

# SIMULTANEOUS CONFIDENCE BANDS IN NONLINEAR REGRESSION MODELS WITH NONSTATIONARITY

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*Abstract:* We consider nonparametric estimation of the regression function  $g(\cdot)$  in a nonlinear regression model  $Y_t = g(X_t) + \sigma(X_t)e_t$ , where the regressor  $(X_t)$  is a nonstationary unit root process and the error  $(e_t)$  is a sequence of independent and identically distributed (i.i.d.) random variables. With proper centering and scaling, the maximum deviation of the local linear estimator of the regression function  $g$  is shown to be asymptotically Gumbel. Based on the latter result, we construct simultaneous confidence bands for  $g$ , which can be used to test patterns of the regression function. Our results extend existing ones that typically require independent or stationary weakly dependent regressors. We examine the finite sample behavior of the proposed approach via simulated and empirical data examples.

*Key words and phrases:* Gumbel convergence, integrated process, local linear estimation, local time limit theory, maximum deviation, simultaneous confidence

bands.

## 1. Introduction

Following Granger (1981) and Engle and Granger (1987), cointegration has become one of the most important topics in econometrics. For nonstationary processes  $(X_t)$  and  $(Y_t)$ , we say that they are (linearly) cointegrated if there exists a nonzero constant  $a$  such that

$$Y_t - aX_t = \varepsilon_t, \quad t = 1, \dots, n, \quad (1.1)$$

where  $(\varepsilon_t)$  is a stationary process. In essence, the nonstationary processes  $(X_t)$  and  $(Y_t)$  have a common stochastic trend over a long time series period. The classic linear cointegration models have been extensively studied in the literature. To account for possible nonlinear relations, we consider the nonlinear and nonstationary regression model

$$Y_t - g(X_t) = \sigma(X_t)e_t, \quad t = 1, \dots, n, \quad (1.2)$$

where  $(e_t)$  is a stationary error process,  $(X_t)$  is a nonstationary regressor,  $g$  and  $\sigma$  are smooth functions. Model (1.2) generalizes (1.1) by allowing possible nonlinear structures on both  $g$  and  $\sigma$ . It provides a flexible tool to analyze nonlinear relationships between two nonstationary processes.

The aim of this paper is to study nonparametric estimates of the function  $g$  in model (1.2). Convergence properties of the conventional Nadaraya-

Watson kernel-based estimator have been widely discussed under the assumption that  $(X_t, Y_t)$  are i.i.d.; see Härdle (1990) and the references therein. In Györfil *et al* (1989), Bosq (1998) and Fan and Yao (2003), the i.i.d. assumption is relaxed and strong mixing stationary processes are allowed. The asymptotic problem is quite challenging if  $(X_t)$  is non-stationary. Karlsen *et al* (2007, 2010) assumed that  $(X_t)$  is a null recurrent Markov chain. Wang and Phillips (2009a, 2009b, 2011) and Cai *et al* (2009) considered the integrated process  $X_t = \sum_{j=1}^t x_j$ , where  $(x_j)$  is a stationary linear process. For other literature on this research area, we refer to Gao *et al* (2009a, 2009b), Chen *et al* (2010, 2012), Kasparis and Phillips (2012), Wang and Phillips (2012), and Wang (2014).

The aforementioned papers deal with the point-wise central limit theorem for the Nadaraya-Watson estimator. This type of result, however, is not useful for testing certain patterns of the regression function  $g$ . In recent years, there has been increasing interest in deriving uniform consistency results for the kernel-based estimator under a nonstationary framework. For instance, Wang and Wang (2013) and Chan and Wang (2014) established uniform consistency of the kernel-based estimator by using local time limit theory, Wang and Chan (2014) obtained uniform convergence rates for a class of martingales and studied their application in nonlinear cointegration

regression, and Gao *et al* (2015) derived some uniform consistency results using the framework of null recurrent Markov chains.

In this paper, we estimate the regression function  $g$  using the local linear smoothing method in which the estimator of  $g(x)$  is

$$\widehat{g}_n(x) = \frac{\sum_{t=1}^n w_t(x) Y_t}{\sum_{t=1}^n w_t(x)}, \quad (1.3)$$

where the weight function is

$$w_t(x) = K[(X_t - x)/h]V_{n2}(x) - K_1[(X_t - x)/h]V_{n1}(x)$$

with  $V_{nj}(x) = \sum_{i=1}^n K_j[(X_i - x)/h]$ ,  $K_j(x) = x^j K(x)$ ,  $K(\cdot)$  being a non-negative real function, and the bandwidth  $h \equiv h_n \rightarrow 0$ . Under the stationarity assumption on the observations, existing literature such as Fan and Gijbels (1996) has shown that local linear estimation has some advantages over the Nadaraya-Watson kernel estimation method. A recent paper by Chan and Wang (2014) further showed that the performance of  $\widehat{g}_n(\cdot)$  is superior to that of the conventional Nadaraya-Watson estimator in uniform asymptotics for nonstationary time series.

