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Title	On the second-order inverse regression methods for a general type of elliptical predictors
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the array of eigenvectors following the same order. When equality exists in the eigenvalues, we allow arbitrariness in $\beta(M)$. We use $\hat{M}_n$, or simply $\hat{M}$, to denote the sample analog of M based on a random sample of size n. The $\sqrt{n}$-consistency of $\hat{M}$ is commonly satisfied in the literature, and is assumed throughout. We define $\{\lambda_1(\hat{M}), \ldots, \lambda_p(\hat{M})\}$ and $\beta(\hat{M})$ similarly.

By the ellipticity of X, $\mathrm{var}(\gamma^\mathsf{T} X|\beta^\mathsf{T} X)$ is a symmetric function with respect to the origin in $\mathbb{R}^d$. Therefore, it is natural to be generalized from a constant function to a quadratic function with the linear term being zero. That is, there exists $a_\gamma \in \mathbb{R}$ and $B_\gamma \in \mathbb{R}^{d\times d}$ such that

$$\mathrm{var}(\gamma^\mathsf{T} X|\beta^\mathsf{T} X) = a_\gamma + (X^\mathsf{T}\beta)B_\gamma(\beta^\mathsf{T} X). \tag{2.1}$$

We call this condition the quadratic variance condition and the induced distribution family the quadratic variance ellipticity family. Although adopting a parametric assumption, this family covers many commonly seen cases. In particular, it reduces to the multivariate normal distribution if B_γ in (2.1) is zero. We give two more examples.

Example 1. Let R be degenerate at 1, then X is uniformly distributed on the sphere centered at the origin and with radius $\sqrt{p}$. This distribution is considered in Luo, Wang, and Tsai (2009) for its advantage in not generating outliers. After simple calculation, we have,

$$\mathrm{var}(\gamma^\mathsf{T} X|\beta^\mathsf{T} X) = (p - \|\beta^\mathsf{T} X\|^2)/(p-d). \tag{2.2}$$

Example 2. Kotz (1975) introduced the p-dimensional Pearson Type II distribution, with density function with respect to the Lebesque measure

$$f(x) = [(2m+p+2)\pi]^{-p/2}\frac{\Gamma(p/2+m+1)}{\Gamma(m+2)}\left[1 - \frac{x^\mathsf{T} x}{2m+p+2}\right]^m$$

on $\{x \in \mathbb{R}^p : x^\mathsf{T} x \le 2m+p+2\}$ and zero elsewhere, $m > -1$ being the shape parameter. As discussed in Chapter 6 of Johnson (2013), this family is closely