Asymptotic Properties of Maximum Likelihood Estimators in Geostatistical Regression Models

By

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1 Introduction

Suppose that we observe spatial data \( \{ (x(s_i), Z(s_i)) \}_{i=1}^{p_n} \) at \( n \) locations in \( D \subset \mathbb{R} \), where \( x(s_i) = (x_1(s_i), \ldots, x_{p_n}(s_i))' \) consists of \( p_n \) explanatory variables observed at \( s_i \in D \), and \( Z(s_i) \) is the corresponding response variable. We consider the following geostatistical regression model:

\[
Z(s_i) = \beta_0 + \sum_{j=1}^{p_n} \beta_j x_j(s_i) + \eta(s_i) + \epsilon(s_i); \quad i = 1, \ldots, n,
\]

where \( \eta(\cdot) \) is a Gaussian Ornstein-Uhlenbeck process (Uhlenbeck and Ornstein 1930), which has an exponential covariance function:

\[
\text{cov}(\eta(s), \eta(s')) = \sigma^2_\eta \exp(-\kappa|s - s'|); \quad s, s' \in D,
\]

with \( \sigma^2_\eta, \kappa > 0 \), \( \epsilon(s_i)'s \) are Gaussian white-noise variables with variance \( \eta(\cdot) \), and \( \{ \beta_0, \beta_1, \ldots, \beta_{p_n} \} \) are the regression parameters.

Throughout this article, we assume that \( D = [0, n^\delta] \) for some \( \delta \in [0, 1] \), and the data are observed regularly at \( s_i = in^{-(1-\delta)}; \ i = 1, \ldots, n, \) in \( D = [0, n^\delta] \). Obviously, when \( \delta > 0 \), the domain \( D \) grows to infinity as \( n \to \infty \) with a faster growing rate for a larger \( \delta \) value. We call the setting increasing domain asymptotics, even though the minimum inter data distance \( n^{-(1-\delta)} \) goes to zero except for \( \delta = 1 \). On the other hand, when \( \delta = 0 \) the domain is fixed at \( D = [0, 1] \), corresponding to fixed domain asymptotics. Re-parameterizing the exponential covariance parameters by \( \theta_2 = \sigma^2_\eta \kappa \) and \( \theta_3 = \kappa \), the covariance parameter vector can be written as \( \theta = (\theta_1, \theta_2, \theta_3)' \), where \( \theta_2 \) is often refereed as a microergodic parameter under the fixed domain asymptotic framework (Stein 1999).

Note that \( \{ \eta(s_1), \ldots, \eta(s_n) \} \) form a stationary AR(1) process:

\[
\eta(s_i) = \rho_n \eta(s_{i-1}) + \zeta_i,
\]

with \( \rho_n \equiv \exp(-\theta_3 n^{-(1-\delta)}) \), where \( \zeta_i \sim N(0, (1 - \rho^2_n) \theta_2/\theta_3) \) is independent of \( \eta(s_{i-1}) \), for \( i = 2, \ldots, n \). Rewrite (1.1) in a matrix form as:

\[
Z = (Z(s_1), \ldots, Z(s_n))' = X \beta + \eta + \epsilon,
\]

where \( X = (1, x(s_1), \ldots, x(s_n))' \) is the design matrix of dimension \( n \times (p_n + 1) \), \( \beta = (\beta_0, \ldots, \beta_{p_n})' \) is the regression parameter vector, \( \eta \equiv (\eta(s_1), \ldots, \eta(s_n))' \sim N(0, \Sigma_\eta(\theta)) \) and \( \epsilon = (\epsilon(s_1), \ldots, \epsilon(s_n))' \sim N(0, \theta_1 I) \). Then

\[
\Sigma(\theta) = \text{var}(\eta + \epsilon) = \Sigma_\eta(\theta) + \theta_1 I.
\]

We denote the true parameter vector of \( \theta \) by \( \theta_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3})' \), and the true mean of \( Z \) by \( \mu_0 \). In other words, the data \( Z \) are generated from \( N(\mu_0, \Sigma(\theta_0)) \). Throughout the paper, the parameters \( \beta \) and \( \theta \) are estimated using
maximum likelihood (ML). The ML estimate of \( \theta \), denoted by \( \hat{\theta} \), is obtained by maximizing the following profile log-likelihood function:

\[
\ell(\theta) = -\frac{1}{2}n \log(2\pi) - \frac{1}{2} \log \det(\Sigma(\theta)) - \frac{1}{2} Z'(I - M(\theta))\Sigma^{-1}(\theta)(I - M(\theta))Z,
\]

over a parameter space \( \Theta \), where

\[
M(\theta) \equiv X(X'\Sigma^{-1}(\theta)X)^{-1}X'\Sigma^{-1}(\theta).
\]

Clearly, \( \ell(\hat{\theta}) = \sup_{\theta \in \Theta} \ell(\theta) \), and the ML estimate of \( \beta \) is given by \( \hat{\beta}(\hat{\theta}) \), where

\[
\beta(\theta) = (X'\Sigma^{-1}(\theta)X)^{-1}X'\Sigma^{-1}(\theta)Z.
\]

Asymptotic properties of \( \hat{\theta} \) have been studied before under some situations. For example, when \( \delta = 1 \), consistency and asymptotic normality of \( \hat{\theta} \) follow from the standard asymptotic theory for ML estimation (see Mardia and Marshall 1984; Yao and Brockwell 2006). On the other hand, under the fixed domain asymptotic framework with \( \delta = 0 \), it has been shown by Ying (1991) (assuming \( \theta_1 = 0 \) known) and Chen, Simpson and Ying (2000) that both \( \sigma^2_\theta = \theta_2/\theta_3 \) and \( \kappa = \theta_3 \) are not consistently estimable, despite that the product, \( \sigma^2_\theta \kappa = \theta_2 \), can be estimated consistently by \( \hat{\theta}_2 \). Although they also derived the asymptotic distribution of \( \hat{\theta}_2 \), the results were established either without the regression term or assuming the regression model is correctly specified (i.e., \( \mu_0 = X\beta \) for some \( \beta \in \mathbb{R}^{p+1} \)). Some fixed domain asymptotic results have also been developed for spatial processes defined on \( D \subset \mathbb{R}^d \) with \( d > 1 \), including Ying (1993) who considered a multiplicative exponential covariance model, Loh (2005) who considered a multiplicative Matérn covariance model, Zhang (2004) who considered the Matérn covariance model for \( d = 2, 3 \), and Andresen (2010) who considered the Matérn covariance model for \( d > 5 \). Again, all the results were established without considering the regression term, and no asymptotic distribution was derived except Ying (1993). Some spatial bootstrapping methods with both fixed and increasing domain asymptotic results have also been developed by Lahiri and Zhu (2006).

In this paper, we shall investigate asymptotic properties of \( \hat{\theta} \) for any \( \delta \in [0, 1) \), corresponding to different growing rates of the spatial domain. We shall further allow the regression model to be mis-specified in the sense that \( \mu_0 \) may not be equal to \( \inf_{\beta \in \mathbb{R}^{p+1}} X\beta \), and explore the impact of

\[
R(\theta) = \max \{ \mu_0'(I - M(\theta))\Sigma^{-1}(\theta)(I - M(\theta))\mu_0, p_n \}
\]

on the asymptotic behaviors of \( \hat{\theta} \). Note that \( \mu_0'(I - M(\theta))\Sigma^{-1}(\theta)(I - M(\theta))\mu_0 \) and \( p_n \) are common measures of model mis-specification and model complexity, respectively, and \( R(\theta) = p_n \) if the model is correctly specified.

It is worth mentioning that \( \ell(\theta) \) can be highly convoluted when the regression term is involved, making it difficult to derive the asymptotic theory of \( \hat{\theta} \). Our strategy is to decompose the nonstochastic part of \( -2\ell(\theta) \) into several layers whose first three leading orders are \( n_1 \equiv n, n_2 \equiv n^{(1+\delta)/2} \) and \( n_3 \equiv n^\delta \), respectively, while express the remainder stochastic part as a sum of three terms: one centered quadratic form in \( \eta + \epsilon, h(\eta) \equiv (\eta + \epsilon)'\Sigma^{-1}(\theta)(\eta + \epsilon) - \text{tr}(\Sigma^{-1}(\theta)\Sigma(\theta_0)), \) one \( O_p(\sup_{\theta \in \Theta} R(\theta)) \) term, and one \( o_p(n^\delta) \) term (see (2.10)). One distinctive characteristic of these nonstochastic layers is that the coefficient associated with the \( i \)th (\( 1 \leq i \leq 3 \)) leading layer is a function of \( \theta_1, \ldots, \theta_{i-1} \). When the order of the magnitude of \( \sup_{\theta \in \Theta} R(\theta) \), say \( O(n^\gamma) \) for some \( \xi \in [0, 1) \), is determined, this hierarchical layer structure together with some uniform probability/moment bounds established for \( h(\theta) \) enables us to derive the convergence rates of \( \hat{\theta} \) in the order of \( \hat{\theta}_1, \hat{\theta}_2 \) and \( \hat{\theta}_3 \) by focusing on one layer and one parameter at a time. Moreover, once the initial convergence rates of \( \hat{\theta} \) are obtained, we can improve the rates further by repeating the same argument one more time, and so on. The final convergence rates can thus be obtained by this iterative procedure. In particular, it is shown in Theorem 2.3 that for \( 1 \leq i \leq 3, \hat{\theta}_i - \theta_{0,i} = O_p(\max\{n^\delta n_i^{-1}, n_i^{-1/2}\}) \) if \( n^\delta = o(n_i) \), and \( n_i^{1/2}(\hat{\theta}_i - \theta_{0,i}) \) has a limiting normal distribution if \( n_i^{1/2} = o(n_i^{1/2}) \).

The results established in this article not only reveal a novel interplay between the the degree of model mis-specification (or complexity) and the growing rate of the domain in the asymptotic behaviors of \( \hat{\theta} \), but also provide a solid mathematical foundation for geostatistical model selection in a companion paper by Chang, Huang and Ing (2012). Although our results are derived when both \( \eta(\cdot) \) and \( \epsilon(\cdot) \) are Gaussian. Similar results can be developed even when either \( \eta(\cdot) \) or \( \epsilon(\cdot) \) is not (but pretended to be) Gaussian, while having the right covariance structure, if some fourth moments information is available. The rest of this article is organized as follows. Our main results regarding consistency, rates of convergence and asymptotic normality of \( \hat{\theta} \) under different growing rates of the domain and orders of the magnitude of \( \sup_{\theta \in \Theta} R(\theta) \) are given in Section 2. Some technical lemmas are given in the appendix. The proofs of the lemmas are deferred to supplemental materials.
2 Main Results

The following proposition provides an uniform bound for the difference between $\Sigma(\theta_0)$ and $\Sigma(\theta)$ over all $\theta \in \Theta$, which plays a key role in establishing asymptotic properties of $\hat{\theta}$.

**Proposition 2.1.** Let $\Sigma_\eta(\theta)$ and $\Sigma(\theta)$ be given by (1.5), where $\theta \in \Theta \subset (0, \infty)^3$ with $\Theta$ a compact set. Then
\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta} \lambda_{\max}(\Sigma_\eta(\theta)) = O(n^{1-\delta}),
\]
and for any $\theta_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3})' \in \Theta$,
\[
0 < \liminf_{n \to \infty} \liminf_{\theta \to \theta_0} \lambda_{\min}(\Sigma^{-1/2}(\theta) \Sigma(\theta_0) \Sigma^{-1/2}(\theta)) \leq \limsup_{n \to \infty} \sup_{\theta \in \Theta} \lambda_{\max}(\Sigma^{-1/2}(\theta) \Sigma(\theta_0) \Sigma^{-1/2}(\theta)) < \infty.
\]

**Proof of Proposition 2.1.** First, we prove (2.1). Consider the spectral density of $\eta()$,
\[
f_\eta(\omega) = \frac{\theta_2(1 - \rho_n^2)}{2\pi \theta_3(1 - 2\rho_n \cos \omega + \rho_n^2)}; \quad \omega \in [0, 2\pi].
\]
By Proposition 4.5.3 of Brockwell and Davis (2009), we have
\[
\limsup_{n \to \infty} \lambda_{\max}(\Sigma_\eta(\theta)) \leq \limsup_{n \to \infty} 2\pi \sup_{\omega \in [0, 2\pi]} f_\eta(\omega) = \limsup_{n \to \infty} \frac{\theta_2(1 + \rho_n)}{\theta_3(1 - \rho_n)}.
\]
Thus (2.1) follows from $1 - \rho_n = \theta_3 n^{-(1-\delta)} + O(n^{-2(1-\delta)})$.

For (2.2), we shall only prove that
\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta} \lambda_{\max}(\Sigma^{-1/2}(\theta) \Sigma(\theta_0) \Sigma^{-1/2}(\theta)) < \infty.
\]
The proof for $0 < \inf_{n \in \mathbb{N}} \inf_{\theta \in \Theta} \lambda_{\min}(\Sigma^{-1/2}(\theta) \Sigma(\theta_0) \Sigma^{-1/2}(\theta))$ is analogous and omitted. Since
\[
\lambda_{\max}(\Sigma^{-1/2}(\theta) \Sigma(\theta_0) \Sigma^{-1/2}(\theta)) \leq \lambda_{\max}(\Sigma^{-1/2}(\theta) \Sigma_\eta(\theta_0) \Sigma^{-1/2}(\theta)) + \theta_{0,1} \lambda_{\max}(\Sigma^{-1}(\theta))
\]
\[
\leq \lambda_{\max}(\Sigma^{-1}(\theta) \Sigma_\eta(\theta_0)) + \theta_{0,1}/\theta_1,
\]
it suffices to show that $\limsup_{n \to \infty} \lambda_{\max}(\Sigma^{-1}_\eta(\theta) \Sigma_\eta(\theta_0)) < \infty$. By (1.3) and (A.1),
\[
\lambda_{\max}(\Sigma^{-1}(\theta) \Sigma_\eta(\theta_0)) = \lambda_{\max}(G_n(\theta)'D^{-1}_n(\theta)G_n(\eta)\text{E}(\eta'))
\]
\[
= \lambda_{\max}(D^{-1}_n(\theta)\text{var}(G_n(\theta)\eta)).
\]
Let $\rho_{0,n} = \exp(-\theta_{0,3} n^{-(1-\delta)})$. We have
\[
1 - \rho_n^k \rho_{0,n}^\ell = (k\theta_3 + \ell\theta_{0,3}) n^{-(1-\delta)} + O(n^{-2(1-\delta)}); \quad k, \ell \in \mathbb{Z},
\]
uniformly in $\Theta$. Rewrite $G_n(\theta)\eta$ in (2.3) as:
\[
G_n(\theta)\eta = \begin{pmatrix}
\eta(1) \\
\eta(2) - \rho_{0,n} \eta(1) \\
\vdots \\
\eta(n) - \rho_{0,n} \eta(n-1)
\end{pmatrix}
+ \rho_{0,n} - \rho_n \begin{pmatrix}
0 \\
\eta(1) \\
\vdots \\
\eta(n-1)
\end{pmatrix}.
\]
Then from (1.3), (2.1), (2.3), (2.4) and (A.4), we have
\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta} \lambda_{\max}(D^{-1}_n(\theta)\text{var}(\eta(1), \eta(2), \ldots, \eta(n))) < \infty,
\]
\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta} (\rho_{0,n} - \rho_n)^2 (1 - \rho_n^2)^{-1} \lambda_{\max}(\Sigma_\eta(\theta_0)) < \infty.
\]
This completes the proof.

The following theorem shows the consistency of $\hat{\theta}$ under a general asymptotic framework with the growing rate of the domain controlled by $\delta$. Here we allow the number of regressors to vary with $n$ and the regression model to be mis-specified.
Theorem 2.1. Consider the model of (1.4). Let $\theta = (\theta_1, \theta_2, \theta_3)' \in \Theta \subset (0, \infty)^3$, where $\Theta$ is compact. Suppose that the data $Z$ are generated from $N(\mu_0, \Sigma(\theta_0))$ with $\theta_0 = (\theta_{0.1}, \theta_{0.2}, \theta_{0.3})' \in \Theta$, and

$$\sup_{\theta \in \Theta} R(\theta) = O(n^\xi),$$

for some $\xi \in [0, 1)$, where $R(\theta)$ is defined in (1.8). Let $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)'$ be the ML estimate of $\theta$. Then, for $\delta \in [0, 1)$,

$$\begin{align*}
\hat{\theta}_1 &= \theta_{0.1} + o_p(1); & 0 \leq \xi < 1, \\
\hat{\theta}_2 &= \theta_{0.2} + o_p(1); & 0 \leq \xi < (1 + \delta)/2, \\
\hat{\theta}_3 &= \theta_{0.3} + o_p(1); & 0 \leq \xi < \delta.
\end{align*}$$

Proof. First, we prove (2.6). By (1.6), we can write

$$-2\ell(\theta) = n \log(2\pi) + \log \det(\Sigma(\theta)) + \mu_0' \Sigma^{-1}(\theta)(I - M(\theta))\mu_0 - 2\mu_0' \Sigma^{-1}(\theta)(I - M(\theta))(\eta + \epsilon) + (\eta + \epsilon)' \Sigma^{-1}(\theta)(I - M(\theta))(\eta + \epsilon) + n \log(2\pi) + \log \det(\Sigma(\theta)) + \text{tr}(\Sigma^{-1}(\theta)\Sigma(\theta_0)) + \mu_0' \Sigma^{-1}(\theta)(I - M(\theta))\mu_0 - 2\mu_0' \Sigma^{-1}(\theta)(I - M(\theta))(\eta + \epsilon) + h(\theta) - (\eta + \epsilon)' \Sigma^{-1}(\theta)M(\theta)(\eta + \epsilon),$$

where $h(\theta)$ is defined in Section 1. Therefore, by (2.5), (A.15), (A.18), (A.28), (A.31) and the fact that $\Sigma^{-1}(\theta)(I - M(\theta)) = (I - M(\theta))' \Sigma^{-1}(\theta)(I - M(\theta))$, we can write

$$-2\ell(\theta) = n \log(2\pi) - \frac{1 - \delta}{2} \log n + \left(\log \theta_1 + \frac{\theta_{0.1}}{\theta_1}\right)n + \left(\frac{2\theta_2}{\theta_1}\right)^{1/2} \left(1 - \frac{\theta_{0.1}}{2\theta_2} + \frac{\theta_{0.2}}{2\theta_2}\right)n^{(1+\delta)/2} - \frac{\theta_2}{\theta_1} + \frac{\theta_{0.2}}{\theta_2} - \frac{\theta_{0.3}}{\theta_2} n^\delta + \sup_{\theta \in \Theta} \left|h(\theta) - h(\theta_0)\right| + o_p(n^\delta),$$

uniformly in $\Theta$. Hence for (2.6) to hold, it suffices to show that for any $\epsilon > 0$,

$$P\left(\inf_{\theta \in \Theta_1(\epsilon)} \{-2\ell(\theta) + 2\ell(\theta_0)\} > 0\right) \to 1,$$

as $n \to \infty$, where $\Theta_1(\epsilon) = \{\theta \in \Theta : |\theta_1 - \theta_{0.1}| > \epsilon\}$. By (2.10),

$$\inf_{\theta \in \Theta_1(\epsilon)} \{-2\ell(\theta) + 2\ell(\theta_0)\} \geq \inf_{\theta \in \Theta_1(\epsilon)} \left\{\log \theta_1 + \frac{\theta_{0.1}}{\theta_1} - \log(\theta_{0.1}) - 1\right\}n - \sup_{\theta \in \Theta_1(\epsilon)} \left|h(\theta) - h(\theta_0)\right| + o_p(n),$$

where $\inf_{\theta \in \Theta_1(\epsilon)} \left\{\log \theta_1 + \frac{\theta_{0.1}}{\theta_1} - \log(\theta_{0.1}) - 1\right\} > 0$. Hence by Chebyshev’s inequality, it is enough to show that

$$E\left(\sup_{\theta \in \Theta} \left|h(\theta) - h(\theta_0)\right|^2\right) = O(n).$$

By Lemma B.1 of Chan and Ing (2011), and the first moment based theorem of Findley and Wei (1993), there exists some constant $C > 0$ such that

$$E\left(\sup_{\theta \in \Theta} \left|h(\theta) - h(\theta_0)\right|^2\right) \leq C \sup_{\theta \in \Theta} \left\{\text{var} \left(\frac{\partial}{\partial \theta_1} h(\theta)\right) + \text{var} \left(\frac{\partial}{\partial \theta_2} h(\theta)\right) + \text{var} \left(\frac{\partial}{\partial \theta_3} h(\theta)\right) \right\} + \text{var} \left(\frac{\partial^2}{\partial \theta_1 \partial \theta_2} h(\theta)\right) + \text{var} \left(\frac{\partial^2}{\partial \theta_1 \partial \theta_3} h(\theta)\right) + \text{var} \left(\frac{\partial^2}{\partial \theta_2 \partial \theta_3} h(\theta)\right) \right\}.$$
We remain to show that each term on the right-hand side of the above inequality is of order $O(n)$. First, we check the first order derivative terms. By (2.2) and (A.17), we have
\[
\sup_{\theta \in \Theta} \var \left( \frac{\partial}{\partial \theta_1} h(\theta) \right) = \sup_{\theta \in \Theta} 2 \text{tr} \left( \left( \frac{\partial}{\partial \theta_1} \Sigma^{-1}(\theta) \Sigma(\theta_0) \right)^2 \right) = \sup_{\theta \in \Theta} 2 \text{tr} \left( (\Sigma^{-2}(\theta) \Sigma(\theta_0))^2 \right) = O(n).
\]
By (2.2) and (A.16), we have
\[
\sup_{\theta \in \Theta} \var \left( \frac{\partial}{\partial \theta_2} h(\theta) \right) = \sup_{\theta \in \Theta} 2 \text{tr} \left( \left( \frac{\partial}{\partial \theta_2} \Sigma^{-1}(\theta) \Sigma(\theta_0) \right)^2 \right) = \sup_{\theta \in \Theta} \frac{1}{\theta_2} \text{tr} \left( (\Sigma^{-1}(\theta) \Sigma_n(\theta) \Sigma^{-1}(\theta) \Sigma(\theta_0))^2 \right) = O(n^{(1+\delta)/2}).
\]
By (2.2) and (A.24), we have
\[
\sup_{\theta \in \Theta} \var \left( \frac{\partial}{\partial \theta_3} h(\theta) \right) = \sup_{\theta \in \Theta} 2 \text{tr} \left( \left( \frac{\partial}{\partial \theta_3} \Sigma^{-1}(\theta) \Sigma(\theta_0) \right)^2 \right) = \sup_{\theta \in \Theta} 2 \text{tr} \left( (\Sigma^{-1}(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta) \Sigma(\theta_0))^2 \right) = O(n^2).
\]
Similarly, the second and the third order derivative terms can be shown to be of order $O(n^{(1+\delta)/2})$. This completes the proof of (2.6).

