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Uniform Four-Level Designs From Two-level Designs: A New Look

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Abstract

Literature reviews reveal that the research on the issue of constructing efficient uniform designs has been very active in the last decade. In addition, coding theory is widely used in the context of constructing good optimal designs. The present paper explores the construction of highly efficient four-level uniform designs via two transformations: a modified Gray map code and a mapping between quaternary codes and the sequence of three binary codes. The efficiency is based on the viewpoint of uniformity measured by the centered $L_2$- and wrap-around $L_2$-discrepancies of the four-level designs’ binary images. Some theoretical results related to the lower bounds of the above uniformity measures for such designs are also considered in this study.

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1 Introduction

Computer experiments have been widely used in engineering and high-technology development because they are often cheaper and faster than physical experiments to perform. Unlike traditional experiments with known models, computer experiments are often conducted with very little knowledge about the model functions. Moreover, many proposed designs of computer experiments may involve more than one model, which could be either linear or nonlinear and either parametric or nonparametric. Among many modeling methods, the uniform design performs well and becomes a central concept that plays a crucial role in the evaluation and construction of space filling designs for computer experiments (Bates et al., 1996). In particular, in the study of model robustness, the uniform design distributes its experimental points evenly throughout the design space and allows practitioners to efficiently perform numerical analyses for their experiments (see, Fang and Wang, 1994, Chapter 5).

The measure of uniformity plays a key role in the construction of uniform designs. An \( s \)-level \( U \)-type design \( U \) that belongs to a design class \( \mathcal{U}(n; s^m) \) is an \( n \times m \) array with entries from the set \( \{1/2s, 3/2s, \ldots, (2s - 1)/2s\} \) such that each entry of the set \( \{1/2s, 3/2s, \ldots, (2s - 1)/2s\} \) appears equally often in each column of the array. It is to be noted that each row (or point) of \( U \) may be regarded as a point \( y = (y_1, y_2, \ldots, y_m) \) belonging to \( \Omega^m = [0, 1]^m \). An \( U \)-type design \( U \in \mathcal{U}(n; s^m) \) is optimal (or uniform) under a given measure of uniformity provided it has the best uniformity measure over \( \mathcal{U}(n; s^m) \). In this paper, we restrict ourselves to two- and four-levels only.

There are many measures to assess the uniformity of various designs. Among them, the centered \( L_2 \)-discrepancy and wrap-around \( L_2 \)-discrepancy possess nice properties. These measures remain invariant under reordering of runs, relabeling coordinates and coordinate shift. For more details, we can refer to Hickernell (1998a, b) and Fang, Li and Sudjianto (2005). Considering the centered and wrap-around \( L_2 \)-discrepancies as the uniformity measures, we concentrated on the evaluation of the efficiency of four-level designs in this paper.

Recent research indicates that designs constructed from quaternary codes are very promising in this regard. Xu and Wong (2007) pioneered research on quaternary codes designs and reported theoretical as well as computational results. Phoa and Xu (2009) obtained comprehensive analytical results on quarter fraction quaternary codes designs and showed that they often have larger resolution and projectivity than regular designs of the same size. Zhang, et al. (2011) extended the work of Phoa and Xu (2009) to more highly fractionated settings and Phoa (2012) proposed some fundamental theorems on designs’ structure and properties. The use of quaternary codes in the context of
constructing nonregular designs are found advantageous because of the simplicity of the construction method and relatively straightforward applications. Moreover, the designs that obtained can be presented and described in a simple manner. More importantly, many designs constructed by quaternary codes have attractive statistical properties, as noted in Xu, Phoa and Wong (2009).

The objective of this paper is to explore the construction of highly efficient four-level uniform designs via two transformations: a modified Gray map code and a mapping between quaternary codes and the sequence of three binary codes. The efficiency is based on the viewpoint of uniformity measured by the centered $L_2$- and wrap-around $L_2$-discrepancies of the four-level designs’ binary images. Some theoretical results related to the lower bounds of the above uniformity measures for such designs are also considered in this study (Fang, Maringer, Tang and Winker, 2005 and Elsawah and Qin, 2014).

In Section 2, some notations and preliminaries are provided. Section 3 deals with type-I replacement rule based on modified Gray map codes for four-level designs as well as the corresponding efficiency measures. In Section 4, we propose type-II replacement rule from mapping between quaternary codes and the sequence of three binary codes for four-level designs. Moreover, we also studied the efficiency measures of such designs in this section. Illustrative examples are provided in Section 5, where numerical studies lend further support to our theoretical results. We close through conclusion and discussion in Section 6.

2 Notations and preliminaries

Consider an experiment involving $m$ factors each at $s$ levels. A typical level combination of the $m$ factors is represented as $x = (x_1, x_2, \ldots, x_m)$, where $x_j = 0, 1, \ldots, s - 1; 1 \leq j \leq m$. Let $V$ be the set of all $v (= s^m)$ level combinations (or runs) written in the lexicographic ordering. Consider a class of designs, denoted by $\mathcal{D}(n; s^m)$, where a design $d$ in $\mathcal{D}(n; s^m)$ is a collection of $n$ level combinations that belong to $V$ such that the levels of every factor occur equally often in $d$. It is trivial to show a one-to-one correspondence between the elements of $\mathcal{D}(n; s^m)$ and $\mathcal{U}(n; s^m)$ through the mapping $f : (x_1, x_2, \ldots, x_m) \rightarrow (y_1, y_2, \ldots, y_m)$, where $y_j = (2x_j + 1)/2s$, $x_j = 0, 1, \ldots, (s - 1)$.

An $U$-type design $U(n; s^m)$ can be viewed as a design with one dimensional uniformity, that is, $n$ points are uniformly distributed in every dimension. As mentioned earlier, here we restrict ourselves to the case of two- and four-level designs.