To assess patterns of the regression function  $g$ , for example, to test whether  $g$  is linear or of other parametric forms, we need to construct a simultaneous confidence band (SCB) for  $g$  over a suitable interval. Neither the point-wise central limit theorem nor the uniform convergence of  $\widehat{g}_n(\cdot)$

is sufficient for testing whether  $g$  has a particular functional form. Here we obtain the asymptotic distribution of the normalized maximum absolute deviation

$$\Delta_n = \sup_{|x| \leq B_n} |V_n(x) [\widehat{g}_n(x) - g(x)] / \sigma(x)|, \quad (1.4)$$

where  $B_n$  is a sequence of positive constants which may diverge to infinity, and the normalizing term is

$$V_n(x) = V_{n2}^{-1}(x) \sum_{t=1}^n w_t(x) / \left( \sum_{t=1}^n K^2[(X_t - x)/h] \right)^{1/2}. \quad (1.5)$$

Such an asymptotic distributional theory refines the existing uniform consistency results such as those obtained by Chan and Wang (2014), and it further enables one to construct a SCB for the unknown regression function  $g$ . In the traditional simultaneous inference theory it is assumed that the regressor process  $(X_t)$  is i.i.d. or stationary; see, for example, Bickel and Rosenblatt (1973), Johnston (1982), Hall and Titterton (1988), Xia (1998), Fan and Zhang (2000), Zhao and Wu (2008), Liu and Wu (2010) and Zhang and Peng (2010). However, in our setting, due to the nonstationarity and the dependence, it is very demanding to establish a limit theory for  $\Delta_n$ . To this end, we introduce new technical mechanisms and obtain a precise characterization of  $V_n(x)$  over an unbounded interval.

The rest of the paper is organized as follows. The assumptions and main theoretical results are stated in Section 2. By using the asymptotic

distribution of  $\Delta_n$ , in Section 3 we construct SCBs for the regression function  $g$  over an expanding interval. In Section 4, we provide simulated and empirical data examples to illustrate the finite sample behavior of the proposed approach. The proofs of the main results are in Section 5. Some technical lemmas with proofs, and some supplemental asymptotic theorems are given in a supplemental document.

## 2. Main results

We need some regularity conditions to establish the asymptotic distribution of  $\Delta_n$  as (1.4). Let  $C$  denote a generic positive constant.

**(C1)** [Regressor process] Let

$$X_t = \sum_{j=1}^t x_j, \quad x_j = \sum_{k=0}^{\infty} \phi_k \eta_{j-k},$$

where  $(\eta_j)$  is a sequence of i.i.d. random variables with  $\mathbf{E}[\eta_1] = 0$ ,  $\mathbf{E}[\eta_1^2] = 1$ , and  $\mathbf{E}[|\eta_1|^{2+\delta}] < \infty$  for some  $\delta > 0$ ; the characteristic function  $\varphi(t)$  of  $\eta_1$  satisfies  $\int_{-\infty}^{\infty} (1 + |t|)|\varphi(t)|dt < \infty$ , and the coefficients  $(\phi_k)_{k \geq 0}$  satisfy  $\sum_{k=0}^{\infty} |\phi_k| < \infty$  and  $\phi \equiv \sum_{k=0}^{\infty} \phi_k \neq 0$ .

**(C2)** [Regression function] The first derivative  $g'(x)$  of  $g(x)$  exists. There is a real positive function  $g_0(x)$  such that

$$|g'(y) - g'(x)| \leq g_0(x)|y - x|, \tag{2.1}$$

uniformly for  $x \in \mathbf{R}$  and  $|y - x|$  sufficiently small.

**(C3)** [Range and Bandwidth] (i)  $B_n = M_0 n^{1/2-\epsilon}$ , where  $M_0$  is a constant and  $\epsilon > 0$  can be arbitrarily small; (ii) for some constant  $0 < \delta_0 < 1/4$ ,  $n^{1/2-\delta_0} h \rightarrow \infty$ ; (iii)  $nh^{10} \sup_{|x| \leq 2B_n} [1 + g_0^4(x)] = O(\log^{-8} n)$ , where  $g_0(x)$  is given in (2.1).

**(C4)** [Errors] (i) The error process  $(e_t)$  is i.i.d. with  $\mathbb{E}[e_1] = 0$ ,  $\mathbb{E}[e_1^2] = 1$  and  $\mathbb{E}[|e_1|^{2p}] < \infty$ , where  $p \geq 1 + [1/\delta_0]$  and  $\delta_0$  is defined as in **(C3)**(ii), and  $(e_t)$  is independent of the process  $(\eta_t)$ . (ii) For the function  $\sigma(\cdot)$ ,

$$\inf_{x \in \mathbb{R}} \sigma(x) > 0, \quad \sup_{x \in \mathbb{R}} \frac{|\sigma(x+y) - \sigma(x)|}{\sigma(x)} \leq C|y| \quad (2.2)$$

for any  $|y|$  sufficiently small.

**(C5)** [Kernel] The kernel function  $K$  is absolutely continuous on a compact support  $[-A, A]$  with  $A > 0$ ,  $\int xK(x)dx = 0$ , and  $|K(x) - K(y)| \leq C|x - y|$  for all  $x, y \in \mathbb{R}$ .

