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# COMPARISON OF EXTENDED EMPIRICAL LIKELIHOOD METHODS: SIZE AND SHAPE OF TEST-BASED CONFIDENCE REGIONS

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*Abstract:* Empirical likelihood is a general non-parametric inference methodology. It uses likelihood principle in a way that is analogous to that of parametric likelihoods. In a wide range of applications the methodology was shown to provide likelihood ratio statistics that have limiting chi-square distributions and yield a nonparametric version of Wilks theorem. Amongst recent extensions of empirical likelihood are the analysis of censored data, longitudinal data and semi-parametric regression models. Wilks theorem remains true in some, but not in others. This motivates our comparison of extended empirical likelihood methods. We evaluate their relative optimality by comparing the confidence regions provided by inverting the likelihood ratio tests. We show that those extension methods with the likelihood ratio statistic observing the Wilks theorem provide

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the smallest confidence regions. Specific examples are provided for the case of censored data analysis and estimating equations involving nuisance parameters.

*Key words and phrases:* Empirical Likelihood, Likelihood Ratio, Wilks Confidence Region, Scaled Chi Square Distribution.

## 1. Introduction

Empirical likelihood is a general non-parametric inference method using likelihood principle in a way that is analogous to that of parametric likelihoods. Since its introduction by Owen (1988, 1990), the methodology has been extended to the analysis of censored data (Li, Li and Zhou (2005)), longitudinal data (e.g., Wang, Qian and Carroll (2010)) and semi-parametric regression model (e.g., Shi and Lau (2000)). In a wide range of applications, as outlined in Owen (2001), empirical likelihood ratio statistics have been shown to have limiting chi-square distributions. This property remains true for some recent extended methods but not for others. We discuss the relative optimality of extended methods. Specific examples are described in Section 3.

We focus our attention on extensions of the empirical likelihood method that have the likelihood function maximized at the same location, thus yielding equivalent maximum empirical likelihood estimators, but provide different likelihood ratio statistics. We discuss the relative optimality of

the methods by comparing the confidence regions provided by inverting the likelihood ratio tests. We show that the extended empirical likelihood that yields the likelihood ratio statistic observing the Wilks theorem provides the smallest confidence region.

The remainder of this paper proceeds as follows. Section 2 formally defines a set-up where multiple empirical likelihood methods can be compared. The section also provides asymptotic results for the comparison of the corresponding confidence regions. Section 3 describes specific examples for the cases of censored data analysis and estimating equations involving nuisance parameters. Section 4 provides empirical results. All the proofs are deferred to the Appendix.

## **2. Comparison of Various Empirical Likelihoods via Its Confidence Regions**

Specifying extended empirical likelihood requires details. For example, various ways of constructing the likelihoods are available for censored data and different extended empirical likelihoods are specified. We defer presentation of specific examples including censored data case to section 3 and provide general results in this section.

Given a random sample  $\mathcal{Z}_n = \{Z_i\}_{i=1}^n$  with a common cumulative distribution function  $F_0$  and  $Z_1 \in R^k$ , we consider an (extended) em-

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empirical likelihood defined for some discrete cumulative distribution function  $F$  (denoted by  $EL(\mathcal{Z}_n, F)$ ). We subsequently define the maximizer  $F_n = \operatorname{argmax}_F \log EL(\mathcal{Z}_n, F)$ . The empirical likelihood gives rise to the empirical likelihood ratio test for a finite parameter  $\theta = T(F) \in R^p$  with the test statistic

$$-2 \log ELR_n(\theta) = -2 \left[ \max_{F \in \mathcal{F}} \log EL(\mathcal{Z}_n, F) - \log EL(\mathcal{Z}_n, F_n) \right],$$

where  $\mathcal{F}$  is a set of discrete cumulative distribution functions that satisfy the constraint  $T(F) = \theta$ . The true value of the parameter is given as  $\theta_0 = T(F_0)$ , and the maximum empirical likelihood estimator is  $\hat{\theta}_n = T(F_n)$ .

We invert the likelihood ratio test and obtain a confidence region  $C_n = \{ \theta \mid -2 \log ELR_n(\theta) \leq c \}$ , where the critical value  $c$  is chosen according to the (asymptotic) distribution of  $-2 \log ELR_n(\theta)$  under the null hypothesis for some desired confidence level. Under some regularity conditions the asymptotic distribution under the null hypothesis is

$$n^{1/2}(\hat{\theta}_n - \theta_0) \rightarrow U \tag{1}$$

$$-2 \log ELR_n(\theta_0) = n(\hat{\theta}_n - \theta_0)^\top V^{-1}(\hat{\theta}_n - \theta_0) + o_p(1),$$

where  $U$  is a  $p$ -variate random variable with  $U \sim N_p(0, W)$ , and  $W$  and  $V$  are  $p \times p$  (semi-)positive definite matrices. Without loss of generality we consider two methods that provide the same maximum empirical likelihood

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estimators,  $\hat{\theta}_{1n} = \hat{\theta}_{2n}$ , but different likelihood ratio statistics  $ELR_{1n}(\theta)$  and  $ELR_{2n}(\theta)$ . We suppose that the asymptotic results in (1) hold with  $V = V_1$  and  $V = V_2$ , and  $WV_1^{-1} = I$  and  $WV_2^{-1} \neq I$  for  $ELR_{1n}(\theta)$  and  $ELR_{2n}(\theta)$ , respectively. By the results of Hjort, McKeague and Keilegom (2009) and Qin and Lawless (1994), the empirical likelihood with  $V = V_1$  admits a chi-square with  $p$  degrees of freedom as the limiting distribution of  $-2\log ELR_{1n}(\theta)$ , whereas the empirical likelihood with  $V = V_2$  admits a limiting distribution characterized by weighted sum of  $p$  independent chi-squares of degree of freedom 1. This distribution is often called “the scaled chi square distribution”, especially in the case of  $p = 1$  (Wang and Jing (2001), Wang and Li (2002), Ren (2008)).