Next, we prove (2.7). It suffices to show that for any $\varepsilon_2 > 0$, there exists an $\varepsilon_1 > 0$ such that
\[
P \left( \inf_{\theta \in \Theta_{(\varepsilon_1)}} \{-2\ell(\theta) + 2\ell((\theta_1, \theta_{0,2}, \theta_{0,3}))\} > 0 \right) \to 1, \quad (2.13)
\]
as $n \to \infty$, where $\Theta_2(\varepsilon) = \{\theta \in \Theta : |\theta_1 - \theta_{0,1}| \leq \varepsilon_1, |\theta_2 - \theta_{0,2}| > \varepsilon_2\}$ and $\varepsilon = (\varepsilon_1, \varepsilon_2)'$. Let $\Theta_0 = (\theta_1, \theta_{0,2}, \theta_{0,3})'$. Since $\xi < (1 + \delta)/2$, by (2.10), we have
\[
\inf_{\theta \in \Theta_2(\varepsilon)} \{-2\ell(\theta) + 2\ell(\theta_0)\} \geq \inf_{\theta \in \Theta_2(\varepsilon)} \frac{1}{2|\theta_1 \theta_2|^{1/2}} \left( \left(\frac{\theta_1}{\theta_2} - \frac{\theta_{0,1}}{\theta_{0,2}}\right)^2 + (\theta_2 - \theta_{0,2}) \left(1 - \frac{\theta_{0,1}}{\theta_1}\right) \right) n^{(1+\delta)/2}
\]}

\[
- \sup_{\theta \in \Theta_2(\varepsilon)} |h(\theta) - h(\theta_0)| + o_p(n^{(1+\delta)/2}).
\]

Hence by Chebyshev's inequality, it is enough to show that
\[
E \left( \sup_{\theta \in \Theta} |h(\theta) - h(\theta_0)|^2 \right) = O_p(n^{(1+\delta)/2}). \quad (2.14)
\]
The desired result then follows from the same argument as in the proof of (2.12) and
\[
\sup_{\theta \in \Theta} \text{tr} \left( \left( \frac{\partial}{\partial \theta_1} (\Sigma^{-1}(\theta) - \Sigma^{-1}(\theta_0)) \Sigma(\theta_0) \right)^2 \right) = \sup_{\theta \in \Theta} \text{tr} \left( (\Sigma^{-2}(\theta) - \Sigma^{-2}(\theta_0)) \Sigma(\theta_0)^2 \right) = \sup_{\theta \in \Theta} \text{tr} \left( (\Sigma^2(\theta) - \Sigma^2(\theta_0)) \Sigma^{-2}(\theta_0) \Sigma(\theta_0)^2 \right) \times \left( (\Sigma^2(\theta_0) - \Sigma^2(\theta_0)) \Sigma^{-2}(\theta_0) \Sigma(\theta_0) \Sigma^{-2}(\theta_0) \right) = \sup_{\theta \in \Theta} \text{tr} \left( (\Sigma(\theta) + \Sigma(\theta_0)) \Sigma^{-2}(\theta_0) \Sigma(\theta_0)^2 \Sigma^{-2}(\theta_0) (\Sigma(\theta) + \Sigma(\theta_0)) \text{tr} \left( (\Sigma^{-1}(\theta) (\Sigma(\theta_0) - \Sigma(\theta)))^2 \right) \leq \sup_{\theta \in \Theta} \text{tr} \left( (\Sigma^{-1}(\theta) (\Sigma(\theta_0) - \Sigma(\theta)))^2 \right) = \sup_{\theta \in \Theta} \text{tr} \left( (\Sigma^{-1}(\theta) (\Sigma(\theta_0) - \Sigma(\theta)))^2 \right) = O(n^{(1+\delta)/2}),
\]
for some constant $C > 0$. This completes the proof of (2.7).
Finally, we prove (2.8). It suffices to show that for any $\varepsilon_3 > 0$, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that
\[
P \left( \inf_{\theta \in \Theta_3(\varepsilon)} \{-2\ell(\theta) + 2\ell((\theta_1, \theta_2, \theta_3, 0)' ) \} > 0 \right) \to 1, \tag{2.16}
\]
as $n \to \infty$, where $\Theta_3(\varepsilon) = \{ \theta \in \Theta : |\theta_1 - \theta_0| \leq \varepsilon_1, |\theta_2 - \theta_0| \leq \varepsilon_2, |\theta - \theta_0, 3| > \varepsilon_3 \}$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)'$. Let $\theta_c = (\theta_1, \theta_2, \theta_3)'$. Since $\xi < \delta$, by (2.10), we have
\[
\inf_{\theta \in \Theta_3(\varepsilon)} \{-2\ell(\theta) + 2\ell(\theta_c) \} \geq \inf_{\theta \in \Theta_3(\varepsilon)} \left\{ \frac{\theta_0 \theta_2 (\theta_3 - \theta_0, 3)}{2\theta_0, 3 \theta_2} - (\theta_3 - \theta_0, 3) (1 - \frac{\theta_0, 2}{\theta_2}) \right\} n^\delta
\]
\[
- \sup_{\theta \in \Theta_3(\varepsilon)} |h(\theta) - h(\theta_c)| + o_p(n^\delta),
\]
Hence by Chebyshev’s inequality, it is enough to show that
\[
E \left( \sup_{\theta \in \Theta} |h(\theta) - h(\theta_c)|^2 \right) = O(n^\delta). \tag{2.17}
\]
By Lemma B.1 of Chan and Ing (2011), and the first moment based theorem of Findley and Wei (1993), there exists some constant $C > 0$ such that
\[
E \left( \sup_{\theta \in \Theta} |h(\theta) - h(\theta_c)|^2 \right) \leq C \sup_{\theta \in \Theta} \left\{ \text{tr} \left( \left( \frac{\partial}{\partial \theta_1} (\Sigma^{-1}(\theta) - \Sigma^{-1}(\theta_c)) \Sigma(\theta_0) \right)^2 \right) \right.
\]
\[
+ \text{tr} \left( \left( \frac{\partial^2}{\partial \theta_2} (\Sigma^{-1}(\theta) - \Sigma^{-1}(\theta_c)) \Sigma(\theta_0) \right)^2 \right) \right.
\]
\[
+ \text{tr} \left( \left( \frac{\partial^2}{\partial \theta_1 \partial \theta_3} (\Sigma^{-1}(\theta) - \Sigma^{-1}(\theta_c)) \Sigma(\theta_0) \right)^2 \right) \right.
\]
\[
+ \left. \text{tr} \left( \left( \frac{\partial^3}{\partial \theta_1 \partial \theta_2 \partial \theta_3} (\Sigma^{-1}(\theta) - \Sigma^{-1}(\theta_c)) \Sigma(\theta_0) \right)^2 \right) \right\}.
\]
Following the arguments as in the proof of (2.12), we remain to show that
\[
\sup_{\theta \in \Theta} \text{tr} \left( \left( \frac{\partial}{\partial \theta_1} (\Sigma^{-1}(\theta) - \Sigma^{-1}(\theta_c)) \Sigma(\theta_0) \right)^2 \right) = O(n^\delta), \tag{2.18}
\]
\[
\sup_{\theta \in \Theta} \text{tr} \left( \left( \frac{\partial^2}{\partial \theta_2} (\Sigma^{-1}(\theta) - \Sigma^{-1}(\theta_c)) \Sigma(\theta_0) \right)^2 \right) = O(n^\delta), \tag{2.19}
\]
\[
\sup_{\theta \in \Theta} \text{tr} \left( \left( \frac{\partial^2}{\partial \theta_1 \partial \theta_3} (\Sigma^{-1}(\theta) - \Sigma^{-1}(\theta_c)) \Sigma(\theta_0) \right)^2 \right) = O(n^\delta). \tag{2.20}
\]
First, we show (2.18). Similar to (2.15), we have
\[
\text{tr} \left( \left( \frac{\partial}{\partial \theta_1} (\Sigma^{-1}(\theta) - \Sigma^{-1}(\theta_c)) \Sigma(\theta_0) \right)^2 \right) \leq C \text{tr} \left( (\Sigma^{-1}(\theta) \Sigma(\theta_c) - \Sigma(\theta_0))^2 \right)
\]
\[
= C \text{tr} \left( (T_{n-1}^{-1}(\theta) G_n(\theta)(\Sigma(\theta_c) - \Sigma(\theta_0)) G_n(\theta)^2) \right),
\]
for some constant $C > 0$, where the last equality follows from (A.1). Since by (A.9),
\[
G_n(\theta)(\Sigma(\theta_c) - \Sigma(\theta)) G_n(\theta)^2
\]
\[
= \left( \frac{\theta_2 \rho_n}{\theta_0, 3 \rho_{0, n}} (1 - \rho_1^2) - \frac{\theta_2}{\theta_3} (1 - \rho_1^2) \right) I + \left( 1 - \frac{\rho_n}{\rho_{0, n}} \right) (1 - \rho_n \rho_{0, n}) \Sigma(\theta_c)
\]
\[
+ \frac{\theta_2}{\theta_0, 3} \left( 1 - \frac{\rho_n}{\rho_{0, n}} \right) (v_0 e_1' + e_1 v_0') + \theta_2 \left( \frac{1}{\theta_0, 3} - \frac{1}{\theta_0, 3} \right) \rho_n e_1 e_1',
\]
therefore by (2.4), for (2.18) to hold, it suffices to show that
\[
\sup_{\theta \in \Theta} n^{-4(1-\delta)} \text{tr} (T_{n-1}^{-2}(\theta)) = O(n^\delta),
\]
\[
\sup_{\theta \in \Theta} n^{-4(1-\delta)} \text{tr} \left( (T_{n-1}^{-1}(\theta) \Sigma(\theta_c))^2 \right) = O(n^\delta),
\]
\[
\sup_{\theta \in \Theta} n^{-2(1-\delta)} \text{tr} \left( (T_{n-1}^{-1}(\theta)(v_0 e_1' + e_1 v_0'))^2 \right) = O(1),
\]
\[
\sup_{\theta \in \Theta} \text{tr} \left( (T_{n-1}^{-1}(\theta) e_1 e_1')^2 \right) = O(1).
\]
where the first equation follows from (A.6), the second equation follows from (A.14) and the last two equations follow from (A.10)-(A.12). Thus (2.18) is established.

Similar arguments can be applied to prove (2.19) and (2.20). Their details are omitted. Thus (2.17) is established. This completes the proof of (2.8).

The following theorem shows that $\hat{\theta}$ is asymptotically normal when the regression model is correctly specified. It’s interesting to see that the convergence rate of $\hat{\theta}_1$ does not depend on $\delta$. On the other hand, the convergence rates of $\hat{\theta}_2$ and $\hat{\theta}_3$ vary linearly with $\delta$, which are in contrast to the convergence rates of $\hat{\theta}_2$ and $\hat{\theta}_3$ in Theorem 2.3, where the regression model may be mis-specified, and the convergence rates are only piecewise linear functions of $\delta$ for a given $\xi$.

**Theorem 2.2.** Consider the model of (1.4) with $p_n$ being fixed. Let $\theta = (\theta_1, \theta_2, \theta_3)' \in \Theta \subset (0, \infty)^3$, where $\Theta$ is compact. Suppose that the data $Z$ are generated from $N(\mu_0, \Sigma(\theta_0))$ with $\theta_0 = (\theta_{01}, \theta_{02}, \theta_{03})'$ being an interior point of $\Theta$ and $\mu_0 = X\beta$ for some $\beta \in \mathbb{R}^{p_n+1}$. Let $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)'$ be the ML estimate of $\theta$. Then for $\delta \in [0, 1)$,

$$n^{1/2}(\hat{\theta}_1 - \theta_{01}) \xrightarrow{d} N(0, 2\theta_{01}^2),$$  

(2.21) $$n^{(1+\delta)/4}(\hat{\theta}_2 - \theta_{02}) \xrightarrow{d} N(0, 2^{5/2}\theta_{01}^2\theta_{02}^2),$$  

(2.22) $$n^{5/2}(\hat{\theta}_3 - \theta_{03}) \xrightarrow{d} N(0, 2\theta_{03}),$$  

(2.23)

and for $\delta \in (0, 1)$,

Proof. We shall prove (2.21)-(2.23) by iterating applying (A.39)-(A.48). For the first iteration, we show that

$$\hat{\theta}_1 - \theta_{01} = O_p(n^{-1/2}), \quad \text{if } \delta \in [0, 1),$$  

(2.24) $$n^{(1+\delta)/4}(\hat{\theta}_2 - \theta_{02}) \xrightarrow{d} N(0, 2^{5/2}\theta_{01}^2\theta_{02}^2); \quad \text{if } \delta \in [0, 1/3),$$  

(2.25) $$\hat{\theta}_2 - \theta_{02} = O_p(n^{-(1-\delta)/2}); \quad \text{if } \delta \in [1/3, 1),$$  

(2.26) $$\hat{\theta}_3 - \theta_{03} = O_p(n^{-(1-\delta)/2}); \quad \text{if } \delta \in [1/2, 1).$$  

Proof of (2.24): Taking the Taylor expansion of $g_1(\theta)$ at $\hat{\theta}_a = (\theta_{01}, \hat{\theta}_2, \hat{\theta}_3)'$ yields

$$0 = g_1(\hat{\theta}) = g_1(\hat{\theta}_a) + g_1(\hat{\theta}_a)'(\hat{\theta}_1 - \theta_{01}),$$  

(2.27)

where $\hat{\theta}_a = (\theta_{01}, \hat{\theta}_2, \hat{\theta}_3)'$ satisfies $|\theta_{01} - \hat{\theta}_1| \leq |\theta_{01} - \theta_{01}|$. Therefore, for (2.24) to hold, it suffices to show that

$$g_1(\hat{\theta}_a) = O_p(n^{1+\delta/2}),$$  

$$g_1(\hat{\theta}_a)' = \frac{\partial}{\partial \theta_{01}},$$

where the first equation follows from (2.7) and (A.39) with $r_2 = 0$, and the last equation follows from (2.6) and (A.46).

Proof of (2.25): Let $\hat{\theta}_b = (\hat{\theta}_1, \theta_{02}, \hat{\theta}_3)'$. Taking the Taylor expansion of $g_2(\theta)$ at $\hat{\theta}_b = (\hat{\theta}_1, \theta_{02}, \hat{\theta}_3)'$ yields

$$0 = g_2(\hat{\theta}) = g_2(\hat{\theta}_b) + g_{22}(\hat{\theta}_b)'(\hat{\theta}_2 - \theta_{02}),$$  

(2.28)

where $\hat{\theta}_b = (\hat{\theta}_1, \theta_{02}, \hat{\theta}_3)'$ satisfies $|\theta_{02} - \hat{\theta}_2| \leq |\theta_{02} - \hat{\theta}_2|$. Therefore, for (2.25) to hold, it suffices to show that

$$n^{-(1+\delta)/4}g_2(\hat{\theta}_b) \xrightarrow{d} N(0, 2^{-1/2}\theta_{01}^{-1/2}\theta_{02}^{-3/2}); \quad \text{if } \delta \in [0, 1/3),$$  

$$g_2(\hat{\theta}_b) = O_p(n^{\delta/2}); \quad \text{if } \delta \in [1/3, 1),$$  

$$g_{22}(\hat{\theta}_b)' = \frac{n^{3/2}g_2(\hat{\theta}_b)}{2^{1/2}\theta_{01}^{3/2}\theta_{02}^{3/2}},$$

where the first two equations follow from (2.24), (A.40) with $r_1 = (1 - \delta)/2$, and (A.43), and the last equation follows from (2.7), (2.24) and (A.47).

Proof of (2.26): Taking the Taylor expansion of $g_3(\theta)$ at $\hat{\theta}_c$ yields

$$0 = g_3(\hat{\theta}) = g_3(\hat{\theta}_c) + g_{33}(\hat{\theta}_c)'(\hat{\theta}_3 - \theta_{03}),$$  

(2.29)
where \( \hat{\theta}_c^* = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_{0,3}) \) satisfies \( |\theta_{0,3} - \hat{\theta}_3| \leq |\theta_{0,3} - \hat{\theta}_3| \). Therefore, for (2.26) to hold, it suffices to show that

\[
n^{-5/2}g_3(\hat{\theta}_c) \xrightarrow{d} N(0, 2\theta_{0,1}^2); \quad \text{if } \delta \in (0, 1/2),
\]

\[
g_3(\hat{\theta}_c) = O_p(n^{-(1-3\delta)/2}); \quad \text{if } \delta \in [1/2, 1),
\]

\[
g_3(\hat{\theta}_c^*) = n^{\delta} \hat{\theta}_{0,3} + o_p(n^{\delta}),
\]

where the first two equations follow from (2.24), (2.25), (A.41) with \( r_1 = r_2 = (1 - \delta)/2 \), and (A.44), and the last equation follows from (A.48). Thus (2.26) is established.

For the second iteration, we show that

\[
n^{1/2}(\hat{\theta}_1 - \theta_{0,1}) \xrightarrow{d} N(0, 2\theta_{0,1}^2); \quad \text{if } \delta \in [0, 1/2),
\]

\[
\hat{\theta}_1 - \theta_{0,1} = O_p(n^{-(1-\delta)}); \quad \text{if } \delta \in [1/2, 1)
\]

(2.30)

\[
n^{(1+\delta)/4}(\hat{\theta}_2 - \theta_{0,2}) \xrightarrow{d} N(0, 2^{5/4}2^{1/2}\theta_{0,1}^2\theta_{0,2}^2); \quad \text{if } \delta \in [0, 3/5),
\]

\[
\hat{\theta}_2 - \theta_{0,2} = O_p(n^{-(1-\delta)}); \quad \text{if } \delta \in [3/5, 1)
\]

(2.31)

\[
n^{5/2}(\hat{\theta}_3 - \theta_{0,3}) \xrightarrow{d} N(0, 2\theta_{0,3}); \quad \text{if } \delta \in (0, 1/2),
\]

\[
\hat{\theta}_3 - \theta_{0,3} = O_p(n^{-(1-\delta)}); \quad \text{if } \delta \in [2/3, 1)
\]

(2.32)

By (2.25), (A.39) with \( r_2 = r_3 = (1 - \delta)/2 \), and (A.42), we have

\[
n^{-1/2}g_1(\hat{\theta}_c) \xrightarrow{d} N(0, 2\theta_{0,1}^2); \quad \text{if } \delta \in [0, 1/2),
\]

\[
g_1(\hat{\theta}_c) = O_p(n^{\delta}); \quad \text{if } \delta \in [1/2, 1).
\]

These together with (2.27) and (A.46) give (2.30). By (2.26), (2.30), (A.40) with \( r_1 = 1 - \delta \) and \( r_3 = (1 - \delta)/2 \), and (A.43), we have

\[
n^{-(1+\delta)/4}g_2(\hat{\theta}_s) \xrightarrow{d} N(0, 2^{-1/2}2\theta_{0,1}^{1/2}\theta_{0,2}^{-3/2}); \quad \text{if } \delta \in (0, 3/5),
\]

\[
g_2(\hat{\theta}_s) = O_p(n^{(1-3\delta)/2}); \quad \text{if } \delta \in [3/5, 1),
\]

These together with (2.28) and (A.47) give (2.31). By (2.30), (2.31), (A.41) with \( r_1 = r_2 = 1 - \delta \), and (A.44), we have

\[
n^{-5/2}g_3(\hat{\theta}_c) \xrightarrow{d} N(0, 2\theta_{0,1}^{-1}); \quad \text{if } \delta \in (0, 1/2),
\]

\[
g_3(\hat{\theta}_c) = O_p(n^{-(1-\delta)/2}); \quad \text{if } \delta \in [2/3, 1)
\]

These together with (2.29) and (A.48) give (2.32).

