Statistica Sinica Preprint No: SS-2017-0507	
Title	Unit Root Testing on Buffered Autoregressive Model
Manuscript ID	SS-2017-0507
URL	http://www.stat.sinica.edu.tw/statistica/
DOI	10.5705/ss.202017.0507

Complete List of Authors	Di Wang and Wai Keung Li
Corresponding Author	Di Wang
E-mail	diwang@connect.hku.hk

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

UNIT ROOT TESTING ON BUFFERED AUTOREGRESSIVE MODEL

Di Wang and Wai Keung Li

University of Hong Kong

Abstract:

A buffered autoregression extends the classical threshold autoregression by allowing a buffer region for regime changes. In this study, we examine asymptotic statistical inferences for the two-regime buffered autoregressive (BAR) model, with autoregressive unit roots. We propose a Sup-LR test for the nonlinear buffer effect in the possible presence of unit roots, and a class of unit root tests to identify the number of nonstationary regimes in the BAR model. The wild bootstrap method is suggested to approximate the critical values of the two tests. Simulation results show that the proposed unit root test outperforms the conventional augmented Dickey–Fuller test, and that the two wild bootstrap tests are robust to unknown heteroscedasticity. Two macroeconomic data examples, based on U.S. unemployment rates and real exchange rates, respectively, are provided to illustrate the methods.

Key words and phrases: Asymptotic theory, buffer effect, nonlinear time series, nonstationary, threshold autoregression, wild bootstrap

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

1. Introduction

As proposed by Tong (1978), the threshold autoregressive (TAR) model has been applied successfully in economics, finance, and other fields; for example, see Tong (1990), Hansen (2011), and Chen, So and Liu (2011). As an extension of the TAR model, Li et al. (2015) proposed the buffered autoregressive (BAR) model, also known as the hysteretic autoregressive (HAR) model. Here, the BAR model extends the TAR model by allowing a buffer region for regime changes. Zhu, Yu and Li (2014) proposed a test for the stationary buffered autoregressive process against the linear process, and BAR models with conditional heteroscedasticity were studied by Chen and Truong (2016) and Zhu, Li and Yu (2017).

However, all previous studies related to the BAR model assume that the data are strictly stationary, geometrically ergodic, and have no unit roots. Thus, the objective of this study is to develop statistical tools to study buffered nonlinearity and nonstationarity simultaneously. Unit root tests have been developed for other nonlinear time series models. These include unit root tests for the two-regime TAR by Caner and Hansen (2001) and Seo (2008), the three-regime TAR by Bec, Ben Salem and Carrasco (2004) and Kapetanios and Shin (1996), and the smooth transitional autoregressive model by Kapetanios, Shin and Snell (2003) and Park and Shintani (2016).

We propose a Sup-LR test for the buffered nonlinear effect in the possible presence of unit roots, and a general class of unit root tests based on t-ratios, taking into account the buffer effect. In order to analyze the possible nonstationarity and nonlinearity si-

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

multaneously, we study the unit root asymptotic theory under two scenarios, namely the linear and buffered nonlinear cases. Because the two-regime TAR model is a special case of the BAR model when the buffer region vanishes, our results extend those of Caner and Hansen (2001). The asymptotic distributions of our proposed tests have a similar form to those of the TAR model. The wild bootstrap method can be employed to approximate the finite-sample critical value, and is robust to unknown heteroscedasticity, according to Monte Carlo simulations.

A rival to the two-regime BAR model is the three-regime TAR model. In many economic applications of the three-regime TAR model, the middle regime is allowed to be a unit root process, while the outer regimes are stationary. In this way, the TAR model can account for ergodicity and allow for local nonstationarity (Bec, Ben Salem and Carrasco, 2004; Maki, 2009; Chen, Chen and Lee, 2013). As pointed in Li et al. (2015) and Truong, Chen and So (2016), the two distinct threshold parameters in the three-regime TAR are interpreted as discontinuous structural change points, whereas those in the BAR model represent asymmetric structural change points with a buffer effect. In this work, we compare these two models with possible unit roots. A combination of nonstationarity and the buffer effect could provide new insights into the behavior of the economic time series, leading to a new interpretation of the business cycle, as shown in our empirical analysis.

The rest of the paper is organized as follows. In Section 2, the buffer effect is introduced and the BAR model is formulated. Section 3 presents the distribution theory of a Sup-LR test for nonlinearity, including the critical values approximated by the wild boot-

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

strap. Section 4 presents the asymptotic theory of the unit root test for cases with and without identified thresholds. The Monte Carlo simulation results are given in Section 5. Section 6 presents an empirical study of macroeconomic data on the U.S. unemployment rate and three real exchange rates. Throughout the paper, $\overset{a.s.}{\to}$, $\overset{p}{\to}$, $\Rightarrow$, and $\overset{p}{\Rightarrow}$ denote convergence almost surely, convergence in probability, weak convergence, and weak convergence in probability, respectively. $I(\cdot)$ is an indicator function. The mathematical proofs are provided in the online Supplementary Material.

2. The BAR Model

The BAR model of order p, BAR(p), is formulated as

$$y_t = \left(\sum_{i=1}^{p} \alpha_i y_{t-i}\right) R_t + \left(\sum_{i=1}^{p} \beta_i y_{t-i}\right)(1 - R_t) + e_t, \quad R_t = \begin{cases} 1, & Z_t \le r_L \\ 0, & Z_t > r_U \\ R_{t-1}, & \text{otherwise}, \end{cases} \tag{2.1}$$

where R_t is a regime-switching indicator with a buffer interval $[r_L, r_U]$, and Z_t is a threshold variable.

If $r_L < r_U$ and the threshold variable Z_t lies in the buffer interval $[r_L, r_U]$, the structure of the autoregressive (AR) model remains unchanged from the previous period. This kind of asymmetric structural change phenomenon is known as the buffer effect, or hysteresis effect in Li et al. (2015). The buffer effect differentiates the conventional TAR model from the BAR model, in which R_t can depend on infinitely many past values of Z_t. From

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

(2.1), we can derive

$$\begin{aligned} R_t &= I(Z_t \le r_L) + I(r_L < Z_t \le r_U) R_{t-1} \\ &= I(Z_t \le r_L) + \sum_{j=0}^{\infty} \prod_{i=0}^{j} I(r_L < Z_{t-i} \le r_U) I(Z_{t-j-1} \le r_L). \end{aligned}$$

Using the Dickey–Fuller reparameterization, we can rewrite (2.1) as

$$\Delta y_t = \left(\phi_0 y_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta y_{t-i} \right) R_t + \left(\psi_0 y_{t-1} + \sum_{i=1}^{p-1} \psi_i \Delta y_{t-i} \right) (1 - R_t) + e_t, \tag{2.2}$$

where $\phi_0 = \sum_{i=1}^{p} \alpha_i - 1$, $\psi_0 = \sum_{i=1}^{p} \beta_i - 1$, $\phi_k = -\sum_{i=k+1}^{p} \alpha_i$, and $\psi_k = -\sum_{i=k+1}^{p} \beta_i$, for $k = 1, 2, \ldots, p-1$. If ϕ_0 or ψ_0 is equal to zero, y_t in the corresponding regime will have a unit root, and hence, is nonstationary.

In this study, we consider only that the buffered process has a self-excited switching mechanism. In addition, except for special cases, we employ the following assumptions throughout the paper.

Assumption 1. *The innovation $\{e_t\}$ is a strictly stationary and ergodic martingale difference sequence, and let $\mathcal{F}_t$ denote the natural filtration associated with this process. In addition, $\mathbb{E}(e_t^2 | \mathcal{F}_{t-1}) = \sigma^2$ and $\mathbb{E}(|e_t|^{4\eta} | \mathcal{F}_{t-1}) < \infty$, for some $\eta > 1$.*

Assumption 2. *The process $\{\Delta y_t\}$ is strictly stationary, ergodic, and absolutely regular with mixing coefficients $\beta(m) = O(m^{-A})$, for some $A > v/(v-1)$ and $r \ge v > 1$; $\mathbb{E}(|\Delta y_t|^{4\eta} < \infty)$, for some $\eta > 1$.*

Assumption 1 requires that the innovations have conditional homoscedasticity. Assumption 2 is from Zhu, Li and Yu (2017), and we assume sufficient conditions $\sum_{i=1}^{p-1} |\phi_i| < 1$ and $\sum_{i=1}^{p-1} |\psi_i| < 1$ for the stationarity and geometric ergodicity of Δy_t. Note that it is

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

necessary to assume that Z_t is stationary in order to preserve the buffer effect; otherwise, if Z_t has a unit root, the long-term probability of Z_t staying within a fixed buffering interval $[r_L, r_U]$ will converge to zero. In this case, the buffer effect will vanish, and the BAR model will reduce to the two-regime TAR model. Because y_t is possibly nonstationary, we may consider the first difference of the process with delay order d; for example, $Z_t^d = \Delta y_{t-d}$ or $Z_t^d = y_{t-1} - y_{t-d-1}$.

3. Test for Buffered Nonlinearity with Possible Unit Roots

3.1 Sup-LR Test Statistic

First, we test whether the series is linear or nonlinear with the buffer effect, regardless of stationarity or nonstationarity. We consider the following null hypothesis H_0, in which buffered nonlinearity does not exist, and the alternative hypothesis H_1:

$$H_0 : \phi = \psi \text{ vs. } H_1 : \phi \neq \psi, \tag{3.3}$$

where $\phi = (\phi_0, \phi_1, \ldots, \phi_{p-1})'$ and $\psi = (\psi_0, \psi_1, \ldots, \psi_{p-1})'$ are coefficients in (2.2).