In order to make easy readability, let us use the symbol $\mathcal{U}(n; 4^m)$ (or $\mathcal{U}^*(n; 2^m)$)
to denote the class of four-level (or two-level) $U$-type designs. Similarly, let us use the symbol $\mathcal{D}(n; 4^m)$ (or $\mathcal{D}^*(n; 2^m)$) to denote the class of four-level (or two-level) $U$-type designs. Here we use the same symbol $V$ to denote the set of all possible level combinations of the design under consideration and the symbol $v$ to denote its number of elements. This means for any design belonging to the class $\mathcal{D}(n; 4^m)$, the number of elements of $V$ is $4^m (= v)$ while for any design belonging to the class $\mathcal{D}^*(n; 2^m)$, the number of elements of $V$ is $2^m (= v)$. Denote $d$ (or $d^*$) as a design belonging to $\mathcal{D}(n; 4^m)$ (or $\mathcal{D}^*(n; 2^m)$). For any $x \in V$ and $d^* \in \mathcal{D}^*(n; 2^m)$, let $n_{d^*}(x)$ be the number of times the level combination $x$ occurs in $d^*$ and let $n_{d^*}$ be the $v \times 1$ vector with elements $n_{d^*}(x)$ arranged in the lexicographic ordering. Moreover, let $c_{ij}$ be the number of entries in the $i$th and $j$th rows of $d^*$ coincide. It is trivial to show that $c_{ii} = m, 1 \leq i \leq m$.

For a design $d \in \mathcal{D}(n; 4^m)$ or equivalently $d \in \mathcal{U}(n; 4^m)$, the centered and wrap-around $L_2$-discrepancy measures of uniformity, denoted as $CD_2(d)$ and $WD_2(d)$, can be expressed respectively in the following closed forms

$$[CD_2(d)]^2 = \left(\frac{13}{12}\right)^m - 2\frac{n}{2} \prod_{i=1}^{n} \left(1 + \frac{1}{2}|y_{il} - \frac{1}{2}| - \frac{1}{2}|y_{il} - \frac{1}{2}|^2\right)$$

$$+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left(1 + \frac{1}{2}|y_{il} - \frac{1}{2}| + \frac{1}{2}|y_{jl} - \frac{1}{2}| - \frac{1}{2}|y_{il} - y_{jl}|\right), 
(1)$$

and

$$[WD_2(d)]^2 = -\left(\frac{4}{3}\right)^m + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left(\frac{3}{2} - |y_{il} - y_{jl}|(1 - |y_{il} - y_{jl}|)\right), 
(2)$$

where, for $1 \leq l \leq m$, $y_{il}, y_{jl} \in \{1/8, 3/8, 5/8, 7/8\}, i, j = 1, 2, \cdots, n$.

Exactly similar formula is there for $CD_2(d^*)$ and $WD_2(d^*)$ for a design $d^* \in \mathcal{D}^*(n; 2^m)$ or equivalently $d^* \in \mathcal{U}^*(n; 2^m)$ with $y_{il}, y_{jl} \in \{1/4, 3/4\}, i, j = 1, 2, \cdots, n, 1 \leq l \leq m$.

The following definitions will be helpful in developing the rest of the paper. These definitions provide optimal $U$-type designs with reference to different discrepancy measures and also the proposed replacement rules.

**Definition 1.** A design $\tilde{d} \in \mathcal{D}(n; 4^m)$ (or $\tilde{d}^* \in \mathcal{D}^*(n; 2^m)$) is optimal with reference to centered $L_2$-discrepancy measure if and only if for any design $d \in \mathcal{D}(n; 4^m)$ (or $d^* \in \mathcal{D}^*(n; 2^m)$)

$$[CD_2(\tilde{d})]^2 \leq [CD_2(d)]^2 \text{ (or } [CD_2(\tilde{d}^*)]^2 \leq [CD_2(d^*)]^2).$$
Definition 2. A design $\tilde{d} \in \mathcal{D}(n; 4^m)$ (or $\tilde{d}^* \in \mathcal{D}^*(n; 2^m)$) is optimal with reference to wrap-around $L_2$-discrepancy measure if and only if for any design $d \in \mathcal{D}(n; 4^m)$ (or $d^* \in \mathcal{D}^*(n; 2^m)$)

$$[WD_2(\tilde{d})]^2 \leq [WD_2(d)]^2 \text{ (or } [WD_2(\tilde{d}^*)]^2 \leq [WD_2(d^*)]^2).$$

Definition 3. A Type-I replacement rule is a rule that replaces two binary columns via a map (it is slightly modified from the Gray map used in Phoa and Xu, 2009) by a quaternary column as explained below:

$$0 \, 0 \rightarrow 0, \ 0 \, 1 \rightarrow 1, \ 1 \, 0 \rightarrow 2, \text{ and } 1 \, 1 \rightarrow 3.$$

Definition 4. A Type-II replacement rule is a rule that replaces three binary columns via the following map (see Mukerjee and Wu, 2006) by a quaternary column as shown below:

$$0 \, 0 \, 0 \rightarrow 0, \ 0 \, 1 \, 1 \rightarrow 1, \ 1 \, 0 \, 1 \rightarrow 2, \text{ and } 1 \, 1 \, 0 \rightarrow 3.$$

Definition 5. An orthogonal array $OA(N, n, s, g)$, having $N$ rows, $n$ columns, $s$ symbols, and strength $g$, is an $N \times n$ array with elements from a set of $s$ symbols in which all possible combinations of symbols appear equally often as rows in every $N \times g$ subarray.

Remark 1. It is to be remarked that under Type-I replacement rule, a design $d^* \in \mathcal{D}^*(n; 2^m)$ can be replaced by a design $\tilde{d} \in \mathcal{D}(n; 4^m)$ and vice versa, i.e., a design $d \in \mathcal{D}(n; 4^m)$ can also be replaced under Type-I replacement rule by a design $d^* \in \mathcal{D}^*(n; 2^m)$. In this replacement procedure, it is to be noted that the columns of $d^*$ are not necessarily distinct and one or more columns may repeat more than once in the entire design. Moreover, the columns can be grouped into $m$ groups such that each group is an $OA(n, 2, 2, 2)$.

Let $\mathcal{D}^{**}(n; 2^m)$ be a class of two-level designs such that the columns of a design belonging to this class are not necessarily distinct and one or more columns may repeat more than once in the entire design. Moreover, these $3m$ columns can be grouped into $m$ groups, say $G_1, G_2, \ldots, G_m$, such that each group has the following possible level combinations (an $OA(4, 3, 2, 2)$)

$$0 \, 0 \, 0, \ 0 \, 1 \, 1, \ 1 \, 0 \, 1, \text{ and } 1 \, 1 \, 0.$$
In addition to the above, the levels 0 and 1 appear equally often in each of the $3m$ columns.

**Remark 2.** It is to be remarked that under Type-II replacement rule, a design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$ can be replaced by a design $d \in \mathcal{D}(n; 4^{m})$ and vice versa.

The following example will illustrate the use of above replacement rules in the construction of designs belonging to $d \in \mathcal{D}(8; 4^{2})$ from designs belonging to $\mathcal{D}(8; 2^{7})$.