We consider the confidence regions with the two methods at the same confidence level as

$$C_{1n} = \{\theta \mid -2\log ELR_{1n}(\theta) \leq c_1\}, \quad C_{2n} = \{\theta \mid -2\log ELR_{2n}(\theta) \leq c_2\},$$

where the critical values  $c_1$  and  $c_2$  are appropriately chosen by the limiting distributions.  $C_{1n}$  and  $C_{2n}$  are centered at the same value in the sense that  $-2\log ELR_{1n}(\hat{\theta}_n) = -2\log ELR_{2n}(\hat{\theta}_n) = 0$  where  $\hat{\theta}_n$  denotes the common maximum empirical likelihood estimator.

**Theorem 1.** *Assume regularity conditions under which the results in (1) holds for  $ELR_{1n}(\theta)$  and  $ELR_{2n}(\theta)$ . Given a confidence level  $1 - \alpha_n \in (0, 1)$*

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with  $\alpha_n \rightarrow 0$ , there exist  $N$  such that for  $n > N$  we have

$$\text{Vol}(C_{1n})/\text{Vol}(C_{2n}) < 1,$$

where  $\text{Vol}(C)$  for any measurable set  $C$  in  $R^p$  is  $\text{Vol}(C) = \int_{R^p} I_{[t \in C]} \mu(dt)$

with  $\mu$  the Lebesgue measure in  $R^p$ .

Theorem 1 implies that the empirical likelihood method that provides the likelihood ratio statistic which is only incompletely self-studentizing is not optimal.

**Remark 1.** In several papers (e.g., Wang and Jing (2001), Wang and Li (2002)), it is suggested that one can *adjust* the empirical likelihood ratio of type  $-2 \log ELR_{2n}$  by multiplying by a factor that turns the asymptotic null distribution into a regular chi square, and may even improve the performance of the resulting confidence regions. While the adjustment may improve the level of confidence of the resulting confidence region, it is clear that if the multiplying factor is (asymptotically) a constant and does not depend on  $\theta$ , then the shape and orientation of the resulting confidence region is the same as before. In later examples, we set the confidence level by simulating the null distribution, eliminating the need for the adjustment. To achieve optimality, the multiplying factor must depend on  $\theta$  and be  $\log ELR_{1n}(\theta)/\log ELR_{2n}(\theta) = (\hat{\theta} - \theta_0)^\top V_1^{-1}(\hat{\theta} - \theta_0)/(\hat{\theta} - \theta_0)^\top V_2^{-1}(\hat{\theta} -$

$\theta_0) + o_p(1)$ . Computing this factor is equivalent to computing the statistic  $-2 \log ELR_{1n}$ , if not more difficult.

### 3. Extended Empirical Likelihoods

In this section we present concrete examples of extended methods that provide likelihood ratio statistics that do not observe the non-parametric version of the Wilks theorem. In each of these examples, the construction of the likelihood does not match the structure of data and hence ill-constructed likelihoods provide the likelihood ratio statistics that are only incompletely self-studentizing. For contrast, we provide a properly defined empirical likelihood that does observe the Wilks theorem.

#### 3.1. Analysis of Estimating Equations Involving Nuisance Parameters

We suppose that the unknown parameters  $\theta \in R^p$  is related to the common cumulative distribution function  $F_0$  via a  $p$ -variate vector of estimating functions involving finite-dimensional nuisance parameters  $\psi \in R^q$ ,  $m(\cdot, \theta, \psi)$ , so  $E\{m(Z_1, \theta_0, \psi_0)\} = 0$  for the true values  $\theta_0$  and  $\psi_0$ . We also suppose a  $q$ -variate vector of estimating functions  $h(\cdot, \theta, \psi)$  exists that defines the true parameter values  $\theta_0$  and  $\psi_0$  uniquely as a solution to the estimating equations,  $E\{m(Z_1, \theta_0, \psi_0)\} = 0$  and  $E\{h(Z_1, \theta_0, \psi_0)\} = 0$ . We

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consider the following empirical likelihood jointly for  $\theta$  and  $\psi$ ;

$$EL_n(\theta, \psi) = \max \left\{ \prod_{i=1}^n w_i \left| \sum_{i=1}^n w_i \begin{pmatrix} m(Z_i, \theta, \psi) \\ h(Z_i, \theta, \psi) \end{pmatrix} = 0, \sum_{i=1}^n w_i = 1, 0 \leq w_i \right. \right\},$$

where  $w_i$  denotes the point mass assigned to the  $i$ -th observation  $Z_i$ . Without the constraints this likelihood is maximized by  $w_i = n^{-1}$ ,  $i = 1, \dots, n$ , and the maximum empirical likelihood estimator  $(\hat{\theta}_n, \hat{\psi}_n)$  is given by the solution to the equations  $n^{-1} \sum_{i=1}^n m(Z_i, \theta, \psi) = 0$  and  $n^{-1} \sum_{i=1}^n h(Z_i, \theta, \psi) = 0$ .