**Remark 1.** Conditions similar to **(C1)** and **(C2)** are common; see, for instance, Wang and Phillip (2009b), Wang and Wang (2013), and Chan and Wang (2014). The smoothness condition (2.1) on  $g'$  is slightly weaker than the existence of second derivative on  $g$ . If the function  $g$  has continuous and bounded second order derivative, we may replace  $g_0(x)$  in (2.1) by a positive constant and **(C3)** (iii) can be simplified to  $nh^{10} \log^8 n = O(1)$ .

In condition **(C3)**(i), we allow  $B_n$  to diverge to infinity, which is an

essential difference from the literature that investigates the SCB for stationary regressor with  $B_n$  usually assumed to be fixed (c.f., Liu and Wu, (2010)). Our main theorem is established for diverging  $B_n$ , which indicates that the SCB for unknown  $g$  with  $I(1)$  regressor may be constructed under an expanding interval whose length is allowed to diverge to infinity; see Remark 4 in Section 3 for more details. **(C3)**(ii) is close to the necessary condition of  $\sqrt{nh} \rightarrow \infty$  by letting  $\delta_0$  be sufficiently close to zero. **(C3)**(iii) implies that there is a trade-off between the function  $g$ , the bandwidth  $h$ , and the range  $B_n$ . If  $B_n$  is assumed to be a positive constant  $B$ , **(C3)**(iii) is satisfied when  $\sup_{|x| \leq 2B} g_0(x) = O(1)$  and  $nh^{10} \log^8 n = O(1)$ . In particular, the condition **(C3)**(iii) ensures that the bias term of the local linear estimator (1.3) with nonstationarity is asymptotically negligible.

In condition **(C4)**,  $E[e_1^2] = 1$  is not necessary. If  $E[e_1^2] = a^2 \neq 1$ , it suffices to standardize the model (1.2) with  $e'_t = e_t/a$  and replace  $\sigma(x)$  by  $a\sigma(x)$ . The moment condition of  $e_1$  depends on the  $\delta_0$  in **(C3)**(ii), which is reasonable. Weaker bandwidth restriction (smaller  $\delta_0$ ) would lead to stronger moment condition on  $e_t$  (larger  $p$ ). We require the independence between the regressor ( $X_t$ ) and the error ( $e_t$ ) by noting that ( $e_t$ ) is independent of ( $\eta_t$ ) in **(C4)**(i). This condition is restrictive, but seems difficult to be relaxed, even with a stationary regressor. The condition imposed on



$\sigma(x)$  is mild and is satisfied by a large class of functions. Typical examples include  $\sigma(x) = 1 + |x|^k$  with  $k \geq 0$  and  $\sigma(x) = e^x/(1 + e^x)$ .

The kernel condition **(C5)** implies that the derivative  $K'(x)$  exists almost everywhere and  $\int_{-A}^A [K'(x)]^2 dx < \infty$ . Let  $\lambda_1 = \int_{-A}^A K(x) dx$ ,  $\lambda_2 = \int_{-A}^A K^2(x) dx$ , and  $\kappa_2 = \int_{-A}^A (K'(x))^2 dx / (2\lambda_2)$ . We further have

$$r(s) \equiv \int_{-A}^A K(x)K(x+s)dx/\lambda_2 = 1 - \kappa_2 s^2 + o(s^2), \quad \text{as } s \rightarrow 0. \quad (2.3)$$

This follows from the Taylor's expansion of  $r(s)$  and Theorems B1–B2 of Bickel and Rosenblatt (1973). As there,  $r(s)$  is used as the covariance function of a certain Gaussian process. The kernel condition **(C5)** is mild and it is satisfied by such commonly-used functions as the Epanechnikov kernel  $K(u) = \max\{3(1 - u^2)/4, 0\}$ .

**Theorem 1.** *Let Conditions (C1)–(C5) be satisfied. Then, for  $z \in R$ ,*

$$P\{(2 \log \bar{h}^{-1})^{1/2} (\Delta_n - d_n) \leq z\} \rightarrow e^{-2e^{-z}}, \quad (2.4)$$

where  $\bar{h} = h/(2B_n)$  and

$$d_n = (2 \log \bar{h}^{-1})^{1/2} + \frac{1}{(2 \log \bar{h}^{-1})^{1/2}} \log \frac{\kappa_2^{1/2}}{\pi 2^{1/2}}.$$

**Remark 2.** If  $\mathcal{G}_n = (2 \log \bar{h}^{-1})^{1/2} (\Delta_n - d_n)$  and  $\mathcal{G}$  is the standard Gumbel distribution with the cumulative distribution function defined as the limit

in (2.4), we can reformulate the above limit result as  $\mathcal{G}_n \xrightarrow{d} \mathcal{G}$ . As we allow that  $B_n$  is divergent, using Theorem 1 we can construct SCBs for the unknown function  $g$  on expanding intervals; the requirement of intervals being expanding is crucial in our functional estimation since the regressor process  $(X_t)$  is integrated and thus is stochastically unbounded, behaving like a random walk. Liu and Wu (2010) obtained Gumbel convergence for the nonlinear regression models with stationary regressors, but assumed that the involved interval is bounded and non-expanding.

