text{ and } N = (2C_1/C_2)^{-1/3} \cdot T^{2/3}.$$

Unfortunately, because $C_1$ and $C_2$ are unknown, we cannot determine the optimal values of $M$ and $N$. However, this suggests $M \approx T^{1/3}$ and $N \approx T^{2/3}$ might be a reasonable choice, given that $(2C_1/C_2)^{1/3} \approx 1$. We provide empirical evidence of this rule in Section 4.

## 4. Finite-sample properties

In this section, we investigate the finite-sample properties of the proposed methods proposed using a simulation study, and illustrate potential applications by analyzing annual temperature curves.

### 4.1 Simulation study

In order to investigate the finite-sample performance of the test (3.4) for the hypothesis $H_0 : m^2 = 0$ using simulated data, we consider a similar setup to that in Aue and van Delft (2019), who used a Fourier basis representation on the interval $[0,1]$ to generate functional data. Specifically, let $\{\psi_l\}_{l=1}^{\infty}$ be the Fourier basis functions. Consider the $p$th-order time-varying functional au-

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toregressive process (tvFAR(p)), $\{X_t\}_{t\in\mathbb{Z}}$, defined as

$$X_t(\tau) = \sum_{t'=1}^{p} A_{t,t'}(X_{t-t'})(\tau) + \epsilon_t(\tau), \qquad \tau \in [0,1], \tag{4.1}$$

where $A_{t,1}, \ldots, A_{t,p}$ are time-varying auto-covariance operators, and $\{\epsilon_t(\tau)\}_{t\in\mathbb{Z}}$ is a sequence of mean zero innovations. We have

$$\begin{aligned}\langle X_t, \psi_l\rangle &= \sum_{l'=1}^{\infty}\sum_{t'=1}^{p} \langle X_{t-t'}, \psi_l\rangle\langle A_{t,t'}(\psi_l), \psi_{l'}\rangle + \langle \epsilon_t, \psi_l\rangle \\ &\approx \sum_{l'=1}^{L_{max}}\sum_{t'=1}^{p} \langle X_{t-t'}, \psi_l\rangle\langle A_{t,t'}(\psi_l), \psi_{l'}\rangle + \langle \epsilon_t, \psi_l\rangle. \end{aligned} \tag{4.2}$$

Therefore, the first $L_{max}$ Fourier coefficients of the process $X_t$ are generated using the $p$th-order vector autoregressive process

$$\widetilde{X}_t = \sum_{t'=1}^{p} \widetilde{A}_{t,t'}\widetilde{X}_{t-t'} + \widetilde{\epsilon}_t,$$

where $\widetilde{X}_t := \big(\langle X_t, \psi_1\rangle, \ldots, \langle X_t, \psi_{L_{max}}\rangle\big)^\top$ is the vector of Fourier coefficients, the $(l, l')$th entry of $\widetilde{A}_{t,j}$ is given by $\langle A_{t,j}(\psi_l), \psi_{l'}\rangle$, and $\widetilde{\epsilon}_t := \big(\langle \epsilon_t, \psi_1\rangle, \ldots, \langle \epsilon_t, \psi_{L_{max}}\rangle\big)^\top$. The entries of the matrix $\widetilde{A}_{t,j}$ are generated as $N\big(0, v_{l,l'}^{(t,j)}\big)$, with $v_{l,l'}^{(t,j)}$ specified below. To ensure stationarity or the existence of a causal solution, the norms $\kappa_{t,j}$ of $A_{t,j}$ are required to satisfy certain conditions (see Bosq (2000) for stationary and van Delft and Eichler (2018) for locally stationary functional time series). If $A_{t,j} \equiv A_j$, for all $t$ in (4.1), and the error sequence $(\epsilon_t, t \in \mathbb{Z})$ is an i.i.d. sequence, we obtain the stationary functional autoregressive (FAR) model of order $p$. In that case, we generate the entries of the operator matrix from $N\big(0, v_{l,l'}^{(j)}\big)$ distributions. Functional white noise can be thought of