**Example 1.** Consider the following regular $OA(8; 7; 2; 2)$ which is an array involving 8 level combinations (or runs), 7 factors each at two-levels and strength 2.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Case 1. Based on the array (3), we construct the following two sets of columns. The first set consists of the columns 4-6 and the second set consists of the columns 2, 3 and 6. Following Type-II replacement rule, we obtain the following four-level design involving two factors.

\[
d_1 = \begin{bmatrix}
0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 \\
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\
\end{bmatrix}^\prime.
\]

Case 2. Based on the array (3), we construct the following two sets of columns. The first set consists of the columns 1, 7 and the second set consists of the columns 2, 3. Using Type-I replacement rule, we obtain the following four-level design involving two factors.

\[
d_2 = \begin{bmatrix}
0 & 1 & 1 & 0 & 3 & 2 & 2 & 3 \\
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\
\end{bmatrix}^\prime.
\]
The following example will illustrate the use of above replacement rules in the construction of designs belonging to \( d \in D(8; 4^3) \) from designs belonging to \( D(8; 2^7) \).

**Example 2.** Consider the array given in (3) again.

Case 1. Based on the array (3), we construct the following three sets of columns. The first set consists of the columns 6, 2, 3, the second set consists of the columns 5, 3, 1 and the third set consists of the columns 7, 3, 4 in the order mentioned. Following Type-II replacement rule, we obtain the following four-level design involving three factors.

\[
d_3 = \begin{bmatrix} 0 & 2 & 3 & 1 & 0 & 2 & 3 & 1 \\ 0 & 3 & 0 & 3 & 2 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 & 2 & 1 & 0 & 3 \end{bmatrix}.
\]

Case 2. Based on the array (3), we construct the following three sets of columns. The first set consists of the columns 1, 2, the second set consists of the columns 3, 4 and the third set consists of the columns 5, 6 in the order mentioned. Using Type-I replacement rule, we obtain the following four-level design involving three factors.

\[
d_4 = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 2 & 1 & 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 1 & 2 & 2 & 1 & 3 & 0 \end{bmatrix}.
\]

**Remark 3.** From the above examples, it is clear that orthogonal array designs will produce four-level uniform or nearly uniform designs through the replacement rules proposed in this paper. The literature study reveals that these replacement rules produce many good optimal designs. It is expected that these rules will also yield good designs from the uniformity point of view. Moreover, it is expected that juxtaposition of orthogonal arrays can be a good choice of original two-level design. The choice of \( k \) in \( D^*(n; 2^k) \) or \( D^{**}(n; 2^k) \) will probably depend upon the array. It is to be noted that similar problem is there in the construction of supersaturated designs.

The following lemmas will be helpful in establishing the lower bounds of the discrepancy measures in the following sections.

**Lemma 1** (Chatterjee et al. 2012). Suppose \( \sum_{i=1}^{q} z_i = c \) and \( z_i \) are nonnegative integers, then

\[
\sum_{i=1}^{q} z_i^g \geq \alpha_1 w^g + \alpha_2 (w + 1)^g,
\]
where $w$ is the largest integer contained in $c/q$; $\alpha_1$ and $\alpha_2$ are integers such that $\alpha_1 + \alpha_2 = q$ and $\alpha_1 w + \alpha_2 (w + 1) = c$.

**Lemma 2** (Chatterjee et al. 2012). Suppose $\sum_{i=1}^{q} z_i = c$ and $z_i$ are non-negative integers, then

$$\sum_{i=1}^{q} \beta^z_i \geq \beta^w (\alpha_1 + \alpha_2),$$

where $w, \alpha_1$ and $\alpha_2$ are as defined in Lemma 1.

The next section provides lower bounds of the discrepancy measures considered in this paper, and also the efficiency of designs based on Type-I replacement rule.

### 3 Lower bounds and efficiency of designs based on Type-I replacement rule

Let us consider a design $d \in \mathcal{D}(n; 4^m)$. The main objective of this section is to develop, as a benchmark of obtaining optimal designs in $\mathcal{D}(n; 4^m)$, a lower bound to such designs under Type-I replacement rule. Following Definition 3, here we replace each quaternary column of $d$ with two binary columns. It is to be noted that each column of the design receives each of the four levels equally often. Hence, it is also to be noted that through this replacement rule, the given design $d \in \mathcal{D}(n; 4^m)$ reduces to a design $d^* \in \mathcal{D}^*(n; 2^{2m})$, where, for any design $d^* \in \mathcal{D}^*(n; 2^{2m})$, the pairs of columns $(i, i+1), i = 1, 3, 5, \ldots$, without loss of generality, receives each of the level combinations 00, 01, 10, 11 equally often. For any design $d^* \in \mathcal{D}^*(n; 2^{2m})$ and for $1 \leq i_1, i_2 \leq n$, let $c_{i_1i_2}$ be the number of entries in the $i_1$th and the $i_2$th rows of $d$ coincide, it is obvious that $c_{ii} = 2m$ for $1 \leq i \leq n$. Moreover,

$$\sum_{i_1=1}^{n} \sum_{i_2(\neq i_1)=1}^{n} c_{i_1i_2} = mn(n - 2).$$

Define

$$A_{11} = \begin{pmatrix} 5/4 & 1 \\ 1 & 5/4 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 3/2 & 5/4 \\ 5/4 & 3/2 \end{pmatrix}, \quad B_{11} = \bigotimes_{j=1}^{2m} A_{11}, \quad B_{12} = \bigotimes_{j=1}^{2m} A_{12},$$
and

\[ \Gamma_1(0) = I'_2 = (1, 1), \quad \Gamma_1(1) = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Note that

\[ A_{11} = b_{11}(\Gamma_1(0))'\Gamma_1(0) + a_{11}(\Gamma_1(1))'\Gamma_1(1), \quad \text{and} \]

\[ A_{12} = b_{12}(\Gamma_1(0))'\Gamma_1(0) + a_{12}(\Gamma_1(1))'\Gamma_1(1), \]

where \( a_{11} = a_{12} = 1/4, \ b_{11} = 1 \) and \( b_{12} = 5/4 \). Let \( \Omega_1 = \{ u = u_1 \cdots u_{2m} | u_j = 0, 1; 1 \leq j \leq 2m \} \). For \( 0 \leq r \leq 2m \), let \( \Omega_{1r} = \{ u \in \Omega_1 | \sum u_j = r \} \). Finally, for any \( u \in \Omega_1 \), define the matrix

\[ \Delta_1(u) = \sum_{j=1}^{2m} \Gamma_1(u_j). \]