In order to make the linear constraint under H_0 a global null hypothesis, we rearrange model (2.2), as follows:

$$\begin{aligned} \Delta y_t &= \psi_0 y_{t-1} + \sum_{i=1}^{p-1} \psi_i \Delta y_{t-i} + \left((\phi_0 - \psi_0) + \sum_{i=1}^{p-1} (\phi_i - \psi_i) \Delta y_{t-i} \right) R_t + e_t \\ &\equiv \psi_0 y_{t-1} + \sum_{i=1}^{p-1} \psi_i \Delta y_{t-i} + \left(\theta_0 + \sum_{i=1}^{p-1} \theta_i \Delta y_{t-i} \right) R_t + e_t, \end{aligned} \tag{3.4}$$

where $\theta_i = \phi_i - \psi_i$, for $i = 0, 1, \ldots, p-1$. Therefore, testing (3.3) is equivalent to the

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

following hypothesis on $\theta = (\theta_0, \theta_1, \ldots, \theta_{p-1})'$:

$$H_0 : \theta = 0 \text{ vs. } H_1 : \theta \neq 0.$$

If the threshold variable Z_t has the delay parameter d, let $\lambda = (\psi', \theta')'$, $\gamma = (r_L, r_U)$, $u_t = \Delta y_t$, $x_t = (y_{t-1}, u_{t-1}, \ldots, u_{t-p+1})$, $Y = (u_s, \ldots, u_T)'$, $\varepsilon = (e_s, \ldots, e_T)'$, $X = (x_s', \ldots, x_T')'$, $X(\gamma) = (R_s(\gamma)x_s', \ldots, R_T(\gamma)x_T')'$, and $Z(\gamma) = (X, X(\gamma))$, where $s = \max(p, d) + 1$ is the starting index of Δy_t on the left-hand side of (3.4), and $n = T - s + 1$ is the effective sample size. Then, the model can be written as $Y = Z(\gamma)\lambda + \varepsilon$.

Following Chan (1990) and Zhu, Yu and Li (2014), for any given γ, if $e_t \sim N(0, \sigma^2)$, conditioning on $\mathcal{F}_t$, we can formulate the likelihood ratio (LR) test as

$$LR_n(\gamma) = \frac{n[\hat{\sigma}_0^2 - \hat{\sigma}^2(\gamma)]}{\hat{\sigma}^2(\gamma)},$$

where $\hat{\sigma}_0^2 = n^{-1}\left[Y'Y - (Y'X)(X'X)^{-1}(X'Y)\right]$ is the estimated residual variance in the fitted AR(p) model, and $\hat{\sigma}^2(\gamma) = n^{-1}\left\{Y'Y - [Y'Z(\gamma)][Z(\gamma)'Z(\gamma)]^{-1}[Z(\gamma)'Y]\right\}$ is that in the BAR(p) model.

Because the nuisance parameter γ exists in the alternative hypothesis only, we develop the Sup-LR statistic

$$LR_n = LR_n(\hat{\gamma}) = \sup_{\gamma \in \Gamma} LR_n(\gamma), \tag{3.5}$$

where $\Gamma \equiv \{(r_L, r_U) : a \leq r_L \leq r_U \leq b\}$, and $[a, b]$ is a predetermined range. Because (3.5) is the same as the nonlinearity test for the stationary BAR model in Zhu, Yu and Li (2014), we can test for buffered nonlinearity, regardless of stationarity or nonstationarity. As discussed in Andrews (1993) and Zhu, Yu and Li (2014), we may choose the empirical

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

quantiles of Z_t as a and b in practice, and optimize (3.5) by searching the observed Z_t values; that is, $\gamma \in \{(Z_t, Z_s) : 1 \leq t, s \leq n, a \leq Z_t \leq Z_s \leq b\}$. In addition, if normality of e_t is not assumed, the proposed statistic can be regarded as a quasi-LR test.

3.2 Asymptotic Distribution

The asymptotic distribution of the Sup-LR test under the null hypothesis has been studied by Zhu, Yu and Li (2014) for stationary y_t. If the process has a unit root, the asymptotic distribution of LR_n is nonstandard. By standard algebra, we obtain that under H_0, for a given γ,

$$LR_n(\gamma) = \frac{S'(\gamma)\left[X'(\gamma)X(\gamma)/n - (X'(\gamma)X/n)\left(X'X/n\right)^{-1}\left(X'X(\gamma)/n\right)\right]^{-1} S(\gamma)}{\hat{\sigma}^2(\gamma)/\sigma^2},$$

where $S(\gamma) = n^{-1/2}[X'(\gamma) - X'(\gamma)X(X'X)^{-1}X']\varepsilon$.

Definition 1. Let $W(s)$ be a Wiener process on $s \in [0, 1]$ with a normal distribution, $W(s) \sim N(0, s)$, and covariance $\mathbb{E}[W(s_1)W(s_2)] = \min(s_1, s_2)$.

Lemma 1. *Under Assumptions 1, 2, and $H_0 : \theta = 0$, when $\psi_0 = 0$, denote $\Sigma = \mathbb{E}(w_t w_t')$ and $\Sigma_\gamma = \mathbb{E}[w_t w_t' R_t(\gamma)]$, where $w_t = (u_{t-1}, \ldots, u_{t-p+1})'$. Then,*

(1.1.) $n^{-1}\sum_{t=s}^{T} w_t w_t' \xrightarrow{a.s.} \Sigma$ *and* $n^{-1}\sum_{t=s}^{T} w_t w_t' R_t(\gamma) \xrightarrow{a.s.} \Sigma_\gamma$

(1.2.) $n^{-3/2}\sum_{t=s}^{T} y_{t-1} u_{t-i} R_t(\gamma) \xrightarrow{p} 0$, *for* $i = 1, 2, \ldots p-1$

(1.3.) $n^{-2}\sum_{t=s}^{T} y_{t-1}^2 R_t(\gamma) \Rightarrow R(\gamma)\int_0^1 W(s)^2 ds$

Lemma 1.1 follows from the stationarity and ergodicity of the process Δy_t, and can be derived using the strong law of large numbers. Lemma 1.2 establishes the asymptotic

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

orthogonality between y_{t-1} and Δy_{t-i}. Lemma 1.3 follows the functional central limit theorem.

Definition 2. Let $W(s,u)$ be a two-parameter Wiener process on $(s,u) \in [0,1]^2$ with a normal distribution, $W(s,u) \sim N(0, su)$, and a covariance kernel $\mathbb{E}[W(s_1,u_1)W(s_2,u_2)] = \min(s_1,s_2)\min(u_1,u_2)$.

Lemma 2. *Under Assumptions 1, 2, and $H_0 : \theta = 0$, when $\psi_0 = 0$,*

(2.1.) $n^{-1/2}\sum_{t=s}^{T} w_t e_t R_t(\gamma) \Rightarrow G(\gamma)$, *where $G(\gamma)$ is a Gaussian process with zero mean and covariance function related to γ.*

(2.2.) $n^{-1}\sum_{t=s}^{T} y_{t-1} e_t R_t(\gamma) \Rightarrow \sigma \int_0^1 W(s)dW(s, R(\gamma))$.

Based on Lemmas 1 and 2, the asymptotic distribution of $LR(\gamma)$ can be derived as follows.

Theorem 1. *Under Assumptions 1, 2, and $H_0 : \theta = 0$, when $\psi_0 = 0$,*

$$LR_n \Rightarrow \sup_{\gamma \in \Gamma} LR(\gamma), \quad \text{as } n \to \infty,$$

where $LR(\gamma) = Q_1(\gamma) + Q_2(\gamma)$, and $Q_1(\gamma)$ and $Q_2(\gamma)$ are independent stochastic processes, defined as

$$Q_1(\gamma) = J_1(R(\gamma))' \left[R(\gamma)(1 - R(\gamma)) \int_0^1 W^2(s)ds \right]^{-1} J_1(R(\gamma)),$$

where $J_1(u) = \int_0^1 W(s)dW(s,u) - u\int_0^1 W(s)dW(s)$, and

$$Q_2(\gamma) = J_2(\gamma)'[\Sigma_\gamma - \Sigma_\gamma \Sigma^{-1} \Sigma_\gamma]^{-1} J_2(\gamma),$$

where $J_2(\gamma) = G(\gamma) - \Sigma_\gamma \Sigma^{-1} G$.

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Theorem 1 gives the asymptotic distribution of the Sup-LR test for testing the existence of a buffer effect in the presence of a nonstationary autoregression. The distribution can be written as the supremum of the sum of two independent processes, $Q_1(\gamma)$ and $Q_2(\gamma)$. Note that it takes a similar form to that of the Sup-Wald test for the TAR model, because under the null hypothesis, there is no buffer effect. Even though the denominator of $LR(\gamma)$ differs from that of the Wald test in Caner and Hansen (2001), it is obvious that these two tests are asymptotically equivalent as the estimated residual variance converges to the true error variance in probability. Here, $Q_2(\gamma)$ represents the contribution of the stationary part, and is very similar to the chi-square process in Zhu, Yu and Li (2014); $Q_1(\gamma)$ represents the effect from the nonstationary regressor y_t. However, compared with the two-regime TAR, which has only one threshold parameter in Caner and Hansen (2001), we have to deal with two parameters (r_L, r_U), which could make the asymptotic distribution more complicated. In addition, note that $Q_1(\gamma)$ does not have a one-to-one correspondence with γ, because $Q_1(\gamma)$ depends on $R(\gamma)$, a function of γ, rather than on γ itself. On the other hand, the chi-square process $Q_2(\gamma)$ has an even more complicated relationship with γ, as shown in Zhu, Yu and Li (2014).

3.3 Wild Bootstrap Critical Value Approximation

We apply the bootstrap method to approximate the critical value of the Sup-LR test. According to Theorem 1, when the true data-generating process is a linear unit root process, the asymptotic distribution of LR_n differs from that in the stationary case. Similarly, the bootstrap approximation is asymptotically valid only when the unit root hypothesis

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

holds for the bootstrap data-generating process. Moreover, heteroscedasticity occurs often in economic and financial time series, which violates Assumption 1. Therefore, the wild bootstrap method is proposed in order to improve the finite-sample performance and allow for unknown heteroscedasticity.