The asymptotic Gumbel distribution result (2.4) also holds when we replace  $\hat{g}_n(\cdot)$  by the Nadaraya-Watson kernel estimation  $\tilde{g}_n(\cdot)$  in the definition of  $\Delta_n$ . We provide some relevant results in Appendix C of the supplemental document.

### 3. Construction of SCBs

This section constructs the SCBs for the regression function  $g$ . Since  $\sigma(x)$  in the definition of  $\Delta_n$  given in (1.4) is unknown, a consistent estimate of  $\sigma(x)$  satisfying certain rates is required over the set  $\{x : |x| \leq B_n\}$ . Using arguments as in Wang and Wang (2013), we construct the kernel estimate

$$\hat{\sigma}_n^2(x) = \frac{\sum_{t=1}^n [Y_t - \hat{g}_n(X_t)]^2 K[(X_t - x)/b]}{\sum_{t=1}^n K[(X_t - x)/b]}, \quad (3.1)$$

where  $\hat{g}_n(x)$  is the local linear estimate defined in (1.3), and  $b$  is a band-

width. Let  $a_n \asymp b_n$  signify that  $a_n$  and  $b_n$  have the same asymptotic order.

**Proposition 1.** *Under the conditions of Theorem 1 and  $b \asymp h$ , we have*

$$\sup_{|x| \leq B_n} \left| \frac{\sigma(x)}{\widehat{\sigma}_n(x)} - 1 \right| = O_P [h + (nh^2)^{-1/4} \log^2 n]. \quad (3.2)$$

**Remark 3.** The proof of Proposition 1 is given in Section 5. Proposition 1 can be seen as an extension of Wang and Wang (2013)'s uniform consistency results from the case of bounded range to the case of diverging range. The uniform convergence rate in (3.2) is close to optimal. Using the bandwidth conditions in **(C3)**, the rate in (3.2) is sufficient for replacing  $\sigma(\cdot)$  by  $\widehat{\sigma}_n(\cdot)$  when constructing the SCBs of the regression function  $g$ .

Let

$$\widehat{\Delta}_n = \sup_{|x| \leq B_n} |V_n(x) [\widehat{g}_n(x) - g(x)] / \widehat{\sigma}_n(x)|.$$

Due to (3.2) and Theorem 1,  $\widehat{\Delta}_n$  and  $\Delta_n$  have the same limit distribution.

Consequently, for given  $\alpha$ , the  $(1 - \alpha)$ -SCB for  $g$  over the set  $\{x : |x| \leq B_n\}$  can be constructed as

$$[\widehat{g}_n(x) - l_\alpha(x), \widehat{g}_n(x) + l_\alpha(x)], \quad (3.3)$$

where

$$\begin{aligned}
 l_\alpha(x) &= [z_\alpha(2 \log \bar{h}^{-1})^{-1/2} + d_n] \hat{\sigma}_n(x) V_n^{-1}(x), \\
 z_\alpha &= -\log \left( -\frac{1}{2} \log(1 - \alpha) \right), \quad \bar{h} = h/(2B_n), \\
 d_n &= (2 \log \bar{h}^{-1})^{1/2} + \frac{1}{(2 \log \bar{h}^{-1})^{1/2}} \log \frac{\kappa_2^{1/2}}{\sqrt{2\pi}}.
 \end{aligned}$$

**Remark 4.** As Theorem 1 implies that the asymptotic bias term is negligible due to the bandwidth condition **(C3)**, we do not need to correct the bias of the local linear estimation when constructing the SCBs in (3.3). The simulation study in Section 4 shows that such construction of SCBs works reasonably well in the finite sample case. When the regression function  $g(x)$  has a thin tail, such as  $g(x) = (\alpha + \beta e^x)/(1 + e^x)$ , some routine calculations show that  $\sup_x |g_0(x)| < \infty$ . As a consequence, the SCB result (3.3) holds over the set  $\{x : |x| \leq M_0 n^{1/2-\epsilon}\}$  whenever  $n^{1/2-\delta_0} h \rightarrow \infty$  and  $nh^{10} \log^8 n \rightarrow 0$ , where  $\epsilon$  can be chosen sufficiently small.

#### 4. Numerical studies

In this section, we report on simulated and empirical data examples to illustrate the finite sample behavior of our SCBs. For the construction, we need to choose an appropriate cut-off value so that the pre-assigned nominal confidence level can be achieved. The Gumbel convergence in Theorem 1 can be quite slow, which implies that the SCB in (3.3) using the

asymptotical cut-off value may not have good finite sample performance. To circumvent such a problem, as in Wu and Zhao (2007) and Liu and Wu (2010), we introduce a simulation-based method.