With the help of above notations, the matrices \( B_{11} \) and \( B_{12} \) can be expressed as

\[
\begin{align*}
B_{11} &= (b_{11})^{2m} \sum_{r=0}^{2m} \left( \frac{a_{11}}{b_{11}} \right)^r \sum_{u \in \Omega_{1r}} (\Delta_1(u))'\Delta_1(u), \\
B_{12} &= (b_{12})^{2m} \sum_{r=0}^{2m} \left( \frac{a_{12}}{b_{12}} \right)^r \sum_{u \in \Omega_{1r}} (\Delta_1(u))'\Delta_1(u).
\end{align*}
\]

(4)

Now, for any design \( d^* \in \mathcal{D}^*(n; 2^{2m}) \), the expressions of \( CD_2(d^*) \) and \( WD_2(d^*) \) stated in (1) and (2) respectively with the choice \( y_{ij}, y_{jl} \in \{1/4, 3/4\} \), \( i, j = 1, 2, \cdots, n \), \( 1 \leq l \leq m \), can be rewritten as

\[
\begin{align*}
[CD_2^{(1)}(d^*)]^2 &= \left( \frac{13}{12} \right)^{2m} - 2 \left( \frac{35}{32} \right)^{2m} + \frac{1}{n^2} n'_d, B_{11} n'_d, \\
[WD_2^{(1)}(d^*)]^2 &= - \left( \frac{4}{5} \right)^{2m} + \frac{1}{n^2} n'_d, B_{12} n'_d.
\end{align*}
\]

(5)

Alternatively, the measures of \( CD_2(d^*) \) and \( WD_2(d^*) \) can be expressed on the basis of \( c_{i_1i_2} \) values as

\[
\begin{align*}
[CD_2^{(1)}(d^*)]^2 &= \left( \frac{13}{12} \right)^{2m} - 2 \left( \frac{35}{32} \right)^{2m} + \frac{1}{n^2} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \left( \frac{5}{4} \right)^{c_{i_1i_2}}, \\
[WD_2^{(1)}(d^*)]^2 &= - \left( \frac{4}{5} \right)^{2m} + \frac{1}{n^2} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \left( \frac{5}{4} \right)^{c_{i_1i_2}}.
\end{align*}
\]

(6)
Lemmas 3 and 4 presented below are helpful in obtaining the lower bounds of the measures given in (5) and (6).

**Lemma 3.** Based on Type-I replacement rule, let \(d \in D(n; 4^m)\) be a design obtained from a design \(d^* \in D^*(n; 2^{2m})\). The uniformity measure of \(d\), measured through \([CD_2^{(1)}(d^*)]^2\) or \([WD_2^{(1)}(d^*)]^2\), has the following lower bound.

\[
\begin{align*}
[CD_2^{(1)}(d^*)]^2 & \geq LB_{11}(CD_2^{(1)}(d^*)), \\
[WD_2^{(1)}(d^*)]^2 & \geq LB_{11}(CD_2^{(1)}(d^*)),
\end{align*}
\]

where

\[
LB_{11}(CD_2^{(1)}(d^*)) = \left(\frac{13}{12}\right)^2 - 2 \left(\frac{3^m}{2^m}\right)^2 + \frac{(b_{11})^2}{n^2} \sum_{r=0}^{2m} \frac{(a_{11})^r}{b_{11}^r} \left(\sum_{r=0}^{2m} \frac{(a_{11})^r}{b_{11}^r}\right) \left[\alpha_{11} w_{1r}^2 + \alpha_{2r} (w_{1r} + 1)^2\right],
\]

\[
LB_{11}(CD_2^{(1)}(d^*)) = -\left(\frac{4}{9}\right)^2m + \frac{(b_{12})^2}{n^2} \sum_{r=0}^{2m} \frac{(a_{12})^r}{b_{12}^r} \left(\sum_{r=0}^{2m} \frac{(a_{12})^r}{b_{12}^r}\right) \left[\alpha_{11} w_{1r}^2 + \alpha_{2r} (w_{1r} + 1)^2\right],
\]

where \(w_{1r}\) is the largest integer contained in \(n/2^r\); \(\alpha_{11}, \alpha_{2r}\) are integers such that \(\alpha_{11} + \alpha_{2r} = 2^r\) and \(\alpha_{11} w_{1r} + \alpha_{2r} (w_{1r} + 1) = n\).

**Proof.** We denote \(1_q\) as the \(q \times 1\) vector with all elements unity. Note that the elements of the \(2^r \times 1\) vector \(\Delta_1(u)n_{d^*}\) are non-negative integers with \(1^r \Delta_1(u)n_{d^*} = n\). Hence, for \(0 \leq r \leq 2m\), and for each \(u \in \Omega_{1r}\), we get from Lemma 1

\[
n'_{d^*}(\Delta_1(u))^{\Delta_1(u)n_{d^*}} \geq \alpha_{1r} w_{1r}^2 + \alpha_{2r} (w_{1r} + 1)^2.
\]

Thus, Lemma 3 follows from Lemma 1, (4) and (5).

As in Lemma 1, let \(w_1\) be the largest integer contained in \(m(n - 2)/(n - 1)\). Let \(\alpha_{11}\) and \(\alpha_{12}\) be integers such that

\[
\alpha_{11} + \alpha_{12} = n(n - 1) \quad \text{and} \quad \alpha_{11} w_1 + \alpha_{12} (1 + w_1) = mn(n - 2).
\]

**Lemma 4.** Based on Type-I replacement rule, let \(d \in D(n; 4^m)\) be a design obtained from a design \(d^* \in D^*(n; 2^{2m})\). The uniformity measure of \(d\), measured through \([CD_2^{(1)}(d^*)]^2\) or \([WD_2^{(1)}(d^*)]^2\), has the following lower bound.

\[
\begin{align*}
[CD_2^{(1)}(d^*)]^2 & \geq LB_{12}(CD_2^{(1)}(d^*)), \\
[WD_2^{(1)}(d^*)]^2 & \geq LB_{12}(WD_2^{(1)}(d^*)),
\end{align*}
\]
where
\[
LB_{12}(CD_2^{(1)}(d^*)) = \left(\frac{13}{12}\right)^{2^m} - 2\left(\frac{35}{32}\right)^{2^m} + \frac{1}{n^2}\left(\frac{5}{4}\right)^{2^m} + \frac{1}{n^2}\left(\frac{3}{4}\right)^{2^m} w_1 \left[\alpha_{11} + \left(\frac{5}{4}\right)\alpha_{12}\right]
\]
\[
LB_{12}(WD_2^{(1)}(d^*)) = -\left(\frac{4}{3}\right)^{2^m} + \frac{1}{n^2}\left(\frac{3}{4}\right)^{2^m} + \left(\frac{5}{4}\right)^{2^m} \frac{1}{n^2}\left(\frac{6}{5}\right)^{2^m} w_1 \left[\alpha_{11} + \left(\frac{6}{5}\right)\alpha_{12}\right]
\]

**Proof.** The proof of Lemma 4 follows from Lemma 2 and (6).

For any design \(d \in D(n; 4^m)\) obtained through Type-I replacement rule, the following theorem provides lower bounds of uniformity measures.