With the unit root constraint, we obtain the OLS estimator in the AR model without a buffer effect, that is, the coefficients $\hat{\lambda}_0 = (0, \hat{\psi}_1, \ldots, \hat{\psi}_{p-1})$ and the estimated residuals $\{\hat{e}_t\}$. The wild bootstrapped residuals $\tilde{e}_1, \ldots, \tilde{e}_T$ are generated using a Rademacher distributed variable, $\tilde{e}_t = \hat{e}_t v_t$, where $\Pr(v_t = 1) = \Pr(v_t = -1) = 0.5$. Then, we use the formula

$$\Delta \tilde{y}_t = \sum_{i=1}^{p-1} \hat{\psi}_i \Delta \tilde{y}_{t-i} + \tilde{e}_t \tag{3.6}$$

to obtain the bootstrapped sample. The initial values of the bootstrap recursion are set to the sample values. For each sample $\tilde{y}_t$, we calculate the test statistic $\widetilde{LR}_n$, and repeat B times to obtain $\{\widetilde{LR}_n^{(1)}, \ldots, \widetilde{LR}_n^{(B)}\}$. The critical value $c_{n,\alpha}^B$ is estimated as the αth upper percentile of $\{\widetilde{LR}_n^{(1)}, \ldots, \widetilde{LR}_n^{(B)}\}$. The following proposition guarantees that the wild bootstrapped critical value is asymptotically valid.

Proposition 1. *If Assumptions 1 and 2 hold, then under H_0,*

$$\widetilde{LR}_n | y_1, \ldots, y_T \xRightarrow{p} \sup_{\gamma \in \Gamma} LR(\gamma),$$

where $LR(\gamma)$ is defined in Theorem 1, and

$$\lim_{n \to \infty} \lim_{B \to \infty} P(LR_n \geq c_{n,\alpha}^B) = \alpha.$$

However, if the true data-generating process is stationary, the critical value given by the unit-root-constrained bootstrap (3.6) is not asymptotically correct. Instead, one

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

might consider the unconstrained estimator $\hat{\lambda}_0^* = (\hat{\psi}_0^*, \hat{\psi}_1^*, \ldots, \hat{\psi}_{p-1}^*)$ and the unconstrained bootstrap

$$\Delta\tilde{y}_t^* = \hat{\psi}_0^* y_{t-1}^* + \sum_{i=1}^{p-1} \hat{\psi}_i^* \Delta\tilde{y}_{t-i}^* + \tilde{e}_t.$$

In practice, if the stationarity property is unknown, it is prudent to calculate the bootstrap critical values in both ways, and then to make inferences based on the more conservative of the two, as suggested by Caner and Hansen (2001). The unit-root-constrained and unconstrained bootstrap are compared in the simulated data analysis in Section 5.

4. Test for Unit Roots with Possible Buffered Nonlinearity

In this section, we propose a class of unit root tests under the BAR(p) model. In model (2.2), the parameters ϕ_0 and ψ_0 control the stationarity of the process. The hypotheses that we are interested in are

$$\begin{cases} H_0 : \phi_0 = \psi_0 = 0, \\ H_1 : (\phi_0 < 0 \text{ and } \psi_0 = 0) \text{ or } (\phi_0 = 0 \text{ and } \psi_0 < 0), \\ H_2 : \phi_0 < 0 \text{ and } \psi_0 < 0. \end{cases}$$

Hypothesis H_k, for $k = 0, 1,$ or 2, indicates that y_t is stationary in k regimes. The BAR(p) model under these three hypotheses will have significantly different properties, enabling us to interpret different patterns in real applications. If both H_0 and H_1 are rejected, it is reasonable to assert that the time series is stationary. It is also interesting to study H_1, because the buffered switching between stationarity and nonstationarity might provide insights into the dynamics in economics and finance. Hence, for real applications, we need to distinguish between the scenarios H_0, H_1, and H_2.

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Similarly to conventional unit root tests, we consider the t-ratios for both ϕ_0 and ψ_0, namely, t_1 and t_2, respectively. In order to discriminate between these three scenarios, we follow Caner and Hansen (2001) and consider a general class of statistics $R_T = R(t_1, t_2)$, which are continuous functions of both t-ratios. For example, if we have negative $\hat{\phi}_0$ and $\hat{\psi}_0$, we may use the one-sided test $R_1 = t_1^2 I(t_1 < 0) + t_2^2 I(t_2 < 0)$ to test H_0 against H_1. Similarly, the two-sided test $R_2 = t_1^2 + t_2^2$ and single-value one-sided tests $R_{t_1} = t_1 I(t_1 < 0)$ and $R_{t_2} = t_2 I(t_2 < 0)$ might also help to differentiate the cases.

In Section 3, we propose a bootstrap Sup-LR test for buffer nonlinearity. Even if we cannot identify significant buffer nonlinearity, we can still fit the data using a BAR model and conduct unit root tests in both regimes. It turns out that the asymptotic distributions do vary depending on the presence of buffered nonlinearity.

4.1 Asymptotic Distribution without Buffered Nonlinearity

First, we consider the case where the true model is linear; that is, $\phi = \psi$.

Theorem 2. *Under Assumptions 1, 2, and $H_0 : \phi_0 = \psi_0 = 0$, when $\phi = \psi$,*

$$(t_1, t_2) \Rightarrow (t_1(\gamma^*), t_2(\gamma^*)) \quad \textit{and} \quad R(t_1, t_2) \Rightarrow R(\gamma^*) = R(t_1(\gamma^*), t_2(\gamma^*)),$$

where $\gamma^* = \arg\max_{\gamma \in \Gamma} LR(\gamma)$,

$$t_1(\gamma) = \frac{\int_0^1 W(s) dW(s, R(\gamma))}{\sqrt{R(\gamma) \int_0^1 W(s)^2 ds}}, \quad \textit{and} \quad t_2(\gamma) = \frac{\int_0^1 W(s) d[W(s,1) - W(s, R(\gamma)]}{\sqrt{(1 - R(\gamma)) \int_0^1 W(s)^2 ds}}.$$

Theorem 2 gives the asymptotic distribution of the t-ratios. It extends the asymptotic theory of Caner and Hansen (2001), which can be viewed as a special case of our results

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

when $r_L = r_U$. The limiting maximizer γ^* is random because the buffered threshold is not identified in advance and depends on the nuisance parameter function $G(\gamma)$ in Theorem 2.

Remark 1. Similarly to the standard unit root tests, it is also important to consider the case of a constant intercept and time trend. Theorem 2 can be generalized to include an intercept and time trend using the standard detrend method in the unit root test literature, namely,

$$t_1^*(\gamma) = \frac{\int_0^1 W^*(s)dW(s, R(\gamma))}{\sqrt{R(\gamma)\int_0^1 W^*(s)^2 ds}}, \quad \text{and} \quad t_2^*(\gamma) = \frac{\int_0^1 W^*(s)d[W(s,1) - W(s, R(\gamma))]}{\sqrt{(1 - R(\gamma)\int_0^1 W^*(s)^2 ds}},$$

where

$$W^*(s) = W(s) - \int_0^1 W(u)r(u)'du \left(\int_0^1 r(u)r(u)du\right)^{-1} r(s),$$

and $r(u) = (1, u)$.

4.2 Asymptotic Distribution with Buffered Nonlinearity

Next, we study the asymptotic distributions of the t-ratios when the true model has genuine buffered nonlinearity; that is, $\phi \neq \psi$, with threshold parameters $\gamma_0 = (r_{L0}, r_{U0})$. The unit root test is based on the null hypothesis $H_0: \ \phi_0 = \psi_0 = 0$.

Because the AR structures in the two regimes are different, Δy_t might have different distributions in these regimes. Therefore, we need to consider the long-run correlation between $u_{t+k} = \Delta y_{t+k}$ and $e_t R_t(\gamma)$ or $e_t(1 - R_t(\gamma))$, which we denote separately as

$$\delta_1(\gamma) = \frac{\sum_{k=-\infty}^{\infty} \mathbb{E}(e_t R_t(\gamma) u_{t+k})}{\sqrt{\mathbb{E}(e_t^2) R(\gamma) \sum_{k=-\infty}^{\infty} \mathbb{E}(u_t u_{t+k})}}, \text{ and } \delta_2(\gamma) = \frac{\sum_{k=-\infty}^{\infty} \mathbb{E}(e_t(1 - R_t(\gamma)) u_{t+k})}{\sqrt{\mathbb{E}(e_t^2)(1 - R(\gamma)) \sum_{k=-\infty}^{\infty} \mathbb{E}(u_t u_{t+k})}},$$

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

respectively.

Note that when y_t is a linear process, Δy_t is a linear function of lagged values of $e_t R(\gamma_0)$ and $e_t(1 - R(\gamma_0))$, such that $\delta_1^2(\gamma_0) + \delta_2^2(\gamma_0) = 1$. However, when there is strong nonlinearity, we may expect that $\delta_1^2(\gamma_0) + \delta_2^2(\gamma_0) < 1$. As a result, when buffered nonlinearity is identified, the asymptotic distributions of the t-ratios are given by the following theorem.

Theorem 3. *Under Assumptions 1, 2, and* $H_0 : \phi_0 = \psi_0 = 0$,

$$\hat{t} = \begin{pmatrix} t_1(\hat{\gamma}) \\ t_2(\hat{\gamma}) \end{pmatrix} \Rightarrow \begin{pmatrix} \sqrt{1 - \delta_1^2(\gamma^*)} Z_1 \\ \sqrt{1 - \delta_2^2(\gamma^*)} Z_2 \end{pmatrix} + \begin{pmatrix} \delta_1(\gamma^*) \\ \delta_2(\gamma^*) \end{pmatrix} DF \equiv \mathcal{T},$$

where $\gamma^* = \arg\max_{\gamma \in \Gamma} LR(\gamma)$,

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \sigma_{21}(\gamma^*) \\ \sigma_{21}(\gamma^*) & 1 \end{pmatrix} \right), \quad DF = \frac{\int_0^1 W(s) dW(s)}{\int_0^1 W(s)^2 ds},$$

and $\sigma_{21}(\gamma) = (-\delta_1(\gamma)\delta_2(\gamma))/[(1 - \delta_1^2(\gamma))(1 - \delta_2^2(\gamma))]$.

Remark 2. If there is a constant intercept and time trend, the standard DF distribution should be replaced by the detrended Dickey–Fuller t-distribution,

$$DF^* = \frac{\int_0^1 W^*(s) dW(s)}{\int_0^1 W^*(s)^2 ds}$$

where W^* is defined in Remark 1.

