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Abstract

We are interested in a multivariate time series that can be expressed as the sum of a constant vector and an infinite-order moving average of a zero mean independent shock with density function obeying a uniform Lipschitz condition. By making use of this latter property, we derive a *negative* moment bound for the minimum eigenvalue of the corresponding normalized sample covariance matrix. Our result generalizes those in Findley and Wei (2002) and Ing and Wei (2003), which focus only on the case where the constant term is known to be zero.

Keywords: Infinite-Order Moving Average, Multivariate Time Series, Negative Moment Bounds, uniform Lipschitz Condition.

1 Main Results

Let

$$\mathbf{x}_t(k) = \mathbf{u} + \sum_{j=0}^{\infty} C_j \boldsymbol{\varepsilon}_{t,j}(k), \qquad t = k, \cdots, n \tag{1.1}$$

where $\mathbf{u}$ is a k-dimensional nonrandom vector, $\boldsymbol{\varepsilon}_{t,j}(k)$ is a k-dimensional random vector with $E(\boldsymbol{\varepsilon}_{t,j}(k)) = \mathbf{0}$ and $E(\boldsymbol{\varepsilon}_{t,j}(k)\boldsymbol{\varepsilon}'_{t,j}(k)) = \sum_{t,j}$ being positive definite, and C_j is a $k \times k$ nonrandom coefficient matrix. We shall also assume that

$$\sup_{\substack{-\infty < t < \infty \\ 0 \le j < \infty}} \|\Sigma_{t,j}\| < \infty, \tag{1.2}$$

$$\sum_{j=0}^{\infty} \|C_j\|^2 < \infty, \tag{1.3}$$

for each fixed t,

$$\{\boldsymbol{\varepsilon}_{t,j}(k), j = 0, 1, 2, \cdots\} \text{ are independent,} \tag{1.4}$$

and there exists a positive integer L s.t. for all $t - s \geq L$,

$$\boldsymbol{\varepsilon}_{t,0}(k) \text{ is independent of } \{\boldsymbol{\varepsilon}_{s,j}(k), j = 0, 1, 2, \cdots\} \tag{1.5}$$

Remark 1. *If* $x_t = v_1 + \sum_{j=0}^{\infty} b_j \delta^*_{t-j}$ *is a univariate stationary time series with* $\sum_{j=0}^{\infty} b_j^2 < \infty$, $E(\delta^*_{t-j}) = 0$ *and* $E(\delta^{*2}_{t-j}) = \sigma^2 < \infty$. *Then*

$$\begin{pmatrix} x_t \\ \vdots \\ x_{t-k+1} \end{pmatrix} = \mathbf{1}_k v_1 + \begin{pmatrix} 1 & b_1 & \cdots & \cdots & b_{k-1} \\ 0 & 1 & b_1 & \cdots & b_{k-2} \\ & \ddots & \ddots & & \vdots \\ 0 & \cdots & & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta^*_t \\ \vdots \\ \delta^*_{t-k+1} \end{pmatrix} + \begin{pmatrix} b_k & \cdots & b_{2k-1} \\ \vdots & & \vdots \\ b_1 & \cdots & b_k \end{pmatrix} \begin{pmatrix} \delta^*_{t-k} \\ \vdots \\ \delta^*_{t-2k+1} \end{pmatrix} + \cdots,$$

where $\mathbf{1}_k$ *is a* k*-dimensional vector of 1's, and hence* (1.1) *holds with* $\mathbf{x}_t(k) = (x_t, \cdots, x_{t-k+1})'$, $\mathbf{u} = \mathbf{1}_k v_1$,

$$C_0 = \begin{pmatrix} 1 & b_1 & \cdots & b_{k-1} \\ & \ddots & & \vdots \\ & & & 1 \end{pmatrix}, \quad C_j = \begin{pmatrix} b_{jk} & \cdots & b_{(j+1)k-1} \\ \vdots & & \vdots \\ b_{(j-1)k+1} & \cdots & b_{jk} \end{pmatrix}, \quad j \geq 1,$$

and $\boldsymbol{\varepsilon}_{t,j}(k) = (\delta^*_{t-jk}, \cdots, \delta^*_{t-(j+1)k+1})'$. *Moreover, it is easy to see that* (1.2)–(1.4) *are satisfied, and* (1.5) *holds with* $L = k$.