**Theorem 1.** Based on Type-I replacement rule, let \(d \in D(n; 4^m)\) be a design obtained from a design \(d^* \in D^*(n; 2^{2m})\). The uniformity measure of \(d\), measured through \(\left[CD_2^{(1)}(d^*)\right]^2\) or \(\left[WD_2^{(1)}(d^*)\right]^2\), has the following lower bound.

\[
\begin{align*}
\left[CD_2^{(1)}(d^*)\right]^2 & \geq LB_1(CD_2^{(1)}(d^*)), \\
\left[WD_2^{(1)}(d^*)\right]^2 & \geq LB_1(WD_2^{(1)}(d^*))
\end{align*}
\]

where
\[
LB_1(CD_2^{(1)}(d^*)) = \max\{LB_{11}(CD_2^{(1)}(d^*)), LB_{12}(CD_2^{(1)}(d^*))\},
\]
\[
LB_1(WD_2^{(1)}(d^*)) = \max\{LB_{11}(WD_2^{(1)}(d^*)), LB_{12}(WD_2^{(1)}(d^*))\}.
\]

**Proof.** The proof of Theorem 1 follows from Lemmas 3 and 4.

**Remark 4.** Apart from the optimality of the design \(d_2\) with respect to the measure stated in (5) and (6), it has been checked that it is also optimal with respect to the measure stated in (1). On the other hand design \(d_4\), even though it is optimal with respect to the measure stated in (5) and (6), it has been checked that it is highly efficient with respect to the efficiency measure provided in (7). It is interesting to note that two-level orthogonal arrays can be used to obtain optimal four-level \(U\)-type design through the application of Type-I replacement rule.
To compare the efficiency of any given design \(d \in \mathcal{D}(n; 4^m)\), derived through the Type-I replacement rule, with reference to the lower bounds stated in Theorem 1, we define

\[
\begin{align*}
Eff_1(CD_2^{(1)}(d^*)) &= \frac{LB_1(CD_2^{(1)}(d^*))}{[CD_2^{(1)}(d^*)]^2}, \\
Eff_1(WD_2^{(1)}(d^*)) &= \frac{LB_1(WD_2^{(1)}(d^*))}{[WD_2^{(1)}(d^*)]^2}.
\end{align*}
\]  

(7)

From equation (7) it is thus clear that, under Type-I replacement rule, the efficiency of a design \(d \in \mathcal{D}(n; 4^m)\) can be measured through the formula proposed in (7). For a design \(d \in \mathcal{D}(n; 4^m)\), if \(Eff_1(CD_2^{(1)}(d^*))\) or \(Eff_1(WD_2^{(1)}(d^*))\) equals or nearly equals to 1, then we can say \(d\) is at least nearly optimal.

The next section deals with the derivation of the lower bounds of the discrepancy measures considered in this paper, and also the efficiency of designs based on Type-II replacement rule.

4 Lower bounds and efficiency of designs based on Type-II replacement rule

Let us consider a design \(d \in \mathcal{D}(n; 4^m)\). Under Type-II replacement rule, a design \(d \in \mathcal{D}(n; 4^m)\) can be replaced by a design belonging to the class of two-level designs \(\mathcal{D}^{**}(n; 2^{3m})\) such that the columns of a design belonging to this class may not be distinct which means one or more columns may be repeated more that once in the entire design. Moreover, these \(3m\) columns can be grouped into \(m\) groups, say \(G_1, G_2, \cdots, G_m\), such that each group has the following possible level combinations (an \(OA(4, 3, 2, 2)\))

\[
0\ 0\ 0,\ 0\ 1\ 1,\ 1\ 0\ 1,\ \text{and}\ 1\ 1\ 0,
\]

and that in each of the \(3m\) columns, the levels 0 and 1 appear equally often. It is to be noted that under Type-II replacement rule, a design \(d^{**} \in \mathcal{D}^{**}(n; 2^{3m})\) can be replaced by a design \(d \in \mathcal{D}(n; 4^m)\) and vice versa.

If we denote the \(3m\) factors of a design \(d^{**} \in \mathcal{D}^{**}(n; 2^{3m})\) as \(F_1, F_2, \cdots, F_{3m}\), then without loss of generality, the grouping scheme can be described below.

\[
\begin{array}{cccccccc}
F_1F_2F_3 & F_4F_5F_6 & F_7F_8F_9 & \cdots & F_{3m-2}F_{3m-1}F_{3m} \\
G_1 & G_2 & G_3 & \cdots & G_m
\end{array}
\]
Let $V$ be the set of all level combinations of the factors $G_1, G_2, \ldots, G_m$, and thus the cardinality of $V$ is $4^m$. For any design $d^{**} \in D^{**}(n; 2^{3m})$, let $n_{d^{**}}(x)$ be the number of times of the level combination $x \in V$ occurs in $d^{**}$ and let $n_{d^{**}}$ be the $4^m \times 1$ vector with elements $n_{d^{**}}(x)$ arranged in the lexicographic order. These notations will help us to establish lower bounds to discrepancy measures. We illustrate the above replacement rule through the following example.

**Example 3.** Consider a design $d \in D(8; 4^2)$ or equivalently a design $d^{**} \in D^{**}(8; 2^6)$. Then the set $V$ consists of the following level combinations presented in a tabular form.

