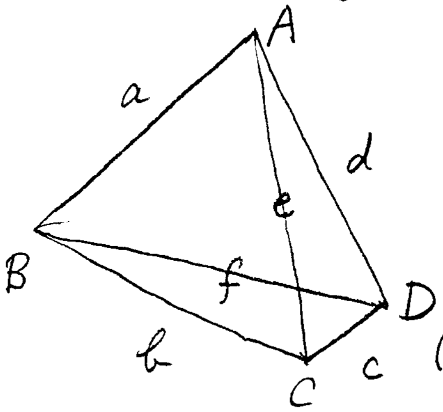
1. (Ptolemy) (1) 凸四邊形 $ABCD$, $AC=e$, $BD=f$, …



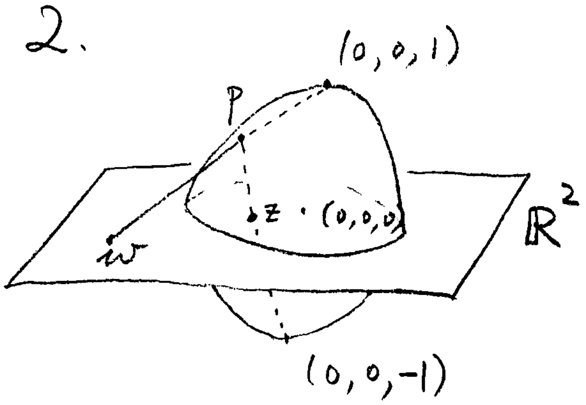
證: $ac+bd \geqslant ef$

"=" $\Longleftrightarrow$ A, B, C, D 共圓.

(2) 空間中四面体 $ABCD$

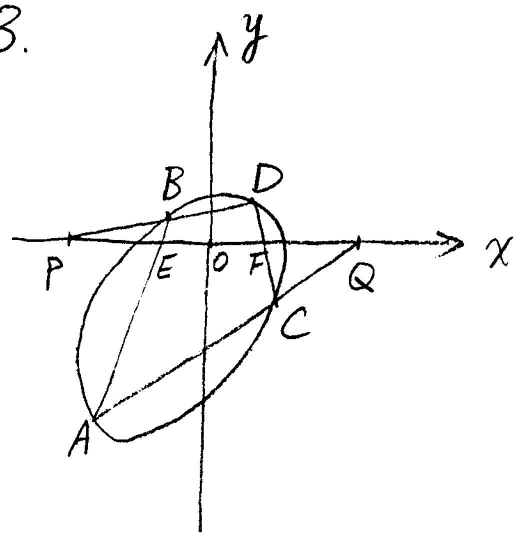
證: $ac+bd > ef$.

2.



(1) 證: $w=\frac{1}{\bar{z}}$

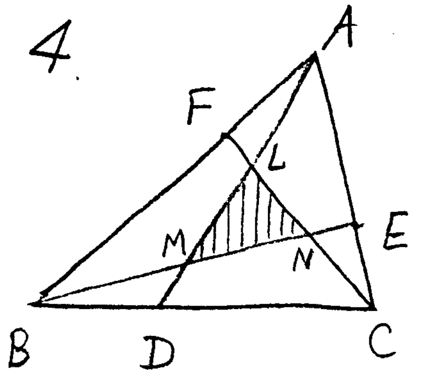
3.



O 為弦中點, $PO=OQ$

證: $OE=OF$.

4.

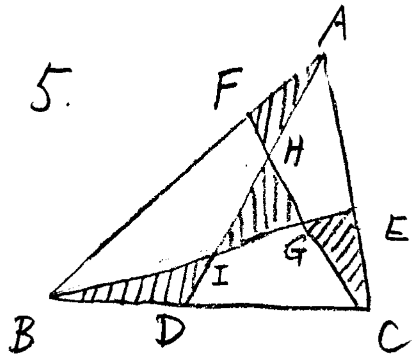


$$\frac{BD}{DC} = \frac{AF}{FB} = \frac{CE}{AE} = \lambda$$

證: Area $\triangle LMN$ / Area $\triangle ABC$

$$= \frac{1-2\lambda+\lambda^2}{1+\lambda+\lambda^2}.$$

5.