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as a FAR model of order zero. Throughout this section, the number of Monte Carlo replications is always 1000. We use the `fda` package from `R` to generate the functional data, where $L_{max}$ is taken to be 15. The periodogram kernels are evaluated on a $100 \times 100$ grid on the square $[0,1]^2$, and their integrals are calculated by averaging the functional values at the grid points. The asymptotic variance under the null hypothesis is estimated using (3.5). In Table 1, we report the simulated nominal levels of the test (3.4) for the hypotheses in (2.7) for the sample sizes $T = 128$, 256, 512, and 1024, where we consider the following three (stationary) data-generating processes:

(I) The functional white noise variables $\epsilon_1, \ldots, \epsilon_T$ are i.i.d., with coefficient variances $\mathrm{Var}(\langle \epsilon_t, \psi_l \rangle) = \exp((l-1)/10)$.

(II) The FAR(2) variables $X_1, \ldots, X_T$, with operators specified by variances $v^{(1)}_{l,l'} = \exp(-l-l')$ and $v^{(2)}_{l,l'} = 1/(l + l'^{3/2})$, with norms $\kappa_1 = 0.75$ and $\kappa_2 = -0.4$ and with innovations $\epsilon_1, \ldots, \epsilon_T$, as in (I).

(III) The FAR(2) variables $X_1, \ldots, X_T$, as in (II), but with $\kappa_1 = 0.4$ and $\kappa_2 = 0.45$.

Recall that the test requires that we choose the number $M$ of blocks, which determines the number $N$ of observations in each block via the equation $T = MN$. As mentioned before, the quantities $M$ and $N$ have to be reasonably large, because they correspond to the number of terms used in the Riemann

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Table 1: *Empirical rejection probabilities (percentage) of the test* (3.4) *for the hypotheses in* (2.7) *under the null hypothesis.*

| | | | I | | | II | | | III | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| T | N | M | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| 128 | 32 | 4 | 6.0 | 2.8 | 0.7 | 7.4 | 3.7 | 0.7 | 6.4 | 2.5 | 0.3 |
| 128 | 16 | 8 | 5.9 | 2.7 | 0.4 | 7.3 | 2.8 | 0.8 | 5.2 | 2.5 | 0.5 |
| 256 | 32 | 8 | 7.0 | 3.2 | 0.5 | 7.1 | 4.1 | 0.7 | 6.8 | 3.5 | 0.7 |
| 256 | 16 | 16 | 7.5 | 2.9 | 0.5 | 7.4 | 3.6 | 0.7 | 7.0 | 3.0 | 0.5 |
| 512 | 64 | 8 | 7.5 | 3.1 | 0.5 | 8.6 | 4.2 | 0.3 | 7.9 | 3.5 | 0.6 |
| 512 | 32 | 16 | 6.7 | 2.4 | 0.4 | 7.1 | 3.3 | 0.7 | 6.4 | 2.4 | 0.2 |
| 1024 | 128 | 8 | 8.8 | 4.2 | 1.0 | 9.6 | 4.1 | 1.0 | 8.9 | 3.9 | 0.9 |
| 1024 | 64 | 16 | 9.7 | 4.7 | 1.1 | 10.0 | 5.3 | 1.4 | 9.8 | 4.6 | 0.9 |
| 1024 | 32 | 32 | 8.0 | 3.3 | 0.5 | 9.3 | 5.2 | 1.3 | 8.0 | 3.6 | 0.5 |

sum that approximates the integral with respect to $du$ and $d\omega$ in (2.10). We investigate the effect of this choice in more detail in the next section. Here, we consider those combinations for which Assumption 2.1 is satisfied. Interestingly, the results reported in Table 1 are rather robust with respect to this choice, and we observe a reasonable approximation of the nominal level in nearly all cases under consideration, albeit the test being slightly undersized for the small samples sizes.