Two methods have been proposed for the inference of  $\theta$  alone. Qin and Lawless (1994) used the profile likelihood  $EL_{1n}(\theta) = \max_{\psi} EL_n(\theta, \psi)$ , while others used the so-called plug-in method (Hjort, McKeague and Keilegom (2009)) with the likelihood

$$EL_{2n}(\theta) = \max \left\{ \prod_{i=1}^n w_i \left| \sum_{i=1}^n w_i m(Z_i, \theta, \tilde{\psi}) = 0, \sum_{i=1}^n w_i = 1, 0 \leq w_i \right. \right\},$$

where  $\tilde{\psi}$  is a  $\sqrt{n}$ -consistent estimator, so that  $E\{m(Z_i, \theta_0, \tilde{\psi})\} = 0$ . Without externally given  $\tilde{\psi}$  available, a common choice is the maximum empirical likelihood estimator  $\hat{\psi}_n$ . With this choice, the plug-in empirical likelihood has  $\hat{\theta}_n$  as the maximum empirical likelihood estimator and  $\max_{\theta} EL_{2n}(\theta) = \prod_{i=1}^n n^{-1}$ , as does the profile likelihood method. The likelihood ratio statis-

tics are given by

$$ELR_{1n}(\theta) = EL_{1n}(\theta) / \prod_{i=1}^n n^{-1}, \quad ELR_{2n}(\theta) = EL_{2n}(\theta) / \prod_{i=1}^n n^{-1}. \quad (2)$$

**Theorem 2.** *Suppose*

$$n^{1/2} \begin{pmatrix} \hat{\theta}_n - \theta_0 \\ \hat{\psi}_n - \psi_0 \end{pmatrix} \rightarrow N_p(0, \Sigma),$$

*in distribution, where  $\Sigma = \{S_{12}S_{11}^{-1}S_{21}\}^{-1}$  with  $S_{12} = S_{21}^\top = E \begin{pmatrix} \partial m / \partial \theta & \partial m / \partial \psi \\ \partial h / \partial \theta & \partial h / \partial \psi \end{pmatrix}$ ,*

$$S_{11} = E \begin{pmatrix} mm^\top & mh^\top \\ hm^\top & hh^\top \end{pmatrix}, \text{ and the expectations taken at } \theta = \theta_0 \text{ and } \psi = \psi_0.$$

*Also suppose that for any  $\tilde{\psi}$  with  $\|\tilde{\psi} - \psi_0\| = O_p(n^{-1/2})$ ,  $n^{-1} \sum_{i=1}^n m(Z_i, \theta_0, \tilde{\psi})m^\top(Z_i, \theta_0, \tilde{\psi})$*

*$\rightarrow E(mm^\top)$ . If  $\Sigma_{(jk)}$ ,  $j, k = 1, 2$ , denote the blocks of  $\Sigma$  that correspond to*

*$\theta$  and  $\psi$ , then, (1) holds for both  $-2 \log ELR_{1n}(\theta_0)$  and  $-2 \log ELR_{2n}(\theta_0)$*

*with*

$$U \sim N_p(0, \Sigma_{(11)}), \quad W = V_1 = \Sigma_{(11)}, \quad V_2 = [E(\partial m / \partial \theta)^\top \{E(mm^\top)\}^{-1} E(\partial m / \partial \theta)]^{-1}.$$

*Furthermore,  $-2 \log ELR_{2n}(\theta_0) \rightarrow \sum_{j=1}^p c_j \xi_j$  in distribution, where  $\xi_j$  are*

*independent chi-squared distributed random variables with degree of freedom*

*1 and  $c_1, \dots, c_p$  are the eigenvalues of  $V_2 W^{-1}$ .*

The weighted  $\chi^2$  asymptotic results stems from the ill-construction of the likelihood: the likelihood of  $EL_{2n}(\theta)$  takes the same form as that of

Owen's likelihood for the case of independent data, whereas  $m(Z_i, \theta, \hat{\psi}_n)$ ,  $i = 1, \dots, n$ , are not independent as they involve the same  $\hat{\psi}_n$ .

### 3.2. Censored Data Analysis

We suppose that  $\{Z_i\}_{i=1}^n = \{(T_i, \delta_i)\}_{i=1}^n$  where  $(T_i, \delta_i)$  denote right censored observations from lifetime variables  $\{Y_i\}$  with a common cumulative distribution function  $F_0$  subject to the random censoring:

$$T_i = \min(Y_i, C_i) \quad \text{and} \quad \delta_i = I_{[Y_i \leq C_i]} \quad \text{for } i = 1, \dots, n.$$

The censoring variable  $C_i$  are assumed independent of the  $Y_i$  with a cumulative distribution function  $G_0$ . We suppose the true value  $\theta_0$  of an unknown parameter  $\theta \in R^p$  is related to  $F_0$  via estimating equations such that

$$E\{g(Y_i)\} = \theta_0,$$

where  $g = (g_1, \dots, g_p)$  are  $p$  functions. We examine three empirical likelihood methods that have been proposed for the inference of  $\theta$ .

**Censored Likelihood Method:** The first method uses the empirical likelihood defined for the censored data. Kaplan and Meier (1958), Owen (2001), and others have defined the empirical likelihood of the censored data  $\{Z_i\}_{i=1}^n$  for some cumulative distribution function  $F$  as

$$EL(\mathcal{Z}_n, F) = \prod_{i=1}^n \{\Delta F(T_i)\}^{\delta_i} \left\{ \sum_{j: T_j > T_i} \Delta F(T_j) \right\}^{1-\delta_i}$$