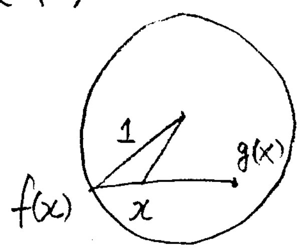
Area $\triangle AFH$ = Area $\triangle BDI$

= Area $\triangle CEG$ = Area $\triangle HIG$ = 1

證 三個四邊形 AHGE, BIHF, CGID 面積也相等，並求其面積。

6. 證明空間中正多面体只有已知的五种：正四、六、八、十二、二十面体。

2 (2)

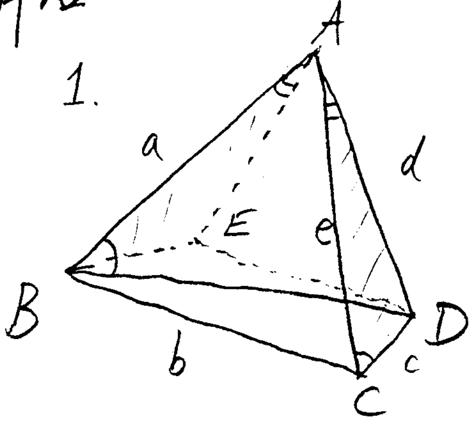


令 $u \equiv \dfrac{x-g(x)}{\|x-g(x)\|}$, $f(x) = x + tu$

求證 $t = -x\cdot u + \sqrt{1 - x\cdot x + (x\cdot u)^2}$

解答

1.



$$\frac{a}{e} \overset{(1)}{=} \frac{BE}{c} = \frac{AE}{d}$$

$$\frac{a}{AE} = \frac{e}{d}$$

$$\to \triangle ABC \sim \triangle AED$$

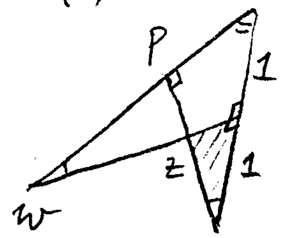
$$\to \frac{b}{ED} \overset{(2)}{=} \frac{e}{d}$$

$$(1) + (2) \Rightarrow ac + bd = e \cdot (BE + ED)$$

$$\geq e \cdot BD = ef$$

易 A, B, C, D 共圆 $\Longleftrightarrow \angle ABD = \angle ACD \Longleftrightarrow$ E 在 BD 上

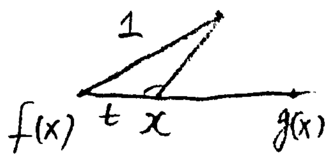
2. (1)



$$\text{AAA} \to \frac{|z|}{1} = \frac{1}{|w|}, \quad w = \lambda z$$

$$\to \lambda = \frac{1}{|z|^2} \quad \therefore w = \frac{1}{\bar{z}}$$

(2)

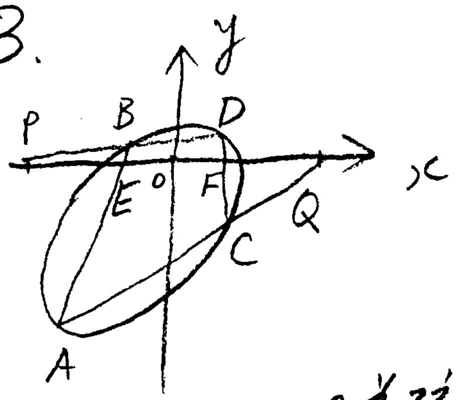


餘弦定理

$$1 = t^2 + \|x\|^2 - (-2t u \cdot x)$$

$$\therefore t = -x \cdot u + \sqrt{1 - \|x\|^2 + (x \cdot u)^2}$$

3.