Following the same arguments as in the second iteration, we can recursively show for \( i = 3, 4, \ldots \), that

\[
n^{1/2}(\hat{\theta}_1 - \theta_{0,1}) \xrightarrow{d} N(0, 2\theta_{0,1}^2); \quad \text{if } \delta \in [0, (i - 1)/i),
\]

\[
\hat{\theta}_1 - \theta_{0,1} = O_p(n^{-(1-\delta)/2}); \quad \text{if } \delta \in [(i - 1)/i, 1),
\]

\[
n^{(1+\delta)/4}(\hat{\theta}_2 - \theta_{0,2}) \xrightarrow{d} N(0, 2^{5/4}2\theta_{0,1}^{1/2}\theta_{0,2}^{3/2}); \quad \text{if } \delta \in (0, (2i - 1)/(2i + 1)),
\]

\[
\hat{\theta}_2 - \theta_{0,2} = O_p(n^{-(1-\delta)/2}); \quad \text{if } \delta \in [(2i - 1)/(2i + 1), 1]
\]

\[
n^{5/2}(\hat{\theta}_3 - \theta_{0,3}) \xrightarrow{d} N(0, 2\theta_{0,3}); \quad \text{if } \delta \in (0, i/(i + 1)),
\]

\[
\hat{\theta}_3 - \theta_{0,3} = O_p(n^{-(1-\delta)/2}); \quad \text{if } \delta \in [i/(i + 1), 1)
\]

Thus we obtain (2.21)-(2.23). This completes the proof. \( \square \)

Note that the diagonal elements of the Fisher information matrix evaluated at \( \theta = \theta_0 \) are given by

\[
-E \left( \frac{\partial^2}{\partial \theta_i^2} \ell(\theta_0) \right) = \frac{n}{2\theta_{0,i}^2} + O(n^{\delta}) + o(n),
\]

\[
-E \left( \frac{\partial^2}{\partial \theta_1^2} \ell(\theta_0) \right) = \frac{n^{(1+\delta)/2}}{2^{5/4}2\theta_{0,1}^{1/2}\theta_{0,2}^{3/2}} + O(n^{\delta}) + o(n^{(1+\delta)/2}),
\]

\[
-E \left( \frac{\partial^2}{\partial \theta_1^2} \ell(\theta_0) \right) = \frac{n^\delta}{2\theta_{0,3}} + O(n^{\delta}) + o(n^{\delta}), \text{ if } 0 < \delta < 1.
\]

It is interesting to point out that the denominators on the right-hand sides of these identities coincide exactly with the limiting variances in (2.21)-(2.23). This is reminiscent of a conventional asymptotic theory for the ML estimate which
2 Main Results

Consider the model of (2.3) Let \( \theta = (\theta_1, \theta_2, \theta_3)' \in \Theta \subset (0, \infty)^3 \), where \( \Theta \) is compact. Suppose that the data \( Z \) are generated from \( N(\mu_0, \Sigma(\theta_0)) \) with \( \theta_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3})' \) being an interior point of \( \Theta \), and (2.5) holds for some \( \xi \in [0, 1) \). Let \( \hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)' \) be the ML estimate of \( \theta \). Then for \( \delta \in [0, 1) \),

\[
\hat{\theta}_1 - \theta_{0,1} = \begin{cases} 
O_p(n^{-1/2}); & \text{if } \xi < 1/2, \\
O_p(n^{-(1-\xi)}); & \text{if } 1/2 \leq \xi < 1,
\end{cases}
\]

(2.33)

\[
\hat{\theta}_2 - \theta_{0,2} = \begin{cases} 
O_p(n^{-(1+\delta)/4}); & \text{if } \xi < (1+\delta)/4, \\
O_p(n^{-(1+\delta)/2-\xi}); & \text{if } (1+\delta)/4 \leq \xi < (1+\delta)/2,
\end{cases}
\]

(2.34)

and for \( \delta \in (0, 1) \),

\[
\hat{\theta}_3 - \theta_{0,3} = \begin{cases} 
O_p(n^{-\delta/2}); & \text{if } \xi < \delta/2, \\
O_p(n^{-(\delta-\xi)}); & \text{if } \delta/2 \leq \xi < \delta.
\end{cases}
\]

(2.35)

In addition, for \( \delta \in [0, 1) \),

\[
n^{1/2}(\hat{\theta}_1 - \theta_{0,1}) \overset{d}{\to} N(0, 2\theta_{0,1}^2); \quad \text{if } \xi < 1/2,
\]

\[
n^{(1+\delta)/4}(\hat{\theta}_2 - \theta_{0,2}) \overset{d}{\to} N(0, 2^{5/2}(1+\delta)\theta_{0,1}^2\theta_{0,2}^{3/2}); \quad \text{if } \xi < (1+\delta)/4,
\]

and for \( \delta \in (0, 1) \),

\[
n^{\delta/2}(\hat{\theta}_3 - \theta_{0,3}) \overset{d}{\to} N(0, 2\theta_{0,3}); \quad \text{if } \xi < \delta/2.
\]

(2.36)

(2.37)

(2.38)

(2.39)

(2.40)

(2.41)

(2.42)

(2.43)

(2.44)

Proof. We divide the proof into three parts corresponding to \( \delta \in [0, 1/3), \delta \in [1/3, 1/2) \) and \( \delta \in [1/2, 1) \).

First, we consider \( \delta \in [0, 1/3) \). We further divide the proof into six subparts with respect to \( \xi \) in terms of a partition of \( [0, 1) \), corresponding to \( \xi \in [0, \delta/2), \xi \in [\delta/2, \delta), \xi \in [\delta, (1+\delta)/4), \xi \in [(1+\delta)/4, 1/2), \xi \in [1/2, (1+\delta)/2) \) and \( \xi \in [(1+\delta)/2, 1) \). We shall prove each of the following six subparts separately:

(a1) For \( \xi \in [(1+\delta)/2, 1) \),

\[
\hat{\theta}_1 - \theta_{0,1} = O_p(n^{\xi-1}).
\]

(2.36)

(a2) For \( \xi \in [1/2, (1+\delta)/2) \),

\[
\hat{\theta}_1 - \theta_{0,1} = O_p(n^{\xi-1}), \\
\hat{\theta}_2 - \theta_{0,2} = O_p(n^{(1+\delta)/2}).
\]

(2.37)

(a3) For \( \xi \in [(1+\delta)/4, 1/2) \),

\[
n^{1/2}(\hat{\theta}_1 - \theta_{0,1}) \overset{d}{\to} N(0, 2\theta_{0,1}^2), \\
\hat{\theta}_2 - \theta_{0,2} = O_p(n^{(1+\delta)/2}).
\]

(2.38)

(2.39)

(a4) For \( \xi \in [\delta, (1+\delta)/4) \),

\[
n^{1/2}(\hat{\theta}_1 - \theta_{0,1}) \overset{d}{\to} N(0, 2\theta_{0,1}^2), \\
n^{(1+\delta)/4}(\hat{\theta}_2 - \theta_{0,2}) \overset{d}{\to} N(0, 2^{5/2}(1+\delta)\theta_{0,1}^2\theta_{0,2}^{3/2}).
\]

(2.40)

(2.41)

(a5) For \( \xi \in [\delta/2, \delta) \),

\[
n^{1/2}(\hat{\theta}_1 - \theta_{0,1}) \overset{d}{\to} N(0, 2\theta_{0,1}^2), \\
n^{(1+\delta)/4}(\hat{\theta}_2 - \theta_{0,2}) \overset{d}{\to} N(0, 2^{5/2}(1+\delta)\theta_{0,1}^2\theta_{0,2}^{3/2}), \\
\hat{\theta}_3 - \theta_{0,3} = O_p(n^{\xi-\delta}).
\]

(2.42)

(2.43)

(2.44)
(a6) For $\xi \in [0, \delta/2)$,
\[
n^{1/2}(\hat{\theta}_1 - \theta_{0,1}) \xrightarrow{d} N(0, 2\theta_{0,1}^2), \tag{2.45}
\]
\[
n^{(1+\delta)/4}(\hat{\theta}_2 - \theta_{0,2}) \xrightarrow{d} N(0, 2^{5/2}\theta_{0,1}^{1/2}\theta_{0,2}^{3/2}), \tag{2.46}
\]
and if in addition $\delta \neq 0$, then
\[
n^{3/2}(\hat{\theta}_3 - \theta_{0,3}) \xrightarrow{d} N(0, 2\theta_{0,3}). \tag{2.47}
\]

Proof of (a1): Applying (A.39) with $r_1 = r_2 = r_3 = 0$ and $\xi \in [(1+\delta)/2, 1)$, we have
\[
g_1(\hat{\theta}_a) = O_p(n^{\xi}). \tag{2.48}
\]
From (2.6) and (A.46), we have
\[
g_{11}(\hat{\theta}_a) = \frac{n}{\theta_{0,1}^2} + o_P(n). \tag{2.49}
\]
The desired result is obtained by plugging (2.48) and (2.49) into (2.27).

Proof of (a2): Applying (A.39) with $r_1 = r_2 = r_3 = 0$ and $\xi \in [1/2, (1+\delta)/2)$, we have
\[
g_1(\hat{\theta}_a) = O_p(n^{(1+\delta)/2}).
\]
Combining this with (2.27) and (2.49) gives
\[
\hat{\theta}_1 - \theta_{0,1} = O_P(n^{-(1-\delta)/2}). \tag{2.50}
\]
Applying (A.40) with $r_1 = (1-\delta)/2$, $r_2 = r_3 = 0$ and $\xi \in [1/2, (1+\delta)/2)$, we obtain
\[
g_2(\hat{\theta}_b) = O_P(n^{\xi}), \tag{2.51}
\]
From (2.7) and (A.47), we have
\[
g_{22}(\hat{\theta}_b^*) = \frac{n^{(1+\delta)/2}}{2^{3/2}\theta_{0,1}^{1/2}\theta_{0,2}^{3/2}} + o_P(n^{(1+\delta)/2}). \tag{2.52}
\]
Combining this with (2.28) and (2.51) leads to (2.37). In addition, applying (A.39) with $r_1 = r_2 = (1-\delta)/2$, $r_3 = 0$ and $\xi \in [1/2, (1+\delta)/2)$, we have
\[
g_1(\hat{\theta}_a) = O_P(n^{1/2}) + O_P(n^{\xi}) = O_P(n^{\xi}).
\]
This together with (2.27) and (2.49) gives (2.36).

Proof of (a3): Following the same arguments as in the proof of (2.37) leads to (2.39). Applying (A.42) with $r_1 = (1-\delta)/2$, $r_2 = (1+\delta)/2 - \xi$, $r_3 = 0$ and $\xi \in [(1+\delta)/4, 1/2)$, we have
\[
n^{-1/2}g_1(\hat{\theta}_a) \xrightarrow{d} N(0, 2\theta_{0,1}^{-2}).
\]
This together with (2.27) and (2.49) gives (2.38).

Proof of (a4): Applying (A.43) with $r_1 = (1-\delta)/2$, $r_2 = r_3 = 0$ and $\xi \in [\delta, (1+\delta)/4)$, we have
\[
n^{-(1+\delta)/4}g_2(\hat{\theta}_b) \xrightarrow{d} N(0, 2^{-1/2}\theta_{0,1}^{-1/2}\theta_{0,2}^{-3/2}).
\]
This together with (2.28) and (2.52) gives (2.41). In addition, (2.40) follows from the same argument as in the proof of (2.38).

Proof of (a5): Following the same arguments as in the proofs of (2.40) and (2.41) leads to (2.42) and (2.43), respectively. Applying (A.41) with $r_1 = r_2 = (1-\delta)/2$, $r_3 = 0$ and $\xi \in [(1+\delta)/4, 1/2)$, we have
\[
g_3(\hat{\theta}_c) = O_P(n^{\xi}). \tag{2.53}
\]
From (2.6)-(2.8) and (A.48), we obtain
\[
g_{33}(\hat{\theta}_c^*) = \frac{n^{\delta}}{\theta_{0,3}} + o(n^{\delta}). \tag{2.54}
\]
Combining this with (2.29) and (2.53) leads to (2.44).

Proof of (a6): Following the same arguments as in the proofs of (2.40) and (2.41) leads to (2.45) and (2.46), respectively. Applying (A.44) with $r_1 = r_2 = (1-\delta)/2$, $r_3 = 0$ and $\xi \in (0, \delta/2)$, we have
\[
n^{-5/2}g_3(\hat{\theta}_c) \xrightarrow{d} N(0, 2\theta_{0,3}^{-1}).
\]
This together with (2.29) and (2.54) gives (2.47). This completes the proof for $\delta \in [0, 1/3)$.

Second, we consider $\delta \in [1/3, 1/2)$. Following similar arguments as those in the first part, we obtain
(b1) For $\xi \in [(1 + \delta)/2, 1)$,
$$\hat{\theta}_1 - \theta_{0,1} = O_p(n^{\xi-1}).$$

(b2) For $\xi \in [1/2, (1 + \delta)/2)$,
$$\hat{\theta}_1 - \theta_{0,1} = O_p(n^{\xi-1}),$$
$$\hat{\theta}_2 - \theta_{0,2} = O_p(n^{\xi-(1+\delta)/2}).$$

(b3) For $\xi \in [\delta, 1/2)$,
$$n^{1/2} (\hat{\theta}_1 - \theta_{0,1}) \xrightarrow{d} N(0, 2\theta_{0,1}^2),$$
$$\hat{\theta}_2 - \theta_{0,2} = O_p(n^{\xi-(1+\delta)/2}).$$

(b4) For $\xi \in [(1 + \delta)/4, \delta)$,
$$n^{1/2} (\hat{\theta}_1 - \theta_{0,1}) \xrightarrow{d} N(0, 2\theta_{0,1}^2),$$
$$\hat{\theta}_2 - \theta_{0,2} = O_p(n^{\xi-(1+\delta)/2}),$$
$$\hat{\theta}_3 - \theta_{0,3} = O_p(n^{\xi-\delta}).$$

(b5) For $\xi \in [\delta/2, (1 + \delta)/4)$,
$$n^{1/2} (\hat{\theta}_1 - \theta_{0,1}) \xrightarrow{d} N(0, 2\theta_{0,1}^2),$$
$$n^{(1+\delta)/4} (\hat{\theta}_2 - \theta_{0,2}) \xrightarrow{d} N(0, 2^{5/2} \theta_{0,1}^{1/2} \theta_{0,2}^{3/2}),$$
$$\hat{\theta}_3 - \theta_{0,3} = O_p(n^{\xi-\delta}).$$

(b6) For $\xi \in [0, \delta/2)$,
$$n^{1/2} (\hat{\theta}_1 - \theta_{0,1}) \xrightarrow{d} N(0, 2\theta_{0,1}^2),$$
$$n^{(1+\delta)/4} (\hat{\theta}_2 - \theta_{0,2}) \xrightarrow{d} N(0, 2^{5/2} \theta_{0,1}^{1/2} \theta_{0,2}^{3/2}),$$
$$n^{\delta/2} (\hat{\theta}_3 - \theta_{0,3}) \xrightarrow{d} N(0, 2\theta_{0,3}).$$

Third, we consider $\delta \in [1/2, 1)$. Following similar arguments as those in the first part, we obtain

(c1) For $\xi \in [(1 + \delta)/2, 1)$,
$$\hat{\theta}_1 - \theta_{0,1} = O_p(n^{\xi-1}).$$

(c2) For $\xi \in [\delta, (1 + \delta)/2)$,
$$\hat{\theta}_1 - \theta_{0,1} = O_p(n^{\xi-1}),$$
$$\hat{\theta}_2 - \theta_{0,2} = O_p(n^{\xi-(1+\delta)/2}).$$

(c3) For $\xi \in [1/2, \delta)$,
$$\hat{\theta}_1 - \theta_{0,1} = O_p(n^{\xi-1}),$$
$$\hat{\theta}_2 - \theta_{0,2} = O_p(n^{\xi-(1+\delta)/2}),$$
$$\hat{\theta}_3 - \theta_{0,3} = O_p(n^{\xi-\delta}).$$

(c4) For $\xi \in [(1 + \delta)/4, 1/2)$,
$$n^{1/2} (\hat{\theta}_1 - \theta_{0,1}) \xrightarrow{d} N(0, 2\theta_{0,1}^2),$$
$$\hat{\theta}_2 - \theta_{0,2} = O_p(n^{\xi-(1+\delta)/2}),$$
$$\hat{\theta}_3 - \theta_{0,3} = O_p(n^{\delta/2}).$$

(c5) For $\xi \in [\delta/2, (1 + \delta)/4)$,
$$n^{1/2} (\hat{\theta}_1 - \theta_{0,1}) \xrightarrow{d} N(0, 2\theta_{0,1}^2),$$
$$n^{(1+\delta)/4} (\hat{\theta}_2 - \theta_{0,2}) \xrightarrow{d} N(0, 2^{5/2} \theta_{0,1}^{1/2} \theta_{0,2}^{3/2}),$$
$$\hat{\theta}_3 - \theta_{0,3} = O_p(n^{\xi-\delta}).$$
2 Main Results

\begin{align*}
\delta \xi_0 \quad 0 \quad 1 \\
O_p(n^{-(1-\xi)}) \\
O_p(n^{-1/2}) \\
\text{(asymptotically normal)}
\end{align*}

(a): Convergence rates of \( \hat{\theta}_1 - \theta_{0,1} \)

\begin{align*}
\delta \xi_0 \quad 0 \quad 1 \\
O_p(n^{-(1+\delta)/2}) \\
O_p(n^{-(1+\delta)/(2-\xi)}) \\
\text{(Asymptotically normal)}
\end{align*}

(b): Convergence rates of \( \hat{\theta}_2 - \theta_{0,2} \)

\begin{align*}
\delta \xi_0 \quad 0 \quad 1 \\
O_p(n^{-(\delta-\xi)}) \\
O_p(n^{-\delta/2}) \\
\text{(Asymptotically normal)}
\end{align*}

(c): Convergence rates of \( \hat{\theta}_3 - \theta_{0,3} \)

Fig. 1: Convergence rates of \( \hat{\theta} - \theta_0 \) with respect to \((\delta, \xi)\), where \( \delta \) is the growing rate of the domain and \( O(n^\xi) = \sup_{\theta \in \Theta} R(\theta) \). Note that the dark gray regions correspond to \((\delta, \xi)\), for which the components of \( \hat{\theta} \) are also asymptotically normal, whereas points on the dash lines and the white regions correspond to \((\delta, \xi)\), for which consistency of \( \hat{\theta} \) may not hold.

(c6) For \( \xi \in [0, \delta/2) \),

\begin{align*}
&n^{1/2}(\hat{\theta}_1 - \theta_{0,1}) \xrightarrow{d} N(0, 2\theta_{0,1}^2), \\
n^{(1+\delta)/4}(\hat{\theta}_2 - \theta_{0,2}) \xrightarrow{d} N(0, 2^{5/2}\theta_{0,1}^{5/2}\theta_{0,2}^{3/2}), \\
n^{\delta/2}(\hat{\theta}_3 - \theta_{0,3}) \xrightarrow{d} N(0, 2\theta_{0,3}).
\end{align*}

Thus the proof of the theorem is complete.

The following corollary provides a specific example of Theorem 2.3 for \( \delta = \xi \). Although both Theorem 2.1 and Theorem 2.3 provide no result for \( \hat{\theta}_3 \) when \( \delta = \xi \), the example shows that \( \hat{\theta}_3 \) is not consistent, suggesting that a more general result for \( \hat{\theta}_3 \) in both Theorem 2.1 and Theorem 2.3 is not possible.