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where  $\Delta F(t) = F(t+) - F(t-)$  is the jump of  $F$  at  $t$  or a probability mass assigned to the point  $t$ . As in the case of uncensored case, this *assumes* a discrete  $F(\cdot)$  with possible jumps only at the observed times. With  $w_i = \Delta F(T_i)$ ,  $i = 1, \dots, n$ , this likelihood can be written in term of the jumps and the log likelihood is

$$\log EL_{1n}(F) = \sum_{i=1}^n \left\{ \delta_i \log w_i + (1 - \delta_i) \log \sum_{j=1}^n w_j I_{[T_j > T_i]} \right\}, \quad (3)$$

where  $I_{[\cdot]}$  denotes an indicator function. The likelihood is maximized at the Kaplan-Meier estimator (Kaplan and Meier (1958)) with  $w_i = \Delta \hat{F}_{KM}(T_i)$ ,  $i = 1, \dots, n$ . Using the empirical likelihood of the censored data in (3), Zhou (2011) proposed the empirical likelihood for  $\theta$  as

$$EL_{1n}(\theta) = \max_{w_i} \left\{ \prod_{i=1}^n w_i^{\delta_i} \left( \sum_{j=1}^n w_j I_{[T_j > T_i]} \right)^{1-\delta_i} \left| \sum_{i=1}^n w_i g(T_i) = \theta, \sum_{i=1}^n w_i = 1, 0 \leq w_i \right. \right\}.$$

The maximum empirical likelihood estimator  $\hat{\theta}_n$  is given by

$$\sum_{i=1}^n g(T_i) \Delta \hat{F}_{KM}(T_i) = \hat{\theta}_n. \quad (4)$$

The empirical likelihood ratio statistic is, accordingly, given as

$$-2 \log ELR_{1n}(\theta) = -2 \log \{ EL_{1n}(\theta) / EL_{1n}(\hat{\theta}_n) \}.$$

**Synthetic data and uncensored empirical likelihood method:** We have  $\theta$  defined as  $Eg(Y_i) = \theta$ . The following lemma is easily proved and motivates this method.

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**Lemma 1.** For any function  $g(\cdot)$  such that  $Eg(Y_i)$  is well defined, if  $X_i = \{g(T_i)\delta_i\}/\{1 - G_0(T_i)\}$ , where  $G_0(\cdot)$  is the cumulative distribution function of censoring variable  $C_i$ , then,  $Eg(Y_i) = E(X_i)$ .

If  $G_0$  were known,  $\{X_i\}_{i=1}^n$  or the synthetic data would be completely observed and the following empirical likelihood could be considered:

$$EL_{2n}(\theta) = \max_{w_i} \left\{ \prod_{i=1}^n w_i \mid \sum_{i=1}^n w_i X_i = \theta, \sum_{i=1}^n w_i = 1, 0 \leq w_i \right\}.$$

The first and second methods utilize different likelihoods.  $EL_{1n}(\theta)$  is based on the likelihood of censored data  $\{(T_i, \delta_i)\}_{i=1}^n$  with  $w_i$  denoting the point mass of a discrete cumulative function of  $Y_i$ , while  $EL_{2n}(\theta)$  is based on the likelihood of completely observed  $\{X_i\}_{i=1}^n$  with  $w_i$  denoting the point mass of a cumulative distribution function of  $X_i$ , assuming the knowledge of the censoring time distribution  $G_0$ .

Usually  $G_0(t)$  is not known, and the  $X_i$  are not available. An alternative method based on the synthetic data uses an estimator  $\hat{G}(t)$ , often the Kaplan-Meier estimator  $\hat{G}_{KM}$ , and defines the likelihood as

$$EL_{2n}^*(\theta) = \max_{w_i} \left\{ \prod_{i=1}^n w_i \mid \sum_{i=1}^n w_i X_i^* = \theta, \sum_{i=1}^n w_i = 1, 0 \leq w_i \right\},$$

where  $X_i^* = \{g(T_i)\delta_i\}/\{1 - \hat{G}_{KM}(T_i)\}$ . Here, however, the  $X_i^*$  are not independent.

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Since  $\hat{G}_{KM}(t)$  is uniformly consistent for  $G_0(t)$ , the mean of the sequence  $\{X_i^*\}$  is still asymptotically same as  $Eg(Y)$ , but the variance is different from the variance of  $X_i$ . Under some regularity conditions, we can show that, as  $n \rightarrow \infty$ ,  $\sqrt{n}(\bar{X} - \theta_0) \rightarrow N(0, \sigma_1^2)$  and  $\sqrt{n}(\bar{X}^* - \theta_0) \rightarrow N(0, \sigma_2^2)$  in distribution with  $\sigma_1^2 > \sigma_2^2$ , for  $g(\cdot)$ , a scalar function ( $p = 1$ ). This result similarly holds with  $p > 1$ . We refer to Srinivasan and Zhou (1994) for the calculation of the asymptotic variances or variance-covariance matrix. Like the plug-in method, the second synthetic data method uses the empirical likelihood method developed for iid sequences, but applies it to the sequence  $\{X_i^*\}$  that is not iid.

If we use the Kaplan-Meier estimator  $\hat{G}_{KM}(t)$  in this case, the empirical likelihood  $EL_{2n}^*(\theta)$  is maximized when  $\theta$  takes the value  $\hat{\theta}_n$  as (4), since  $\Delta\hat{F}_{KM}(T_i) = \delta_i / [n\{1 - \hat{G}_{KM}(T_i)\}]$ . Therefore, even though the likelihoods  $EL_{1n}(\theta)$  and  $EL_{2n}^*(\theta)$  look different, they are maximized at the same  $F$ , and thus the confidence regions obtained by inverting the tests, are centered at the same point  $\hat{\theta}_n$ . Accordingly the likelihood ratio statistic is given as

$$-2 \log ELR_{2n}^*(\theta) = -2 \log \{EL_{2n}^*(\theta) / EL_{2n}^*(\hat{\theta}_n)\}.$$

This method is also a plug-in method as we substitute  $\hat{G}_{KM}(t)$  for  $G_0(t)$ . It differs from the plug-in method described in Section 3.1, however,

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in the sense that the problem has been switched from original data to synthetic data before plug-in, and the parameter that got plugged is infinite dimensional ( $G_0(t)$ ).