1. Choose an appropriate bandwidth  $h$  and a kernel function  $K$  satisfying **(C5)**. Then using (1.3), compute  $\widehat{g}_n(x)$  and the estimated residuals  $\widehat{e}_t = Y_t - \widehat{g}_n(X_t)$ . Based on the latter, with another bandwidth  $b$ , estimate  $\sigma^2(\cdot)$  by using (3.1) and denote the estimate by  $\widehat{\sigma}_n^2(\cdot)$ .
2. Generate i.i.d. standard normal random variables  $e_1^*, \dots, e_n^*$  independent of  $x_1, \dots, x_n$ ; compute  $Y_t^* = \widehat{\sigma}_n(X_t)e_t^*$ ; perform a local linear regression of  $(Y_t^*)$  on  $(X_t)$  with bandwidth  $h$  and kernel  $K$ , and let  $\widehat{g}_n^*(\cdot)$  be the estimated regression function. Then compute the normalized maximum deviation

$$\Delta_n^* = \max_{|x| \leq B_n} |V_n(x)\widehat{g}_n^*(x)/\widehat{\sigma}_n(x)|. \quad (4.1)$$

3. Repeat Step 2 for  $N$  times, and then compute the  $(1 - \alpha)$ th sample quantile, denoted by  $\Delta_{n,1-\alpha}^*$ .
4. Construct the  $(1 - \alpha)$ th SCB for  $g(x)$  over the interval  $x \in [-B_n, B_n]$  as

$$\left[ \widehat{g}_n(x) - \widehat{\sigma}_n(x)\Delta_{n,1-\alpha}^*/V_n(x), \widehat{g}_n(x) + \widehat{\sigma}_n(x)\Delta_{n,1-\alpha}^*/V_n(x) \right].$$

From Theorem 1, with the normalization in (2.4),  $\Delta_n^*$  and  $\Delta_n$  have the same asymptotic Gumbel distribution. Then the empirical quantile of the former can approximate that of the latter. In comparison with the asymptotic Gumbel distribution, the distribution of  $\Delta_n^*$  better approximates that of  $\Delta_n$ . Our simulation-based construction has an important practical convenience: the procedure is the same as when data are stationary. We do not have to be concerned with whether the true data generating process is  $I(0)$  or  $I(1)$  for constructing SCBs with asymptotically correct coverage probabilities.

**Example 1.** We took the nonlinear regression model

$$Y_t = g(X_t) + \sigma(X_t)e_t, \quad X_t = X_{t-1} + x_t, \quad t = 1, 2, \dots, n, \quad (4.2)$$

with  $g(x) = \log(10 + 0.25x^{1.2})$ ,  $\sigma(x) = (6 + 0.2x^2)^{0.5}$ ,  $(x_t)$  generated by the AR(1) process  $x_t = 0.4x_{t-1} + v_t$ ,  $v_t, t \in \mathbf{Z}$ , i.i.d.  $\mathbf{N}(0, 1)$ ,  $e_t, t \in \mathbf{Z}$ , i.i.d.  $\text{uniform}(-3^{1/2}, 3^{1/2})$ , and  $(v_t)_{t \in \mathbf{Z}}$  and  $(e_t)_{t \in \mathbf{Z}}$  independent. We chose  $n = 100, 200, 500$  and considered SCBs over the interval  $x \in [B_l, B_u]$ , where  $B_l$  (resp.  $B_u$ ) is the 0.1th (resp. 0.9th) sample quantile of the  $I(1)$  regressor  $(X_t)$ .

In our simulation, we used the Gaussian kernel function  $K(u) = (2\pi)^{-1/2} \exp(-u^2/2)$ , since it has a very thin tail and is the default choice in R package `KernSmooth`. We used the function `dpill` (Ruppert *et al*, (1995)) in the pack-

age `KernSmooth` for choosing the bandwidth and the function `locpoly` for performing the local linear regression. Table 1 shows the coverage probabilities as proportions of the SCBs that cover the true function  $g(x)$  over  $x \in [B_l, B_u]$  based on  $10^5$  repetitions. The levels considered were:  $\alpha = 0.05$  and  $\alpha = 0.01$ . Coverage probabilities based on theoretical cut-off values computed from Theorem 1 are also shown. In the left panel we use the true variance function  $\sigma^2(\cdot)$ , while in the right one the estimated  $\widehat{\sigma}_n^2(\cdot)$  is used. Using the true variance function  $\sigma^2(\cdot)$ , with  $\alpha = 5\%$  and  $n = 500$ , the coverage probability for the simulation-based method is 0.9495. For the theoretical cut-off value based on Theorem 1, the coverage probability is 0.9622. As expected, larger  $n$  leads to more accurate coverage probabilities. A similar claim can be made for the SCBs with  $\alpha = 1\%$ . Hence the simulation-based method has a more accurate finite sample performance, in particular when  $n = 500$ . The accuracy of the coverage probabilities can be slightly affected if the estimated variance function  $\widehat{\sigma}_n^2(\cdot)$  is used.