$$ax^2+bxy+cy^2+dx+ey+f=0$$

$$y=0,\ ax^2+dx+f=0$$

O為弦中點 $\rightarrow x_1+x_2=0 \rightarrow d=0$

$P(-g,0)$, $Q(g,0)$

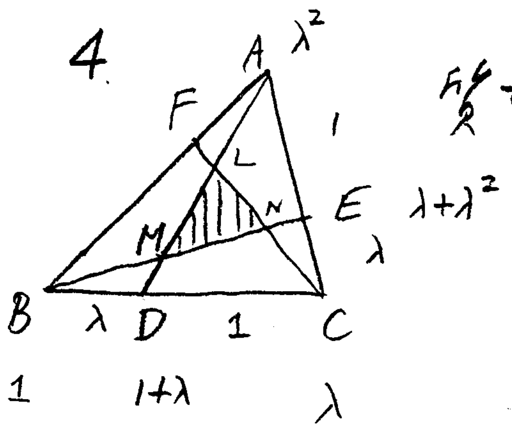
過ABCD四點的二次曲線系

$$0=\lambda_1(ax^2+bxy+cy^2+ey+f)+\lambda_2[y-k_1(x-g)][y-k_2(x+g)]$$

令 $y=0$, $\lambda_1 ax^2+\lambda_1 f+\lambda_2 k_1 k_2(x^2-g^2)=0\ \forall \lambda_1,\lambda_2$

無一次項 $\therefore x_E+x_F=0$ ($\overleftrightarrow{AB}\cup\overleftrightarrow{CD}$ for some λ_1,λ_2)

4.



質量重心法：

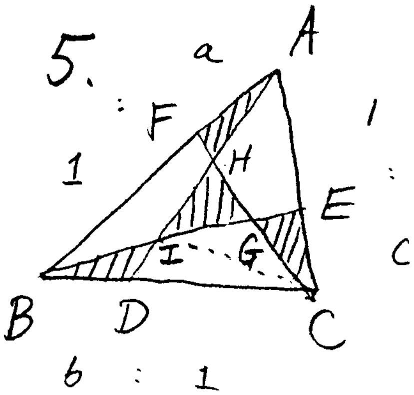
$$\frac{AM}{MD}=\frac{1+\lambda}{\lambda^2}$$

$$\frac{\text{Area}\triangle AMB}{\text{Area}\triangle ABC}=\frac{1+\lambda}{1+\lambda+\lambda^2}\cdot\frac{\lambda}{1+\lambda}=\frac{\lambda}{1+\lambda+\lambda^2}$$

$$\text{Area}\triangle LMN=\text{Area}\triangle ABC-\text{Area}\triangle ABM-\text{Area}\triangle BCN-\text{Area}\triangle ALC$$

$$\frac{\text{Area}\triangle LMN}{\text{Area}\triangle ABC}=1-\frac{3\lambda}{1+\lambda+\lambda^2}=\frac{1-2\lambda+\lambda^2}{1+\lambda+\lambda^2}$$

5.



$$\frac{AH}{HI} = \frac{AE}{EC} = \frac{1}{c}, \quad \frac{HI}{ID} = \frac{BI}{IG} = \frac{1}{a}$$

$$\rightarrow AH : HI : ID$$
$$= 1 : c : ca$$

$$\frac{\text{Area}\,\triangle AFH}{\text{Area}\,\triangle ABD} = \frac{\text{Area}\,\triangle BDI}{\text{Area}\,\triangle ABD}$$

$$\rightarrow \frac{1 \cdot a}{(1+c+ca)(a+1)} = \frac{ca}{1+c+ca} \rightarrow c = \frac{1}{1+a}$$

同理 $b = \frac{1}{1+c}$, $a = \frac{1}{1+b}$ $\therefore a^2 + a - 1 = 0$

$\rightarrow a = b = c = \frac{\sqrt{5}-1}{2}$, $BD : DC = HG : GC = b : 1$

$$\text{Area}\,\square CGID = \text{Area}\,\triangle CGI + \text{Area}\,\triangle CID$$
$$= \frac{1}{b} + \frac{1}{b}$$
$$= \frac{2}{b} = \sqrt{5} + 1$$

同理 $\text{Area}\,\square AEGH = \text{Area}\,\square FHIB = \sqrt{5} + 1$.