**Weighted empirical likelihood:** This third method was proposed by Ren (2001, 2008). Given *fixed* weights  $v_i \geq 0$ , a weighted empirical likelihood is defined

$$EL_{3n}(w_1, \dots, w_n) = \prod_{i=1}^n (w_i)^{v_i}, \quad \sum_i w_i = 1.$$

It is not hard to show that  $EL_{3n}$  is maximized when  $w_i = v_i / \sum_j v_j$ . With the particular choice of  $v_i = \Delta \hat{F}_{KM}(T_i)$  the maximizer of  $EL_{3n}$  is  $w_i^* = \Delta \hat{F}_{KM}(T_i)$ , the jump of Kaplan-Meier. Thus, with this choice of weights, the confidence regions obtained by inverting the likelihood ratio test below are also centered at the same value  $\hat{\theta}_n$  in (4).

The weighted empirical likelihood under the null hypothesis can be computed by solving the constrained maximization problem; given  $v_i = \Delta \hat{F}_{KM}(T_i)$ ,

$$EL_{3n}(\theta) = \max_{w_i} \left\{ \prod_{i=1}^n (w_i)^{v_i} \mid \sum_{i=1}^n w_i \delta_i g(T_i) = \theta, \sum_{i=1}^n w_i = 1, 0 \leq w_i \right\}.$$

The likelihood ratio statistic is given as  $-2 \log ELR_{3n}(\theta) = -2 \log \{EL_{3n}(\theta) / EL_{3n}(\hat{\theta}_n)\}$ .

Here observations  $i$  and  $j$  are supposed to be independent, from which the

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product form of the empirical likelihood arises. The weights  $v_i$  are supposed to be fixed and non-random, but with  $v_i = \Delta \hat{F}_{KM}(T_i)$ , this does not hold.

**Theorem 3.** *Assume conditions under which the Kaplan-Meier integral is consistent and asymptotically normal. Then,*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow U, \quad (5)$$

in distribution, where  $U \sim N(0, \Sigma)$ . Furthermore,

$$-2 \log ELR_{1n}(\theta_0) = n(\hat{\theta}_n - \theta_0)^\top V_{1n}^{-1}(\hat{\theta}_n - \theta_0) + o_p(1), \quad (6)$$

$$-2 \log ELR_{2n}^*(\theta_0) = n(\hat{\theta}_n - \theta_0)^\top V_{2n}^{-1}(\hat{\theta}_n - \theta_0) + o_p(1), \quad (7)$$

$$-2 \log ELR_{3n}(\theta_0) = n(\hat{\theta}_n - \theta_0)^\top V_{3n}^{-1}(\hat{\theta}_n - \theta_0) + o_p(1), \quad (8)$$

where  $\lim_{n \rightarrow \infty} V_{1n} = \Sigma$ ,  $\lim_{n \rightarrow \infty} V_{2n} \neq \Sigma$  and  $\lim_{n \rightarrow \infty} V_{3n} \neq \Sigma$ .

It follows from Theorem 3 and Hjort, McKeague and Keilegom (2009) that  $-2 \log ELR_{1n}(\theta_0) \rightarrow \chi_p^2$  in distribution, whereas the Wilk's theorem does not hold for  $-2 \log ELR_{2n}^*(\theta_0)$  and  $-2 \log ELR_{3n}(\theta_0)$ . Theorems 1 and 3 together show that the confidence region provided by inverting the test  $-2 \log ELR_{1n}(\theta_0)$  by the censored empirical likelihood method is asymptotically optimal and, in particular, better than those confidence regions obtained from  $ELR_{2n}^*$  and  $ELR_{3n}$ .

#### 4. Empirical Studies

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We illustrate the relative optimality of confidence regions using two simulation studies. Confidence regions calibrated to yield same coverage probabilities were obtained by inverting these likelihood ratio tests.

In each case the extended likelihood methods which yield the likelihood ratio statistics that do not observe the Wilk's theorem are computationally challenging since the asymptotic distributions need to be estimated. For example, Theorem 2 implies that the extended likelihood method using  $ELR_{2n}(\theta)$  requires estimating the scale parameters  $c_1, \dots, c_p$  or the eigenvalues of  $V_2 W^{-1}$ . In applications the bootstrap may be used to calibrate coverage probabilities. In our simulations we used 10,000 Monte Carlo samples to calibrate the associated confidence regions to have 90% coverage probabilities. The confidence regions obtained by inverting the likelihood ratio statistics that observe the Wilk's theorem do not need numerical calibration and were constructed based on the 90-th percentiles of the corresponding chi-square limiting distributions. The comparisons provide empirical confirmation of Theorem 1.

#### 4.1. Simulation Studies 1

We used a regression model with heteroscedastic errors, and applied the profile likelihood and the plug-in method described in Section 3.1 for

the inference of two slope parameters:

$$y_i = \alpha + \beta_1 x_{1i} + \beta_2 x_{2i} + e_i x_{2i},$$

where  $x_{1i} \sim N(0, 1)$ ,  $x_{2i} \sim \chi_1^2$  and  $e_i \sim N(0, 1)$ . Here the parameters of interest was  $\theta = (\beta_1, \beta_2)$  with the true value  $\theta_0 = (1, 0.5)$  and the intercept was the nuisance parameter ( $\psi = \alpha$ ). Consider  $Z_i = (Z_{1i}, Z_{2i})$  with  $Z_{1i} = y_i$  and  $Z_{2i} = (x_{1i}, x_{2i})$ . The estimating functions  $m(Z_i, \theta, \psi)$  and  $h(Z_i, \theta, \psi)$  are given, respectively, as  $m(Z_i, \theta, \psi) = (Z_{1i} - \theta^\top Z_{2i})Z_{2i}$  and  $h(Z_i, \theta, \psi) = (Z_{1i} - \theta^\top Z_{2i})$ . Figure 1 shows empirically estimated null distributions of the profile likelihood ratio statistic  $-2 \log ELR_{1n}(\theta_0)$  and the plug-in likelihood ratio statistic  $-2 \log ELR_{2n}(\theta_0)$  in (2) based on 5,000 Monte Carlo samples. The Q-Q plots show that the null distribution of the plug-in likelihood ratio statistic remains apart from a chi-squared distribution with degrees of freedom 2, even in a large sample case.