**Corollary 2.1.** Consider the intercept-only model of (1.4) with \( p_n = 0 \) and \( \delta \in [0, 1) \). Suppose that \( \mu_0(s) = \beta_{0,0} + \beta_{0,1}n^{-\delta}s \) and \( \theta = (\theta_1, \theta_2, \theta_3)' \in \Theta \subset (0, \infty)^3 \), where \( \beta_{0,0} \) and \( \beta_{0,1} \) are nonzero constants, \( \Theta \) is compact, and \( \theta_0 = \)

\begin{align*}
\theta_0 = & \begin{cases}
(0, 0, \theta_0) & \text{if } \delta < \xi > 0 \\
(0, 0, 0) & \text{if } \delta < 0 < \xi \\
(0, 0, \infty) & \text{if } \delta > 0 > \xi \\
\infty & \text{if } \delta > \xi.
\end{cases}
\end{align*}
\((\theta_{0,1}, \theta_{0,2}, \theta_{0,3})\) is an interior point of \(\Theta\). Let \(\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)\)' be the ML estimate of \(\theta\). Then

\[
\begin{align*}
n^{1/2} (\hat{\theta}_1 - \theta_{0,1}) & \overset{d}{\to} N(0, 2\theta_{0,1}^2); \quad \delta \in [0, 1/2), \\
n^{1-\delta} (\hat{\theta}_1 - \theta_{0,1}) & = O_p(1); \quad \delta \in [1/2, 1), \\
n^{(1+\delta)/4} (\hat{\theta}_2 - \theta_{0,2}) & \overset{d}{\to} N(0, 2^{\delta/2} \theta_{0,1}^{(1/2)} \theta_{0,2}^{(3/2)}); \quad \delta \in [0, 1/3), \\
n^{(1-\delta)/2} (\hat{\theta}_2 - \theta_{0,2}) & = O_p(1); \quad \delta \in [1/3, 1), \\
\hat{\theta}_3 & = \frac{12\theta_{0,2}}{12\theta_{0,2} + \beta_{0,1}^2 \theta_{0,3}} \theta_{0,3} + o_p(1); \quad \delta \in (0, 1).
\end{align*}
\] (2.55)

(2.56)

(2.57)

Proof. First, we check (2.5):

\[
\mu_0^* \Sigma^{-1}(\theta) (I - M(\theta)) \mu_0 = \mu_0^* \Sigma^{-1}(\theta) \mu_0 - \mu_0^* \Sigma^{-1}(\theta) M(\theta) \mu_0 = \beta_0^2 \Sigma^{-1}(\theta) x - \beta_0^2 \Sigma^{-1}(\theta) 1 \epsilon_1 \Sigma^{-1}(\theta) 1^{-1} \epsilon_2 \Sigma^{-1}(\theta) x
\]

\[
= \frac{\beta_0^2 \Sigma^{-1}(\theta)}{\theta_0} n^{\delta} - \frac{\beta_0^2 \Sigma^{-1}(\theta)}{\theta_0} n^{\delta} + o(n^{\delta})
\]

\[
= \frac{\beta_0^2}{24\theta_2} n^{\delta} + o(n^{\delta}),
\] (2.58)

uniformly in \(\Theta\), where \(x = n^{-1}(1, \ldots, n)'\), and the second last equality is obtained from (A.49)-(A.51). Hence (2.5) holds for \(\xi = \delta\). Thus (2.55) and (2.56) follow directly from Theorem 2.3.

We remain to prove (2.57). By (2.9), (2.58) and (A.31), we have

\[
-2\ell(\theta) = n \log(2\pi) + \log \det(\Sigma(\theta)) + \text{tr}(\Sigma^{-1}(\theta) \Sigma(\theta_0))
\]

\[
+ \frac{\beta_0^2}{24\theta_2} n^{\delta} + h(\theta) + o_p(n^{\delta}) + O_p(1),
\]

uniformly in \(\Theta\), where \(h(\theta) = (\eta + \epsilon) \Sigma^{-1}(\theta)(\eta + \epsilon) - \text{tr}(\Sigma^{-1}(\theta) \Sigma(\theta_0))\). Therefore, by (A.15) and (A.18),

\[
-2\ell(\theta) = n \log(2\pi) - \frac{1 - \delta}{2} \log n + \left( \log \theta_1 + \frac{\theta_{0,1}}{\theta_1} \right) n
\]

\[
+ \frac{2\theta_2}{\theta_1} \left( \frac{\theta_{0,1}}{2\theta_1} + \frac{\theta_{0,2}}{2\theta_2} \right) n^{(1+\delta)/2}
\]

\[
- \left( \frac{\theta_2}{\theta_1} + \frac{\theta_{0,2}}{\theta_3} \right) n^{\delta} + o(n^{\delta})
\]

\[
+ h(\theta) + o_p(n^{\delta}) + O_p(1)
\]

\[
= n \log(2\pi) - \frac{1 - \delta}{2} \log n + \left( \log \theta_1 + \frac{\theta_{0,1}}{\theta_1} \right) n
\]

\[
+ \frac{2\theta_2}{\theta_1} \left( \frac{\theta_{0,1}}{2\theta_1} + \frac{\theta_{0,2}}{2\theta_2} \right) n^{(1+\delta)/2}
\]

\[
- \left( \frac{\theta_2}{\theta_1} + \frac{\theta_{0,2}}{\theta_3} \right) \left( 1 - \frac{\theta_{0,1}}{\theta_{0,2}} \right) n^{\delta} + o(n^{\delta}) + O_p(1),
\] (2.59)

uniformly in \(\Theta\), where \(\theta_{0,3}^* = \frac{12\theta_{0,2}}{12\theta_{0,2} + \beta_{0,1}^2 \theta_{0,3}} \theta_{0,3}\). We can now show that for any \(\varepsilon_3 > 0\), there exist \(\varepsilon_1, \varepsilon_2 > 0\) such that

\[
P\left( \inf_{\theta \in \Theta_2(\varepsilon)} \{ -2\ell(\theta) + 2\ell(\theta_1, \theta_2, \theta_{0,3}^*) \} > 0 \right) \to 1,
\]

as \(n \to \infty\), where \(\Theta_2(\varepsilon) = \{ \theta \in \Theta : |\theta_1 - \theta_{0,1}| \leq \varepsilon_1, |\theta_2 - \theta_{0,2}| \leq \varepsilon_2, |\theta_3 - \theta_{0,3}^*| > \varepsilon_3 \} \) and \(\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)'\), which follows from the same arguments as in the proof of (2.16). Thus we obtain (2.57). This completes the proof.\(\square\)

Note that the scaling factor \(n^{-\delta}\) is introduced for the linear term, \(x_1(s) = n^{-\delta} s\), so that \(\frac{1}{n^{\delta}} \int_0^{n^{\delta}} (x_1(s) - \bar{x}_1)^2 ds\) does not depend on \(n\), where \(\bar{x}_1 = \frac{1}{n^{\delta}} \int_0^{n^{\delta}} x_1(s) ds\).
The following corollary provides a specific example of Theorem 2.3 for \( \xi = (1 + \delta)/2 \). Although both Theorem 2.1 and Theorem 2.3 provide no result for \( \theta_2 \) when \( \xi = (1 + \delta)/2 \), the example shows that \( \theta_2 \) is not consistent, suggesting that a more general result for \( \theta_2 \) in both Theorem 2.1 and Theorem 2.3 is not possible.

**Corollary 2.2.** Consider the same setup as in Corollary ?? except that \( \mu_0(s) = \beta_{0,0} + \beta_{0,1} x(s) \), where \( x(\cdot) \) is generated from a zero-mean Gaussian spatial process with covariance function

\[
\text{cov}(x(s), x(s')) = \frac{\theta_{1,2}}{\theta_{1,3}} \exp \left( -\theta_{1,3}|s - s'| \right); \quad s, s' \in [0, \ell],
\]

for some constants \( \theta_{1,2}, \theta_{1,3} > 0 \). Then

\[
\begin{align*}
\hat{\theta}_1 &= \theta_{0,1} + O_p(n^{-1-(1-\delta)/2}); \quad \delta \in [0, 1), \\
\hat{\theta}_2 &= \theta_{0,2} + \theta_{0,3}\beta_{1,2} + o_p(1); \quad \delta \in [0, 1),
\end{align*}
\]

\[
\hat{\theta}_3 = \frac{\theta_{0,2} + \theta_{0,3}\theta_1}{\beta_{0,1}\theta_{1,2}\theta_{1,3} + \theta_{0,3}\theta_{1,3}} + o_p(1); \quad \delta \in (0, 1).
\]

**Proof.** First, we check (2.5):

\[
\begin{align*}
\mu_0^*(\Sigma^{-1}(\theta)(I - M(\theta)) \mu_0 \\
= \mu_0^*(\Sigma^{-1}(\theta) \mu_0 - \mu_0^*(\Sigma^{-1}(\theta) M(\theta)) \mu_0 \\
= \beta_{0,1}' x\Sigma^{-1}(\theta) x - \beta_{0,1}' x\Sigma^{-1}(\theta) 1(1'\Sigma^{-1}(\theta) 1)^{-1} 1'\Sigma^{-1}(\theta) x \\
= \beta_{0,1}' \text{tr}(\Sigma^{-1}(\theta) \Sigma_\eta(\theta_1, \theta_{1,2}, \theta_{1,3})) + h_x(\theta) + O_p(1) \\
= \frac{\beta_{0,1}' \theta_{1,2}}{(2\theta_{1,2})^{1/2}} n^{(1+\delta)/2} + \frac{\beta_{0,1}' \theta_{1,2} (\theta_{1,2} - \theta_{1,3})}{2\theta_{1,2}} n^{\delta} \\
+ h_x(\theta) + o(n^{\delta}) + O_p(1),
\end{align*}
\]

uniformly in \( \Theta \), where \( x = (x(s_1), \ldots, x(s_n))' \),

\[
h_x(\theta) = \beta_{0,1}' (x\Sigma^{-1}(\theta) x - \text{tr}(\Sigma^{-1}(\theta) \Sigma_\eta(\theta_1, \theta_{1,2}, \theta_{1,3})))
\]

the second last equality follows from (A.51) and \( \sup \theta \in \Theta n^{-\delta/2} x' \Sigma^{-1}(\theta) 1 = O_p(1) \), and the last equality follows from (A.17). In addition, from (A.16),

\[
\sup \theta \in \Theta h_x(\theta) = o_p(n^{(1+\delta)/2}).
\]

Hence (2.5) holds for \( \xi = (1 + \delta)/2 \). This gives (2.61).

Next, we prove (2.62). Similar to (2.59), it follows that

\[
-2\ell(\theta) = n \log(2\pi) - \frac{1}{2} \log n + \left( \log \theta_1 + \frac{\theta_{0,1}}{\theta_1} \right) n \\
+ \frac{\beta_{0,1}' \theta_{1,2}}{(2\theta_{1,2})^{1/2}} n^{(1+\delta)/2} \\
- \frac{\beta_{0,1}' \theta_{1,2} (\theta_{1,2} - \theta_{1,3})}{2\theta_{1,2}} n^{\delta} \\
- \frac{\theta_{0,2}' \theta_{0,3} (\theta_{3} - \theta_{0,3})^2}{2\theta_{0,3}} n^{\delta} \\
+ h_x(\theta) + h(\theta) + o_p(n^{\delta}) + O_p(1),
\]

uniformly in \( \Theta \), where \( \theta_{0,2}' = \theta_{0,2} + \beta_{0,1}' \theta_{1,2} \), \( \theta_{0,3}' = \theta_{0,3} + \beta_{0,1}' \theta_{1,2} \), and recall that \( h(\theta) = (\eta + \epsilon)' \Sigma^{-1}(\theta) (\eta + \epsilon) - \text{tr}(\Sigma^{-1}(\theta) \Sigma(\theta_0)) \). Therefore, for (2.62) to hold, it suffices to show that for any \( \epsilon_2 > 0 \), there exists an \( \epsilon_1 > 0 \) such that

\[
P \left( \inf_{\theta \in \Theta_2(\epsilon)} \{-2\ell(\theta) + 2\ell((\theta_1, \theta_{0,2}', \theta_{0,3}')) \} > 0 \right) \to 1,
\]

as \( n \to \infty \), where \( \Theta_2(\epsilon) = \{ \theta \in \Theta : |\theta_1 - \theta_{0,1}| \leq \epsilon_1, |\theta_2 - \theta_{0,2}'| > \epsilon_2 \} \) and \( \epsilon = (\epsilon_1, \epsilon_2)' \), which follows the same arguments as in the proof of (2.13).

Finally, we prove (2.63). It suffices to show that for any \( \epsilon_3 > 0 \), there exist \( \epsilon_1, \epsilon_2 > 0 \) such that

\[
P \left( \inf_{\theta \in \Theta_3(\epsilon)} \{-2\ell(\theta) + 2\ell((\theta_1, \theta_{0,2}', \theta_{0,3}')) \} > 0 \right) \to 1,
\]

as \( n \to \infty \), where \( \Theta_3(\epsilon) = \{ \theta \in \Theta : |\theta_1 - \theta_{0,1}| \leq \epsilon_1, |\theta_2 - \theta_{0,2}'| \leq \epsilon_2, |\theta_3 - \theta_{0,3}'| > \epsilon_3 \} \) and \( \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)' \), which follows from the same arguments as in the proof of (2.16). This completes the proof. \(\square\)
A Auxiliary Lemmas

The appendix contains thirteen lemmas, which are needed to establish the main results in Section 2. Their proofs are given in supplemental materials.

Lemma A.1. Let \( \Sigma(\theta) \) be given by (1.5) with \( \theta_1 \geq 0, \theta_2 > 0 \) and \( \theta_3 > 0 \). Then

\[
\Sigma^{-1}(\theta) = G_n(\theta)^T T_n^{-1}(\theta) G_n(\theta),
\]

(A.1)

where

\[
G_n(\theta) \equiv \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\rho_n & 1 & 0 & \cdots & 0 \\ 0 & -\rho_n & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & -\rho_n & 1 \end{pmatrix}_{n \times n},
\]

(A.2)

\[
T_n(\theta) \equiv D_n(\theta) + \theta_1 G_n(\theta) G_n(\theta)^T,
\]

(A.3)

\[
\rho_n \equiv \exp(-\theta_3 n^{-(1-\delta)}), \text{ and}
\]

\[
D_n(\theta) \equiv \frac{\theta_2}{\theta_3} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 - \rho_n^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 1 - \rho_n^2 \end{pmatrix}_{n \times n}.
\]

(A.4)

Lemma A.2. Under the setup of Lemma A.1, suppose that \( \theta \in \Theta \subset (0, \infty)^3 \) with \( \Theta \) a compact set. Then, for any \( \delta \in [0,1) \), the following equations hold uniformly in \( \Theta \):

\[
\text{tr}(T_n^{-1}(\theta)) = \frac{n^{(3-\delta)/2}}{\theta_1^{1/2}} + O(n^{(1+\delta)/2}) + O(n^{1-\delta}),
\]

(A.5)

\[
\text{tr}(T_n^{-2}(\theta)) = \frac{n^{(5-3\delta)/2}}{2^{7/2} \theta_2^{1/2} \theta_3^{1/2}} + o(n^{(5-3\delta)/2}).
\]

(A.6)

In addition,

\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta} \lambda_{\text{max}}(T_n^{-1}(\theta)) = O(n^{1-\delta}),
\]

(A.7)

\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta} \lambda_{\text{max}}(G_n^{-1}(\theta) G_n^{-1}(\theta)^T) = O(n^{1-\delta}).
\]

(A.8)

Lemma A.3. Under the setup of Lemma A.2, for any \( \theta_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3})' \in \Theta,

\[
G_n(\theta_0) \Sigma_q(\theta_0) G_n(\theta_0)^T
\]

\[
= \frac{\theta_{0,2} \rho_0}{\theta_{0,3} \rho_{0,n}} (1 - \rho_n^2) I + \left(1 - \frac{\rho_n}{\rho_{0,n}}\right)(1 - \rho_n \rho_{0,n}) \Sigma_q(\theta_0)
\]

(A.9)

\[
+ \frac{\theta_{0,2}}{\theta_{0,3}} \left\{(1 - \frac{\rho_n}{\rho_{0,n}})(v_0 e_1' + e_1 v_0') + \rho_n^2 e_1 e_1'\right\}.
\]

where \( e_1 = (1,0,\ldots,0)' \), \( v_0 \equiv (1, \rho_{0,n}, \ldots, \rho_{0,n}^{n-1}) \) and \( \rho_{0,n} \equiv \exp(-\theta_{0,3} n^{-(1-\delta)}) \). In addition, for \( \delta \in [0,1) \),

\[
\sup_{\theta \in \Theta} v_0^T T_n^{-1}(\theta) v_0 = O(n^{2(1-\delta)}),
\]

(A.10)

\[
\sup_{\theta \in \Theta} v_0^T T_n^{-1}(\theta) e_1 = O(n^{1-\delta}),
\]

(A.11)

\[
\sup_{\theta \in \Theta} e_1^T T_n^{-1}(\theta) e_1 = O(1),
\]

(A.12)

\[
\sup_{\theta \in \Theta} \text{tr}(T_n^{-1}(\theta) \Sigma_q(\theta_0)) = \frac{\theta_{0,2}}{2 \theta_{0,3}} n^{2-\delta} + o(n^{2-\delta}).
\]

(A.13)

Furthermore, for \( \delta \in (0,1) \),

\[
\sup_{\theta \in \Theta} \text{tr}\left((T_n^{-1}(\theta) \Sigma_q(\theta))^2\right) = \frac{1}{40} n^{4-3\delta} + o(n^{4-3\delta}).
\]

(A.14)
Lemma A.4. Under the setup of Lemma A.2, for any $\delta \in [0, 1)$ and $\boldsymbol{\theta}_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3})' \in \Theta$, the following equations hold uniformly in $\Theta$:

\[
\log(\det(\Sigma(\boldsymbol{\theta}))) = n \log \theta_1 + \left( \frac{2\theta_2}{\theta_1} \right)^{1/2} n^{(1+\delta)/2} - \left( \frac{\theta_2}{\theta_1} + \theta_3 \right) n^\delta - \frac{1-\delta}{2} \log n + o(n^\delta) + O(1),
\]

(A.15)

\[
\text{tr}\left( (\Sigma_n(\boldsymbol{\theta}))\Sigma_n^{-1}(\boldsymbol{\theta}) \right)^2 = \left( \frac{\theta_2}{8\theta_1} \right)^{1/2} n^{(1+\delta)/2} + o(n^{(1+\delta)/2}),
\]

(A.16)

\[
\text{tr}\left( \Sigma_n(\boldsymbol{\theta}_0)\Sigma_n^{-1}(\boldsymbol{\theta}) \right) = \frac{\theta_{0,2}}{n} - \frac{\theta_{0,1}}{\theta_1} \left( \frac{2\theta_2}{\theta_1} \right)^{1/2} n^{(1+\delta)/2} + o(n^\delta) + O(1),
\]

(A.17)

\[
\text{tr}\left( \Sigma(\boldsymbol{\theta}_0)\Sigma^{-1}(\boldsymbol{\theta}) \right) = \frac{\theta_{0,1}}{n} - \frac{\theta_{0,1}}{\theta_1} \left( \frac{2\theta_2}{\theta_1} \right)^{1/2} n^{(1+\delta)/2} + o(n^\delta) + O(1).
\]

(A.18)

Lemma A.5. Under the setup of Lemma A.2, let $\mathbf{F}(\boldsymbol{\theta})$ be an $n \times n$ matrix with the $(k, \ell)$th entry $|k - \ell| \theta_n^{k-\ell} n^{-(1-\delta)}$. Then

\[
\frac{\partial}{\partial \theta_3} \Sigma(\boldsymbol{\theta}) = - \frac{1}{\theta_3} \Sigma_n(\boldsymbol{\theta}) - \frac{\theta_2}{\theta_3^2} \mathbf{F}(\boldsymbol{\theta}).
\]

(A.19)

In addition,

\[
\mathbf{F}(\boldsymbol{\theta}) = \frac{\theta_3}{\theta_2} (1 - \rho_n^2) n^{-(1-\delta)} G_n^{-1}(\boldsymbol{\theta}) \Sigma_n(\boldsymbol{\theta}) G_n^{-1}(\boldsymbol{\theta})' - n^{-(1-\delta)} G_n^{-1}(\boldsymbol{\theta}) \left( (1 + \rho_n^2) I - \rho_n^2 (\mathbf{v} \mathbf{v}' + \mathbf{e}_1 \mathbf{e}_1') \right) G_n^{-1}(\boldsymbol{\theta})',
\]

(A.20)

where $\mathbf{v} = (1, \rho_n, \ldots, \rho_n^{n-1})'$ and $\mathbf{e}_1 = (1, 0, \ldots, 0)'$. Furthermore,

\[
\frac{\partial^2}{\partial \theta_3^2} \Sigma(\boldsymbol{\theta}) = \frac{2\theta_3}{\theta_2^2} \Sigma_n(\boldsymbol{\theta}) + \frac{2\theta_2}{\theta_3^2} \mathbf{F}(\boldsymbol{\theta}) - \frac{\theta_2}{\theta_3} \frac{\partial}{\partial \theta_3} \mathbf{F}(\boldsymbol{\theta}),
\]

(A.21)

where

\[
\frac{\partial}{\partial \theta_3} \mathbf{F}(\boldsymbol{\theta}) = \frac{\theta_3}{\theta_2} (3 + \rho_n^2) n^{2(1-\delta)} G_n^{-1}(\boldsymbol{\theta}) \Sigma_n(\boldsymbol{\theta}) G_n^{-1}(\boldsymbol{\theta})' - \frac{\theta_3}{\theta_2} (1 - \rho_n^2) n^{2(1-\delta)} G_n^{-2}(\boldsymbol{\theta}) \Sigma_n(\boldsymbol{\theta}) G_n^{-1}(\boldsymbol{\theta})' - \frac{\theta_3}{\theta_2} (1 - \rho_n^2) n^{2(1-\delta)} G_n^{-1}(\boldsymbol{\theta}) \Sigma_n(\boldsymbol{\theta}) G_n^{-2}(\boldsymbol{\theta})' + (1 - \rho_n^2) n^{-(1-\delta)} G_n^{-1}(\boldsymbol{\theta}) \mathbf{F}(\boldsymbol{\theta}) G_n^{-1}(\boldsymbol{\theta})' - (1 - \rho_n^2) n^{2(1-\delta)} G_n^{-1}(\boldsymbol{\theta}) G_n^{-1}(\boldsymbol{\theta})' + \rho_n^2 n^{2(1-\delta)} G_n^{-1}(\boldsymbol{\theta}) (\mathbf{v} \mathbf{v}' + \mathbf{e}_1 \mathbf{e}_1') G_n^{-1}(\boldsymbol{\theta})' - 2 \rho_n^2 n^{2(1-\delta)} G_n^{-2}(\boldsymbol{\theta}) \mathbf{v} \mathbf{v}' G_n^{-1}(\boldsymbol{\theta})' - 2 \rho_n^2 n^{2(1-\delta)} G_n^{-1}(\boldsymbol{\theta}) \mathbf{v} \mathbf{v}' G_n^{-1}(\boldsymbol{\theta})'.
\]

(A.22)

Lemma A.6. Under the setup of Lemma A.2, for any $\boldsymbol{\theta}_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3})' \in \Theta$,\n
\[
0 < \liminf_{n \to \infty} \inf_{\theta \in \Theta} \lambda_{\min} \left( \left( \Sigma_n^{-1}(\boldsymbol{\theta}) \frac{\partial}{\partial \theta_3} \Sigma(\boldsymbol{\theta}) \right)^2 \right) \leq \limsup_{n \to \infty} \sup_{\theta \in \Theta} \lambda_{\max} \left( \left( \Sigma_n^{-1}(\boldsymbol{\theta}) \frac{\partial}{\partial \theta_3} \Sigma(\boldsymbol{\theta}) \right)^2 \right) < \infty.
\]

(A.23)
In addition, the following equations hold uniformly in $\Theta$:

\[
\begin{align*}
\text{tr}\left(\left(\Sigma^{-1}(\theta) \frac{\partial}{\partial \theta^3} \Sigma(\theta)\right)^2\right) &= \frac{1}{\theta^3} n^\delta + o(n^\delta), \quad \text{(A.24)} \\
\text{tr}\left(\left(\Sigma^{-1}(\theta) \frac{\partial^2}{\partial \theta^3} \Sigma(\theta)\right)^2\right) &= O(n^\delta), \quad \text{(A.25)} \\
\text{tr}\left(\left(\Sigma^{-2}(\theta) \frac{\partial}{\partial \theta^3} \Sigma(\theta)\right)\right) &= O(n^\delta), \quad \text{(A.26)} \\
\text{tr}\left(\left(\Sigma^{-2}(\theta) \frac{\partial^2}{\partial \theta^3} \Sigma(\theta)\right)\right) &= O(n^\delta). \quad \text{(A.27)}
\end{align*}
\]

**Lemma A.7.** Consider the model of (1.4). Suppose that $\theta = (\theta_1, \theta_2, \theta_3)' \in \Theta \subset (0, \infty)^3$ with $\Theta$ a compact set. Then

\[
\sup_{\theta \in \Theta} \{ (\eta + \epsilon)' \Sigma^{-1}(\theta) M(\theta) (\eta + \epsilon) \} = O_p(p_n), \quad \text{(A.28)}
\]

where $M(\theta)$ is defined in (1.7).