Table 1. Coverage Probabilities for SCBs

( $\alpha; n$ )	True $\sigma^2(\cdot)$		Estimated $\sigma^2(\cdot)$	
	Simulation	Theoretical	Simulation	Theoretical
0.05; 100	0.9411	0.9568	0.8859	0.9069
0.01; 100	0.9874	0.9971	0.9564	0.9805
0.05; 200	0.9376	0.9530	0.9098	0.9271
0.01; 200	0.9875	0.9960	0.9723	0.9878
0.05; 500	0.9495	0.9622	0.9447	0.9573
0.01; 500	0.9901	0.9974	0.9880	0.9963

**Example 2.** We considered the monthly US share price indices and treasury bill rates for the period January/1957–December/2009, downloaded from International Monetary Fund’s website, as in Chen *et al* (2012). The upper two plots in Figure 1 give the two series with the log transformation applied to the share price data. The augmented Dickey-Fuller test indicates that the treasury bill rates are the  $I(1)$  process, and the log-transformed share price indices are the  $I(1)$  process with drift. To ensure that our methodology and theory are applicable to the data, we removed the drift from the share price indices and the adjusted share price indices are plotted in the lower-left plot of Figure 1. The lower-right plot in Figure 1 gives the scatter plot of the adjusted share price indices against the treasury bill



rates, which indicates the existence of heteroskedasticity. The aim of this example is to analyze the relationship between the share price series and the treasury bill rates. Let  $(Y_t)$  be the adjusted share price series, and  $(X_t)$  be the treasury bill series. We first fit the data with the linear regression model  $Y_t = a + bX_t + e_t$ , and obtained the least squares estimation  $\hat{a}$  and  $\hat{b}$  of the two parameters:  $-0.1582$  and  $-0.0514$ , respectively. The cointegration test shows that the null hypothesis of the  $I(1)$  error process cannot be rejected. This indicates that the linear cointegration model is not appropriate for the data.

We considered the nonlinear regression model

$$Y_t = g(X_t) + \varepsilon_t^*, \quad \varepsilon_t^* = \sigma(X_t)e_t, \quad t = 1, \dots, 636, \quad (4.3)$$

where  $(e_t)$  is assumed to be independent of  $(X_t)$ ,  $g$  and  $\sigma$  are two nonlinear functions as in model (1.2). The structure of  $\varepsilon_t^* = \sigma(X_t)e_t$  allows the existence of nonstationarity and heteroskedasticity for the model error term. The local linear estimation method was used to estimate the regression function  $g$ , with resulting estimate  $\hat{g}_n(\cdot)$ , where the bandwidth was chosen by the commonly-used cross-validation method. Let  $\hat{\varepsilon}_t^* = Y_t - \hat{g}_n(X_t)$ . Figure 2 plots the residual  $(\hat{\varepsilon}_t^*)$  against the regressor  $(X_t)$ ; it shows that our nonlinear model (4.3) with heteroskedasticity on errors is appropriate for the data.

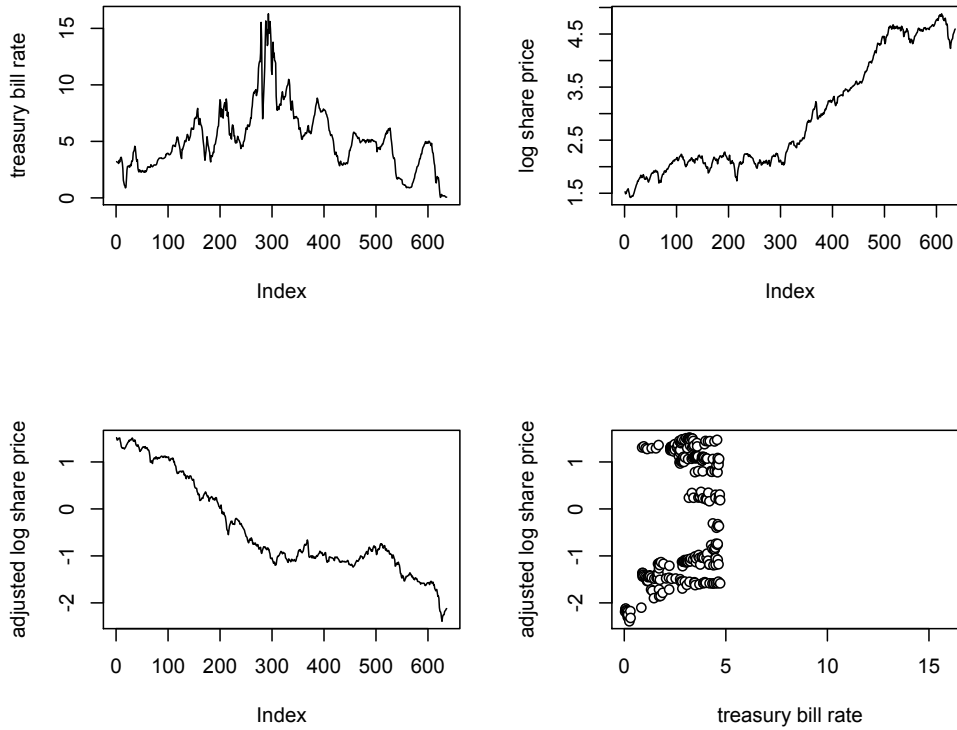


Figure 1: The data plots over period of January/1959–December/2009.

The SCB for the regression function is plotted in Figure 3 with the cut-off value chosen by the simulation-based method. The upper and lower dashed lines are the 95%-SCB for the regression function  $g$ . The solid line is the local linear estimated regression function and the dotted line is the estimated linear regression function. From the figure, we have to reject the hypothesis that  $g$  is linear at 5% level as part of the dotted line lies outside the SCB. This again shows that  $(X_t)$  and  $(Y_t)$  are not linearly cointegrated

and the traditional linear model is not suitable for the data.