Next, we investigate the performance of the test (3.4) under the alternative, where we consider the following (nonstationary) data-generating processes:

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(IV) The tvFAR(1) variables $X_1, \ldots, X_T$, with the operator specified by the variances $v_{l,l'}^{(t,1)} = v_{l,l'}^{(1)} = \exp(-l - l')$ and the norm $\kappa_1 = 0.8$, and with innovations as in (I), with a multiplicative time-varying variance

$$\sigma^2(t) = \cos\Big(\frac{1}{2} + \cos\big(\frac{2\pi t}{1024}\big) + 0.3\sin\big(\frac{2\pi t}{1024}\big)\Big).$$

(V) The tvFAR(2) variables $X_1, \ldots, X_T$, with operators as in (IV), but with the time-varying norm

$$\kappa_{1,t} = 1.8\cos\left(1.5 - \cos\left(\frac{4\pi t}{T}\right)\right),$$

constant norm $\kappa_2 = -0.81$, and innovations as in (I).

(VI) The structural break FAR(2) variables $X_1, \ldots, X_T$, generated as follows:

- for $t \leq 3T/8$, the operators are as in (II), with norms $\kappa_1 = 0.7$ and $\kappa_2 = 0.2$, and innovations as in (I).
- for $t > 3T/8$, the operators are as in (II), with norms $\kappa_1 = 0$ and $\kappa_2 = -0.2$, and innovations as in (I), but with coefficient variances $\mathrm{Var}(\langle \epsilon_t, \psi_l \rangle) = 2\exp((l-1)/10)$.

The results of the test (3.4) under the alternative are displayed in Table 2. We observe that the test has very good power for models IV and VI, even for small sample sizes. For model V, the power is lower than that of the other two models, but is still very good, and not completely unintuitive, because it can be

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explained by its data-generating mechanism. Depending on the draw of the operators, the resulting process in finite samples can be highly dependent, or may show barely any dependence at all.

Table 2: *Empirical rejection probabilities (percentage) of the test* (3.4) *for the hypotheses in* (2.7) *under the alternative hypothesis.*

| | | | IV | | | V | | | VI | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| T | N | M | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| 128 | 32 | 4 | 65.8 | 55.8 | 31.8 | 55.2 | 43.1 | 19.1 | 72.8 | 59.7 | 34.2 |
| 128 | 16 | 8 | 66.7 | 57.1 | 36.9 | 46.4 | 37.9 | 24.1 | 41.6 | 30.1 | 12.6 |
| 256 | 32 | 8 | 99.9 | 99.8 | 99.7 | 73.1 | 65.2 | 46.3 | 65.1 | 53.6 | 30.2 |
| 256 | 16 | 16 | 99.5 | 99.4 | 99.2 | 54.2 | 48.8 | 37.6 | 70.8 | 59.0 | 34.0 |
| 512 | 64 | 8 | 99.9 | 99.9 | 99.9 | 89.3 | 85.1 | 71.6 | 90.6 | 82.5 | 62.2 |
| 512 | 32 | 16 | 100.0 | 100.0 | 100.0 | 80.2 | 75.3 | 66.6 | 92.2 | 87.8 | 70.1 |
| 1024 | 128 | 8 | 100.0 | 100.0 | 100.0 | 92.2 | 90.1 | 83.9 | 99.6 | 98.4 | 92.9 |
| 1024 | 64 | 16 | 100.0 | 100.0 | 99.9 | 90.2 | 88.2 | 83.5 | 99.7 | 99.1 | 96.5 |
| 1024 | 32 | 32 | 99.9 | 99.9 | 99.9 | 81.4 | 79.8 | 74.6 | 99.3 | 98.5 | 95.9 |

### 4.2 Choice of $M$ and $N$

In order to examine how the choice of $M$ and $N$ affects the test's performance, we considered a simulation study with a sample size equal to $T = 4096$, because this allows us to vary $M$ from $M = 4, 8, \ldots, 1024$. Note that we thus also include choices of $M$ for which assumption (2.1) does not hold. The study was again performed over 1000 replications for each of the above models. Figure 1(a)–(c) provides the estimated densities for each $M$ for model I, II, and