**Lemma A.8.** Consider the model of (1.4). Suppose that $\theta = (\theta_1, \theta_2, \theta_3)' \in \Theta \subset (0, \infty)^3$ with $\Theta$ a compact set, and the data $Z$ are generated from $N(\mu_0, \Sigma(\theta_0))$ with $\theta_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3})' \in \Theta$ and

\[
\sup_{\theta \in \Theta} \{ \mu_0' \Sigma^{-1}(\theta) (I - M(\theta)) \mu_0 \} = O(n^\delta), \quad \text{(A.29)}
\]

for some $\xi \in \mathbb{R}$, where $M(\theta)$ is defined in (1.7). Then for $j = 1, 2, 3$,

\[
\begin{align*}
\sup_{\theta \in \Theta} \frac{\partial}{\partial \theta_j} \{ \mu_0' \Sigma^{-1}(\theta) (I - M(\theta)) \mu_0 \} &= O(n^\delta), \quad \text{(A.30)} \\
\sup_{\theta \in \Theta} \frac{\partial^2}{\partial \theta_j^2} \{ \mu_0' \Sigma^{-1}(\theta) (I - M(\theta)) \mu_0 \} &= O(n^\delta). \quad \text{(A.31)}
\end{align*}
\]

**Lemma A.9.** Under the setup of Lemma A.2, let $\hat{\theta}_1 = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)'$ be an estimate of $\theta$ with $\hat{\theta}_1 = \theta_{0,1} + O_p(n^{-r_1})$, $\hat{\theta}_2 = \theta_{0,2} + O_p(n^{-r_2})$ and $\hat{\theta}_3 = \theta_{0,3} + O_p(n^{-r_3})$, for some constants $r_1, r_2, r_3 \geq 0$. Then for any $\delta \in [0, 1)$ and $\theta_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3})' \in \Theta$,

\[
\begin{align*}
\text{tr}\left(\Sigma^{-2}(\hat{\theta}) \left(\Sigma_{\eta}(\hat{\theta}) - \Sigma_{\eta}(\theta_0)\right)\right) &= O_p(n^{(1+\delta)/2-r_2}) + O_p(n^{3-r_3}) + O_p(1), \quad \text{(A.32)} \\
\text{tr}\left(\Sigma^{-1}(\hat{\theta}) \Sigma_{\eta}(\hat{\theta}) \Sigma^{-1}(\hat{\theta}) \left(\Sigma_{\eta}(\hat{\theta}) - \Sigma_{\eta}(\theta_0)\right)\right) &= O_p(n^{1+\delta/2-r_2}) + O_p(n^{3-r_3}) + O_p(1), \quad \text{(A.33)} \\
\text{tr}\left(\Sigma^{-1}(\hat{\theta}) \left(\frac{\partial}{\partial \theta_3} \Sigma_{\eta}(\hat{\theta})\right) \Sigma^{-1}(\hat{\theta}) \left(\Sigma_{\eta}(\hat{\theta}) - \Sigma_{\eta}(\theta_0)\right)\right) &= O_p(n^{3-r_2}) + O_p(n^{3-r_3}) + O_p(1). \quad \text{(A.34)}
\end{align*}
\]

**Lemma A.10.** Under the setup of Lemma A.2, let $\hat{\theta}_2 = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)'$ be an estimate of $\theta$ with $\hat{\theta}_2 = \theta_{0,2} + o_p(1)$ and $\hat{\theta}_3 = \theta_{0,3} + o_p(1)$. Then for any $\delta \in [0, 1)$ and $\theta_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3})' \in \Theta$,

\[
\begin{align*}
\text{tr}\left(\left(\Sigma^{-1}(\hat{\theta}) \Sigma_{\eta}(\hat{\theta})\right)^2 \Sigma^{-1}(\hat{\theta}) \left(\Sigma_{\eta}(\hat{\theta}) - \Sigma_{\eta}(\theta_0)\right)\right) &= o_p(n^{(1+\delta)/2}), \quad \text{(A.35)} \\
\text{tr}\left(\left(\Sigma^{-1}(\hat{\theta}) \frac{\partial}{\partial \theta_3} \Sigma_{\eta}(\hat{\theta})\right)^2 \Sigma^{-1}(\hat{\theta}) \left(\Sigma_{\eta}(\hat{\theta}) - \Sigma_{\eta}(\theta_0)\right)\right) &= o_p(n^\delta), \quad \text{(A.36)} \\
\text{tr}\left(\Sigma^{-1}(\hat{\theta}) \left(\frac{\partial^2}{\partial \theta^2_3} \Sigma_{\eta}(\hat{\theta})\right) \Sigma^{-1}(\hat{\theta}) \left(\Sigma_{\eta}(\hat{\theta}) - \Sigma_{\eta}(\theta_0)\right)\right) &= o_p(n^\delta). \quad \text{(A.37)}
\end{align*}
\]
Lemma A.11. Under the setup of Lemma A.8, define for $k = 1, 2, 3$,

$$g_k(\theta) = -\frac{\partial}{\partial \theta_k} 2\ell(\theta),$$  \hspace{1cm} (A.38)

where $\ell(\theta)$ is given by (1.6). Let $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)'$ be an estimate of $\theta$ with $\hat{\theta}_1 = \theta_{0,1} + O_p(n^{-1})$, $\hat{\theta}_2 = \theta_{0,2} + O_p(n^{-2})$ and $\hat{\theta}_3 = \theta_{0,3} + O_p(n^{-3})$ for some constants $r_1, r_2, r_3 \geq 0$. Then for any $\delta \in [0,1)$,

$$g_1((\hat{\theta}_{0,1}, \hat{\theta}_2, \hat{\theta}_3)') = O_p(n^{1/2}) + O_p(n^{(1+\delta)/2-r_2}) + O_p(n^{\delta-r_2}),$$  \hspace{1cm} (A.39)

and for $\delta \in (0,1)$,

$$g_3((\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_{0,3})') = O_p(n^{\delta/2}) + O_p(n^{\delta-r_2}) + O_p(n^{\delta-r_2}).$$  \hspace{1cm} (A.40)

In addition, for $\delta \in [0,1)$

$$n^{-1/2}g_1((\hat{\theta}_{0,1}, \hat{\theta}_2, \hat{\theta}_3)') \overset{d}{\rightarrow} N(0, 2\theta_{0,1}^{-2}),$$  \hspace{1cm} (A.41)

if $\xi < 1/2$ and $r_2 \geq \delta$,

$$n^{-1+\delta/2}g_2((\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)') \overset{d}{\rightarrow} N(0, 2^{-1/2}n^{1/2}\theta_{0,1}^{-1/2}\theta_{0,2}^{-3/2}),$$  \hspace{1cm} (A.42)

if $\xi < (1+\delta)/4$, $r_1 > (1+\delta)/4$ and $r_2 > -(1-3\delta)/4$. Furthermore, for $\delta \in (0,1)$

$$n^{-\delta/2}g_3((\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_{0,3})') \overset{d}{\rightarrow} N(0, 2\theta_{0,3}^{-1}),$$  \hspace{1cm} (A.43)

if $\xi < \delta/2$, $r_1 > \delta/2$ and $r_2 > \delta/2$.

Lemma A.12. Under the setup of Lemma A.8, let

$$g_{kk}(\theta) = -\frac{\partial^2}{\partial \theta_k^2} 2\ell(\theta); \quad k = 1, 2, 3,$$  \hspace{1cm} (A.44)

where $\ell(\theta)$ is given by (1.6). Let $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)'$ be an estimate of $\theta$. Suppose that $\hat{\theta}_1 = \theta_{0,1} + o_p(1)$. Then for $\delta \in [0,1)$, there exists a constant $\theta_{0,1}^* > 0$ satisfying $|\theta_{0,1}^* - \hat{\theta}_1| \leq |\theta_{0,1} - \hat{\theta}_1|$ such that

$$g_{11}((\theta_{0,1}^*, \hat{\theta}_2, \hat{\theta}_3)') = \frac{n}{\theta_{0,1}^*} + o_p(n).$$  \hspace{1cm} (A.45)

In addition, suppose that $\hat{\theta}_1 = \theta_{0,1} + o_p(1)$ and $\hat{\theta}_2 = \theta_{0,2} + o_p(1)$, then for $\delta \in [0,1)$, there exists a constant $\theta_{0,2}^* > 0$ satisfying $|\theta_{0,2}^* - \hat{\theta}_2| \leq |\theta_{0,2} - \hat{\theta}_2|$ such that

$$g_{22}((\hat{\theta}_1, \theta_{0,2}^*, \hat{\theta}_3)') = \frac{n^{1+\delta/2}}{2^{1/2}\theta_{0,1}^{1/2}\theta_{0,2}^{1/2}} + O_p(n^{\delta}) + o_p(n^{(1+\delta)/2}).$$  \hspace{1cm} (A.46)

Furthermore, suppose that $\hat{\theta} = \theta_0 + o_p(1)$, then for $\delta \in (0,1)$, there exists a constant $\theta_{0,3}^* > 0$ satisfying $|\theta_{0,3}^* - \hat{\theta}_3| \leq |\theta_{0,3} - \hat{\theta}_3|$ such that

$$g_{33}((\hat{\theta}_1, \hat{\theta}_2, \theta_{0,3}^*)') = \frac{n^{\delta}}{\theta_{0,3}^*} + O_p(n^{\delta}) + o_p(n^{\delta}).$$  \hspace{1cm} (A.47)

Lemma A.13. Under the setup of Lemma A.2, let $x = n^{-1}(1, 2, \ldots, n)'$ and $1 = (1, \ldots, 1)'$. Then for any $\delta \in [0,1)$, the following equations hold uniformly in $\Theta$:

$$\begin{align*}
1'\Sigma^{-1}(\theta)1 &= \frac{\theta_3^2}{2\theta_2}n^\delta + o(n^\delta) + O(1), \\
x'\Sigma^{-1}(\theta)x &= \frac{\theta_3^2}{4\theta_2}n^\delta + o(n^\delta) + O(1), \\
x'\Sigma^{-1}(\theta)x &= \frac{\theta_3^2}{6\theta_2}n^\delta + o(n^\delta) + O(1).
\end{align*}$$  \hspace{1cm} (A.48-51)
A Auxiliary Lemmas

References


Supplement to “Asymptotic Properties of Maximum Likelihood Estimators in Geostatistical Regression Models”

This supplement materials contain the proofs of all lemmas given in Appendix.

Proof of Lemma A.1. By (1.3), \( G_n(\theta) \eta = (\eta(s_1), \eta(s_2), \ldots, \eta(s_n)) \), which implies that \( \eta = G_n^{-1}(\theta)(\eta(s_1), \eta(s_2), \ldots, \eta(s_n)) \).

Taking the variance on both sides, we have \( \Sigma_n(\theta) = G_n^{-1}(\theta)D_n(\theta)(G_n^{-1}(\theta))' \) implying \( G_n(\theta)\Sigma(\theta)G_n(\theta)' = D_n(\theta) + \theta_1G_n(\theta)G_n(\theta)' = T_n(\theta) \), which gives (A.1). This completes the proof.

Proof of Lemma A.2. For notational simplicity, we suppress the dependence of \( \theta \) for all matrices. First, we show that for any \( \delta \in [0, 1] \), there exists a constant \( \tau > 0 \) such that

\[
\theta_1^{-n} \det(T_n) = \frac{f_2^{-1}(\theta)}{(f_1'(\theta) - 4\rho^2 \theta^2_1)^{1/2}} \left\{ \left( \frac{\theta_2}{\theta_3} + \theta_1 \right)f_2(\theta) - \rho^2 \theta_1 \right\} + o(\exp(-\tau n^{(1+\delta)/2})),
\]

uniformly in \( \Theta \), where \( \rho_n \) is defined in (A.3),

\[
f_1(\theta) = \frac{\theta_2}{\theta_3}(1 - \rho_n^2) + (1 + \rho_n^2)\theta_1
\]

\[
= 2\theta_1 + 2(\theta_2 - \theta_1 \theta_3)n^{-(1-\delta)} + O(n^{-2(1-\delta)}),
\]

\[
f_2(\theta) = \frac{1}{2\theta_1} \left\{ f_1(\theta) + (f_1'(\theta) - 4\rho^2 \theta^2_1)^{1/2} \right\}
\]

\[
= 1 + \left( \frac{2\theta_2}{\theta_1} \right)^{1/2} n^{-(1-\delta)/2} + \left( \frac{\theta_2}{\theta_1} - \theta_3 \right)n^{-(1-\delta)} + O(n^{-3(1-\delta)/2}),
\]

uniformly in \( \Theta \). Let

\[
B_n = \begin{pmatrix}
    f_1(\theta) & -\rho_n \theta_1 & 0 & \cdots & 0 \\
    -\rho_n \theta_1 & f_1(\theta) & -\rho_n \theta_1 & \ddots & \vdots \\
    0 & -\rho_n \theta_1 & f_1(\theta) & \ddots & 0 \\
    \vdots & \ddots & \ddots & \ddots & -\rho_n \theta_1 \\
    0 & \cdots & 0 & -\rho_n \theta_1 & f_1(\theta)
\end{pmatrix}
\]

It follows from (A.3) that

\[
T_k = \begin{pmatrix}
    \theta_2 \theta_3^{-1} + \theta_1 & -\rho_n \theta_1 & 0 & \cdots & 0 \\
    -\rho_n \theta_1 & f_1(\theta) & -\rho_n \theta_1 & \ddots & \vdots \\
    0 & -\rho_n \theta_1 & f_1(\theta) & \ddots & 0 \\
    \vdots & \ddots & \ddots & \ddots & -\rho_n \theta_1 \\
    0 & \cdots & 0 & -\rho_n \theta_1 & f_1(\theta)
\end{pmatrix}
\]

and hence

\[
\det(T_k) = \left( \frac{\theta_2}{\theta_3} + \theta_1 \right) \det(B_{k-1}) - \rho^2 \theta^2_1 \det(B_{k-2}).
\]

Since

\[
\det(B_{k-1}) = f_1(\theta) \det(B_{k-2}) - \rho^2 \theta^2_1 \det(B_{k-3}),
\]

for \( k \geq 3 \), where \( \det(B_{0}) \equiv 1 \), solving the difference equation, we have

\[
\det(B_{k-1}) = \frac{\theta_2}{\theta_1} \left( f_2^2(\theta) - f_2^2(\theta) \right) \left( f_1'(\theta) - 4\rho^2 \theta^2_1 \right)^{1/2},
\]

where \( f_2(\theta) \) is defined in (A.54) and

\[
f_3(\theta) = \frac{1}{2\theta_1} \left\{ f_1(\theta) + (f_1'(\theta) - 4\rho^2 \theta^2_1)^{1/2} \right\}
\]

\[
= 1 - \left( \frac{2\theta_2}{\theta_1} \right)^{1/2} n^{-(1-\delta)/2} + O(n^{-(1-\delta)}),
\]

\[
(A.52)
\]

\[
(A.53)
\]

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(A.54)
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(A.55)
\]

\[
(A.56)
\]

\[
(A.57)
\]

\[
(A.58)
\]

\[
(A.59)
\]
uniformly in $\Theta$. Hence by (A.56),

\[
\theta_i \frac{n}{\theta_1} \det(T_n) = \frac{(\theta_2 \theta_3^{-1} + \theta_1) f_2^n(\theta) - \rho_n^2 \theta_1 f_3^{n-1}(\theta)}{(f_1^n(\theta) - 4 \rho_n^2 \theta_1)^{1/2}} \tag{A.60}
\]

Since

\[
\frac{1}{(f_1^n(\theta) - 4 \rho_n^2 \theta_1)^{1/2}} = \frac{n^{(1-\delta)/2}}{(8 \theta_1 \theta_2)^{1/2}} + O(n^{-(1-\delta)/2}), \tag{A.61}
\]

uniformly in $\Theta$, for (A.52) to hold, we remain to show that

\[
f_3^n(\theta) = (\rho_2^2 f_2^{-1}(\theta))^{n} = o\left(\exp(-2/2^{1/2} \theta_1^{-1/2} \theta_2^{1/2} n^{(1+\delta)/2})\right), \tag{A.62}
\]

uniformly in $\Theta$, which follows from

\[
\log f_3^n(\theta) = n \log f_3(\theta) = -\frac{2 \theta_2}{\theta_1}^{1/2} n^{(1+\delta)/2} + O(n^4),
\]

uniformly in $\Theta$, after applying $\log(1-x) = -x + O(x^2)$, as $x \to 0$. This completes the proof of (A.52).

Next, we compute the entries of $T_n^{-1}$. Let $C_n(k, \ell)$ the $(k, \ell)$th entry of $T_n^{-1}$; $1 \leq k, \ell \leq n$. Then

\[
C_n(k, \ell) = (-1)^{k+\ell} \frac{\det(W_{n-1}(k, \ell))}{\det(T_n)}; \quad 1 \leq k, \ell \leq n, \tag{A.63}
\]

where

\[
W_{n-1}(k, \ell) = W_{n-1}(k, \ell)^t = \begin{pmatrix}
T_{k-1} & * & * \\
0 & P_{\ell-k} & * \\
0 & 0 & B_{n-\ell}
\end{pmatrix},
\]

is the matrix resulting from deleting row $k$ and column $\ell$ of $T_n$, and

\[
P_m \equiv \begin{pmatrix}
-\rho_n \theta_1 & f_1(\theta) & -\rho_n \theta_1 & 0 & \ldots & 0 \\
0 & -\rho_n \theta_1 & f_1(\theta) & -\rho_n \theta_1 & \ddots & \vdots \\
0 & 0 & -\rho_n \theta_1 & f_1(\theta) & \ddots & 0 \\
0 & 0 & 0 & -\rho_n \theta_1 & \ddots & -\rho_n \theta_1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & f_1(\theta) \\
0 & 0 & \ldots & 0 & 0 & -\rho_n \theta_1
\end{pmatrix},
\]

is an $m \times m$ matrix, and $P_0 = T_0 \equiv B_0 \equiv \emptyset$. It follows that for $1 \leq k, \ell \leq n$,

\[
\det(W_{n}(k, \ell)) = \det(T_{\min(k,\ell)-1}) \det(P_{\ell-k}) \det(B_{n-\max(k,\ell)}). \tag{A.64}
\]

By (A.52), (A.58), (A.60), (A.63) and (A.64), we have for $1 \leq k, \ell \leq n$,

\[
C_n(k, \ell) = \left(\frac{f_2^{k-2}(\theta) - f_3(\theta) f_2^{k-2}(\theta)}{f_2^{k-1}(\theta)}\right) \rho_n^{\ell-k} \left(\frac{f_2^{n-\ell+1}(\theta) - f_3^{n-\ell+1}(\theta)}{f_2^{n}(\theta) - 4 \rho_n^2 \theta_1^{1/2}}\right)
\]

\[
+ o\left(\exp(-\tau n^{(1+\delta)/2})\right), \tag{A.65}
\]

uniformly in $\Theta$, where

\[
f_3(\theta) = \frac{(\theta_2 \theta_3^{-1} + \theta_1) f_2(\theta) - \rho_n \theta_1}{(\theta_2 \theta_3^{-1} + \theta_1) f_3(\theta) - \rho_n \theta_1} = 1 + O(n^{-(1-\delta)/2}), \tag{A.66}
\]

uniformly in $\Theta$. Using $f_2(\theta)f_3(\theta) = \rho_n^2$, we obtain for $1 \leq k, \ell \leq n$,

\[
C_n(k, \ell) = \frac{\rho_n^{[\ell-k]}}{(f_2^{n}(\theta) - 4 \rho_n^2 \theta_1^{1/2})^{1/2}} \left(\frac{f_2^{k-\ell}(\theta)}{f_2^{k}(\theta)}\right)
\]

\[
- f_2(\theta) \rho_n^{\ell-k} f_2^{k-\ell}(\theta) + o\left(\exp(-\tau n^{(1+\delta)/2})\right), \tag{A.67}
\]
uniformly in \( \Theta \).