Theorem 3 gives the asymptotic distribution of the t-ratios in both regimes when buffered nonlinearity is identified. This asymptotic theory extends the results in Caner

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

and Hansen (2001) and their result can be viewed as a special case of ours. When there is nonlinear buffer effect, the asymptotic joint distribution of the t-ratios is a linear combination of a bivariate normal distribution and a Dickey–Fuller t-distribution. Therefore, we can expect that the quantiles of the asymptotic distribution are larger than those of Dickey–Fuller t-distribution. In other words, for the BAR model, the augmented Dickey–Fuller (ADF) test is a much less powerful statistical test. It is therefore important for us to develop new unit root tests for the BAR and other nonlinear time series models.

Because the proposed unit root test $R(t_1, t_2)$ is a function of t_1 and t_2, we can approximate the critical values of $R(t_1, t_2)$ from the asymptotic joint distribution of t_1 and t_2. However, this is complicated in practice because (1) we have to identify whether or not the time series is linear, because the asymptotic distributions differ, and (2) it is relatively difficult to estimate δ_1, δ_2, and σ_{21}. Therefore, we propose the following bootstrap unit root test, which can be adapted to both linear and buffered nonlinear cases.

4.3 Wild Bootstrap Critical Value Approximation

The asymptotic distributions of t_1 and t_2 are different in the linear and buffered nonlinear cases. However, we propose approximating the p-value of the test statistic using the wild bootstrap method directly, regardless of whether a buffered nonlinear effect has been identified. In the BAR(p) model, we need to estimate the parameter $\hat{\lambda} = (\hat{\psi}_0, \hat{\psi}_1, \ldots, \hat{\psi}_{p-1}, \hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_{p-1})$, estimated residuals $\{\hat{e}_s, \ldots, \hat{e}_T\}$, and threshold parameter $\hat{\gamma} = (\hat{r}_L, \hat{r}_U)$. Then, we can generate the bootstrapped sample using the unit-

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

root-constrained data-generating process

$$\Delta\tilde{y}_t = \left(\sum_{i=1}^{p-1}\hat{\psi}_i\Delta\tilde{y}_{t-i}\right)R(\hat{\gamma}) + \left(\sum_{i=1}^{p-1}\hat{\phi}_i\Delta\tilde{y}_{t-i}\right)(1-R(\hat{\gamma})) + \tilde{e}_t, \tag{4.7}$$

where $\tilde{e}_t$ is generated using the wild bootstrap method as $\tilde{e}_t = \hat{e}_t v_t$, where v_t is independent and Rademacher distributed.

For each $\tilde{y}_t$, we calculate the t-ratio for ψ_0 and ϕ_0 and $R(\tilde{t}_1, \tilde{t}_2)$. Then, we repeat this B times to obtain $\{R(\tilde{t}_1^{(1)}, \tilde{t}_2^{(1)}), \ldots, R(\tilde{t}_1^{(B)}, \tilde{t}_2^{(B)})\}$. The estimated critical value is the αth upper percentage of the bootstrapped $\{R(\tilde{t}_1^{(1)}, \tilde{t}_2^{(1)}), \ldots, R(\tilde{t}_1^{(B)}, \tilde{t}_2^{(B)})\}$. If an intercept and/or a time trend are included in the BAR(p) model, they can be estimated and then substituted into the bootstrap data-generating process to approximate the critical value. If the true process is a buffered nonlinear process, the bootstrap method is valid.

Proposition 2. *Under Assumptions 1, 2, and* $H_0 : \phi_0 = \psi_0 = 0$,

$$\tilde{t} = (\tilde{t}_1, \tilde{t}_2)^T | y_t, \ldots, y_T \xrightarrow{p} \mathcal{T},$$

where $\mathcal{T}$ *is defined in Theorem 3.*

If the true data-generating process is linear, the bootstrap method is valid only when we use a unit-root-constrained linear generating process (Park, 2003). The performance of the bootstrap test based on (4.7) for a linear process is discussed in Section 5.

5. Monte Carlo Simulation

In this section, we examine the finite-sample performance of our Sup-LR test and the unit root test, which we then compare with that of the Wald test in Caner and Hansen (2001)

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

and the ADF test. In addition, because the wild bootstrap is proposed for the possible unknown heteroscedasticity, we also check for GARCH errors.

5.1 Sup-LR Test

5.1.1 Size

We first study the size of the Sup-LR test at the nominal 5% level. The data are simulated from the null model

$$\Delta y_t = \rho y_{t-1} + \alpha \Delta y_{t-1} + e_t, \tag{5.8}$$

and we investigate how the size is affected by ρ and α. The results are presented in Table 1.

Table 1: Size of 5% Sup-LR Test (%)

	Unconstrained bootstrap						Constrained bootstrap					
$\alpha =$	-0.5			0.5			-0.5			0.5		
$T =$	100	200	500	100	200	500	100	200	500	100	200	500
$\rho = -0.25$	5.3	5.5	5.6	4.9	4.3	4.4	3.0	3.1	3.2	2.3	3.8	3.7
$\rho = 0$	5.5	5.4	4.9	5.3	4.8	5.0	4.2	4.1	4.8	4.0	3.9	4.9

$B = 500, \sigma = 1$. *Rejection frequencies from 1000 replications.*

The sizes of the unconstrained bootstrap tests are all acceptable, regardless of whether the true data generating process is stationary. In addition, the unit-root-constrained bootstrap test has a smaller size when the true model is stationary, suggesting that it is a more conservative test. Because the unconstrained bootstrap test has a respectable size in both cases, and because we are more concerned about the power against the nonlinear alternatives, we propose to use the unconstrained threshold test for both cases.

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

5.1.2 Power

Next, we explore the power of the 5% unconstrained Sup-LR test against various alternative models. We consider the following four cases:

Case 1: $\Delta y_t = \rho_1 y_{t-1} R_t(\gamma) + \rho_2 y_{t-1}(1 - R_t(\gamma)) + \alpha \Delta y_{t-1} + e_t.$ (5.9)

The stationarity properties in both regimes might not be the same. Denote $\Delta\rho = \rho_1 - \rho_2$.

Case 2: $\Delta y_t = \rho y_{t-1} + \alpha_1 \Delta y_{t-1} R_t(\gamma) + \alpha_2 \Delta y_{t-1}(1 - R_t(\gamma)) + e_t.$

The stationarity properties in both regimes are the same, but the AR structures differ. Denote $\Delta\alpha = \alpha_1 - \alpha_2$.

Case 3: $\Delta y_t = (\rho_1 y_{t-1} + \alpha_1 \Delta y_{t-1}) R_t(\gamma) + (\rho_2 y_{t-1} + \alpha_2 \Delta y_{t-1})(1 - R_t(\gamma)) + e_t$ with a genuine buffer effect, where $r_L < r_U$.

Case 4: $\Delta y_t = (\rho_1 y_{t-1} + \alpha_1 \Delta y_{t-1}) R_t(\gamma) + (\rho_2 y_{t-1} + \alpha_2 \Delta y_{t-1})(1 - R_t(\gamma)) + e_t$ in the degenerated threshold case, where $r_L = r_U$.

The simulation results are summarized in Table 2. Not surprisingly, the Sup-LR test becomes more powerful, even close to 100%, when the coefficients in the two regimes move farther apart, or when the sample size increases. Note that the test becomes more powerful when the process is stationary in one regime and nonstationary in the other. Intuitively, this is because it is easier to distinguish between two processes with different stationarity properties. We also consider the threshold test for the TAR model by (Caner and Hansen, 2001). Suppose we ignore the buffer effect and fit the data using a two-regime TAR model. Then, we can test for nonlinearity using a Sup-Wald test (Caner and Hansen, 2001). The simulation results of the TAR threshold tests are also summarized

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Table 2: Power of 5% Sup-LR Test (%)

		BAR Sup-LR test									TAR Sup-Wald test								
$T =$		100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500
Case 1: $B = 500$, $\alpha = 0.5$, $\sigma = 1$, $\gamma = (-0.5, 0.5)$																			
		$\Delta\rho = 0.05$			$\Delta\rho = 0.1$			$\Delta\rho = 0.2$			$\Delta\rho = 0.05$			$\Delta\rho = 0.1$			$\Delta\rho = 0.2$		
$\rho_1 = 0$		19	39	64	38	70	96	75	98	100	14	23	49	29	59	75	67	91	94
$\rho_1 = 0.1$		10	15	17	16	29	55	39	76	98	5	7	10	9	25	44	31	59	72
Case 2: $B = 500$, $\alpha_1 = 0.5$, $\sigma = 1$, $\gamma = (-0.5, 0.5)$.																			
		$\Delta\alpha = 0.2$			$\Delta\alpha = 0.6$			$\Delta\alpha = 1$			$\Delta\alpha = 0.2$			$\Delta\alpha = 0.6$			$\Delta\alpha = 1$		
$\rho = 0$		9	12	32	41	77	98	84	98	100	5	8	19	37	64	87	72	97	99
$\rho = 0.25$		12	24	42	67	94	100	96	100	100	9	11	28	47	74	92	94	99	100
Case 3: $B = 500$, $\alpha_1 = 0.5$, $\sigma = 1$, $\gamma = (-0.5, 0.5)$																			
ρ_1	ρ_2	$\Delta\alpha = 0.2$			$\Delta\alpha = 0.6$			$\Delta\alpha = 1$			$\Delta\alpha = 0.2$			$\Delta\alpha = 0.6$			$\Delta\alpha = 1$		
0	-0.1	43	78	96	79	98	100	96	100	100	31	66	88	73	95	97	92	99	100
0	-0.2	78	99	100	95	100	100	99	100	100	67	91	95	89	100	100	98	100	100
0	-0.4	98	100	100	99	100	99	100	100	100	97	99	98	96	100	100	100	99	100
-0.1	-0.2	22	51	79	72	82	88	96	99	100	15	46	64	58	75	83	92	98	100
-0.1	-0.3	47	98	100	91	99	100	97	100	100	37	93	97	71	97	100	95	99	99
-0.1	-0.5	92	100	100	96	100	100	100	100	100	78	99	99	94	100	100	100	99	100
Case 4: $B = 500$, $\alpha_1 = 0.5$, $\sigma = 1$, $\gamma = (0, 0)$																			
ρ_1	ρ_2	$\Delta\alpha = 0.2$			$\Delta\alpha = 0.6$			$\Delta\alpha = 1$			$\Delta\alpha = 0.2$			$\Delta\alpha = 0.6$			$\Delta\alpha = 1$		
0	-0.1	36	77	95	77	96	98	98	100	100	33	78	92	74	98	100	96	100	100
0	-0.2	66	95	100	91	100	100	100	100	100	74	91	99	90	100	100	98	100	100
0	-0.4	95	100	100	99	100	100	99	100	100	96	99	100	99	99	100	100	100	100
-0.1	-0.2	16	35	58	62	73	91	94	99	100	15	30	66	57	69	90	88	96	100
-0.1	-0.3	45	93	98	82	99	100	97	100	100	44	94	99	76	100	100	97	100	100
-0.1	-0.5	85	100	100	97	100	100	100	100	100	87	100	100	97	100	100	100	100	100

Rejection frequencies from 200 replications.