<table>
<thead>
<tr>
<th></th>
<th>000000</th>
<th>011000</th>
<th>101000</th>
<th>110000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>000011</td>
<td>011011</td>
<td>101011</td>
<td>110011</td>
</tr>
<tr>
<td>3</td>
<td>001011</td>
<td>011101</td>
<td>101101</td>
<td>110101</td>
</tr>
<tr>
<td>4</td>
<td>001110</td>
<td>011110</td>
<td>101110</td>
<td>110110</td>
</tr>
</tbody>
</table>

For any design $d^{**} \in D^{**}(n; 2^{3m})$ and for $1 \leq i_1, i_2 \leq n$, let $c_{i_1i_2}$ be the number of entries in the $i_1$th and the $i_2$th rows of $d^{**}$ coincide. Then it is obvious that $c_{ii} = 3m$ for $1 \leq i \leq n$. Moreover,

$$
\sum_{i_1=1}^{n} \sum_{i_2(\neq i_1)=1}^{n} c_{i_1i_2} = \frac{3mn(n-2)}{2}.
$$

Define

$$
A_{21} = \left( \begin{array}{cccc}
  a_1^3 & a_1b_2^2 & a_1b_1^2 & a_1b_1^2 \\
  a_1b_1^2 & a_2^3 & a_1b_1^2 & a_1b_1^2 \\
  a_1b_1^2 & a_1b_1^2 & a_2^3 & a_1b_1^2 \\
  a_1b_1^2 & a_1b_1^2 & a_1b_1^2 & a_1^3
\end{array} \right), \quad B_{21} = \bigotimes_{j=1}^{m} A_{21},
$$

and

$$
A_{22} = \left( \begin{array}{cccc}
  a_2^3 & a_2b_2^2 & a_2b_2^2 & a_2b_2^2 \\
  a_2b_2^2 & a_2^3 & a_2b_2^2 & a_2b_2^2 \\
  a_2b_2^2 & a_2b_2^2 & a_2^3 & a_2b_2^2 \\
  a_2b_2^2 & a_2b_2^2 & a_2b_2^2 & a_2^3
\end{array} \right), \quad B_{22} = \bigotimes_{j=1}^{m} A_{22},
$$

where $a_1 = 5/4, b_1 = 1, a_2 = 3/2$ and $b_2 = 5/4$. 

13
Similar to (5) and (6), the $CD_2(d^{**})$ and $WD_2(d^{**})$ discrepancy measures for any design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$ can be expressed as

\[
[CD_2^{(2)}(d^{**})]^2 = \left(\frac{13}{12}\right)^{3m} - 2 \left(\frac{35}{32}\right)^{3m} + \frac{1}{n'} n'_d, B_{21} n_{d^{**}} \\
[WD_2^{(2)}(d^{**})]^2 = -\left(\frac{4}{3}\right)^{3m} + \frac{1}{n'} n'_d, B_{22} n_{d^{**}}
\]

and

\[
[CD_2^{(2)}(d^{**})]^2 = \left(\frac{13}{12}\right)^{3m} - 2 \left(\frac{35}{32}\right)^{3m} + \frac{1}{n} \left(\frac{5}{4}\right)^{3m} + \frac{1}{n'} \sum_{i_1=1}^n \sum_{i_2\neq i_1=1}^n \left(\frac{5}{4}\right)^{c_{i_1i_2}} \\
[WD_2^{(2)}(d^{**})]^2 = -\left(\frac{4}{3}\right)^{3m} + \frac{1}{n} \left(\frac{3}{2}\right)^{3m} + \frac{1}{n} \sum_{i_1=1}^n \sum_{i_2\neq i_1=1}^n \left(\frac{5}{4}\right)^{c_{i_1i_2}}
\]

Lemmas 5 and 6 presented below are helpful in obtaining the lower bounds of the measures given in (8) and (9).

**Lemma 5.** Based on Type-II replacement rule, let $d \in \mathcal{D}(n; 4^m)$ be a design obtained from a design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$. The uniformity measure of $d$, measured through $[CD_2^{(2)}(d^{**})]^2$ or $[WD_2^{(2)}(d^{**})]^2$, has the following lower bound.

\[
\begin{align*}
[CD_2^{(2)}(d^{**})]^2 & \geq LB_{21}(CD_2^{(2)}(d^{**})) \\
[WD_2^{(2)}(d^{**})]^2 & \geq LB_{21}(WD_2^{(2)}(d^{**}))
\end{align*}
\]

where

\[
LB_{21}(CD_2^{(2)}(d^{**})) = \left(\frac{13}{12}\right)^{3m} - 2 \left(\frac{35}{32}\right)^{3m} \\
+ \frac{\delta_1}{n^2} \sum_{r=0}^m \frac{\delta_{1r}^2}{\delta_{2r}}^{r} \left(\begin{array}{c} m \\ r \end{array}\right) \left[\alpha_1^{(2)} w_2^2 + \alpha_2^{(2)} (w_2 + 1)^2\right]
\]

\[
LB_{21}(WD_2^{(2)}(d^{**})) = -\left(\frac{4}{3}\right)^{3m} + \frac{\delta_{12}^2}{n^2} \sum_{r=0}^m \frac{\delta_{22}^2}{\delta_{21}}^{r} \left(\begin{array}{c} m \\ r \end{array}\right) \left[\alpha_1^{(2)} w_2^2 + \alpha_2^{(2)} (w_2 + 1)^2\right]
\]

and $w_2$ is the largest integer contained in $n/4^r$; $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ are integers such that $\alpha_1^{(2)} + \alpha_2^{(2)} = 4^r$ and $\alpha_1^{(2)} w_2 + \alpha_2^{(2)} (w_2 + 1) = n$; and $\delta_{11} = a_1 b_1^2$, $\delta_{12} = a_1 (a_1^2 - b_1^2)$, $\delta_{21} = a_2 b_2^2$, $\delta_{22} = a_2 (a_2^2 - b_2^2)$.
The rest vector $\Delta_2$ and $\Delta_2$ can respectively express the matrices $B_1$ and $B_2$ as

$$B_{21} = (\delta_{11})^m \sum_{r=0}^m \left( \delta_{21} \right)^r \sum_{u \in \Omega_2} \left( \Delta_2(u) \right)^1 \Delta_2(u),$$

$$B_{22} = (\delta_{21})^m \sum_{r=0}^m \left( \delta_{22} \right)^r \sum_{u \in \Omega_2} \left( \Delta_2(u) \right)^1 \Delta_2(u).$$

The rest of the proof follows from Lemma 3 with a note that the elements of the $4^r \times 1$ vector $\Delta_2(u)n_{d^*}$ are nonnegative integers with $1_{4}^r, \Delta_2(u)n_{d^*} = n$.