We are now ready to prove (A.5). By (A.61),

\[
\text{tr}(T_n^{-1}) = \sum_{k=1}^{n} C_n(k, k)
= \frac{1}{(f_1^2(\theta) - 4\rho_n^2 \theta_1^2)^{1/2}} \sum_{k=1}^{n} \left( 1 - f_4(\theta) \rho_n^{2(k-2)} f_2^{2(k-2)}(\theta) - \rho_n^{2(n-k+1)} f_2^{2k-2n-2}(\theta) \right) + o\left( \exp(-\tau n^{(1+\delta)/3}) \right)
= \frac{n}{(f_1^2(\theta) - 4\rho_n^2 \theta_1^2)^{1/2}} + O\left(n^{(1-\delta)/2} \sum_{k=1}^{n} f_2^{k-n}(\theta) \right)
+ O\left(n^{(1-\delta)/2} \sum_{k=1}^{n} f_2^{k-n}(\theta) \right),
\]

where the last equality follows from (A.61), (A.65), \( \rho_n < 1 \) and \( f_2^{-1}(\theta) < 1 \). It follows from

\[
\sum_{k=1}^{n} f_2^{k-n}(\theta) = O(n^{(1-\delta)/2}),
\]

(A.67)

uniformly in \( \Theta \) that

\[
\text{tr}(T_n^{-1}) = \frac{n}{(f_1^2(\theta) - 4\rho_n^2 \theta_1^2)^{1/2}} + O(n^{1-\delta}),
\]

uniformly in \( \Theta \). This together with (A.61) implies (A.5).

Next, we prove (A.6). By (A.66),

\[
\text{tr}(T_n^{-2}) = \sum_{k=1}^{n} \sum_{\ell=1}^{k} C_n(\ell, k)^2
= \sum_{k=1}^{n} \sum_{\ell=1}^{k} \rho_n^{2(k-\ell)} f_2^{2(k-\ell)}(\theta) + O\left( \sum_{k=1}^{n} \sum_{\ell=1}^{k} f_2^{-\ell}(\theta) \right)
+ O\left( \sum_{k=1}^{n} \sum_{\ell=1}^{k} f_2^{2k-2\ell}(\theta) \right)
= \sum_{k=1}^{n} \sum_{\ell=1}^{k} \rho_n^{2(k-\ell)} f_2^{2(k-\ell)}(\theta) + O(n^{2(1-\delta)}),
\]

uniformly in \( \Theta \), where the second equality follows from (A.65), \( \rho_n < 1 \) and \( f_2^{-1}(\theta) < 1 \), and the last equality follows from (A.67) and

\[
\frac{1}{f_1^2(\theta) - 4\rho_n^2 \theta_1^2} = n^{1-\delta} + O(1),
\]

(A.68)

uniformly in \( \Theta \). It follows that

\[
\text{tr}(T_n^{-2}) = \sum_{k=1}^{n} \left\{ \sum_{\ell=1}^{k} \rho_n^{2(k-\ell)} f_2^{2(k-\ell)}(\theta) + \sum_{\ell=k+1}^{n} \rho_n^{2(\ell-k)} f_2^{2(\ell-k)}(\theta) \right\}
= \frac{1}{(f_1^2(\theta) - 4\rho_n^2 \theta_1^2)} \sum_{k=1}^{n} \left( \frac{f_2^{2}(\theta) - \rho_n^{2k} f_2^{2-2k}(\theta)}{f_2^{2}(\theta) - \rho_n^{2}} \right)
+ \frac{\rho_n^{2n-2k+2} f_2^{2k-2n}(\theta)}{f_2^{2}(\theta) - \rho_n^{2}} \right\}
+ O\left(n^{2(1-\delta)} \right)
= \sum_{k=1}^{n} \left( \frac{f_2^{2}(\theta) + \rho_n^{2}}{f_2^{2}(\theta) - \rho_n^{2}} \right) + O\left(n^{3(1-\delta)/2} \sum_{k=1}^{n} f_2^{k}(\theta) \right)
+ O\left(n^{3(1-\delta)/2} \sum_{k=1}^{n} f_2^{k-n}(\theta) \right) + O(n^{2(1-\delta)})
= \frac{n(f_2^{2}(\theta) + \rho_n^{2})}{(f_1^2(\theta) - 4\rho_n^2 \theta_1^2)(f_2^{2}(\theta) - \rho_n^{2})} + O(n^{2(1-\delta)}),
\]
uniformly in \( \Theta \), where the second last equality follows from \((2.4), (A.54), (A.68), \rho_n < 1 \) and \( f_2^{-1}(\theta) < 1 \), and the last equality follows from \((A.67)\). This together with \((2.4), (A.54) \) and \((A.68) \) gives \((A.6)\).

Finally, we prove \((A.7) \) and \((A.8) \). Notice that \( T_n(\theta) \) defined in \((A.3) \) corresponds to the variance-covariance matrix of a moving average (MA) process \( \{\nu(s_1), \ldots, \nu(s_n)\} \) of order \( 1 \):

\[
\text{var}(\nu_n) = T_n(\theta),
\]

where \( \nu_n \equiv (\nu(s_1), \ldots, \nu(s_n))' \),

\[
\nu(s_i) = u(s_i) - \frac{\rho_n}{f_2(\theta)}u(s_{i-1}); \quad i = 1, 2, \ldots, n,
\]

with \( u(s_0) \sim N(0, \rho_n^2(\theta^2_2\theta^2_3^{-1} + 1 - f_2(\theta)\theta_1) f_2^2(\theta)) \) and \( u(s_i) \sim N(0, f_2(\theta)\theta_1); \quad i = 1, 2, \ldots, n, \) and recall that \( f_2(\theta) \) is defined in \((A.54) \). Consider the spectral density of \( \nu(\cdot) \),

\[
f_\nu(\omega) = \frac{f_2(\theta)\theta_1}{2\pi} \left( 1 + \frac{2\rho_n}{f_2(\theta)} \cos \omega + \frac{\rho_n^2}{f_2^2(\theta)} \right); \quad \omega \in [0, 2\pi].
\]

By Proposition 4.5.3 of Brockwell and Davis (2009), we have

\[
\liminf_{n \to \infty} \lambda_{\min}(T_n(\theta)) \geq \liminf_{n \to \infty} 2\pi \inf_{\omega \in [0, 2\pi]} f_\nu(\omega) = \liminf_{n \to \infty} \frac{\theta_1}{f_2(\theta)}(f_2(\theta) - \rho_n)^2.
\]

Thus \((A.7) \) follows from \((2.4) \) and \((A.54) \). Clearly, \( G_n(\theta)G_n(\theta)' \) also corresponds to the variance-covariance matrix of \( \nu(\cdot) \) with \( \theta_1 = 1, \theta_2 \theta_3^{-1} = 0 \). Thus, \((A.8) \) follows similarly. This completes the proof. \( \square \)

**Proof of Lemma A.3.** First, we prove \((A.9) \). Clearly,

\[
G_n^{-1}(\theta) = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \rho_n & 0 & \cdots & 0 \\
0 & 0 & \rho_n & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \rho_n^{-1}
\end{pmatrix}
\]

and hence

\[
\frac{\theta_1^2}{\theta_2} \Sigma_n(\theta) = G_n^{-1}(\theta) + G_n^{-1}(\theta)' - I.
\]

It follows that

\[
G_n(\theta)\Sigma_n(\theta_0)G_n(\theta)' = \frac{\theta_{0,2}}{\theta_{0,3}} G_n(\theta)(G_n^{-1}(\theta_0) + G_n^{-1}(\theta_0)' - I)G_n(\theta)'
\]

\[
= \frac{\theta_{0,2}}{\theta_{0,3}} (G_n(\theta)G_n^{-1}(\theta_0)G_n(\theta)')
\]

\[
= G_n(\theta)G_n^{-1}(\theta_0)G_n(\theta)'
\]

\[+ G_n(\theta)G_n^{-1}(\theta_0)'G_n(\theta)' - G_n(\theta)G_n(\theta)'\).
\]

It follows from

\[
G_n(\theta)G_n^{-1}(\theta_0) = I + \frac{\rho_n}{\rho_{0,n}}(G_n^{-1}(\theta_0) - I)
\]

\[
= \frac{\rho_n}{\rho_{0,n}} I + \left(1 - \frac{\rho_n}{\rho_{0,n}}\right)G_n^{-1}(\theta_0),
\]

that

\[
G_n(\theta)\Sigma_n(\theta_0)G_n(\theta)' = \frac{\theta_{0,2}}{\theta_{0,3}} \left\{ \frac{\rho_n}{\rho_{0,n}}(G_n(\theta) + G_n(\theta))
\]

\[+ \left(1 - \frac{\rho_n}{\rho_{0,n}}\right)(G_n^{-1}(\theta_0)G_n(\theta)' + G_n(\theta)G_n^{-1}(\theta_0)')
\]

\[= G_n(\theta)'\right\}.
\]

Finally, it follows from

\[
G_n^{-1}(\theta_0)G_n(\theta)' = G_n(\theta)' + (G_n^{-1}(\theta_0) - I) - \rho_n\rho_{0,n}(G_n^{-1}(\theta_0) - \nu e_1' ),
\]
that

\[ G_n(\theta)\Sigma_n(\theta_0)G_n(\theta)' \]

\[ = \frac{\theta_{0.2}}{\theta_{0.3}} \left\{ G_n(\theta) + G_n(\theta)' - G_n(\theta)G_n(\theta)' \right\} \]

\[ + \left( 1 - \frac{\rho_n}{\rho_{0,n}} \right) (1 - \rho_n\rho_{0,n}) (G_n^{-1}(\theta_0) + G_n^{-1}(\theta_0)') - I \]

\[ = \left( 1 - \frac{\rho_n}{\rho_{0,n}} \right) (1 + \rho_n\rho_{0,n}) I - \left( 1 - \frac{\rho_n}{\rho_{0,n}} \right) (ve_1 + e_1v') \]

This completes the proof of (A.9).

Next, we prove (A.10), (A.11) and (A.12). Clearly, (A.12) follows from (A.66) that \( sup_{\theta \in \Theta} C_n(1,1) = O(1) \). It follows from (2.4) that \( v_0'v_0 = O((1 - \rho_0^2)^{-1}) = O(n^{1-\delta}) \). This together gives (A.10). Finally, (A.11) follows from (A.10) and (A.12).

Next, we prove (A.13). By (A.63),

\[ \text{tr}(T_n^{-1}(\theta)\Sigma_n(\theta_0)) = \frac{\theta_{0.2}}{\theta_{0.3}} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \rho_n^{k-\ell} \rho_{0,n}^{k-\ell} f_2^{\ell-k}(\theta) \]

\[ = \frac{\theta_{0.2}}{\theta_{0.3}} \sum_{k=1}^{n} \rho_n f_2^{2-k}(\theta) + O(n(3(1-\delta)/2)) \]

uniformly in \( \Theta \), where the second last equality follows from (A.65), \( \rho_{0,n} < 1 \), \( \rho_n < 1 \) and \( f_2^{-1}(\theta) < 1 \), and the last equality follows from (A.61) and (A.67). It follows that

\[ \text{tr}(T_n^{-1}(\theta)\Sigma_n(\theta_0)) = \frac{\theta_{0.3}}{\theta_{0.2}(f_2^2(\theta) - 4\rho_n^2\theta_2^2)^{1/2}} \sum_{k=1}^{n} \left\{ \sum_{\ell=1}^{k} \rho_n^{k-\ell} \rho_{0,n}^{k-\ell} f_2^{\ell-k}(\theta) \right\} + O(n^{3(1-\delta)/2}) \]

\[ = \frac{\theta_{0.3}}{\theta_{0.2}(f_2^2(\theta) - 4\rho_n^2\theta_2^2)^{1/2}} \sum_{k=1}^{n} \left\{ \frac{f_2(\theta)}{f_2(\theta) - \rho_n\rho_{0,n}} \sum_{\ell=1}^{k} \rho_n^{k-\ell} \rho_{0,n}^{k-\ell} f_2^{\ell-k}(\theta) \right\} + O(n^{3(1-\delta)/2}) \]

\[ = \frac{\theta_{0.3} f_2(\theta) + \rho_n\rho_{0,n}}{\theta_{0.3}(f_2^2(\theta) - 4\rho_n^2\theta_2^2)^{1/2}(f_2(\theta) - \rho_n\rho_{0,n})} + O(n^{3(1-\delta)/2}) \]

\[ = \frac{n\theta_{0.2} f_2(\theta) + \rho_n\rho_{0,n}}{\theta_{0.3}(f_2^2(\theta) - 4\rho_n^2\theta_2^2)^{1/2}(f_2(\theta) - \rho_n\rho_{0,n})} + O(n^{3(1-\delta)/2}), \]
uniformly in $\Theta$, where the second last equality follows from (2.4), (A.54), (A.61), $\rho_n < 1$ and $f_2^{-1}(\theta) < 1$, and the last equality follows from (A.67). This together with (2.4), (A.54) and (A.61) gives (A.13).

Finally, we proof (A.14). By (A.63), we have

\[
\text{tr}\left((T_n^{-1}(\theta)\Sigma_n(\theta))^2\right) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(\frac{\theta^2}{\theta_1^2} \sum_{i=1}^{n} C_n(k, i) \rho_n^{[\ell-i]}\right)^2
\]
\[= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(\frac{\theta^2}{\theta_1^2} \sum_{i=1}^{n} \rho_n^{[\ell-k-i]} f_2^{-1} f_2^{-1}(\theta) - \rho_n^{[\ell-i]} f_2^{-1}(\theta)\right)^2
\]
\[+ O\left(\sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(\frac{f_2^{-1}(\theta)}{f_2^{-1}(\theta) - 4 \rho_n^2 \theta_1^2} \right)^2\right)
\]
\[= \frac{\theta^2}{\theta_1^2} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(\sum_{i=1}^{n} \rho_n^{[\ell-i]} f_2^{-1} f_2^{-1}(\theta) - \rho_n^{[\ell-i]} f_2^{-1}(\theta)\right)^2 + O(n(7-\delta)/2),
\]

uniformly in $\Theta$, where the second equality follows from (A.65), $\rho_n < 1$ and $f_2^{-1}(\theta) < 1$, and the last equality follows from (2.4), (A.54), (A.67) and (A.68). It follows that

\[
\frac{\theta^2}{\theta_1^2} (f_2^{-1}(\theta) - 4 \rho_n^2 \theta_1^2) \text{tr}\left((T_n^{-1}(\theta)\Sigma_n(\theta))^2\right)
\]
\[= n \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(\sum_{i=1}^{n} \rho_n^{[\ell-k-i]} f_2^{-1} f_2^{-1}(\theta) - \rho_n^{[\ell-i]} f_2^{-1}(\theta)\right)^2
\]
\[+ O\left(\sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(\frac{f_2^{-1}(\theta)}{f_2^{-1}(\theta) - 4 \rho_n^2 \theta_1^2} \right)^2\right)\equiv \phi_1(\theta) + \phi_2(\theta).
\]

First, we deal with $\phi_1(\theta)$. That is

\[
\phi_1(\theta) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \rho_n^{[\ell-k-i]} f_2^{-1} f_2^{-1}(\theta) - \rho_n^{[\ell-i]} f_2^{-1}(\theta)
\]
\[= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(\sum_{i=1}^{n} \rho_n^{[\ell-k-i]} f_2^{-1} f_2^{-1}(\theta) - \rho_n^{[\ell-i]} f_2^{-1}(\theta)\right)^2
\]
\[+ O\left(\sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(\frac{f_2^{-1}(\theta)}{f_2^{-1}(\theta) - 4 \rho_n^2 \theta_1^2} \right)^2\right).
\]

unifomly in $\Theta$, where the second last equality follows from (2.4), (A.54), $\rho_n < 1$ and $f_2^{-1}(\theta) < 1$, and the last equality follows from (2.4) and (A.54). It follows from (A.67) that

\[
\phi_1(\theta) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{\rho_n^{[\ell-k-i]} f_2^{-1}(\theta) - \rho_n^{[\ell-i]} f_2^{-1}(\theta)}{(f_2^{-1}(\theta) - \rho_n^2)^2(f_2^{-1}(\theta) - 1)^2} + O(n(5-\delta)/2),
\]
uniformly in \( \Theta \). We remain to deal with \( \phi_2(\theta) \). It follows from an argument similar to the proof of \((A.72)\) that
\[
\phi_2(\theta) = \sum_{k=1}^{n} \sum_{\ell=k+1}^{n} \frac{\rho_{n,\ell}^{2(l-k)}(f_2^2(\theta) - \rho_{n,\ell}^2)}{(f_2(\theta) - \rho_{n,\ell}^2)^2(f_2(\theta) - 1)^2} + O(n^{(5-3\delta)/2}),
\]
(A.73)
uniformly in \( \Theta \). It follows from \((A.71)\), \((A.72)\) and \((A.73)\) that for \( \delta \in (0, 1) \),
\[
\text{tr}((T_n^{-1}(\theta) \Sigma_0(\theta)))^2
= \frac{\theta_2^2 f_2^2(\theta) - \rho_{n,\ell}^2}{(f_1^2(\theta) - 4\rho_{n,\ell}^2 \theta_1^2)^1/2} \left( \frac{\rho_{n,\ell}^2}{\theta_1} f_2(\theta) - \rho_{n,\ell}^2 \theta_1 \right) + O(\exp(-\tau n^{(1+\delta)/3}))
\]
uniformly in \( \Theta \). It follows from \( \log(x + \Delta_x) = \log x + O(\Delta_x/x) \) as \( \Delta_x \to 0 \) that
\[
\log(\det(\Sigma(\theta)))
= \log \left( \frac{\theta_1^2 f_2^2(\theta)}{(f_1^2(\theta) - 4\rho_{n,\ell}^2 \theta_1^2)^1/2} \left( \frac{\rho_{n,\ell}^2}{\theta_1} f_2(\theta) - \rho_{n,\ell}^2 \theta_1 \right) \right) + o(n^{(1+\delta)/3})
\]
uniformly in \( \Theta \), where the third equality follows from \((2.4)\) and \((A.54)\), the fourth equality follows from \((A.61)\) that
\[
\log((f_1^2(\theta) - 4\rho_{n,\ell}^2 \theta_1^2)^1/2) = \frac{1-\delta}{2} \log n + O(1),
\]
uniformly in \( \Theta \), and the last equality follows from a Taylor expansion of \( \log f_2(\theta) \) at \( f_2(\theta) = 1 \) and together with \((A.54)\) that
\[
\log f_2(\theta) = \left( \frac{\rho_{n,\ell}^2}{\theta_1} \right)^{1/2} n^{-(1-\delta)/2} - \left( \frac{\rho_{n,\ell}^2}{\theta_1} \right) n^{-(1-\delta)}
\]
uniformly in \( \Theta \). Hence, \((A.15)\) is obtained.

Next, we prove \((A.16)\). By \((A.1)\) and \((A.9)\) with \( \theta_0 = \theta \),
\[
\text{tr}((\Sigma^{-1}(\theta) \Sigma_n(\theta))^2)
= \text{tr}((T_n^{-1}(\theta)((1 - \rho_{n,\ell}^2)I + \rho_{n,\ell}^2 e_1 e_1'))^2)
\]
\[
= (1 - \rho_{n,\ell}^2)^2 \text{tr}(T_n^{-2}(\theta)) + 2\rho_{n,\ell}^2(1 - \rho_{n,\ell}^2)e_1^T T_n^{-2} e_1 + \rho_{n,\ell}^2(e_1^T T_n^{-1}(\theta))e_1
\]
\[
= (1 - \rho_{n,\ell}^2)^2 \text{tr}(T_n^{-2}(\theta)) + O(1),
\]
uniformly in $\Theta$, where the last equality follows from (A.7) and (A.12). Thus (A.16) follows from (A.6) and (2.4).