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

in Table 2. In general, when the buffer region is absent in case 4, our proposed test exhibits similar power to that of the TAR threshold test. However, the TAR threshold test is less powerful, in general, in the presence of the buffer effect in other three cases. In summary, the proposed bootstrap Sup-LR test possesses appropriate size and power, and outperforms the TAR threshold test.

5.1.3 Conditional heteroscedasticity

In reality, the assumption of homoscedasticity does not hold in many economic and financial data. Therefore, we propose a wild bootstrap test that is robust to unknown heteroscedasticity. Because we focus on testing nonlinearity and stationarity, heteroscedasticity is not modeled in the BAR(p) model. Here, we consider the following BAR model with GARCH(1,1) errors:

$$h_t = 0.01 + 0.19e_{t-1}^2 + 0.8h_{t-1}, \quad \varepsilon_t \overset{iid}{\sim} N(0,1), \quad e_t = \varepsilon_t\sqrt{h_t}. \tag{5.10}$$

In the standard bootstrap approach, the residuals can be drawn randomly, with replacement, from the estimated residuals. We compare the performance of both methods under conditional heteroscedasticity.

First, we consider that the true generating process is a linear AR process (5.8), with GARCH errors given by (5.10); the results are summarized in Table 3. The size of the test is distorted if the standard bootstrap method is applied, whereas the wild bootstrap method solves the problem of over-rejection and corrects the size.

Next, we consider that the true generating process is a BAR(p) process (5.9), with

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Table 3: Size of 5% Sup-LR Test with GARCH Errors (%)

	Wild bootstrap				Standard bootstrap			
$\alpha =$	-0.5		0.5		-0.5		0.5	
$T =$	100	200	100	200	100	200	100	200
$\rho = -0.25$	5.8	5.2	6.1	5.5	15.0	16.7	9.7	10.1
$\rho = 0$	5.4	5.5	4.4	4.6	11.7	11.9	10.1	12.4

$B = 500$, $\sigma = 1$, $\gamma = (-0.5, 0.5)$. *Rejection frequencies from 1000 replications.*

GARCH errors given by (5.10); the results are summarized in Table 4. Because the standard bootstrap method suffers from the problem of over-rejection, it is reasonable that the standard bootstrap method is more powerful than the wild bootstrap method.

Table 4: Power of 5% Sup-LR Test with GARCH Errors (%)

	Wild bootstrap						Standard bootstrap					
$\Delta\rho =$	-0.05		-0.1		-0.2		-0.05		-0.1		-0.2	
$T =$	100	200	100	200	100	200	100	200	100	200	100	200
$\rho_1 = 0$	13.0	18.5	33.0	64.0	60.5	94.5	16.0	23.5	34.5	78.5	71.0	98.0
$\rho_1 = -0.1$	9.5	14.0	13.0	30.5	29.5	69.5	17.5	20.0	19.0	39.5	45.0	80.5

$B = 500$, $\alpha = 0.5$, $\sigma = 1$, $\gamma = (-0.5, 0.5)$. *Rejection frequencies from 200 replications.*

In summary, the simulation results show that the proposed wild bootstrap method can largely alleviate the oversize problem caused by conditional heteroscedasticity, with appropriate size and good power. Based on its good performance under homoscedasticity or conditional heteroscedasticity, the wild bootstrap Sup-LR test is recommended as a robust test under unknown heteroscedasticity.

5.2 Unit Root Test

The asymptotic distributions of the t-ratios differ depending on the presence of buffered nonlinearity. Simulations are conducted in both cases.

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

5.2.1 Unit Root Test without Buffered Nonlinearity

First, we consider the case without buffered nonlinearity; in other words, the data are generated from an AR(2) model

$$\Delta y_t = \rho y_{t-1} + \alpha \Delta y_{t-1} + e_t.$$

We nevertheless fit the data using a BAR(2) model, and consider $R_2 = t_1^2 + t_2^2$ as the test statistic. The simulated results are shown in Table 5, and are based on 1000 replications for each case. When the true model is a linear unit root test ($\rho = 0$), the size of the unit root test is very close to the nominal significance level, even though we cannot derive the validity of the bootstrapped test in this case. As ρ gets farther from zero, the unit root test becomes increasingly powerful. However, because of the model misspecification, the BAR unit root test is less powerful than the ADF test for the AR(2) model. Nevertheless, with a larger sample size, the discrepancy becomes much smaller. In addition, we can fit the data using a TAR(2) model, and thus obtain the TAR R_2 statistics. The simulated results show that BAR and TAR R_2 perform similarly.

Table 5: Size and Power of 5% Unit Root Test without Buffered Nonlinearity (%)

	BAR R_2				ADF test				TAR R_2			
$\alpha =$	-0.5		0.5		-0.5		0.5		-0.5		0.5	
$T =$	200	500	200	500	200	500	200	500	200	500	200	500
$\rho = 0$	4.4	5.1	5.5	5.3	4.3	5.0	5.7	4.9	4.6	4.4	5.4	4.9
$\rho = -0.1$	50.1	89.6	88.7	100.0	92.1	99.8	100.0	99.9	55.0	90.4	83.6	97.8
$\rho = -0.25$	93.4	99.9	99.8	100.0	100.0	100.0	100.0	100.0	94.3	99.8	99.9	100.0

$B = 500$, $\sigma = 1$. *Rejection frequencies from 1000 replications.*

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

5.2.2 Unit Root Test with Buffered Nonlinearity

Next, we consider the case with buffered nonlinearity in a BAR(3):

$$\Delta y_t = \left(\psi_0 y_{t-1} + \sum_{i=1}^{2} \psi_i \Delta y_{t-i}\right) R(\gamma) + \left(\phi_0 y_{t-1} + \sum_{i=1}^{2} \phi_i \Delta y_{t-i}\right)(1 - R(\gamma)) + e_t. \quad (5.11)$$

Because we need to distinguish between three cases (i.e., unit roots in both regimes, a unit root in only one regime, and no unit roots), we consider the one-sided tests $t_1^2 I(t_1 \leq 0)$ and $t_2^2 I(t_2 \leq 0)$, and the combined test $R_1 = t_1^2 I(t_1 \leq 0) + t_2^2 I(t_2 \leq 0)$. The simulated results for their size and power are shown in Tables 6 and 7, respectively.

According to Tables 6 and 7, we can identify the nonstationarity in each regime using these three tests. If we obtain large p-values for all three statistics, it is very likely that there are unit roots in both regimes. If the p-value for R_1 is small and one of the p-values for t_1 or t_2 is small, then there could be a unit root in one regime and no unit root in the other. If the p-values for the three statistics are all smaller than the significance level, we can reject the unit root null hypotheses, and conclude that there is strong evidence of stationarity in the data.

In addition, as we have discussed, the asymptotic distribution of the t-ratio is a linear combination of a Dickey–Fuller t-distribution and a standard normal distribution. Therefore, when there is a strong nonlinear buffer effect, the actual distribution is far from the Dickey–Fuller distribution; hence, the ADF test is much less powerful than the bootstrap unit root test, as shown for all of the cases in Table 7.

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Table 6: Size of 5% Bootstrap Unit Root Tests with Buffered Nonlinearity (%)

	R_1		$t_1^2 I(t_1 \le 0)$		$t_2^2 I(t_2 \le 0)$		ADF(t_1)		ADF(t_2)	
ψ_0	T =100	200	100	200	100	200	100	200	100	200
0	0.5	0.5	5.9	3.1	3.8	3.0	3.5	2.1	0.9	0.5
−0.1	75.3	96.4	5.6	2.9	94.5	99.1	2.4	1.7	92.6	98.5
−0.25	99.2	99.8	5.4	2.8	99.5	100.0	3.2	1.3	99.2	99.6

$B = 500$, $\phi_0 = 0$, $\phi_1 = -0.4$, $\phi_2 = -0.2$, $\psi_1 = \psi_2 = 0.3$, $\sigma = 1$, $\gamma = (-0.5, 0.5)$. *Rejection frequencies from 1000 replications.*

Table 7: Power of 5% Bootstrap Unit Root Tests with Buffered Nonlinearity (%)

		R_1		$t_1^2 I(t_1 \le 0)$		$t_2^2 I(t_2 \le 0)$		ADF(t_1)		ADF(t_2)	
ϕ_0	ψ_0	T =100	200	100	200	100	200	100	200	100	200
	-0.05	18.5	64.0	39.0	65.0	36.0	53.5	29.5	45.0	24.0	39.0
-0.05	-0.1	32.0	91.5	26.5	53.5	54.5	83.0	21.5	35.5	48.0	80.0
	-0.25	89.5	100.0	22.0	31.0	94.0	99.0	10.5	17.0	93.5	100.0
	-0.05	23.0	83.0	55.0	82.5	25.5	40.5	47.0	78.5	14.0	26.5
-0.1	-0.1	30.5	85.5	48.0	79.5	46.0	67.0	39.5	67.0	35.0	52.5
	-0.25	83.0	99.5	37.5	62.5	90.0	96.5	27.0	42.5	88.5	96.0
	-0.05	58.5	96.0	86.5	95.0	14.0	20.5	85.5	94.5	07.0	09.5
-0.25	-0.1	49.5	97.0	78.0	97.0	25.5	32.5	74.5	94.5	13.5	18.5
	-0.25	76.5	98.0	70.0	91.0	63.0	76.0	62.0	88.5	50.5	71.0

$B = 500$, $\psi_1 = \psi_2 = 0.3$, $\phi_1 = -0.4$, $\phi_2 = -0.2$, $\sigma = 1$, $\gamma = (-0.5, 0.5)$. *Rejection frequencies from 200 replications.*

5.2.3 Conditional heteroscedasticity

Similarly to the Sup-LR test, we use Monte Carlo simulations to study the wild bootstrap unit root test with unknown heteroscedasticity. We also consider the BAR model with GARCH(1,1) errors, as in (5.10), and compare the performance of the wild bootstrap with that of the standard residual bootstrap.