Remark 2. *If*

$$\mathbf{x}_t(k) = \boldsymbol{\mu} + \mathbf{y}_t,$$

where $\mathbf{y}_t$ *is zero-mean* k*-variate time series obeying* (52)–(54) *of Findley and Wei* (2002) (*except that "d" is replaced by "k"*), *then we have* $\boldsymbol{\varepsilon}_{t,j}(k) = \boldsymbol{\varepsilon}_{t-j}$, *where* $\boldsymbol{\varepsilon}_{t-j}$ *is defined in Findley and Wei* (2002), *and* (1.2)–(1.5) *hold with* L *in* (1.5) *equal to 1.*

Theorem 1. *Assume*

$$\sup_{\substack{-\infty < t < \infty \\ j \geq 0}} E\left(\boldsymbol{\varepsilon}'_{t,j}(k)\boldsymbol{\varepsilon}_{t,j}(k)\right)^{\gamma/2} < \infty \quad \text{for some} \quad \gamma > 2q, \tag{1.6}$$

and there exist positive constants $K, v,$ *and* δ *s.t.*

$$\boldsymbol{\varepsilon}_{t,j}(k) \in UL_{ip_k}(K, \delta, v), \tag{1.7}$$

where $UL_{ip_k}(K, \delta, v)$ *is defined in Findley and Wei (2002, p.432). Then, for any* $q \geq 1$,

$$E\left(\lambda_{\min}^{-q}\left(\frac{1}{N}\mathbf{X}'(I - \frac{E}{N})\mathbf{X}\right)\right) = O(1), \tag{1.8}$$

where

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_k'(k) \\ \vdots \\ \mathbf{x}_n'(k) \end{pmatrix}, \ N = n - k \ \textit{and} \ E = \mathbf{1}_N \mathbf{1}_N'.$$

2 Proof of Theorem 1.

Note first that (1.8) is implied by

$$E\left(\lambda_{\min}^{-q}\left(\frac{1}{N}\sum_{t=k}^{n-1}\boldsymbol{\phi}_t(k)\boldsymbol{\phi}_t'(k)\right)\right) = O(1), \tag{2.1}$$

where $\boldsymbol{\phi}_t(k) = (1, \mathbf{x}_t'(k))'$. Assume

$$C_0 \quad \text{is nonsingular.} \tag{2.2}$$

Then,

$$D = \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & C_0^{-1} \end{pmatrix}\begin{pmatrix} 1 & \mathbf{0}' \\ -\boldsymbol{\mu} & I \end{pmatrix} \quad \text{is nonsingular.}$$

Hence, (2.1) is equivalent to

$$E\left(\lambda_{\min}^{-q}\left(\frac{1}{N}\sum_{t=k}^{n-1}\boldsymbol{\phi}_t^*(k)\boldsymbol{\phi}_t^{*'}(k)\right)\right) = O(1), \tag{2.3}$$

where $\boldsymbol{\phi}_t^*(k) = D\boldsymbol{\phi}_t(k) = (1, \mathbf{x}_t^{*'}(k))'$ with

$$\mathbf{x}_t^*(k) = \boldsymbol{\varepsilon}_{t,0}(k) + \sum_{j=1}^{\infty} C_j^* \boldsymbol{\varepsilon}_{t,j}(k)$$

and $C_j^* = C_0^{-1} C_j$, noting that by (2.2) and (1.3), we have

$$\sum_{j=1}^{\infty} \left\| C_j^* \right\|^2 < \infty. \tag{2.4}$$

By an argument similar to that used in (2.8) and (2.9) of Ing and Wei (2003, JMVA), we have

$$\lambda_{\min}^{-q} \left(\frac{1}{N} \sum_{t=k}^{n-1} \boldsymbol{\phi}_t^*(k) \boldsymbol{\phi}_t^{*'}(k) \right) \leq \frac{1}{L} \sum_{j=0}^{L-1} (\frac{N}{L})^q \lambda_{\min}^{-q} \left(\sum_{i=0}^{N/L-1} \boldsymbol{\phi}_{iL+k+j}^*(k) \boldsymbol{\phi}_{iL+k+j}^{*'}(k) \right), \tag{2.5}$$

where N/L is assumed to be an integer for simplicity, and

$$\begin{aligned} &(\frac{N}{L})^q \lambda_{\min}^{-q} \left(\sum_{i=0}^{N/L-1} \boldsymbol{\phi}_{iL+k+j}^*(k) \boldsymbol{\phi}_{iL+k+j}^{*'}(k) \right) \\ &\qquad \leq \frac{C}{C_N} \sum_{s_1=0}^{C_N-1} (tL)^q \lambda_{\min}^{-q} \left(\sum_{i=0}^{tL-1} \boldsymbol{\phi}_{(i+s_1 tL)L+k+j}^*(k) \boldsymbol{\phi}_{(i+s_1 tL)L+k+j}^{*'}(k) \right), \end{aligned} \tag{2.6}$$

where C is a generic positive constant independent of n, $C_N = N/tL^2$ and $t \geq 1$ is a positive constant independent of n and will be specified later.