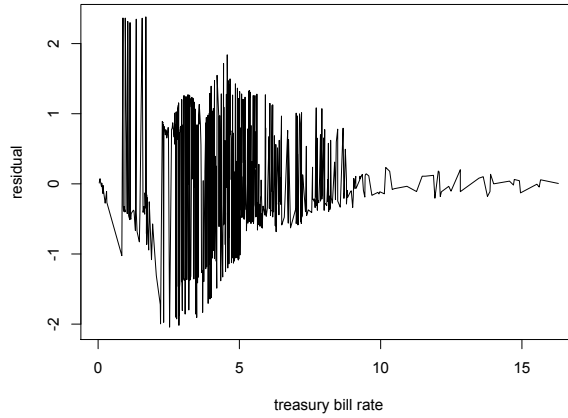


Figure 2: The residuals from the nonlinear regression model against the treasury bill rates.

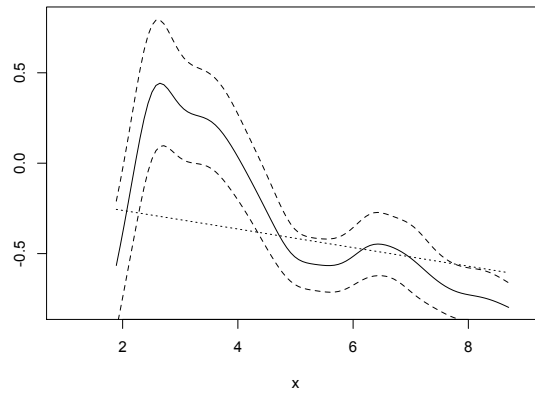


Figure 3: The SCB for the regression function  $g$ .

It is difficult to derive the local time limit theory for local linear estimation when  $(X_t)$  is multivariate  $I(1)$ , which limits the empirical applicability

of the proposed SCB construction. While Figure 3 suggests a nonlinear relationship between the adjusted share price series and the treasury bill series, the true relationship may also depend on some other macroeconomic variables such as the long-term bond yields (Chen *et al*, (2012)). To address the latter problem, we could extend the nonlinear regression model (4.3) to the nonlinear varying-coefficient models (c.f., Cai *et al*, (2009)) and include some other macroeconomic time series in the empirical analysis. This is beyond the scope of this paper.

## 5. Proofs of the main results

In this section, we give the proofs of Theorem 1 in Section 2 and Proposition 1 in Section 3. We prove Theorem 1 for  $B_n = \sqrt{n}/(\log^{c_0} n)$  with  $c_0$  being a positive constant, which is asymptotically wider than  $B_n = M_0 n^{1/2-\epsilon}$ . Some of the arguments used in this section and the supplemental material are similar to those in Bickel and Rosenblatt (1973) and Liu and Wu (2010). However, due to the presence of non-stationary series, our proofs are quite challenging.

**Proof of Theorem 1.** Let

$$\begin{aligned}\Gamma_{1n}(x) &= \frac{1}{S_n(x)V_{n2}(x)} \cdot \sum_{t=1}^n w_t(x)[g(X_t) - g(x)], \\ \Gamma_{2n}(x) &= \frac{V_{n1}(x)}{S_n(x)V_{n2}(x)} \cdot \sum_{t=1}^n K_1[(X_t - x)/h]e_t, \\ \Gamma_{3n}(x) &= \frac{1}{S_n(x)V_{n2}(x)} \cdot \sum_{t=1}^n w_t(x)[\sigma(X_t) - \sigma(x)]e_t/\sigma(x),\end{aligned}$$

where  $S_n^2(x) = \sum_{t=1}^n K^2[(X_t - x)/h]$ ,  $V_{nj}(\cdot)$ ,  $w_t(\cdot)$  and  $K_1(\cdot)$  are defined as in Section 1. Similarly to Liu and Wu (2010), we split the  $V_n(x)[\widehat{g}_n(x) - g(x)]/\sigma(x)$  as

$$\begin{aligned}& V_n(x)[\widehat{g}_n(x) - g(x)]/\sigma(x) \\ &= V_n(x) \left\{ \sum_{t=1}^n w_t(x)\sigma(X_t)e_t + \sum_{t=1}^n w_t(x)[g(X_t) - g(x)] \right\} / \left[ \sigma(x) \sum_{t=1}^n w_t(x) \right] \\ &= \frac{1}{S_n(x)} \cdot \sum_{t=1}^n K[(X_t - x)/h]e_t + \Gamma_{1n}(x)/\sigma(x) + \Gamma_{2n}(x) + \Gamma_{3n}(x).\end{aligned}\quad (5.1)$$

We apply truncation to deal with the first term on the right hand side of

(5.1). Let  $\mathcal{I}_n = [-h^{-1}B_n, h^{-1}B_n]$  and  $e'_t = \tilde{e}_t/(\mathbf{E}\tilde{e}_t^2)^{1/2}$ , where

$$\tilde{e}_t = e_t I\{|e_t| \leq \log n\} - \mathbf{E}[e_t I\{|e_t| \leq \log n\}], \quad 1 \leq t \leq n. \quad (5.2)$$