Next, we prove (A.17). By (A.1) and (A.9),
\[
\text{tr}(\Sigma^{-1}(\theta)\Sigma_n(\theta_0)) = \text{tr}(T_n^{-1}(\theta)G_n(\theta)\Sigma_n(\theta_0)G_n(\theta)')
\]
\[
= \frac{\theta_0,2\rho_n}{\theta_0,3\rho_n} (1 - \rho_n^2) \text{tr}(T_n^{-1}(\theta))
\]
\[
+ \left(1 - \frac{\rho_n}{\rho_0,n}\right)(1 - \rho_n,\rho_0,n) \text{tr}(T_n^{-1}(\theta)\Sigma_n(\theta_0))
\]
\[
+ \frac{\theta_0,2}{\theta_0,3} \left(1 - \frac{\rho_n}{\rho_0,n}\right) \text{tr}(T_n^{-1}(\theta)(v_0e_1 + e_1v'_0))
\]
\[
+ \frac{\theta_0,2}{\theta_0,3} \rho_n^2 \text{tr}(T_n^{-1}(\theta)e_1e_1').
\]
Therefore, for (A.17) to hold, it suffices to show the following equations hold uniformly in $\Theta$:
\[
\frac{\theta_0,2\rho_n}{\theta_0,3\rho_n} (1 - \rho_n^2) \text{tr}(T_n^{-1}(\theta))
\]
\[
= \frac{\theta_0,2}{2(\theta_1,\theta_2)^1/2} n^{(1+\delta)/2} + O(n^{-3/2}),
\] (A.74)
\[
\left(1 - \frac{\rho_n}{\rho_0,n}\right)(1 - \rho_n,\rho_0,n) \text{tr}(T_n^{-1}(\theta)\Sigma_n(\theta_0))
\]
\[
= \frac{\theta_0,2(\theta_1^2 - \theta_0,3^2)}{2\theta_2,\theta_3} n^\delta + o(n^\delta),
\] (A.75)
and
\[
\frac{\theta_0,2}{\theta_0,3} \left(1 - \frac{\rho_n}{\rho_0,n}\right) \text{tr}(T_n^{-1}(\theta)(v_0e_1 + e_1v'_0)) = O(1),
\] (A.76)
\[
\frac{\theta_0,2}{\theta_0,3} \rho_n^2 \text{tr}(T_n^{-1}(\theta)e_1e_1') = O(1),
\] (A.77)
uniformly in $\Theta$, where (A.74) and (A.75) follow from (2.4), (A.5) and (A.13), and (A.76)-(A.77) follow from (A.11) and (A.12). Thus (A.17) is established.

Finally, we prove (A.18). By (A.17), we have
\[
\text{tr}(\Sigma(\theta)\Sigma^{-1}(\theta)) = \left(\frac{\theta_2}{2\theta_1}\right)^{1/2} n^{(1+\delta)/2} + o(n^\delta) + O(1),
\]
uniformly in $\Theta$. It follows that
\[
\text{tr}(\Sigma^{-1}(\theta)) = \frac{1}{\theta_1} (\text{tr}(I) - \text{tr}(\Sigma(\theta)\Sigma^{-1}(\theta)))
\]
\[
= \frac{n}{\theta_1} - \frac{1}{2\theta_1} \left(\frac{\theta_2}{2\theta_1}\right)^{1/2} n^{(1+\delta)/2} + o(n^\delta) + O(1),
\]
uniformly in $\Theta$. This together with (A.17) gives that
\[
\text{tr}(\Sigma(\theta_0)\Sigma^{-1}(\theta)) = \theta_0,1 \text{tr}(\Sigma^{-1}(\theta)) + \text{tr}(\Sigma(\theta)\Sigma^{-1}(\theta))
\]
\[
= \frac{\theta_0,1}{\theta_1} n - \frac{\theta_0,1}{2\theta_1} (\frac{\theta_2}{\theta_1})^{1/2} n^{(1+\delta)/2}
\]
\[
+ \frac{\theta_0,2}{2\theta_2,\theta_3} n^{(1+\delta)/2} + \frac{\theta_0,2(\theta_1^2 - \theta_0,3^2)}{2\theta_2,\theta_3} n^\delta
\]
\[
+ o(n^\delta) + O(1),
\]
uniformly in $\Theta$. Thus (A.18) is established. This completes the proof.

Proof of Lemma A.5. The proof of (A.19) and (A.20) are straightforward and can be verified easily and hence are omitted here. We remain to show (A.21) and (A.22). By (A.19) and $F(\theta) = \frac{\partial}{\partial \theta_3} \Sigma(\theta)$, we have
\[
\frac{\partial^2}{\partial \theta_3^2} \Sigma(\theta) = \frac{\partial}{\partial \theta_3} \left\{ \frac{\theta_2}{\theta_3} \left( \frac{\theta_3}{\theta_2} \Sigma_n(\theta) - \frac{\theta_2}{\theta_3} F(\theta) \right) - \frac{\theta_2}{\theta_3} F(\theta) \right\}
\]
\[
= \frac{2\theta_2}{\theta_3^3} \left( \frac{\theta_3}{\theta_2} \Sigma_n(\theta) \right) + \frac{\theta_2}{\theta_3^2} F(\theta) + \frac{\theta_2}{\theta_3^2} F(\theta) - \frac{\theta_2}{\theta_3} \frac{\partial}{\partial \theta_3} F(\theta).
\]
Thus (A.20) and
\[
\frac{\partial}{\partial \theta_3} G_n^{-1}(\theta) = - G_n^{-1}(\theta) \frac{\partial}{\partial \theta_3} G_n(\theta) G_n^{-1}(\theta)
\]
\[
= n^{-(1-\delta)} G_n^{-1}(\theta) (G_n(\theta) - I) G_n^{-1}(\theta) = n^{-(1-\delta)} (G_n^{-1}(\theta) - G_n^{-2}(\theta)),
\]
that
\[
\frac{\partial}{\partial \theta_3} F(\theta) = 2 \rho_n^2 n^{-2(1-\delta)} G_n^{-1}(\theta) \left( \frac{\partial}{\partial \theta_2} \Sigma(\theta) \right) G_n^{-1}(\theta)'
\]
\[
+ (1 - \rho_n^2) n^{-2(1-\delta)} \left( G_n^{-1}(\theta) - G_n^{-2}(\theta) \right) \left( \frac{\partial}{\partial \theta_2} \Sigma(\theta) \right) G_n^{-1}(\theta)'
\]
\[
+ (1 - \rho_n^2) n^{-2(1-\delta)} G_n^{-1}(\theta) \left( \frac{\partial}{\partial \theta_2} \Sigma(\theta) \right) \left( G_n^{-1}(\theta)' - G_n^{-2}(\theta)' \right)
\]
\[
+ (1 - \rho_n^2) n^{-2(1-\delta)} G_n^{-1}(\theta) F(\theta) G_n^{-1}(\theta)'
\]
\[
- n^{-2(1-\delta)} (G_n^{-1}(\theta) - G_n^{-2}(\theta)) \left( (1 + \rho_n^2) I - \rho_n^2 (\gamma' + e_1 v') \right) G_n^{-1}(\theta)'
\]
\[
- n^{-2(1-\delta)} G_n^{-1}(\theta) \left( (1 + \rho_n^2) I - \rho_n^2 (\gamma' + e_1 v') \right) \left( G_n^{-1}(\theta)' - G_n^{-2}(\theta)' \right)
\]
\[
+ \rho_n^2 n^{-2(1-\delta)} G_n^{-1}(\theta) (2I - v' e_1' - e_1 v') G_n^{-1}(\theta)',
\]
where $v^* = (2, \ldots, (n + 1) \rho_n^{n-1})'$. This together with (A.69) that
\[
G_n^{-1}(\theta) v = v^* - v,
\]
\[
G_n^{-1}(\theta) e_1 = v,
\]
follows that
\[
\frac{\partial}{\partial \theta_3} F(\theta) = 2n^{-2(1-\delta)} G_n^{-1}(\theta) \left( \frac{\partial}{\partial \theta_2} \Sigma(\theta) \right) G_n^{-1}(\theta)
\]
\[
- (1 - \rho_n^2) n^{-2(1-\delta)} G_n^{-2}(\theta) \left( \frac{\partial}{\partial \theta_2} \Sigma(\theta) \right) G_n^{-1}(\theta)'
\]
\[
- (1 - \rho_n^2) n^{-2(1-\delta)} G_n^{-1}(\theta) \left( \frac{\partial}{\partial \theta_2} \Sigma(\theta) \right) G_n^{-2}(\theta)'
\]
\[
+ (1 - \rho_n^2) n^{-2(1-\delta)} G_n^{-1}(\theta) F(\theta) G_n^{-1}(\theta)'
\]
\[
- 2n^{-2(1-\delta)} G_n^{-1}(\theta) G_n^{-1}(\theta)'
\]
\[
+ (1 + \rho_n^2) n^{-2(1-\delta)} G_n^{-1}(\theta) (G_n^{-1}(\theta) + G_n^{-1}(\theta)') G_n^{-1}(\theta)'
\]
\[
+ \rho_n^2 n^{-2(1-\delta)} G_n^{-1}(\theta) (\gamma' + e_1 v') G_n^{-1}(\theta)'
\]
\[
- 2 \rho_n^2 n^{-2(1-\delta)} G_n^{-1}(\theta) (\gamma' + e_1 v') G_n^{-1}(\theta)'
\]
\[
- 2 \rho_n^2 n^{-2(1-\delta)} G_n^{-1}(\theta) v' G_n^{-1}(\theta)',
\]
It follows from (A.70) that
\[
\frac{\partial}{\partial \theta_3} F(\theta) = (3 + \rho_n^2) n^{-2(1-\delta)} G_n^{-1}(\theta) \left( \frac{\partial}{\partial \theta_2} \Sigma(\theta) \right) G_n^{-1}(\theta)
\]
\[
- (1 - \rho_n^2) n^{-2(1-\delta)} G_n^{-2}(\theta) \left( \frac{\partial}{\partial \theta_2} \Sigma(\theta) \right) G_n^{-1}(\theta)'
\]
\[
- (1 - \rho_n^2) n^{-2(1-\delta)} G_n^{-1}(\theta) \left( \frac{\partial}{\partial \theta_2} \Sigma(\theta) \right) G_n^{-2}(\theta)'
\]
\[
+ (1 - \rho_n^2) n^{-2(1-\delta)} G_n^{-1}(\theta) F(\theta) G_n^{-1}(\theta)'
\]
\[
- (1 - \rho_n^2) n^{-2(1-\delta)} G_n^{-1}(\theta) F(\theta) G_n^{-1}(\theta)'
\]
\[
+ \rho_n^2 n^{-2(1-\delta)} G_n^{-1}(\theta) (\gamma' + e_1 v') G_n^{-1}(\theta)'
\]
\[
- 2 \rho_n^2 n^{-2(1-\delta)} G_n^{-1}(\theta) (\gamma' + e_1 v') G_n^{-1}(\theta)'
\]
\[
- 2 \rho_n^2 n^{-2(1-\delta)} G_n^{-1}(\theta) v' G_n^{-1}(\theta)'.
\]
Thus (A.22) is established. This completes the proof. \qed

Proof of Lemma A.6. First, we prove (A.23). Clearly, all the eigenvalues of \( \left( \Sigma^{-1}(\theta) \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right)^2 \) are positive, we remain to show the right-hand side inequality of (A.23). It follows from (2.2) and (A.19) that for (A.23) to hold, it
suffices to show that
\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta} \lambda_{\max}\left((\Sigma^{-1}(\theta)F(\theta))^2\right)
\]
\[
= \limsup_{n \to \infty} \sup_{\theta \in \Theta} \lambda_{\max}\left((T_n^{-1}(\theta)G_n(\theta)F(\theta)G_n(\theta)^\prime)^2\right) < \infty. \tag{A.78}
\]

By (A.20), we have
\[
G_n(\theta)F(\theta)G_n(\theta)^\prime = \frac{\theta_2}{\theta_3}(1 - \rho_n^2)n^{-(1-\delta)}\Sigma_n(\theta)
\]
\[- \frac{\rho_2}{\theta_3}(1 + \rho_n^2)n^{-(1-\delta)}I + \rho_n^2n^{-(1-\delta)}(ve_1^\prime + e_1v'). \tag{A.79}
\]

Therefore, for (A.78) to hold, it suffices to show that
\[
\lambda_{\max}\left((T_n^{-1}(\theta)\Sigma_n(\theta))^2\right) = O(n^{4(1-\delta)}),
\]
\[
\lambda_{\max}(T_n^{-2}(\theta)) = O(n^{2(1-\delta)}),
\]
\[
\lambda_{\max}\left((T_n^{-1}(\theta)(ve_1^\prime + e_1v'))^2\right) = O(n^{2(1-\delta)}),
\]
uniformly in \(\Theta\), which follow from (2.1), (A.7) and (A.10)-(A.11). Thus (A.23) is established and hence completes the proof of (A.23).

Next, we prove (A.24). By (A.1), we have
\[
\text{tr}\left((\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} \Sigma(\theta))^2\right) = \text{tr}\left((T_n^{-1}(\theta)G_n(\theta)\left(\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)G_n(\theta)^\prime)^2\right).
\]

Also, by (A.9) with \(\theta_0 = \theta\), (A.19) and (A.20), we have
\[
G_n(\theta) \left(\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)G_n(\theta)^\prime = -\frac{\theta_2}{\theta_3}(1 - \rho_n^2)I - \frac{\rho_2}{\theta_3}(1 + \rho_n^2)n^{-(1-\delta)}I - \frac{\rho_2}{\theta_3}n^{-(1-\delta)}(ve_1^\prime + e_1v'), \tag{A.79}
\]
\[
= \frac{\theta_2}{\theta_3}\left((1 + \rho_n^2)n^{-(1-\delta)} - \frac{1 - \rho_n^2}{\theta_3}\right)I
\]
\[- \left(1 - \rho_n^2\right)n^{-(1-\delta)}\frac{\partial}{\partial \theta_3} \Sigma_n(\theta) - \frac{\theta_2}{\theta_3}n^{-(1-\delta)}(ve_1^\prime + e_1v').
\]

Therefore, by (2.4), for (A.24) to hold, it suffices to show that
\[
n^{-4(1-\delta)}\text{tr}(T_n^{-2}(\theta)) = o(n^\delta),
\]
\[
n^{-4(1-\delta)}\text{tr}(T_n^{-2}(\theta)\Sigma_n(\theta)) = o(n^\delta),
\]
\[
n^{-2(1-\delta)}\text{tr}\left((T_n^{-1}(\theta)(ve_1^\prime + e_1v'))^2\right) = O(1),
\]
\[
n^{-3(1-\delta)}\text{tr}(T_n^{-1}(\theta)\Sigma_n(\theta)\text{tr}(T_n^{-1}(\theta)(ve_1^\prime + e_1v')) = O(1),
\]
and
\[
(1 - \rho_n^2)n^{-2(1-\delta)}\text{tr}\left((T_n^{-1}(\theta)\Sigma_n(\theta))^2\right) = \frac{1}{\theta_3}n^\delta + o(n^\delta),
\]
uniformly in \(\Theta\), where the first four equations follow from (2.1), (A.7) and (A.10)-(A.12), and the last equation follows from (2.4) and (A.14). Thus (A.24) is established.

Next, we prove (A.25). By (A.21), for (A.25) to hold, it suffices to show that
\[
\text{tr}\left((\Sigma^{-1}(\theta)\left(\frac{\partial}{\partial \theta_3} \Sigma_n(\theta) + \frac{\theta_2}{\theta_3}F(\theta)\right)^2\right) = O(n^\delta), \tag{A.80}
\]
\[
\text{tr}\left((\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} F(\theta))^2\right) = O(n^\delta), \tag{A.81}
\]
uniformly in \(\Theta\), where (A.80) follows from (A.19) and (A.24). We remain to prove (A.81). By (A.1), we have
\[
\text{tr}\left((\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} F(\theta))^2\right) = \text{tr}\left((T_n^{-1}(\theta)G_n(\theta)\left(\frac{\partial}{\partial \theta_3} F(\theta)\right)G_n(\theta)^\prime)^2\right).
\]
Therefore, by (2.4) and (A.22), for (A.81) to hold, it suffices to show that

\[
\begin{align*}
&n^{-4(1-\delta)}\text{tr}\left((T_n^{-1}(\theta)\Sigma_n(\theta))^2\right) = O(n^5), \\
&n^{-8(1-\delta)}\text{tr}\left((T_n^{-1}(\theta)G_n^{-1}(\theta)\Sigma_n(\theta)G_n^{-1}(\theta))\right) = O(n^5), \\
&n^{-6(1-\delta)}\text{tr}\left(T_n^{-1}(\theta)\right) = O(n^5), \\
&n^{-4(1-\delta)}\text{tr}\left((T_n^{-1}(\theta)(ve'_1 + e_1 v))^2\right) = O(n^5), \\
&n^{-4(1-\delta)}\text{tr}\left((T_n^{-1}(\theta)vv')^2\right) = O(n^5),
\end{align*}
\]

and

\[
\text{tr}\left((T_n^{-1}(\theta)F(\theta))^2\right) = O(n^5),
\]
(A.82)

uniformly in \(\Theta\), where the first six equations follow from (2.1), (A.7), (A.8), (A.10)-(A.12) and (A.14). We remain to prove (A.82). By (2.4) and (A.20), for (A.82) to hold, it suffices to show that

\[
\begin{align*}
&n^{-8(1-\delta)}\text{tr}\left((T_n^{-1}(\theta)G_n^{-1}(\theta)\Sigma_n(\theta)G_n^{-1}(\theta))\right) = O(n^5), \\
&n^{-6(1-\delta)}\text{tr}\left((T_n^{-1}(\theta)G_n^{-1}(\theta)G_n^{-1}(\theta))^2\right) = O(n^5), \\
&n^{-6(1-\delta)}\text{tr}\left((T_n^{-1}(\theta)G_n^{-1}(\theta)(ve'_1 + e_1 v'))^2\right) = O(n^5),
\end{align*}
\]

uniformly in \(\Theta\), which follow from (2.1), (A.7), (A.8), and (A.10)-(A.12). Thus (A.25) is established.

Next, we prove (A.26). By (A.1), we have

\[
\begin{align*}
&\text{tr}\left(\Sigma^{-2}(\theta)\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right) \\
&= \text{tr}\left(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)T_n^{-1}(\theta)G_n(\theta)\left(\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)G_n(\theta)\right).
\end{align*}
\]

Therefore, by (2.4) and (A.79), for (A.26) to hold, it suffices to show that

\[
\begin{align*}
&n^{-2(1-\delta)}\text{tr}\left(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)\right) = O(n^5), \\
&n^{-2(1-\delta)}\text{tr}\left(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)\Sigma_n(\theta)\right) = O(n^5), \\
&n^{-4(1-\delta)}\text{tr}\left(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)(ve'_1 + e_1 v')\right) = O(n^5),
\end{align*}
\]

which follow from (A.5), (A.10)-(A.12), (A.13) with \(\theta_0 = \theta\) and

\[
\sup_{\theta \in \Theta} \lambda_{\max}\left(T_n^{-1/2}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1/2}(\theta)\right) = \sup_{\theta \in \Theta} \lambda_{\max}(\Sigma^{-1}(\theta)) = O(1),
\]
(A.83)

uniformly in \(\Theta\). Thus (A.26) is established.