As in Lemma 1, let $w_2$ be the largest integer contained in $3m(n-2)/[2(n-1)]$. Let $\alpha_{21}$ and $\alpha_{22}$ be integers such that

$$\alpha_{21} + \alpha_{22} = n(n-1) \quad \text{and} \quad \alpha_{21}w_2 + \alpha_{22}(1 + w_2) = \frac{3mn(n-2)}{2}.$$
Lemma 6. Based on Type-II replacement rule, let $d \in \mathcal{D}(n; 4^m)$ be a design obtained from a design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$. The uniformity measure of $d$, measured through $\left[ CD_2^{(2)}(d^{**}) \right]^2$ or $\left[ WD_2^{(2)}(d^{**}) \right]^2$, has the following lower bound.

\[
\begin{align*}
\left[ CD_2^{(2)}(d^{**}) \right]^2 & \geq LB_{22}(CD_2^{(2)}(d^{**})), \\
\left[ WD_2^{(2)}(d^{**}) \right]^2 & \geq LB_{22}(WD_2^{(2)}(d^{**})),
\end{align*}
\]

where

\[
\begin{align*}
LB_{22}(CD_2^{(2)}(d^{**})) & = \left( \frac{13}{12} \right)^{3m} - 2 \left( \frac{35}{32} \right)^{3m} + \frac{1}{n} \left( \frac{5}{4} \right)^{3m} \frac{1}{n^2} \left( \frac{3}{4} \right)^{w_2} \left( \alpha_{21} + \left( \frac{5}{4} \right) \alpha_{22} \right), \\
LB_{22}(WD_2^{(2)}(d^{**})) & = - \left( \frac{5}{3} \right)^{3m} + \frac{1}{n} \left( \frac{3}{2} \right)^{3m} + \left( \frac{5}{4} \right)^{3m} \frac{1}{n^2} \left( \frac{6}{5} \right)^{w_2} \left( \alpha_{21} + \left( \frac{6}{5} \right) \alpha_{22} \right).
\end{align*}
\]

For any design $d \in \mathcal{D}(n; 4^m)$ obtained through Type-II replacement rule, the following theorem provides lower bounds of uniformity measures.

Theorem 2. Based on Type-II replacement rule, let $d \in \mathcal{D}(n; 4^m)$ be a design obtained from a design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$. The uniformity measure of $d$, measured through $\left[ CD_2^{(2)}(d^{**}) \right]^2$ or $\left[ WD_2^{(2)}(d^{**}) \right]^2$, has the following lower bound.

\[
\begin{align*}
\left[ CD_2^{(2)}(d^{**}) \right]^2 & \geq LB_{2}(CD_2^{(2)}(d^{**})), \\
\left[ WD_2^{(2)}(d^{**}) \right]^2 & \geq LB_{2}(WD_2^{(2)}(d^{**})),
\end{align*}
\]

where

\[
\begin{align*}
LB_{2}(CD_2^{(2)}(d^{**})) & = \max\{LB_{21}(CD_2^{(2)}(d^{**})), LB_{22}(CD_2^{(2)}(d^{**}))\}, \\
LB_{2}(WD_2^{(2)}(d^{**})) & = \max\{LB_{21}(WD_2^{(2)}(d^{**})), LB_{22}(WD_2^{(2)}(d^{**}))\}.
\end{align*}
\]

Proof. The proof of Theorem 2 follows from Lemmas 5 and 6.

Remark 5. Apart from the optimality of the designs $d_1$ and $d_3$ with respect to the measure stated in (5) and (6), it has been checked that these designs are also optimal with respect to the measure stated in (1). It is interesting to note that two-level orthogonal arrays can be used to obtain optimal four-level $U$-type design through the application of Type-II replacement rule.
To compare the efficiency of any given design \(d \in \mathcal{D}(n;4^m)\), derived following the Type-II replacement rule, with reference to the lower bounds stated in Theorem 2, we define

\[
\begin{align*}
\text{Eff}_2(CD_2(d^{**})) &= \frac{\text{LB}_2(CD_2(d^{**}))}{[CD_2(d^{**})]^2}, \\
\text{Eff}_2(WD_2(d^{**})) &= \frac{\text{LB}_2(WD_2(d^{**}))}{[WD_2(d^{**})]^2}.
\end{align*}
\]  

(11)

With the help of the efficiency measures proposed in (11), we can compare the efficiency of a design \(d \in \mathcal{D}(n;4^m)\) under Type-II replacement rule. For a design \(d \in \mathcal{D}(n;4^m)\), if \(\text{Eff}_2(CD_2(d^{**}))\) or \(\text{Eff}_2(WD_2(d^{**}))\) equals or nearly equals to 1, then we can say \(d\) is at least nearly optimal.

5 Illustrative examples

In this section we give some examples to illustrate our theoretical results. For convenience, we denote the squared centered \(L_2\)-discrepancy values for a given design that are given in (5) and (8) as \(CD\) in the following Tables. Similarly, we denote the squared wrap-around \(L_2\)-discrepancy values for a given design that are given in (5) and (8) as \(WD\) in the following Tables, and denote \(LB\), \(Eff\) as the corresponding lower bounds and efficiency, respectively.

In order to measure the efficiency of the four-level designs constructed from the replacement rules introduced in this paper, we can compare the \(CD\) and \(WD\) values of the derived four-level designs with respect to the available lower bounds \(LB(CD_2(d))\) and \(LB(WD_2(d))\) (Fang, Maringer, Tang and Winker, 2005; Elsawah and Qin, 2014) for four level designs. We define, for any derived four-level design \(d \in \mathcal{D}(n;4^m)\)

\[
\begin{align*}
\text{Eff}(CD_2(d)) &= \frac{\text{LB}(CD_2(d))}{[CD_2(d)]^2}, \\
\text{Eff}(WD_2(d)) &= \frac{\text{LB}(WD_2(d))}{[WD_2(d)]^2}.
\end{align*}
\]

(12)

Example 4. Consider the design \(d_5 \in \mathcal{D}(8;4^3)\), given below, with \(n = 8\) and \(m = 3\).

\[
d_4 = \\
\begin{bmatrix}
1 & 2 & 1 & 3 & 0 & 2 & 3 & 0 \\
3 & 3 & 0 & 2 & 2 & 0 & 1 & 1 \\
3 & 1 & 1 & 2 & 0 & 3 & 0 & 2 \\
\end{bmatrix}'.
\]
Table 1: Numerical results of $d_5$