In view of (2.5) and (2.6), it suffices for (2.3) to show that for all $k \leq j \leq L+k-1$ and $0 \leq l_0 \leq N/L - tL$,

$$E \left(\lambda_{\min}^{-q} \left(\sum_{i=l_0}^{tL+l_0-1} \boldsymbol{\phi}_{iL+j}^*(k) \boldsymbol{\phi}_{iL+j}^{*'}(k) \right) \right) \leq C. \tag{2.7}$$

In the following, we only prove (2.7) for $j = k$ and $l_0 = 0$ since other cases can be proved similarly. Define

$$\mathbf{z}_i = \boldsymbol{\phi}_{iL+k}^*(k).$$

Then, the left-hand sides of (2.7) (with $l_0 = 0$ and $j = k$) is bounded by

$$k^* + \int_{k^*}^{\infty} P(D(\mu)) d\mu + \int_{k^*}^{\infty} P \left(T(\mu) \bigcap D^c(\mu) \right) d\mu := k^* + \text{(I)} + \text{(II)}, \tag{2.8}$$

where k^* satisfies $6{k^*}^{-(1/2q-1/\gamma)} \le \delta$ with δ defined in (1.7),

$$D(\mu) = \left\{ \sum_{i=0}^{tL-1} \|\mathbf{z}_i\|^2 > \mu^{2l/q}/(k+1) \right\}, \qquad l > q/2,$$

and

$$T(\mu) = \left\{ \inf_{\substack{\|\mathbf{y}\|=1 \\ \mathbf{y}\in R^k}} \sum_{i=0}^{tL-1} (\mathbf{z}_i'\mathbf{y})^2 < \mu^{-1/q} \right\}.$$

By (2.4), (1.4) and (1.2), it is easy to see that

$$(\mathrm{I}) = \int_{k^*}^{\infty} P(D(\mu))d\mu < \infty. \tag{2.9}$$

To deal with (II), define

$$\boldsymbol{\eta}_i = \mathbf{x}^*_{iL+k}(k) - \boldsymbol{\varepsilon}_{iL+k,0}(k).$$

Then, by (1.4) and (1.5),

$$\boldsymbol{\varepsilon}_{iL+k,0}(k) \text{ is independent of } \boldsymbol{\eta}_i \text{ and } \{\mathbf{z}_j,\ j \le i-1\}. \tag{2.10}$$

Now,

$$\begin{aligned} (\mathrm{II}) \ \le\ & \int_{k^*}^{\infty} P\left(\max_{0\le i\le tL-1} \|\boldsymbol{\eta}_i\| > s\mu^{1/\gamma} \right) d\mu \\ +\ & \int_{k^*}^{\infty} P\left(\max_{0\le i\le tL-1} \|\boldsymbol{\eta}_i\| \le s\mu^{1/\gamma}, Q(\mu) \right) d\mu := (\mathrm{III}) + (\mathrm{IV}), \end{aligned} \tag{2.11}$$

where $Q(\mu) = T(\mu) \bigcap D^c(\mu)$. Since

$$(\mathrm{III}) \le \sum_{i=0}^{tL-1} \int_{k^*}^{\infty} P\left(\|\boldsymbol{\eta}_i\|^\gamma s^{-\gamma} > \mu \right) d\mu \le \sum_{i=0}^{tL-1} s^{-\gamma} E\|\boldsymbol{\eta}_i\|^\gamma,$$

it follows from Burkhölder's and Minkowski's inequalities, (1.6) and (2.4) that

$$E\|\boldsymbol{\eta}_i\|^\gamma \le C,$$

yielding

$$(\text{III}) \leq Cs^{-\gamma} + L \leq C. \tag{2.12}$$

Following an argument similar to that used in p.137 of Ing and Wei (2003), there exist a positive integer $m^* \leq C\mu^{(l+1/2)q^{-1}(k+1)}$ and

$$\mathbf{l}_j = (l_{j_1}, l_{j_2}, \cdots, l_{j_{(k+1)}})' \equiv (l_{j_1}, \bar{\mathbf{l}}_j')' \in B = \{\mathbf{b} \in R^{k+1} : \mathbf{b}'\mathbf{b} = 1\},\ j = 1, \cdots, m^*,$$