Figure 2 shows an example of confidence regions. The confidence regions provided by the plug-in empirical likelihood method are larger relative to those by the profile likelihood, and shaped differently.

## 4.2. Simulation Studies 2

We used censored data  $T_i$  and  $\delta_i$  from the censored data setting with lifetime  $Y_i$  and censoring times  $C_i$ ;  $Y_i \sim \exp(1)$  ,  $C_i \sim 0.1 + \exp(0.6)$  .

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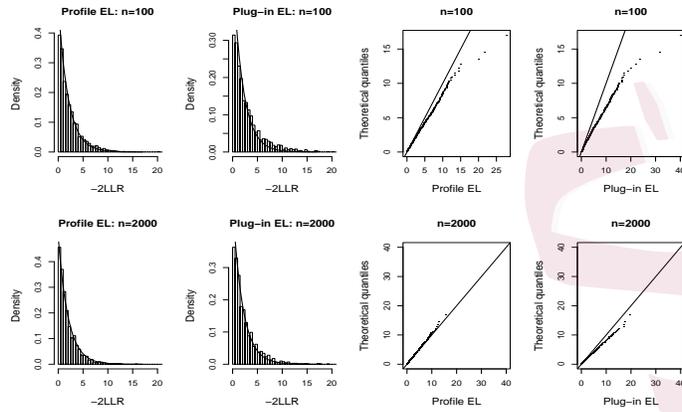


Figure 1: Empirically estimated null distributions of the profile and the plug-in likelihood ratio statistic for the case of estimating equations with nuisance parameters

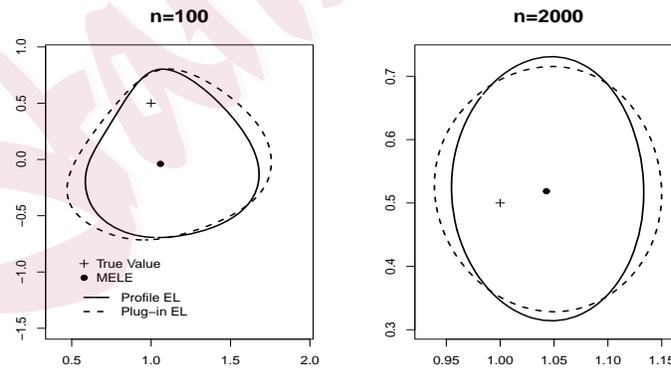


Figure 2: Confidence regions drawn for a sample of the case of estimating equations with nuisance parameters

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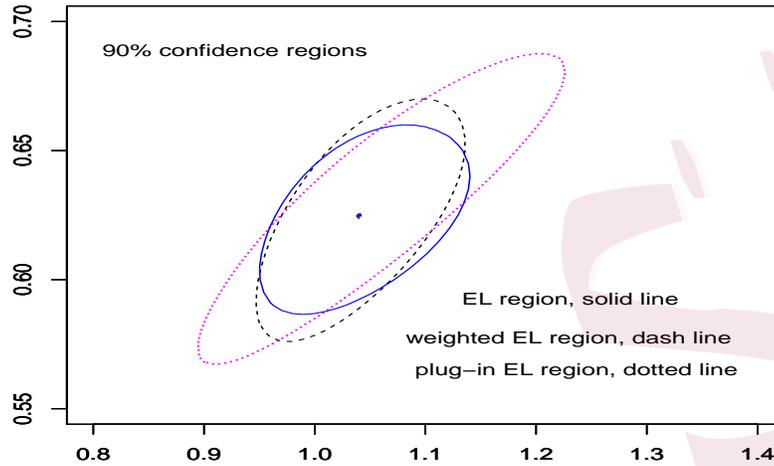


Figure 3: Confidence regions drawn for a sample data of  $n = 1,000$  subject to 37% censoring

The two expectations tested  $EY_i = a_1$ , and  $EI_{[Y_i > 0.5]} = a_2$ , where the parameter of interest was  $\theta = (a_1, a_2)$  with the true value  $\theta_0 = (1, e^{-0.5})$ .

Figure 3 shows three confidence regions drawn for a sample data with the sample size  $n = 1,000$  subject to 37% censoring. As implied by Theorem 3, the orientation and shape of the confidence regions by the plug-in (using  $-2 \log ELR_{2n}^*(\theta_0)$ ) and the weighted empirical likelihood methods (using  $-2 \log ELR_{3n}(\theta_0)$ ) are different from the confidence region with the censored empirical likelihood method ( $-2 \log ELR_{1n}(\theta_0)$ ). Also, the confidence region by the censored empirical likelihood method is the one with

the smallest area.

## 5. Conclusion

In this paper we consider several extensions of empirical likelihood method. Limiting to the extensions that provide the same maximum empirical likelihood estimators, we evaluate relative optimality of the methods by comparing the confidence regions obtained by inverting the likelihood ratio tests. We show that the extension method which yields the likelihood ratio statistic observing the Wilks theorem provides the smallest confidence region.