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III, respectively. The estimated densities for model I appear well-aligned with the standard normal for all values of $M$. However, the best fit appears to be for $16 \leq M \leq 128$. For models II and III, we clearly observe that for $M > N$, the distribution becomes skewed and flatter. This is intuitive, because the assumptions underlying Theorem 3.1 do not hold. The difference with the standard normal curve seems to become more pronounced as the dependence increases. From these three models, the dependence is strongest for model II. In order to quantify our observations, we computed the mean absolute error to measure the difference between the estimated density of the test statistic and the standard normal density (see Figure 1(d)). The results indicate that a relatively small value of $M$ compared to $N$ leads to the best approximation. However, $M$ should not be too small. Specifically, a minimal error is attained with $M = 32$ for model I and model III, and with $M = 16$ for model II.

Figure 2 shows the rejection probabilities for $\alpha = 0.1, 0.05, 0.01$ under the three alternatives. For model IV and model VI, we find perfect power for all choices of $M$ and all critical values. For model V, there is some sensitivity, and the power seems best for $8 \leq M \leq 32$. As previously remarked, the sensitivity for model V is due to its data-generating mechanism. To summarize, it appears that our test is very robust to different choices of $M$ that satisfy Assumption 2.1. This empirical study indicates particularly good performance

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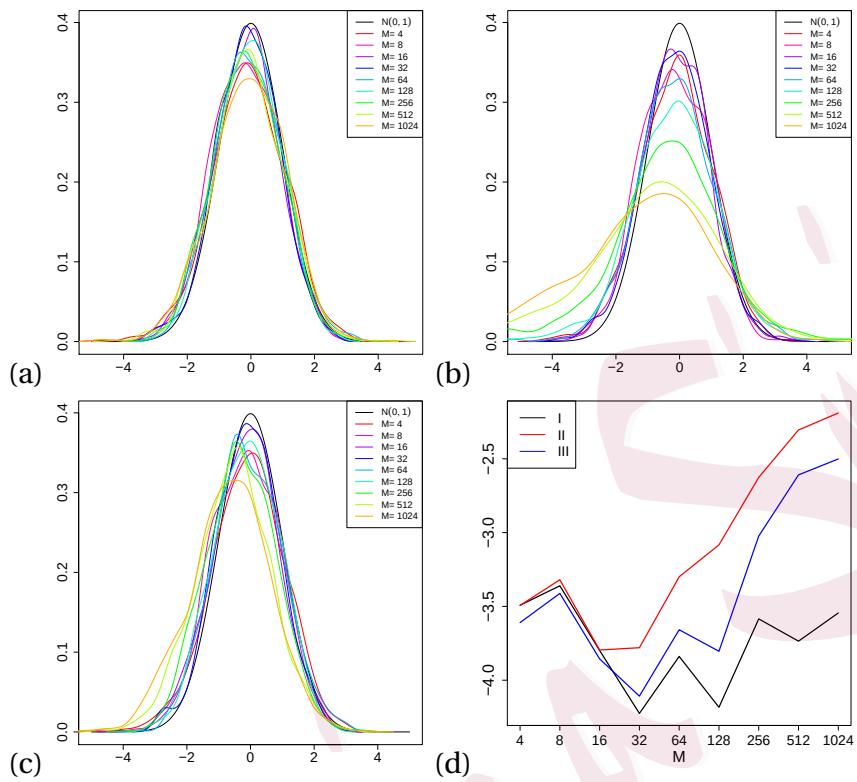


Figure 1: *(a)–(c) Estimated densities for different choices of $M$ with $T = 4096$ compared to a standard normal distribution (black); (d) Natural logarithm of the mean absolute error compared to the standard normal distribution.*

for the range $16 < M < 64$ for $T = 4096$, which corroborates our findings in Section 3.