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

First, the true data-generating process is a BAR(3) model, as in (5.11), with at least one unit root process in the two regimes. The rejection frequencies for the wild bootstrap and standard bootstrap are summarized in Table 8. The over-rejection problem is again observed for the standard bootstrap, and the wild bootstrap remains robust under conditional heteroscedasticity, producing quite acceptable sizes.

Table 8: Size of 5% Bootstrap Unit Root Tests with GARCH Errors (%)

	Wild bootstrap						Standard bootstrap					
	R_1		$t_1^2 I(t_1 \le 0)$		$t_2^2 I(t_2 \le 0)$		R_1		$t_1^2 I(t_1 \le 0)$		$t_2^2 I(t_2 \le 0)$	
ψ_0	$T=$100	200	100	200	100	200	100	200	100	200	100	200
0	0.6	0.8	4.6	5.1	5.0	5.0	0.8	1.0	7.5	8.6	7.6	9.2
-0.1	76.3	97.9	4.9	5.2	90.2	99.0	72.4	98.2	6.9	7.1	93.7	99.0
-0.25	96.2	99.8	5.2	3.9	99.0	100.0	97.5	99.9	8.4	7.9	99.0	100.0

$B = 500$, $\phi_0 = 0$, $\phi_1 = -0.4$, $\phi_2 = -0.2$, $\psi_1 = \psi_2 = 0.3$, $\sigma = 1$, $\gamma = (-0.5, 0.5)$. *Rejection frequencies from 1000 replications.*

Table 9: Power of 5% Bootstrap Unit Root Tests with GARCH Errors (%)

		Wild bootstrap			Standard bootstrap		
ϕ_0	ψ_0	R_1	$t_1^2 I(t_1 \le 0)$	$t_2^2 I(t_2 \le 0)$	R_1	$t_1^2 I(t_1 \le 0)$	$t_2^2 I(t_2 \le 0)$
	-0.05	27.0	41.5	41.0	30.5	43.0	51.5
-0.05	-0.1	50.0	33.0	65.0	46.5	28.0	66.5
	-0.25	89.0	20.0	94.0	89.5	21.0	94.0
	-0.05	35.0	59.5	35.0	30.5	55.0	36.0
-0.1	-0.1	45.5	47.5	57.0	36.5	51.5	50.5
	-0.25	83.5	41.5	88.0	82.5	38.5	91.5
	-0.05	61.0	80.5	23.0	60.0	86.0	23.5
-0.25	-0.1	62.5	83.0	32.0	65.5	80.5	33.0
	-0.25	77.5	67.5	72.0	73.5	65.0	71.5

$T = 100$, $B = 500$, $\psi_1 = \psi_2 = 0.3$, $\phi_1 = -0.4$, $\phi_2 = -0.2$, $\sigma = 1$, $\gamma = (-0.5, 0.5)$. *Rejection frequencies from 200 replications.*

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Next, we consider the case that the true data-generating process is a stationary BAR process with GARCH errors; the results are summarized in Table 9. Similarly to the Sup-LR test under conditional heteroscedasticity, the standard bootstrap unit root test is sometimes slightly more powerful, owing to its over-rejection phenomenon, whereas the wild bootstrap unit root test exhibits acceptable power compared with that of the case with homoscedasticity. In summary, the wild bootstrap unit root test shows good size and power, suggesting that it is robust to unknown heteroscedasticity.

6. Real-Data Analysis

We apply the BAR model to macroeconomic data on U.S. unemployment rates and real exchange rates. In what follows, we first obtain the OLS estimators for different orders p; then we select appropriate values for p and d based on information criteria such as the Akaike information criterion (AIC) and Bayesian information criterion (BIC). After selecting the lag order and delay order of the threshold variable, we approximate the p-value of the proposed Sup-LR test and the unit root tests to verify the presence of nonlinearity and stationarity using the bootstrap approach. In addition, we compare the BAR model with other linear and nonlinear time series models.

6.1 U.S. Unemployment Rate

Nonlinear time series models have been applied to characterize the business cycle in the literature; see Hamilton (1989) and Potter (1995). The U.S. monthly unemployment rates among adult males were studied using a two-regime TAR model by Caner and Hansen

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

(2001). We consider a two-regime BAR model for U.S. seasonally adjusted monthly unemployment rates among the total population, from January 1955 to December 2016, from the Bureau of Labor Statistics; the data are plotted in the upper panel of Figure 1.

By definition, a change (and, particularly, an increase) in the unemployment rate might imply a structural change in an industry or the start of a long-term economic depression. However, considering that the length of a contract in reality is usually at least one year, it is natural to consider the cumulative unemployment rate change during the past d months as the threshold variable if there is any structural change. Therefore, we propose the following BAR model:

$$\begin{aligned}\Delta y_t = &\left(C_1 + \phi_0 y_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta y_{t-i}\right) R(Z_{t-1}^d) \\ &+ \left(C_2 + \psi_0 y_{t-1} + \sum_{i=1}^{p-1} \psi_i \Delta y_{t-i}\right)(1 - R(Z_{t-1}^d)) + e_t,\end{aligned}$$

where $Z_t^d = y_t - y_{t-d}$ is a stationary threshold variable. Based on the information criteria, we select $d = 13$ and $p = 15$. As suggested by a referee, we consider a subset BAR(15) based on the AIC, because not all estimates are significant. The estimates for the BAR(15) and subset BAR(15) models are shown in Table 10. In addition, the fitted regimes in the BAR(15) model are shown in the lower panel of Figure 1. We can classify the fitted regimes into four categories: absolute high, absolute low, buffer high, and buffer low. The first two refer to cases in which Z_{t-1}^d lies outside the buffer interval; the other two indicate that Z_{t-1}^d lies inside the buffer interval, and the regime is maintained as before.

The fitted BAR(15) model provides a clear interpretation of the long-term business

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

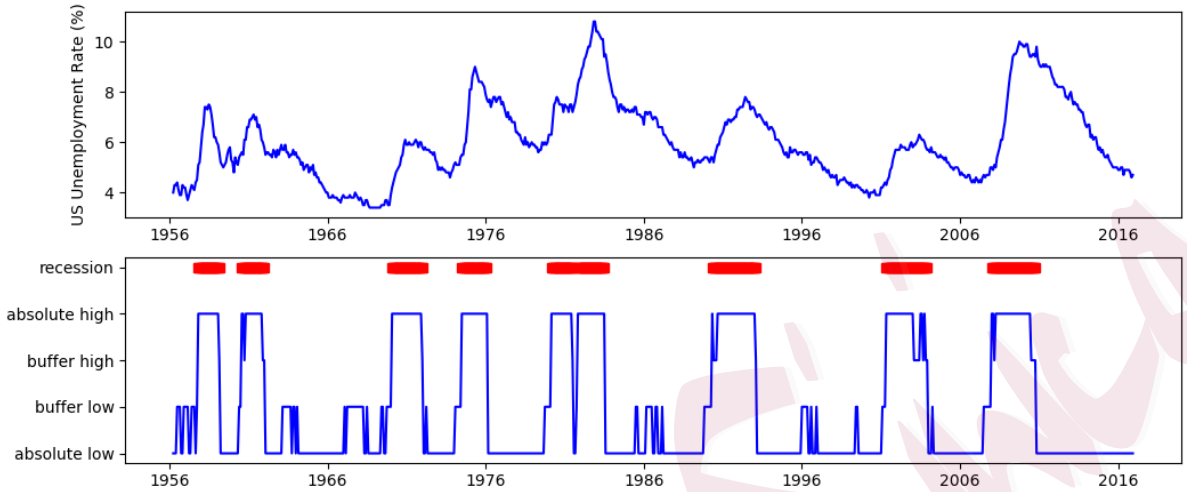

Figure 1: *Panel 1: U.S. monthly unemployment rates from April 1956 to December 2016. Panel 2: Trajectory of estimated regime in BAR(15) and the corresponding recession periods.*

cycle. First, strong consistency is evident between unemployment rate changes and regime switching, as shown in Figure 1. If we consider the absolute high and buffer high regimes as recession periods, there are in eight recession periods within the 62-year period. These periods are highly consistent with the business cycle reported by the National Bureau of Economic Research (NBER, 2010), implying that the autoregressive structure of unemployment differs between recession and expansion periods. Second, there is evidence of a buffer effect, because 112 out of 727 data points lie in the buffer regime, and most are in the buffer low regime. The buffer interval $[-0.1, 0.3](\%)$ guarantees that a small-scale unemployment increase during an expansion period will not trigger a regime switch to a recession. However, the recession period will end if the unemployment rate reaches a level 0.1% below that 13 months previously. Third, the recession periods obtained using the BAR model are slightly delayed compared with those reported by the NBER, because the

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

unemployment rate is widely known as a lagging indicator of an economy (Smith, 2009).

According to Table 10, there is a strong evidence of nonlinearity in the fitted BAR(15) model, because the estimates in each regime are significantly different. In addition, in the fitted model, the estimates $\hat{\phi}_0$ and $\hat{\psi}_0$ are negative and close to zero. Therefore, we implement the proposed Sup-LR test for nonlinearity and the one-sided t-test for nonstationarity based on 2000 bootstrap samples; the results are summarized in Table 11.

The Sup-LR test statistic is highly significant, with a p-value of nearly zero, indicating strong nonlinearity. Therefore, we choose the one-sided unit root test $R_1(t_1, t_2) = t_1^2 I(t_1 < 0) + t_2^2 I(t_2 < 0)$ with buffered nonlinearity. Note that the statistic $\hat{t}_2$ (-1.8695) is not very significant, because the 1% and 5% critical values of the ADF test with a constant intercept are -3.43 and -2.86, respectively. However, we can obtain a much more smaller p-value for the proposed test, which also indicates that our proposed test is much more powerful than the ADF test. The results in Table 11 imply the existence of strong nonlinearity and stationarity in the unemployment rate, consistent with the findings of Caner and Hansen (2001) and Roed (2002).

