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Title	Sensitivity Analysis for Unmeasured Confounding in Coarse Structural Nested Mean Models
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where the second equality follows from (3.2).

$$
\begin{aligned}
& E[q(\bar{L}_m)q^{\text{opt}}(\bar{L}_l)^T H^a_{m,(\psi,\eta)} H^a_{l,(\psi,\eta)} 1_{\bar{A}_{m-1}=\bar{0}} \\
& \times\{A_m - Pr(A_m = 1 \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0})\}\{A_l - Pr(A_l = 1 \mid \bar{L}_l, \bar{A}_{l-1} = \bar{0})\}] \\
= \; & E[q(\bar{L}_m)q^{\text{opt}}(\bar{L}_l)^T E\{H^a_{m,(\psi,\eta)} H^a_{l,(\psi,\eta)} \mid \bar{L}_m, \bar{A}_m\} 1_{\bar{A}_{m-1}=\bar{0}} \\
& \times\{A_m - Pr(A_m = 1 \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0})\}\{A_l - Pr(A_l = 1 \mid \bar{L}_l, \bar{A}_{l-1} = \bar{0})\}] \\
= \; & E[q(\bar{L}_m)q^{\text{opt}}(\bar{L}_l)^T E\{H^a_{m,(\psi,\eta)} H^a_{l,(\psi,\eta)} \mid \bar{L}_m, \bar{A}_{m-1}\} 1_{\bar{A}_{m-1}=\bar{0}} \\
& \times\{A_m - Pr(A_m = 1 \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0})\}\{A_l - Pr(A_l = 1 \mid \bar{L}_l, \bar{A}_{l-1} = \bar{0})\}] \\
= \; & E[q(\bar{L}_m)q^{\text{opt}}(\bar{L}_l)^T E\{H^a_{m,(\psi,\eta)} H^a_{l,(\psi,\eta)} \mid \bar{L}_m, \bar{A}_{m-1}\} 1_{\bar{A}_{m-1}=\bar{0}} \\
& \times E[\{A_m - Pr(A_m = 1 \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0})\} 1_{\bar{A}_{m-1}=\bar{0}} \mid \bar{L}_m, \bar{A}_{m-1}] \\
& \times\{A_l - Pr(A_l = 1 \mid \bar{L}_l, \bar{A}_{l-1} = \bar{0})\}] = 0,
\end{aligned}
$$

where the third equality follows assuming $E\{H^a_{m,(\psi,\eta)} H^a_{l,(\psi,\eta)} \mid \bar{L}_m, \bar{A}_m\} = E\{H^a_{m,(\psi,\eta)} H^a_{l,(\psi,\eta)} \mid \bar{L}_m, \bar{A}_{m-1}\}$. Since (A4) equals (A4) for any q, the solution of q^{opt} is (3.6), proving Theorem 3.

Appendix E. Exploring the connection between the two approaches to sensitivity analysis

The approach of Schlesselman (1978) and Rosenbaum and Rubin (1983) can be used to motivate the specification of the selection bias function. As-

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sume that there is one unmeasured confounder U, and we have no unmeasured confounding if U is taken into account. For $0 \leq m \leq K$,

$$A_m \coprod Y^{(\infty)} \mid \bar{L}_m, \bar{U}_m, \bar{A}_{m-1},$$

which implies that

$$E(Y^{(\infty)} \mid \bar{L}_m, \bar{U}_m, \bar{A}_{m-1} = \bar{0}, A_m = 1) = E(Y^{(\infty)} \mid \bar{L}_m, \bar{U}_m, \bar{A}_m = \bar{0}).$$

To motivate the selection bias function $g(\bar{L}_m)$ due to the unmeasured confounder U, assuming that we have $E(Y^{(\infty)} \mid \bar{L}_m, \bar{U}_m, \bar{A}_{m-1} = \bar{0}) = \beta_0 + \beta_L^T L_m + \beta_U U_m + \psi(K-m)$, note that in this case,

$$\begin{aligned} g(\bar{L}_m) &= E(Y^{(\infty)} \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 1) - E(Y^{(\infty)} \mid \bar{L}_m, \bar{A}_m = \bar{0}) \\ &= E\{E(Y^{(\infty)} \mid \bar{L}_m, \bar{U}_m, \bar{A}_{m-1} = \bar{0}, A_m = 1) \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 1\} \\ &\quad - E\{E(Y^{(\infty)} \mid \bar{L}_m, \bar{U}_m, \bar{A}_{m-1} = \bar{0}, A_m = 0) \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 0\} \\ &= E\{E(Y^{(\infty)} \mid \bar{L}_m, \bar{U}_m, \bar{A}_{m-1} = \bar{0}) \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 1\} \\ &\quad - E\{E(Y^{(\infty)} \mid \bar{L}_m, \bar{U}_m, \bar{A}_{m-1} = \bar{0}) \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 0\} \\ &= \beta_U\{E(U_m \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 1) - E(U_m \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 0)\}. \end{aligned}$$

Considering U to be a variable like a measured confounder inspires the specification of $E(U_m \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 1) - E(U_m \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 0)$, and therefore of $g(\bar{L}_m)$. For example, in our application, it could be that $E(U_m \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 1) - E(U_m \mid \bar{L}_m, \bar{A}_{m-1} = \bar{0}, A_m = 0) = \alpha_0 + \alpha_1 CD4_m$, which corresponds to the third scenario for specification of $g(\bar{L}_m)$ in Section 5.

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