A Framework For Estimation of Convex Functions

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Abstract

A general non-asymptotic framework, which evaluates the performance of any procedure at individual functions, is introduced in the context of estimating convex functions at a point. This framework, which is significantly different from the conventional minimax theory, is also applicable to other problems in shape constrained inference.

A benchmark is provided for the mean squared error of any estimate for each convex function in the same way that Fisher Information depends on the unknown parameter in a regular parametric model. A local modulus of continuity is introduced and is shown to capture the difficulty of estimating individual convex functions. A fully data-driven estimator is proposed and is shown to perform uniformly within a constant factor of the ideal benchmark for every convex function. Such an estimator is thus adaptive to every unknown function instead of to a collection of function classes as is typical in the nonparametric function estimation literature.

Keywords: Adaptive estimation, convex function, local modulus of continuity, minimax estimation, nonparametric regression, shape constrained inference, white noise model.

1 Introduction

The problem of estimating functions under assumptions of convexity or monotonicity has a long history dating back at least to Grenander (1956). The extensive literature on this topic has partly been motivated by specific applications but also by the fact that these shape constrained problems have features that are shared with regular parametric models and other features that are shared with nonparametric function estimation.

One connection with parametric models is that the least squares and maximum likelihood estimates perform well. On the other hand, as is typical in nonparametric function estimation, the rates of convergence in these models are slower than the usual root $n$ rate. For example Mammen (1991) established rates of convergence of the least squares estimate for the value of a piecewise convex/concave function at a point. Precise analysis of the asymptotic distributions for estimating a convex function has also been given in Groeneboom, Jongbloed and Wellner

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