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Table 10: Least Squares Estimates of BAR(15) and Three-Regime TAR(15) Model

	BAR(15) with $d = 13$, $\gamma = (-0.1, 0.3)$				Subset BAR(15) with $d = 13$, $\gamma = (-0.1, 0.3)$				Three-regime TAR(15) with $d = 13$, $\gamma = (-0.1, 0.3)$					
	Regime 1		Regime 2		Regime 1		Regime 2		Regime 1		Regime 2		Regime 3	
Regressor	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.
Constant	0.0799	0.0303	0.1644	0.0517	0.0800	0.0303	0.1628	0.0508	0.0738	0.0334	0.0038	0.0677	0.1638	0.0550
y_{t-1}	-0.0209	0.0056	-0.0166	0.0084	-0.0206	0.0055	-0.0162	0.0077	-0.0238	0.0059	0.0003	0.0122	-0.0172	0.0091
Δy_{t-1}	-0.2095	0.0485	0.1848	0.0576	-0.2105	0.0484	0.1824	0.0555	-0.2747	0.0563	-0.1055	0.1501	0.1992	0.0609
Δy_{t-2}	-0.0648	0.0505	0.2610	0.0581	-0.0683	0.0502	0.2637	0.0572	-0.1088	0.0568	-0.1122	0.1546	0.3148	0.0616
Δy_{t-3}	-0.0470	0.0484	0.1201	0.0593	-0.0484	0.0483	0.1258	0.0570	-0.0601	0.0550	0.0828	0.1443	0.0814	0.0629
Δy_{t-4}	0.0482	0.0486	0.0593	0.0586	0.0453	0.0484	0.0632	0.0565	0.0578	0.0558	0.0687	0.1336	0.0054	0.0621
Δy_{t-5}	0.0730	0.0484	0.0208	0.0578	0.0704	0.0483	-	-	0.0455	0.0544	0.0928	0.1460	0.0358	0.0597
Δy_{t-6}	0.0636	0.0484	-0.0190	0.0579	0.0640	0.0474	-	-	0.0350	0.0532	0.0987	0.1521	0.0266	0.0591
Δy_{t-7}	-0.0110	0.0484	0.0215	0.0586	-	-	-	-	-0.0316	0.0522	0.1360	0.1627	-0.0326	0.0615
Δy_{t-8}	0.1139	0.0487	-0.1530	0.0581	0.1162	0.0477	-0.1528	0.0557	0.1022	0.0529	-0.0427	0.1815	-0.1205	0.0598
Δy_{t-9}	0.0475	0.0479	0.0155	0.0590	0.0530	0.0473	-	-	0.0172	0.0536	-0.1232	0.1798	0.0310	0.0603
Δy_{t-10}	-0.0314	0.0474	-0.0051	0.0586	-	-	-	-	-0.0230	0.0524	-0.1453	0.1582	-0.0006	0.0614
Δy_{t-11}	0.0625	0.0467	0.0096	0.0585	0.0664	0.0462	-	-	0.0493	0.0512	0.0015	0.1533	-0.0298	0.0609
Δy_{t-12}	-0.2023	0.0456	-0.1084	0.0599	-0.2041	0.0456	-0.1081	0.0588	-0.2449	0.0497	-0.1252	0.1635	-0.1175	0.0618
Δy_{t-13}	-0.0458	0.0462	-0.0003	0.0605	-0.0472	0.0462	-	-	-0.0647	0.0519	-0.1429	0.1486	0.0049	0.0629
Δy_{t-14}	-0.0535	0.0439	-0.1045	0.0616	-0.0563	0.0435	-0.1025	0.0608	-0.0408	0.0476	-0.1163	0.0879	-0.0546	0.0664
T	510		217		510		217		421		112		194	
$\hat{\sigma}^2$	0.0205		0.0371		0.0207		0.0372		0.0197		0.0237		0.0374	
AIC	-2657.83				-2681.41				-2640.64					
BIC	-2505.72				-2570.72				-2414.78					

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Table 11: Test Statistics and p-values

Test statistic	value	bootstrapped p-value
LR_T	102.5545	0.000
t_1	-3.7527	0.000
t_2	-1.8695	0.042
$R_1(t_1, t_2)$	17.5781	0.002

Many empirical economic studies employ the three-regime TAR model, and tests for unit roots in such models are used to verify the stationarity of economic series; for example, see Taylor (2001), Bec, Ben Salem and Carrasco (2004) and Chen, Chen and Lee (2013). In most works on three-regime TAR models, the middle regime, also known as the corridor regime, is assumed to have unit roots, while the other two regimes are assumed to be stationary. This kind of structure preserves global ergodicity and local nonstationarity, which has also been studied for other nonlinear time series models, such as Chen, Lee and Chen (2016). The three-regime TAR model is a natural alternative to the two-regime BAR model, and a Bayesian selection between these two models is conducted by Truong, Chen and So (2016).

In this work, to facilitate real economic data analysis, we focus particularly on the comparison between the globally stationary BAR and the three-regime TAR model with local nonstationarity. First, the three-regime TAR model exhibits global ergodicity property, even though the middle regime exhibits local nonstationary. Note that geometric ergodicity is an important property in nonlinear time series because, as discussed in Tong (1990), geometric ergodicity implies the existence of a unique stationary distribution for

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

y_t, such that y_t converges to stationarity exponentially fast. Thus, it is preferable to use a globally stationary process to model an ergodic process. Second, it is highly possible for a globally stationary two-regime BAR model to be misspecified as a locally nonstationary three-regime TAR model. In the BAR model, the process in the buffer regime can be viewed as a two-component mixture model in the long term, and the heterogeneity from different regimes might lead to over-dispersion if we try to fit the process in the buffer region using an AR process, as we have done in the three-regime TAR. In other words, if we fit data generated from a stationary BAR model using a three-regime TAR model, we might mistakenly conclude that the middle regime has a unit root, owing to the misspecification. Third, compared with the three-regime TAR model, the two-regime BAR model is more parsimonious, and provides a more concise regime-switching trajectory. This might enable better interpretation of a country's long-term economic status or the business cycle.

For example, if we fit the three-regime TAR model to the U.S. unemployment rate data, the fitted coefficient for y_{t-1} in regime 2 is positive and very close to zero, whereas those in the other two regimes suggest stationarity, as shown in Table 10. Based on the AIC and BIC, the two-regime globally stationary (subset) BAR(15) is preferable to the three-regime TAR(15).

6.2 Real Exchange Rates

Our second example is based on real exchange rate data. Many studies have applied the unit root test to real exchange rates, largely owing to the theory of purchasing power parity

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

(PPP), which states that a bilateral exchange rate is in equilibrium when the purchasing power of each currency is the same. By definition, the real exchange rate is defined as $E \times P^*/P$, where E is the nominal exchange rate, and P^* and P are the purchasing power of the foreign and local currencies, respectively. Under the theory of PPP, the real exchange rate cannot have a unit root, because deviations should not be permanent (Steigarwald, 1996). Zhu, Li and Yu (2017) model exchange rates using a BAR-GARCH model, finding evidence of conditional heteroscedasticity in exchange rate data. Our primary interest is buffered nonlinearity and nonstationarity; thus, we ignore conditional heteroscedasticity and apply wild bootstrap tests, which are shown to be robust under unknown conditional heteroscedasticity. The data include monthly exchange rates for the Canadian dollar (CAD), Japanese yen (JPY) and British pound (GBP) to the U.S. dollar (USD) for the period January 1977 to September 2017; see Figure 2.

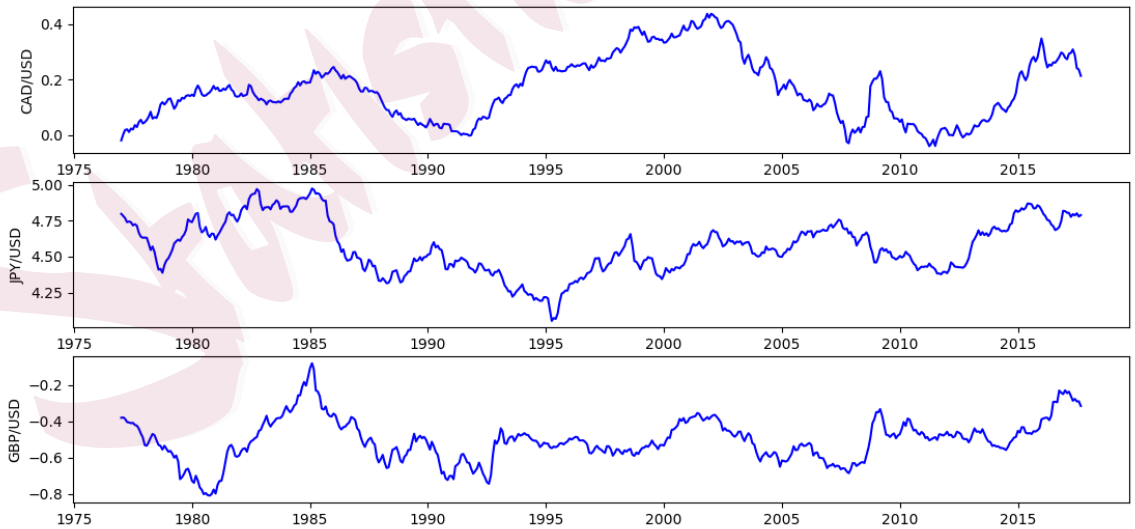

Figure 2: *Real exchange rates from January 1977 to September 2017.*

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Trade between the United States and these three countries has been significant and consistent for the past forty years, which supports the PPP theory. The nominal exchange rates are adjusted by the consumer indices in both countries; that is, $y_t = \log(E_t) + \log(P_t^*) - \log(P_t)$. We fit the BAR($p$) model in the data using the threshold variable $y_t - y_{t-d}$, and select the order p and delay parameter d based on the AIC. The estimates of the BAR(p) model and its linear and nonlinear alternatives, such as the AR(p) and the three-regime TAR(p) models, are summarized in Table 12.