<table>
<thead>
<tr>
<th>Type</th>
<th>CD</th>
<th>LB1</th>
<th>LB2</th>
<th>Eff</th>
<th>WD</th>
<th>LB1</th>
<th>LB2</th>
<th>Eff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type I</td>
<td>0.2318</td>
<td>0.2240</td>
<td>0.2318</td>
<td>1.0000</td>
<td>1.1610</td>
<td>1.1457</td>
<td>1.1610</td>
<td>1.0000</td>
</tr>
<tr>
<td>Type II</td>
<td>0.6274</td>
<td>0.6274</td>
<td>0.5816</td>
<td>1.0000</td>
<td>4.8767</td>
<td>4.8767</td>
<td>4.6836</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

From above, it appears that $d_5$ is optimal from the optimality point of view under both replacement rules. This fact is reflected from Table 1 under Type-I and Type-II replacement rules. In fact, $d_5$ is uniform design measured by centered $L_2$-discrepancy, which is given in the Uniform Design website: http://sites.stat.psu.edu/rli/DMCE/UniformDesign/.

Example 5. Consider the design $d_6 \in \mathcal{D}(12;4^3)$, given below, with $n = 12$ and $m = 3$.

$$d_6 = \begin{bmatrix} 2 & 2 & 3 & 1 & 0 & 1 & 3 & 0 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 & 1 & 2 & 0 & 1 & 0 & 2 & 2 & 3 \\ 1 & 3 & 1 & 3 & 2 & 3 & 0 & 0 & 2 & 2 & 1 & 0 \end{bmatrix}'.$$

Table 2: Numerical results of $d_6$

<table>
<thead>
<tr>
<th>Type</th>
<th>CD</th>
<th>LB1</th>
<th>LB2</th>
<th>Eff</th>
<th>WD</th>
<th>LB1</th>
<th>LB2</th>
<th>Eff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type I</td>
<td>0.2396</td>
<td>0.2256</td>
<td>0.2031</td>
<td>0.9416</td>
<td>1.1821</td>
<td>1.1501</td>
<td>1.0984</td>
<td>0.9729</td>
</tr>
<tr>
<td>Type II</td>
<td>0.5356</td>
<td>0.5356</td>
<td>0.4848</td>
<td>1.0000</td>
<td>4.5195</td>
<td>4.5195</td>
<td>4.3049</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

From above, it appears that $d_6$ is optimal from the optimality point of view under Type-II replacement rule. This fact is reflected from Table 2 under Type-II replacement rule. In fact, $d_6$ is uniform design measured by centered $L_2$-discrepancy, which is given in the Uniform Design website: http://sites.stat.psu.edu/rli/DMCE/UniformDesign/.
Example 6. Consider the design \( d_7 \in \mathcal{D}(8; 4^6) \), given below, with \( n = 8 \) and \( m = 6 \):

\[
d_7 = \begin{bmatrix}
1 & 2 & 0 & 3 & 1 & 0 & 2 & 3 \\
2 & 3 & 1 & 1 & 0 & 3 & 2 & 0 \\
3 & 2 & 1 & 0 & 0 & 3 & 1 & 2 \\
0 & 2 & 2 & 0 & 1 & 3 & 1 & 3 \\
3 & 0 & 3 & 2 & 0 & 2 & 1 & 1 \\
2 & 1 & 0 & 1 & 0 & 3 & 3 & 2
\end{bmatrix}.
\]

Table 3: Numerical results of \( d_7 \)

<table>
<thead>
<tr>
<th>Type</th>
<th>CD</th>
<th>LB1</th>
<th>LB2</th>
<th>Eff</th>
<th>WD</th>
<th>LB1</th>
<th>LB2</th>
<th>Eff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type I</td>
<td>1.4244</td>
<td>1.0856</td>
<td>1.3357</td>
<td>0.9377</td>
<td>17.9344</td>
<td>15.2808</td>
<td>17.2378</td>
<td>0.9612</td>
</tr>
<tr>
<td>Type II</td>
<td>6.0741</td>
<td>6.0741</td>
<td>6.0443</td>
<td>1.0000</td>
<td>207.0948</td>
<td>207.0948</td>
<td>206.2660</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

From above, it appears that \( d_7 \) is optimal from the optimality point of view under Type-II replacement rule. This fact is reflected from Table 3 under Type-II replacement rule. In fact, \( d_7 \) is a supersaturated design, which is given in the Supersaturated Design website: http://www.iasri.res.in/design/Supersaturated_Design/SSD/Supersaturated.html.

6 Concluding remarks

Using coding theory, our main objective here is to introduce a new direction to the theory of uniform designs. In this paper, we implemented modified Gray map codes and mapping between quaternary codes and the sequence of three binary codes to obtain four-level designs with high efficiency. Based on these codes, we proposed two types of replacement rules for four-level designs. The centered \( L_2 \)-discrepancy and wrap-around \( L_2 \)-discrepancy measures of uniformity are used for obtaining the efficiency of the designs. Some theoretical results related to the lower bounds of uniformity measures for such designs are also obtained in this paper. Based on illustrative examples (Examples 1-2), it is shown that the proposed replacement rules can be efficiently used to obtain four-level uniform designs with at least high efficiency.
As a concluding remark, it may be concluded that the optimality measure under replacement rule II justifies the introduction of the discrete discrepancy measure by Qin and Fang (2004).

Based on Hadamard matrices, as mentioned in Theorem 4.3.1 of Dey and Mukerjee (1999), Kronecker Calculus and using the replacement rules one can efficiently construct optimal designs belonging to the class \( D(n, 2^{m_1} \times 4^{m_2}) \) for some suitable choices of \( n, m_1 \) and \( m_2 \). For example, consider the following designs belonging to the classes \( D(8, 2^1 \times 4^3) \) and \( D(8, 2^2 \times 4^2) \) derived from array (3) using respective replacement rules Type-I and Type-II

\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\
0 & 2 & 1 & 3 & 1 & 3 & 0 & 2 \\
0 & 3 & 1 & 2 & 2 & 1 & 3 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 \\
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3
\end{bmatrix}
\]

The replacement rules can be efficiently used to obtain partial foldover four-level optimal designs. Moreover, these rules can be used to obtain partial foldover two- and four-level mixed optimal designs.

The replacement rules considered in this paper presents a novel direction to develop four-level uniform designs with high efficiency and deserves further attention. The construction of efficient uniform designs is also an interesting issue in this line, and will be studied in the future work.

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