Take  $Z_t(x) = K(X_t/h - x)/S_n(xh)$  and

$$\begin{aligned}M_n(x) &= \sum_{t=1}^n Z_t(x)e_t, & M_n &= \sup_{x \in \mathcal{I}_n} |M_n(x)|; \\ \widetilde{M}_n(x) &= \sum_{t=1}^n Z_t(x)e'_t, & \widetilde{M}_n &= \sup_{x \in \mathcal{I}_n} |\widetilde{M}_n(x)|.\end{aligned}$$

The main idea is to show that  $\Gamma_{jn}(x)$ ,  $j = 1, 2, 3$ , are asymptotically dominated by  $M_n(x)$  uniformly over  $|x| \leq B_n$ , and  $M_n(x)$  is asymptotically equivalent to its truncated version  $\widetilde{M}_n(x)$ . Due to (5.1) and  $\inf_{x \in \mathbb{R}} \sigma(x) > 0$ , Theorem 1 follows from the three propositions. Their proofs are in the supplemental materials. □

**Proposition 2.** *Under the conditions of Theorem 1, we have*

$$\sup_{|x| \leq B_n} |\Gamma_{jn}(x)| = O_P(\log^{-2} n), \quad j = 1, 2, 3.$$

**Proposition 3.** *Under the conditions of Theorem 1, we have*

$$\sup_{x \in \mathcal{I}_n} |M_n(x) - \widetilde{M}_n(x)| = O_P(\log^{-2} n).$$

**Proposition 4.** *Under the conditions of Theorem 1, we have for any  $z \in \mathbb{R}$ ,*

$$P \left\{ (2 \log \bar{h}^{-1})^{1/2} (\widetilde{M}_n - d_n) \leq z \right\} \rightarrow e^{-2e^{-z}},$$

where  $d_n$  is defined as in Theorem 1.

**Proof of Proposition 1.** Due to the condition (C4)(ii) and  $K(s) = 0$  if

$$|s| \geq A,$$

$$\frac{|\sigma^i(X_t) - \sigma^i(x)|}{\sigma^i(x)} \cdot K[(X_t - x)/b] \leq CbK[(X_t - x)/b],$$

for  $i = 1$  and  $2$ , all  $x \in \mathbb{R}$ , and a sufficiently small  $b$ . Similarly, we have

$$|\widehat{g}_n(X_k) - g(X_k)|^i K[(X_k - x)/b] \leq CK[(X_k - x)/b] \cdot \sup_{|x| \leq 2B_n} |\widehat{g}_n(x) - g(x)|^i,$$

for  $i = 1$  and  $2$ ,  $|x| \leq B_n$ , and a sufficiently small  $b$ . Then, we have

$$\begin{aligned} \frac{|\widehat{\sigma}_n^2(x) - \sigma^2(x)|}{\sigma^2(x)} &\leq C \left[ b + \sigma^{-2}(x) \sup_{|z| \leq 2B_n} |\widehat{g}_n(z) - g(z)|^2 \right] + \\ &C \left| \frac{\sum_{t=1}^n K[(X_t - x)/b] (e_t^2 - 1)}{\sum_{t=1}^n K[(X_t - x)/b]} \right| + \\ &C \left[ \frac{\sum_{t=1}^n K[(X_t - x)/b] |e_t|}{\sigma(x) \sum_{t=1}^n K[(X_t - x)/b]} \cdot \sup_{|z| \leq 2B_n} |\widehat{g}_n(z) - g(z)| \right], \end{aligned}$$

for  $|x| \leq B_n$  and a sufficiently small  $b$ . By Theorem 4.1 of Chan and Wang (2014) and (C3)(iii), we have

$$\begin{aligned} \sup_{|x| \leq 2B_n} |\widehat{g}_n(x) - g(x)| &= O_P[(nh^2)^{-1/4} \log^{1/2} n + h^2 \sup_{|x| \leq 2B_n} |g_0(x)|] \\ &= O_P[(nh^2)^{-1/4} \log^2 n]. \end{aligned}$$

The above arguments, together with Lemma A.4 given in the supplemental document,  $b \asymp h$ , and  $\inf_{x \in \mathbb{R}} \sigma(x) > 0$ , lead to

$$\sup_{|x| \leq B_n} \frac{|\widehat{\sigma}_n^2(x) - \sigma^2(x)|}{\sigma^2(x)} = O_P[h + (nh^2)^{-1/4} \log^2 n].$$

Hence, we have

$$\sup_{|x| \leq B_n} \left| \frac{\sigma(x)}{\widehat{\sigma}(x)} - 1 \right| \leq \sup_{|x| \leq B_n} \frac{\sigma(x)}{\widehat{\sigma}(x)} \frac{|\widehat{\sigma}^2(x) - \sigma^2(x)|}{\sigma^2(x)} = O_P[h + (nh^2)^{-1/4} \log^2 n].$$

This completes the proof of Proposition 1.  $\square$

## Supplementary materials

The supplementary materials contain the proofs of Propositions 2–4, some technical lemmas with proofs, and the discussion on the asymptotic Gumbel distribution for the Nadaraya-Watson kernel estimator.

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