We first perform the Sup-LR test for buffered nonlinearity for the three series. Strong buffered nonlinearity is detected in the CAD/USD and GRB/USD series; however, JPY/USD appears to be a linear process, because the Sup-LR test is not statistically significant, and the AR(p) is preferred to the BAR(p) and the three-regime TAR(p), based on the information criteria. To test for stationarity, we apply the one-sided unit root test to the three series, because the estimates of the coefficients of y_{t-1} are all negative and close to zero. The 5% one-sided unit root test only rejects the null hypothesis for GBP/USD. The ADF test does not draw the same conclusion because the t-values for both regimes (-2.20 and -2.21) are not statistically significant for the ADF test. In summary, the theory of PPP seems to hold only for the GBP/USD, while the JPY/USD rate appears to be linear and nonstationary, and the CAD/USD rate appears to be buffered nonlinear and nonstationary.

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Table 12: Estimates of BAR(p), AR(p), and the Three-Regime TAR(p) Models

		BAR(3), $d=2$, $\gamma=(-0.0084, 0.0021)$				AR(3)		Three-regime TAR(3), $d=2$, $\gamma=(-0.0084, 0.0021)$					
	Regressor	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.
CAD/USD	Constant	-0.0021	0.0022	0.0041	0.0016	0.0023	0.0012	-0.0021	0.0027	0.0036	0.0027	0.0035	0.0021
	y_{t-1}	-0.0106	0.0091	-0.0132	0.0072	-0.0118	0.0058	-0.0138	0.0102	-0.0151	0.0125	-0.0089	0.0082
	Δy_{t-1}	0.3571	0.0824	0.1345	0.0625	0.2685	0.0453	0.3546	0.0966	0.2237	0.4402	0.1483	0.0749
	Δy_{t-2}	-0.3172	0.0892	0.0089	0.0673	-0.0225	0.0453	-0.3453	0.0965	0.1662	0.4577	-0.0142	0.0768
	T	180		306		486		145		109		232	
	$\hat{\sigma}^2$	0.000199		0.000213		0.000216		0.000209		0.000160		0.000230	
	AIC	-4130.64				-4117.02		-4120.89					
	BIC	-4092.91				-4096.06		-4066.39					

		BAR(2), $d=1$, $\gamma=(-0.0195, 0.0067)$				AR(2)		Three-regime TAR(2), $d=2$, $\gamma=(-0.0005, 0.0067)$					
	Regressor	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.
JPY/USD	Constant	0.0176	0.0527	0.0974	0.0369	0.0711	0.0304	0.0692	0.0445	0.1293	0.0795	0.0640	0.0480
	y_{t-1}	-0.0052	0.0115	-0.0210	0.0081	-0.0156	0.0067	-0.0155	0.0097	-0.0272	0.0173	-0.0127	0.0104
	Δy_{t-1}	0.1647	0.0886	0.2940	0.0785	0.3156	0.0430	0.2675	0.0859	-3.3215	1.6569	0.1715	0.1287
	T	172		315		487		227		60		184	
	$\hat{\sigma}^2$	0.000798		0.000676		0.000726		0.000743		0.000876		0.000631	
	AIC	-3527.15				-3526.14		-3525.21					
	BIC	-3497.81				-3509.37		-3483.28					

		BAR(3), $d=2$, $\gamma=(-0.0236, 0.0021)$				AR(3)		Three-regime TAR(3), $d=2$, $\gamma=(-0.0236, 0.0021)$					
	Regressor	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.	Est	s.e.
GBP/USD	Constant	-0.0346	0.0097	-0.0103	0.0055	-0.0131	0.0048	-0.0420	0.0138	-0.0123	0.0080	-0.0136	0.0071
	y_{t-1}	-0.0408	0.0169	-0.0227	0.0106	-0.0261	0.0093	-0.0400	0.0214	-0.0224	0.0152	-0.0247	0.0133
	Δy_{t-1}	0.0834	0.1143	0.2659	0.0613	0.3123	0.0451	-0.0645	0.1767	0.0242	0.2382	0.3648	0.0960
	Δy_{t-2}	-0.3252	0.0898	-	-	-0.0568	0.0455	-0.3817	0.1253	0.0221	0.0684	-0.0431	0.0667
	T	121		365		486		72		187		227	
	$\hat{\sigma}^2$	0.000601		0.000515		0.000554		0.000571		0.000550		0.000504	
	AIC	-3668.19				-3656.61		-3662.79					
	BIC	-3634.65				-3635.65		-3608.29					

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Table 13: Test Statistics and p-values

	CAD/USD		JPY/USD		GRB/USD	
Test statistic	value	p-value	value	p-value	value	p-value
$\sup LR_n$	20.92	0.021	6.68	0.589	17.94	0.043
t_1	-1.13	0.279	-0.45	0.483	-2.20	0.021
t_2	-1.85	0.073	-2.59	0.012	-2.21	0.029
R_1	4.71	0.223	6.93	0.129	9.71	0.025

7. Conclusion

This work studied the asymptotic theory for a BAR model with possible unit roots, and proposed two types of tests to identify nonstationarity and nonlinearity. Our asymptotic results extend the theory related to the TAR model (Caner and Hansen, 2001) to a more general setting, but with a similar form. Thus, the additional buffer effect does not necessarily make the asymptotic theory much more complicated. The results of the simulation experiments show that the bootstrap tests perform well and that the proposed unit root tests are more powerful than the conventional ADF test.

For real economic and financial data analyses, combining the unit root test in the BAR model and conditional heteroscedasticity would be interesting and important, and we leave it for future research.

Supplementary Material

The online Supplementary Material includes auxiliary lemmas and proofs of the theorems presented here.

Acknowledgments

The authors would like to thank two anonymous referees, the Associate Editor, and the Co-Editor for their constructive suggestions and comments. This work was partially supported by the Research Grants Council of the Hong Kong SAR Government (GRF grant 17304417). The computations were performed using research computing facilities provided by Information Technology Services, the University of Hong Kong.

References

Andrews, D. W. K. (1993). Tests for parameter instability and structural change with unknown change point. *Econometrica* **61**, 821-856.

Bec, F., Ben Salem, M. and Carrasco, M. (2004). Tests for unit-root versus threshold specification with an application to the purchasing power parity relationship. *Journal of Business and Economic Statistics* **22**, 382–395.

Caner, M. and Hansen, B. (2001). Threshold autoregression with a unit root. *Econometrica* **69**, 1555-1596.

Chan, K. S. (1990). Testing for threshold autoregression. *Annals of Statistics* **18**, 1886-1894.

Chen, C. W., Chen, S. Y., and Lee, S. (2013). Bayesian unit root test in double threshold heteroskedastic models. *Computational Economics* **42**, 471-490.

Chen, C. W., Lee, S., and Chen, S. Y. (2016). Local non-stationarity test in mean for Markov switching GARCH models: an approximate Bayesian approach. *Computational Statistics* **31**, 1-24.

Chen, C. W. S., So, M. K. P. and Liu. F. (2011). A review of threshold time series models in finance. *Statistics and Its Interface* **4**, 167-181.

Statistica Sinica: Preprint
doi:10.5705/ss.202017.0507

Chen, C. W., and Truong, B. C. (2016). On double hysteretic heteroskedastic model. *Journal of Statistical Computation and Simulation* **86**, 2684-2705.

Hamilton, J. (1989), A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica* **5**, 357-384.

Hansen, B. (2011). Threshold autoregression in economics. *Statistics and Its Interface* **4**, 123-127.

Kapetanios, G., Shin, Y. and Snell, A. (2003). Testing for a unit root in the nonlinear STAR framework. *Journal of Econometrics* **112**, 359-379.

Kapetanios, G., and Shin, Y. (2006). Unit root tests in three-regime SETAR models. *Econometrics Journal* **9**, 252-278.

Li, G., Guan, B., Li, W. K., and Yu. P. L. H. (2015). Hysteretic autoregressive time series models. *Biometrika* **102**, 717-723.

Maki, D. (2009) Tests for a unit root using three-regime TAR models: power comparison and some applications, *Econometric Reviews* **28** (4), 335-363.

National Bureau of Economic Research. (2010). US business cycle expansions and contractions. Retrieved from http://www.nber.org/cycles/.

Park, J. Y. (2003). Bootstrap unit root tests. *Econometrica* **71**, 1845-1895.

Park, J. Y. and Shintani, M. (2016). Testing for a unit root against transitional autoregressive models. *International Economic Review* **57**, 635-664.

Potter, S. M. (1995). A nonlinear approach to U.S. GNP. *Journal of Applied Econometrics* **2**, 109-125.

Roed, K. (2002). Unemployment hysteresis and the natural rate of vacancies, *Empirical Economics* **27**, 687-704.

Seo, M. H. (2008). Unit root test in a threshold autoregression: asymptotic theory and residual-based block

bootstrap. *Econometric Theory* **24**, 1699-1716.

Smith, C. E. (2009). Economic indicators. In Wankel C. (ed.) *Encyclopedia of Business in Today's World*. California, USA.

Steigerwald, D. G. (1996). Purchasing power parity, unit roots, and dynamic structure. *Journal of Empirical Finance* **2**, 343-357.

Taylor, A. M. (2001). Potential pitfalls for the purchasing-power-parity puzzle? Sampling and specification biases in mean-reversion tests of the law of one price. *Econometrica* **69**, 473-498.

Tong, H. (1978). On a threshold model. In Chen C. H. (ed.), *Pattern Recognition and Signal Processing*, Sijthoff & Noordhoff, Amsterdam.

Tong, H. (1990). *Non-linear Time Series. A Dynamical System Approach.* Clarendon Press, Oxford.

Truong, B. C., Chen, C. W., and So, M. K. (2016). Model selection of a switching mechanism for financial time series. *Applied Stochastic Models in Business and Industry* **32**, 836-851.

Zhu, K., Yu, P. L. H. and Li, W. K. (2014). Testing for the buffered autoregressive processes. *Statistica Sinica* **24**, 971-984.

Zhu, K., Li, W. K. and Yu, P. L. H. (2017). Buffered autoregressive models with conditional heteroscedasticity: An application to exchange rates. *Journal of Business and Economic Statistics* **35**, 528-542.

Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam Road, Hong Kong.

E-mail: diwang@connect.hku.hk

Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam Road, Hong Kong.

E-mail: hrntlwk@hku.hk