### 4.3 Data example

We illustrate the proposed methodology by analyzing annual temperature curve data, recorded at several measuring stations across Australia. The recorded daily minimum temperatures for each year are treated as functional data. The locations of the measuring stations and the lengths of the time series are re-

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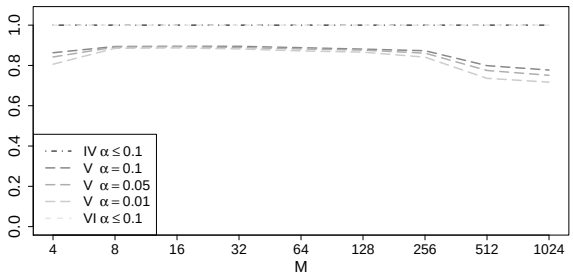
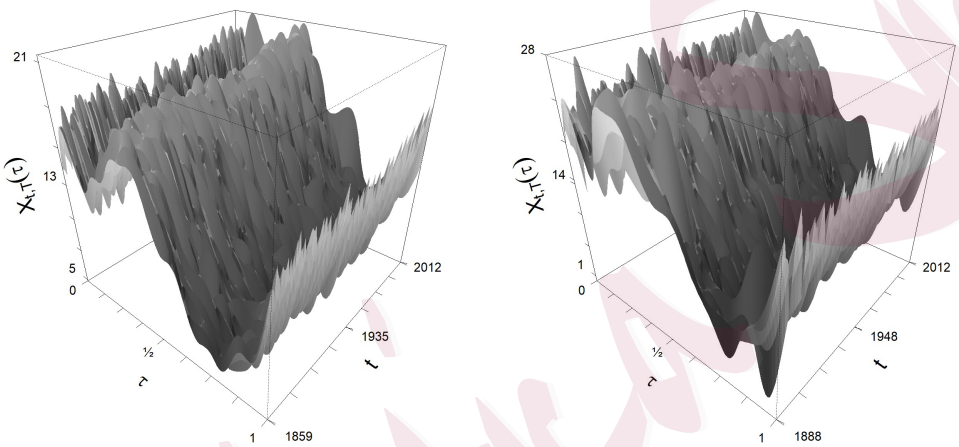



Figure 2: Rejection probabilities for the three alternative models for $T = 4096$.




(a) Sydney (b) Boulia Airport

Figure 3: *Minimum temperature curves, plotted by year.*

ported in Table 3. Figure 3 depicts the minimum temperature curves for Sydney and Boulia Airport as three-dimensional plots, visualizing also the annual dynamics.

We use the proposed test in (3.4) to investigate whether these temperature curves are realizations of a stationary process. For the the number of blocks, we use the above findings; that is, $M = \lceil T^{1/3} \rceil$. Given the number of curves for

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Table 3: *Values of the test statistic* (3.4) *for the hypothesis of stationarity of the annual temperature curve data.*

| Measuring Station | T | $M = \lceil T^{1/3} \rceil$ | M=8 | $M = \lfloor T^{1/2} \rfloor$ |
|---|---|---|---|---|
| Boulia Airport | 120 | 3.21 | 2.95 | 4.55 |
| Cape Otway | 149 | 3.87 | 4.42 | 4.48 |
| Gayndah Post Office | 117 | 3.19 | 4.46 | 4.16 |
| Gunnedah Pool | 133 | 4.33 | 3.72 | 5.04 |
| Hobart | 121 | 4.99 | 4.60 | 5.13 |
| Melbourne | 158 | 2.88 | 3.68 | 4.36 |
| Robe | 129 | 2.88 | 2.91 | 3.65 |
| Sydney | 154 | 3.30 | 3.71 | 4.30 |

each station, this value can be rather small thus, we also consider $\lfloor T^{1/2} \rfloor$. As a fixed comparison for all curves, we take $M = 8$, because the sample length is closest to $T = 128$ (and often slightly larger). The data are centered, as explained in Remark 3.1. The corresponding values of the test statistic (3.4) for the hypothesis of stationarity are reported in Table 3. It is clear that we reject the null of stationarity in all cases at the 1% significance level. Therefore, the test provides strong evidence against the null hypothesis of stationarity for all measuring stations.

**Supplementary Material** The file online Supplementary Material contains proofs and additional background information.

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