such that

$$\begin{aligned}
& P\left(Q(\mu), \max_{0\leq i\leq tL-1} \|\boldsymbol{\eta}_i\| \leq s\mu^{1/\gamma}\right) \\
\leq\ & \sum_{j=1}^{m^*} P\left(\bigcap_{i=0}^{tL-1} \left\{|\mathbf{l}_j'\mathbf{z}_i| \leq 3\mu^{-1/2q}\right\},\ \max_{0\leq i\leq tL-1} \|\boldsymbol{\eta}_i\| \leq s\mu^{1/\gamma}\right) \\
\leq\ & \sum_{\substack{j=1 \\ \|\bar{\mathbf{l}}_j\|\geq \mu^{-1/\gamma}}}^{m^*} E\left(\prod_{i=0}^{tL-1} I_{\{|\mathbf{l}_j'\mathbf{z}_i|\leq 3\mu^{-1/2q}\}}\right) + \sum_{\substack{j=1 \\ \|\bar{\mathbf{l}}_j\|< \mu^{-1/\gamma}}}^{m^*} E\left(\prod_{i=0}^{tL-1} I_{\{|\mathbf{l}_j'\mathbf{z}_i|\leq 3\mu^{-1/2q}\}} \cdot I_{\max\limits_{0\leq i\leq tL-1}\|\boldsymbol{\eta}_i\| \leq s\mu^{1/\gamma}}\right) \\
:=\ & (\text{V}) + (\text{VI})
\end{aligned} \tag{2.13}$$

By (1.7) and (2.10),

$$\begin{aligned}
& E\left(\prod_{i=0}^{tL-1} I_{\{|\mathbf{l}_j'\mathbf{z}_i|<3\mu^{-1/2q}\}}\right) \\
=\ & E\left\{\prod_{i=0}^{tL-2} I_{\{|\mathbf{l}_j'\mathbf{z}_i|\leq 3\mu^{-1/2q}\}}\right. \\
& \left.\times P\left(-3\mu^{-1/2q} \leq l_{j_1} + \bar{\mathbf{l}}_j'\boldsymbol{\varepsilon}_{(tL-1)L+k,0}(k) + \bar{\mathbf{l}}_j'\boldsymbol{\eta}_{tL-1} < 3\mu^{-1/2q} \,\middle|\, \boldsymbol{\eta}_{tL-1},\, \mathbf{z}_i,\, 0\leq i\leq tL-2\right)\right\} \\
\leq\ & K(6\mu^{-(1/2q-1/\gamma)})^v E\left(\prod_{i=0}^{tL-2} I_{\{|\mathbf{l}_j'\mathbf{z}_i|\leq 3\mu^{-1/2q}\}}\right) \\
& \qquad \vdots \\
\leq\ & K^{tL} 6^{vtL} \mu^{-(1/2q-1/\gamma)vtL},
\end{aligned}$$

and hence

$$(\text{V}) \leq k^{tL} 6^{vtL} \mu^{-\{(1/2q-1/\gamma)vtL-(l+1/2)q^{-1}(k+1)\}}.$$

By choosing t large enough such that $(\frac{1}{2q} - \frac{1}{\gamma})vtL > (l + \frac{1}{2})q^{-1}(k+1) + 1$, it follows that

$$\text{(V) is integrable.} \tag{2.14}$$

On the other hand, when $\|\bar{\mathbf{l}}_j\| < \mu^{-1/\gamma}$,

$$\begin{aligned} & \left\{|\mathbf{l}_j'\mathbf{z}_i| \leq 3\mu^{-1/2q}\right\} \bigcap \left\{\max_{0\leq i\leq tL-1} \|\boldsymbol{\eta}_i\| \leq s\mu^{1/\gamma}\right\} \\ \subseteq \; & \left\{\mu^{-1/\gamma}\|\boldsymbol{\varepsilon}_{iL+k,0}(k)\| \geq (1-\mu^{-2/\gamma})^{1/2} - 3\mu^{-1/2q} - s\right\}, \; i = 0, \cdots, tL-1. \end{aligned} \tag{2.15}$$

By choosing $0 < s < 1/2$, there exists positive number k_2^* such that for all $\mu \geq k_2^*$,

$$(1-\mu^{-2/\gamma})^{1/2} - 3\mu^{-1/2q} - s \geq 1/4. \tag{2.16}$$

It follows from (1.2), (2.15), (2.16) and the independence of $\boldsymbol{\varepsilon}_{iL+k,0}(k), i = 0, \cdots, tL-1$ that

$$\text{(VI)} \leq C\mu^{-\{tL/\gamma-(l+1/2)q^{-1}(k+1)\}},$$

provided $\mu \geq k_2^*$. Moreover, by letting t large enough such that $\frac{tL}{\gamma} > (l+\frac{1}{2})q^{-1}(k+1)+1$, one obtains

$$\text{(VI) is integrable.} \tag{2.17}$$

Define $k_3^* = \max\{k^*, k_2^*\}$. Then, (2.13), (2.14) and (2.17) yield

$$\text{(IV)} \leq k_3^* + \int_{k_3^*} \text{(V)}d\mu + \int_{k_3^*} \text{(VI)}d\mu \leq C. \tag{2.18}$$

Combining (2.18), (2.12), (2.11), (2.9) and (2.8) gives the desired conclusion (2.7).

References

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