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## 6. Appendix

*Proof of Theorem 1* Consider

$$C_{1n}^* = \{ \theta | n(\hat{\theta} - \theta)^\top V_1^{-1}(\hat{\theta} - \theta) \leq c_1 \}, \quad C_{2n}^* = \{ \theta | n(\hat{\theta} - \theta)^\top V_2^{-1}(\hat{\theta} - \theta) \leq c_2 \},$$

where  $n^{1/2}(\hat{\theta} - \theta) \rightarrow N_p(u, W)$  for  $\theta = \theta_0 + n^{-1/2}u$  with a  $p$ - variate constant vector  $u$ . It follows from (1) that  $Vol(C_{1n}^*) = Vol(C_{1n}) + \epsilon_{1n}$  and  $Vol(C_{2n}^*) = Vol(C_{2n}) + \epsilon_{2n}$  where  $\epsilon_{1n}, \epsilon_{2n} \rightarrow 0$ . WLOG, we take the  $\alpha_n$  such that  $\Phi^{-1}(1 - \alpha_n) = O(\sqrt{n})$ , so the limiting volumes do not vanish as  $n \rightarrow \infty$ .

By Lemma 2, as  $n \rightarrow \infty$ ,  $Vol(C_{1n}^*)/Vol(C_{2n}^*) \approx Vol(C_{1n}^*)/Vol(C_{1n}^*) < 1$

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unless  $V_2 = W$ .

**Remark 2.** If we let  $\alpha_n = \alpha$  (fixed), both confidence regions would have volumes shrink to zero as  $n$  grows, and hence the relative optimality is in the asymptotic sense of the confidence level increasing with  $n$ .

**Lemma 2.** (i) Consider the confidence regions based on an estimator  $\hat{\theta}$  that has distribution  $N(\theta_0, I)$ :  $C_1 = \{\theta : (\hat{\theta} - \theta)^\top (\hat{\theta} - \theta) < c\}$ ,  $C_2 = \{\theta : (\hat{\theta} - \theta)^\top B(\hat{\theta} - \theta) < c^*\}$ , where  $B \neq I$  is a non-negative definite matrix. If both confidence region have the coverage probability  $1 - \alpha \in (0, 1)$ ,  $C_1$  is superior to  $C_2$  in the sense that  $Vol(C_1) < Vol(C_2)$ .

(ii) If  $\hat{\theta}$  is normally distributed with  $N(\theta_0, \Sigma)$ , then the optimal confidence region for the parameter  $\theta_0$  is  $\{\theta : (\hat{\theta} - \theta)^\top \Sigma^{-1}(\hat{\theta} - \theta) < c\}$ .

**Remark 3.** If  $B = \sigma^2 I$ , then the two confidence regions in Lemma 2 (i) are identical. The constant  $\sigma^2$  cancels out when adjusts the cut-off level  $c^*$ . Lemma @ should be understood to use any non-negative definite matrix other than  $\sigma^2 I$ .

**Remark 4.** If confidence regions are allowed not to center at  $\hat{\theta}$  then there may be better regions available (Stein phenomena) than those in Lemma 2. We only consider a limited class of confidence regions that are centered at the same  $\hat{\theta}$  since they are derived from extended empirical likelihood ratio

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tests.

*Proof of Lemma 2* As (ii) is an easy consequence of (i), proving (i) suffices. The coverage probability of  $C_1$  is  $\int_{\|\theta_0-x\|_2 < c} f(x)dx$ , where  $f(x)$  denotes the density of  $\hat{\theta}$ . Define a distance by  $d_B(\theta_0, \hat{\theta}) = (\theta_0 - \hat{\theta})^\top B(\theta_0 - \hat{\theta})$ . The coverage probability of  $C_2$  is  $\int_{d_B(\theta_0-x) < c^*} f(x)dx$ . By substituting  $y = \hat{\theta} - \theta_0$  with  $y \sim N(0, I)$ , the two integrals become

$$\int_{\|y\|_2 < c} f(y)dy \quad \text{and} \quad \int_{d_B(y) < c^*} f(y)dy .$$

Let the regions of the two integrations be  $R_1 = \{y : \|y\|_2 < c\}$  and  $R_2 = \{y : d_B(y) < c^*\}$ . Since the normal density  $f(y)$  is a strict monotone decreasing function of  $\|y\|$ , the requirement of equal coverage probability

$$\int_{R_1} f(y)dy = \int_{R_2} f(y)dy \tag{9}$$

forces  $Vol(R_2) > Vol(R_1)$ . Thus,  $R_1$  is the region with highest  $f(y)$  values that integrates to a given  $\alpha$  value, and any other region that integrates to the same  $\alpha$  value includes regions with some lower  $f(y)$  values, thus a larger  $Vol(R_2)$ . It is clear that the inequality is strict unless  $R_2$  coincide with  $R_1$ .

*Proof of Theorem 2* Let  $Q_n = (n^{-1} \sum_{i=1}^n m(Z_i, \theta_0, \psi_0), n^{-1} \sum_{i=1}^n h(Z_i, \theta_0, \psi_0))$ .

From the proofs of Theorem 1 and Corollary 5 of Qin and Lawless (1994)

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we have

$$\begin{aligned} n^{1/2}(\hat{\theta}_n - \theta_0) &= \begin{pmatrix} \Sigma_{(11)} & \Sigma_{(12)} \end{pmatrix} S_{21} S_{11}^{-1} (n^{1/2} Q_n) + o_p(1), \\ n^{1/2}(\hat{\psi}_n - \psi_0) &= \begin{pmatrix} \Sigma_{(21)} & \Sigma_{(22)} \end{pmatrix} S_{21} S_{11}^{-1} (n^{1/2} Q_n) + o_p(1). \end{aligned} \quad (10)$$

where  $S_{21(2)} = S_{12(2)}^\top = E \{ (\partial m / \partial \psi)^\top, (\partial h / \partial \psi)^\top \}$ . After some matrix manipulation we have

$$-2 \log ELR_{1n}(\theta_0) = (n^{1/2} Q_n)^\top S_{11}^{-1} S_{12} \begin{pmatrix} \Sigma_{(11)} \\ \Sigma_{(21)} \end{pmatrix} (\Sigma_{(11)})^{-1} [\Sigma_{(11)}, \Sigma_{(12)}] S_{21} S_{11}^{-1} (n^{1/2} Q_n).$$

On the other hand, by the standard arguments of the empirical likelihood method,

$$-2 \log ELR_{2n}(\theta_0) = \tilde{\lambda}^\top \{E(mm^\top)\} \tilde{\lambda} - \hat{\lambda}^\top \{E(mm^\top)\} \hat{\lambda} + o_p(1), \quad (11)$$

where  $\hat{\lambda}$  and  $\tilde{\lambda}$  are defined by solutions to the equations

$$0 = n^{-1} \sum_{i=1}^n \frac{m(Z_i, \hat{\theta}_n, \hat{\psi}_n)}{1 + \hat{\lambda}^\top m(Z_i, \hat{\theta}_n, \hat{\psi}_n)}, \quad 0 = n^{-1} \sum_{i=1}^n \frac{m(Z_i, \theta_0, \hat{\psi}_n)}{1 + \tilde{\lambda}^\top m(Z_i, \theta_0, \hat{\psi}_n)}.$$

From (10) and (11) we have

$$-2 \log ELR_{2n}(\theta_0) = n(\hat{\theta}_n - \theta_0)^\top \left( E \frac{\partial m}{\partial \theta} \right)^\top \{E(mm^\top)\}^{-1} \left( E \frac{\partial m}{\partial \theta} \right) (\hat{\theta}_n - \theta_0) + o_p(1).$$

Definitions of  $S_{12}$ ,  $S_{21}$ ,  $S_{11}$  and  $\Sigma_{(11)}$  complete the proof.

*Proof of Theorem 3* When  $p = 1$  the result (5) is given by Akritas (2001). Zhou (2011) shows that, for  $p > 1$ , (5) also holds with the  $jk$ th

element of  $\Sigma$  as

$$\sigma_{jk} = \int [g_j(x) - \bar{g}_j(x)][g_k(x) - \bar{g}_k(x)] \frac{dF_0(x)}{1 - G_0(x-)}, \quad (12)$$

where  $\bar{g}_j(t)$  denoting the ‘advanced time transformation’ of  $g_j$  in Efron and Johnstone (1990).

Zhou (2011) also showed that (6) holds with the  $jk$ th element of  $V_{1n}$  given by

$$\hat{\sigma}_{jk} = \sum_{i=1}^n [g_j(T_i) - \bar{g}_j(T_i)][g_k(T_i) - \bar{g}_k(T_i)] \frac{\Delta \hat{F}_{KM}(T_i)}{1 - \hat{G}_{KM}(T_i)}.$$

Since  $V_{1n}$  is just  $\Sigma$  with the unknown  $F_0$  and  $G_0$  replaced by their Kaplan-Meier estimators and the fact that the Kaplan-Meier estimators are uniformly consistent,  $V_{1n} \rightarrow \Sigma$  in probability as  $n \rightarrow \infty$ . We refer to Zhou (2011) for details.

It follows from Owen (2001, p. 220-221) (with  $X_i^*$  in place of  $X_i$  there)

that

$$-2 \log ELR_{2n}^*(\theta_0) = U_{2n}^\top V_{2n}^{-1} U_{2n} + o_p(1)$$

where

$$U_{2n} = \sqrt{n}[\bar{X}^* - \theta_0] = \sqrt{n}(\hat{\theta}_n - \theta_0), \quad V_{2n} = n^{-1} \sum_{i=1}^n (X_i^* - \theta_0)(X_i^* - \theta_0)^\top.$$

Since  $\hat{G}_{KM}(t)$  is uniformly consistent for  $G_0(t)$ ,

$$V_{2n} = n^{-1} \sum_{i=1}^n (X_i - \theta_0)(X_i - \theta_0)^\top + o_p(1) = Var(X_1) + o_p(1).$$

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By the results in Zhou (1992) and Srinivasan and Zhou (1994),  $\text{var}(X_1) \neq \Sigma$  however. This suffices to show (7).

As for (8), when  $p = 1$ , Ren (2001) showed that the limiting distribution of this test statistic with  $\theta = \theta_0$  is a scaled chi-square distribution with degree of freedom 1. To obtain a multivariate version, we modify the proof of Ren along the lines of Owen (2001, p. 220-221) to incorporate weights, and find

$$-2 \log \text{ELR}_{3n}(\theta) = U_n^\top V_{3n}^{-1} U_n + o_p(1)$$

where

$$U_n = \sqrt{n} \left[ \sum_{i=1}^n g(T_i) \Delta \hat{F}_{KM}(T_i) - \theta_0 \right] = \sqrt{n} (\hat{\theta}_n - \theta_0),$$

and  $V_{3n}$  is a matrix with the  $jk$ th element given by

$$v_{jk} = \sum_{i=1}^n (g_j(T_i) - \theta_0)(g_k(T_i) - \theta_0) \Delta \hat{F}_{KM}(T_i).$$

An application of the law of large numbers for the Kaplan-Meier integral gives

$$v_{jk} = \int (g_j(t) - \theta_0)(g_k(t) - \theta_0) d\hat{F}_{KM}(t) \rightarrow \int (g_j(t) - \theta_0)(g_k(t) - \theta_0) dF_0(t)$$

in probability as  $n \rightarrow \infty$ . Since  $\Sigma$  depends on the censoring distribution  $G_0$ , while  $\lim V_{3n}$  does not,  $\lim_{n \rightarrow \infty} V_{3n} \neq \Sigma$ .

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