Finally, we prove (A.27). By (A.21), for (A.27) to hold, it suffices to show that

\[
\begin{align*}
&\text{tr}\left(\Sigma^{-2}(\theta)\left(\frac{1}{\theta_3} \Sigma_n(\theta) + \frac{\theta_3}{\theta_3} F(\theta)\right)\right) = O(n^5), \\
&\text{tr}\left(\Sigma^{-2}(\theta)\frac{\partial}{\partial \theta_3} F(\theta)\right) = O(n^5),
\end{align*}
\]

uniformly in \(\Theta\), where (A.84) follows from (A.26). We remain to prove (A.85). By (A.1), we have

\[
\begin{align*}
&\text{tr}\left(\Sigma^{-2}(\theta)\frac{\partial}{\partial \theta_3} F(\theta)\right) \\
&= \text{tr}\left(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)G_n(\theta)^\prime\left(\frac{\partial}{\partial \theta_3} F(\theta)\right)G_n(\theta)\right).
\end{align*}
\]

Therefore, by (2.4) and (A.22), for (A.85) to hold, it suffices to show that

\[
\begin{align*}
&n^{-2(1-\delta)}\text{tr}\left(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)\Sigma_n(\theta)\right) = O(n^5), \\
&n^{-4(1-\delta)}\text{tr}(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)G_n^{-1}(\theta)\Sigma_n(\theta)G_n^{-1}(\theta)^\prime) = O(n^5), \\
&n^{-4(1-\delta)}\text{tr}(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)(ve'_1 + e_1 v')) = O(n^5),
\end{align*}
\]

(A.85)
and
\[ n^{-2(1 - \delta)} \text{tr}(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)F(\theta)) = O(n^\delta), \tag{A.86} \]
uniformly in \( \Theta \), where the first six equations follow from (2.1), (A.7), (A.10)-(A.13) and (A.83). We remain to prove (A.86). By (2.4) and (A.20), for (A.86) to hold, it suffices to show that
\[ \begin{align*}
&n^{-4(1 - \delta)} \text{tr}(T_n^{-2}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-2}(\theta)G_n(\theta)\Sigma_n(\theta)G_n(\theta)^\prime) = O(n^\delta), \\
&n^{-3(1 - \delta)} \text{tr}(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)G_n(\theta)\Sigma_n(\theta)G_n(\theta)^\prime) = O(n^\delta), \\
&n^{-3(1 - \delta)} \text{tr}(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)G_n(\theta)\Sigma_n(\theta)G_n(\theta)^\prime) = O(n^\delta),
\end{align*} \]
uniformly in \( \Theta \), which follow from (2.1), (A.7), (A.8). Thus (A.27) is established. This completes the proof. \( \square \)

**Proof of Lemma A.7.**

Let
\[ g(\theta) = (\eta + \epsilon)^\prime \Sigma^{-1}(\theta)M(\theta)(\eta + \epsilon) - E\{(\eta + \epsilon)^\prime \Sigma^{-1}(\theta)M(\theta)(\eta + \epsilon)\}. \]
Clearly, \( g(\theta_0) = p_n + O_p(p_n/2) \) and
\[ \sup_{\theta \in \Theta} E\{(\eta + \epsilon)^\prime \Sigma^{-1}(\theta)M(\theta)(\eta + \epsilon)\} = O(p_n). \]
Therefore, for (A.28) to hold, it suffices to show that to show that
\[ \sup_{\theta \in \Theta} |g(\theta) - g(\theta_0)| = o_p(p_n). \tag{A.87} \]
By Lemma B.1 of Chan and Ing (2011), the first moment based theorem of Findley and Wei (1993), and an argument similar to the proof of (2.12), we check the first derivative terms and the other derivatives terms are similar. That is
\[ \begin{align*}
&\sup_{\theta \in \Theta} \text{var}\left(\frac{\partial}{\partial \theta_1} g(\theta)\right) = O(p_n), \\
&\sup_{\theta \in \Theta} \text{var}\left(\frac{\partial}{\partial \theta_2} g(\theta)\right) = O(p_n), \\
&\sup_{\theta \in \Theta} \text{var}\left(\frac{\partial}{\partial \theta_3} g(\theta)\right) = O(p_n).
\end{align*} \tag{A.88, A.89, A.90} \]

First, we prove (A.88). We have
\[ \frac{\partial g(\theta)}{\partial \theta_1} = - (\eta + \epsilon)^\prime \Sigma^{-2}(\theta)M(\theta)(\eta + \epsilon) - \Sigma^{-1}(\theta)M(\theta)\Sigma^{-1}(\theta)M(\theta)(\eta + \epsilon) + (\eta + \epsilon)^\prime (\Sigma^{-2}(\theta)M(\theta))(\eta + \epsilon). \]
Therefore, for (A.88) to hold, it suffices to show that
\[ \begin{align*}
&\text{tr}\left(\left(\Sigma^{-2}(\theta)M(\theta) + \Sigma^{-2}(\theta)M(\theta)\Sigma(\theta_0)\right)^2\right) = O(p_n), \\
&\text{tr}\left(\left(\Sigma^{-2}(\theta)M(\theta)\Sigma(\theta_0)\right)^2\right) = O(p_n),
\end{align*} \]
uniformly in \( \Theta \), which follow from (2.2).

Next, we prove (A.89). We have
\[ \frac{\partial g(\theta)}{\partial \theta_2} = - \frac{1}{\theta_3} (\eta + \epsilon)^\prime \Sigma^{-1}(\theta)\Sigma^{-1}(\theta)M(\theta)(\eta + \epsilon) + \frac{1}{\theta_3} (\eta + \epsilon)^\prime (\Sigma^{-2}(\theta)M(\theta))(\eta + \epsilon) + (\eta + \epsilon)^\prime (\Sigma^{-2}(\theta)M(\theta))(\eta + \epsilon). \]

By an argument similar to the proof of (A.88), it gives (A.89).

Finally, we prove (A.90). We have
\[ \begin{align*}
&\frac{\partial g(\theta)}{\partial \theta_3} = - (\eta + \epsilon)^\prime \Sigma^{-1}(\theta)\left(\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)\Sigma^{-1}(\theta)M(\theta)(\eta + \epsilon) - (\eta + \epsilon)^\prime (\Sigma^{-2}(\theta)M(\theta))\left(\frac{\partial}{\partial \theta_3} \Sigma^{-1}(\theta)\right)\Sigma^{-1}(\theta)M(\theta)(\eta + \epsilon) + (\eta + \epsilon)^\prime (\Sigma^{-2}(\theta)M(\theta))\left(\frac{\partial}{\partial \theta_3} \Sigma^{-1}(\theta)\right)\Sigma^{-1}(\theta)M(\theta)(\eta + \epsilon).
\end{align*} \]
Therefore, for (A.90) to hold, it suffices to show that
\[
\text{tr}\left( \left( M(\theta)'\Sigma^{-1}(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta) M(\theta) \Sigma(\theta_0) \right)^2 \right) = O(p_n),
\] (A.91)
\[
\text{tr}\left( \left( \Sigma^{-1}(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta) M(\theta) \Sigma(\theta_0) + M(\theta)'\Sigma^{-1}(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta) \Sigma(\theta_0) \right)^2 \right) = O(p_n),
\] (A.92)
uniformly in \( \Theta \). For (A.91), we have
\[
\text{tr}\left( \left( M(\theta)'\Sigma^{-1}(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta) M(\theta) \Sigma(\theta_0) \right)^2 \right)
= \text{tr}\left( \Sigma^{1/2}(\theta_0) M(\theta)'\Sigma^{-1}(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta) M(\theta) \Sigma(\theta_0) M(\theta)'\Sigma^{-1}(\theta) \right)
\times \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta) M(\theta) \Sigma^{1/2}(\theta_0)
= O\left( \text{tr}\left( \Sigma^{1/2}(\theta_0) M(\theta)'\Sigma^{-1}(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta) \right) \times \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta) M(\theta) \Sigma^{1/2}(\theta_0) \right)
= O\left( \text{tr}\left( \Sigma^{1/2}(\theta_0) M(\theta)'\Sigma^{-1}(\theta) M(\theta) \Sigma^{1/2}(\theta_0) \right) \right) = O(p_n),
\]
uniformly in \( \Theta \), where the second equality follows from (2.2) and \( \Sigma^{1/2}(\theta) M(\theta) \Sigma^{1/2}(\theta) \leq I \), the third equality follows from (A.23), and the fourth equality follows from (2.2). Similar to the proof of (A.91), (A.92) is established. Thus (A.90) is established. This completes the proof. \( \square \)

**Proof of Lemma A.8.** First, we prove (A.30). Here we check the first order derivative terms where the proofs of the second order derivative terms are similar and are omitted here. For \( j = 1 \), we have
\[
\left| \frac{\partial}{\partial \theta_1} \{ \mu_0' \Sigma^{-1}(\theta)(I - M(\theta))\mu_0 \} \right|
= \bigg| - \mu_0' \Sigma^{-2}(\theta)(I - M(\theta))\mu_0 + \mu_0 M(\theta)'\Sigma^{-2}(\theta)(I - M(\theta))\mu_0 \bigg|
= \mu_0' (I - M(\theta))'\Sigma^{-2}(\theta)(I - M(\theta))\mu_0
\leq C \mu_0' (I - M(\theta))'\Sigma^{-1}(\theta)(I - M(\theta))\mu_0
= C \mu_0' \Sigma^{-1}(\theta)(I - M(\theta))\mu_0 = O(n^5),
\]
uniformly in \( \Theta \), where \( C > 0 \) is some constant and the first inequality follows from (2.2). For \( j = 2 \), similarly, we have
\[
\left| \frac{\partial}{\partial \theta_2} \{ \mu_0' \Sigma^{-1}(\theta)(I - M(\theta))\mu_0 \} \right|
= \mu_0' (I - M(\theta))'\Sigma^{-1}(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta)(I - M(\theta))\mu_0
\leq C \mu_0' (I - M(\theta))'\Sigma^{-1}(\theta)(I - M(\theta))\mu_0
= C \mu_0 \Sigma^{-1}(\theta)(I - M(\theta))\mu_0 = O(n^5),
\]
uniformly in \( \Theta \), where \( C > 0 \) is some constant and the first inequality follows from (2.2). For \( j = 3 \), similarly, we have
\[
\left| \frac{\partial}{\partial \theta_3} \{ \mu_0' \Sigma^{-1}(\theta)(I - M(\theta))\mu_0 \} \right|
= \left| \mu_0' (I - M(\theta))'\Sigma^{-1}(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta)(I - M(\theta))\mu_0 \right|
\leq C \mu_0' (I - M(\theta))'\Sigma^{-1}(\theta)(I - M(\theta))\mu_0
= C \mu_0 \Sigma^{-1}(\theta)(I - M(\theta))\mu_0 = O(n^5),
\]
uniformly in \( \Theta \), where \( C > 0 \) is some constant and the first inequality follows from (2.4) and (A.19) that for any vector \( \mu \),
\[
n^{-2(1-\delta)} \mu' \Sigma^{-1}(\theta) G_n^{-1}(\theta) \Sigma_\eta(\theta) G_n^{-1}(\theta)' \Sigma^{-1}(\theta) \mu = O(\mu' \Sigma^{-1}(\theta) \mu),
n^{-(1-\delta)} \mu' \Sigma^{-1}(\theta) G_n^{-1}(\theta) G_n^{-1}(\theta)' \Sigma^{-1}(\theta) \mu = O(\mu' \Sigma^{-1}(\theta) \mu),\]
\[
n^{-2(1-\delta)} \mu' \Sigma^{-1}(\theta) G_n^{-1}(\theta) (\nu \epsilon_n + e_n v') G_n^{-1}(\theta)' \Sigma^{-1}(\theta) \mu = O(\mu' \Sigma^{-1}(\theta) \mu),
\]
uniformly in $\Theta$, which follow from (2.1) and (A.8).

Finally, we prove (A.31). Let $g(\theta) = \mu_0(\Sigma_0^{-1}(\theta)(I - M(\theta))(\eta + \epsilon)$. Thus for (A.31) to hold, it suffices to prove that

$$E\left(\sup_{\theta \in \Theta} |g(\theta)|^2\right) = O(n^5). \quad (A.93)$$

It follows by Lemma B.1 of Chan and Ing (2011), and the first moment based theorem of Findley and Wei (1993) that

$$E\left(\sup_{\theta \in \Theta} |g(\theta)|^2\right) \leq C \sup_{\theta \in \Theta} \left\{ \text{var}(g(\theta_0)) + \text{var}\left(\frac{\partial}{\partial \theta_1} g(\theta)\right) + \text{var}\left(\frac{\partial^2}{\partial \theta_2 g(\theta)}\right) + \text{var}\left(\frac{\partial^2}{\partial \theta_2 \partial \theta_3 g(\theta)}\right) + \text{var}\left(\frac{\partial^3}{\partial \theta_2 \partial \theta_3 \partial \theta_3 g(\theta)}\right) \right\}. \quad \text{terms and other derivatives terms are omitted. It follows from the proof of (A.30) that }$$

Clearly, it follows from (A.30) that $\text{var}(g(\theta_0)) = O(n^5)$. Similar to the proof of (2.12), we check the first derivative terms and other derivatives terms are omitted. It follows from the proof of (A.30) for $j = 1$ that

$$\text{var}\left(\frac{\partial}{\partial \theta_1} g(\theta)\right) = E\left((\mu_0(\Sigma_0^{-1}(\theta)(I - M(\theta))(\eta + \epsilon))^2\right) = \mu_0(\Sigma_0^{-1}(\theta)(I - M(\theta))(\eta + \epsilon).$$

uniformly in $\Theta$, where the second last equality follows from (2.2) and $I - \Sigma^{-1/2}(\theta)M(\theta)\Sigma^{1/2}(\theta) \leq I$. Similarly to the proofs of (A.30) for $j = 2, 3$, it gives that

$$\sup_{\theta \in \Theta} \text{var}\left(\frac{\partial}{\partial \theta_2} g(\theta)\right) = O(n^5),$$

$$\sup_{\theta \in \Theta} \text{var}\left(\frac{\partial}{\partial \theta_3} g(\theta)\right) = O(n^5).$$

Thus (A.93) is established. This completes the proof. \hfill \Box

Proof of Lemma A.9. First, we prove (A.32). By (A.9) with $\theta_0 = \theta$, we have

$$G_n(\theta)\Sigma_{\eta}(\theta)G_n(\theta)' = \frac{\theta_2}{\theta_3}(1 - \rho_n^2)I + \rho_n^2 e_1 e_1', \quad \text{and hence}$$

$$H(\theta) \equiv G_n(\theta)(\Sigma_{\eta}(\theta) - \Sigma_{\eta}(\theta_0))G_n(\theta)' = \left\{ \frac{\theta_2}{\theta_3}(1 - \rho_n^2) - \frac{\theta_0 \rho_n}{\theta_0 \rho_n}(1 - \rho_n^2) \right\}I$$

$$- \left\{ 1 - \rho_n^2 \right\} \rho_n \rho_n \Sigma_{\eta}(\theta_0)$$

$$- \frac{\theta_0}{\theta_0 \rho_n}(1 - \rho_n^2) \left\{ (e_n' + e_1'n) + \left( \frac{\theta_2}{\theta_3} - \frac{\theta_0}{\theta_0 \rho_n} \right) \rho_n^2 e_1 e_1' \right\}. \quad (A.94)$$

Note that

$$\frac{\theta_2}{\theta_3}(1 - \rho_n^2) - \frac{\theta_0 \rho_n}{\theta_0 \rho_n}(1 - \rho_n^2)$$

$$= \left\{ \frac{\theta_2 - \theta_0 \rho_n}{\theta_3} (1 - \rho_n^2) \right\}$$

$$+ \frac{\theta_0}{\theta_3 \theta_0 \rho_n}(1 - \rho_n^2) \left\{ \left( \theta_0 \rho_n (1 - \rho_n^2) - \theta_3 \rho_n (1 - \rho_n^2) \right) \right\} \quad (A.95)$$

$$= \left( \frac{\theta_2 - \theta_0 \rho_n}{\theta_3} (1 - \rho_n^2) + O((\theta_3 - \theta_0 \rho_n) n^{-2(1 - \delta)}), \right)$$
uniformly in $\Theta$, where the last equality follows from a Taylor expansion of $f(\theta_3) = \theta_{0,3}\rho_{0,n}(1 - \rho_n^2) - \theta_3\rho_n(1 - \rho_{0,n}^2)$ at $\theta_3 = \theta_{0,3}$ that for $k = 1, 2, \ldots$,

$$f(\theta_3) = 0,$$

$$\frac{\partial^k}{\partial^{k}\theta_3} f(\theta_3) = O(n^{-2(1-\delta)}),$$

uniformly in $\Theta$. It then follows from (A.1), (A.83) and (A.94) that

$$\text{tr}(\Sigma^{-1}(\theta)(\Sigma_0(\theta) - \Sigma_0(\theta_0)))$$

$$= \text{tr}(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)H(\theta))$$

$$= \left\{ \frac{\theta_2}{\theta_3}(1 - \rho_n^2) - \frac{\theta_{0,2}\rho_n}{\theta_{0,3}\rho_0,n}(1 - \rho_0^2) \right\} \text{tr}(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta))$$

$$- \left(1 - \frac{\rho_n}{\rho_0,n}\right)(1 - \rho_n\rho_0,n)\text{tr}(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)\Sigma_\eta(\theta_0))$$

$$+ \frac{\theta_{0,2}}{\theta_{0,3}}\text{tr}(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)e_1e_1^\prime)$$

$$\leq C\left\{ \frac{\theta_2}{\theta_3}(1 - \rho_n^2) - \frac{\theta_{0,2}\rho_n}{\theta_{0,3}\rho_0,n}(1 - \rho_0^2) \right\} \text{tr}(T_n^{-1}(\theta))$$

$$= \left(1 - \frac{\rho_n}{\rho_0,n}\right)(1 - \rho_n\rho_0,n)\text{tr}(T_n^{-1}(\theta)\Sigma_\eta(\theta_0))$$

$$- \theta_{0,2}\text{tr}(T_n^{-1}(\theta)(ve_1 + e_1v'))$$

$$+ \frac{\theta_{2}}{\theta_{3}}\text{tr}(T_n^{-1}(\theta)e_1e_1^\prime)$$

where $C > 0$ is some constant. Therefore, for (A.32) to hold, it suffices to show that

$$\left\{ \frac{\theta_2}{\theta_3}(1 - \rho_n^2) - \frac{\theta_{0,2}\rho_n}{\theta_{0,3}\rho_0,n}(1 - \rho_0^2) \right\} \text{tr}(T_n^{-1}(\theta)) = O((\theta_2 - \theta_{0,2})n^{(1+\delta)/2})$$

$$+ O((\theta_3 - \theta_{0,3})n^{-(1-3\delta)/2}),$$

$$\left(1 - \frac{\rho_n}{\rho_0,n}\right)(1 - \rho_n\rho_0,n)\text{tr}(T_n^{-1}(\theta)\Sigma_\eta(\theta_0)) = O((\theta_3 - \theta_{0,3})n^\delta),$$

$$\frac{\theta_{0,2}}{\theta_{0,3}}\text{tr}(T_n^{-1}(\theta)(ve_1 + e_1v')) = O(1),$$

$$\frac{\theta_{2}}{\theta_{3}}\text{tr}(T_n^{-1}(\theta)e_1e_1^\prime) = O(1),$$

uniformly in $\Theta$, where the first equation follows from (2.4), (A.5) and (A.95), the second equation follows from (2.4) and (A.13), and the last two equations follow from (A.11) and (A.12). Thus (A.32) is established.

Note that (A.33) follows by an argument similar to the proof of (A.32). We remain to prove (A.34). By (A.1), (A.79) and (A.94), we have

$$\text{tr}\left(\Sigma^{-1}(\theta)\left(\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)\Sigma^{-1}(\theta)(\Sigma_\eta(\theta) - \Sigma_\eta(\theta_0))\right)$$

$$= \text{tr}\left(T_n^{-1}(\theta)G_n(\theta)\left(\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)G_n(\theta)^\prime T_n^{-1}H(\theta)\right)$$

$$= \frac{\theta_2}{\theta_3}\{(\theta_1 + \rho_n^2)n^{-(1-\delta)} - (1 - \rho_n^2)\}\text{tr}(T_n^{-1}(\theta)G_n(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)H(\theta))$$

$$- (1 - \rho_n^2)n^{-(1-\delta)}\text{tr}(T_n^{-1}(\theta)G_n(\theta)\Sigma_\eta(\theta)G_n(\theta)^\prime T_n^{-1}(\theta)H(\theta))$$

$$- \frac{\theta_2}{\theta_3}\rho_n^2n^{-(1-\delta)}\text{tr}(T_n^{-1}(\theta)G_n(\theta)(ve_1' + e_1v')G_n(\theta)^\prime T_n^{-1}(\theta)H(\theta)).$$
Therefore, by (2.4), for (A.34) to hold, it suffices to show that
\[
\begin{align*}
    n^{-2(1-\delta)} & \text{tr}\left( T_n^{-1}(\theta)G_n(\theta)G_n(\theta)'T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1), \\
    n^{-2(1-\delta)} & \text{tr}\left( T_n^{-1}(\theta)G_n(\theta)\Sigma_\eta(\theta)G_n(\theta)'T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1), \\
    n^{-1-\delta} & \text{tr}\left( T_n^{-1}(\theta)G_n(\theta)(ve_1 + e_1v')G_n(\theta)'T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1),
\end{align*}
\]
uniformly in \( \Theta \), which follow by an argument similar to the proof of (A.32). Thus (A.34) is established. This completes the proof.

\textbf{Proof of Lemma A.10.} Clearly, (A.35) follows from (2.2) and an argument similar to the proof of (A.32), and (A.36) follows from (A.23) and an argument similar to the proof of and (A.34). We remain to prove (A.37). By (A.1), (A.21) and (A.94), we have
\[
\begin{align*}
    \text{tr}\left( \Sigma^{-1}(\theta) \left( \frac{\partial^2}{\partial \theta_3^2} \Sigma(\theta) \right) \Sigma^{-1}(\theta)(\Sigma_\eta(\theta) - \Sigma_n(\theta)) \right) \\
    = \text{tr}\left( T_n^{-1}(\theta)G_n(\theta) \left( \frac{\partial^2}{\partial \theta_3^2} \Sigma(\theta) \right) G_n(\theta)'T_n^{-1}(\theta)H(\theta) \right) \\
    = -\frac{\theta_2}{\theta_3} \text{tr}\left( T_n^{-1}(\theta)G_n(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) G_n(\theta)'T_n^{-1}(\theta)H(\theta) \right) \\
    - \frac{\theta_2}{\theta_3} \text{tr}\left( T_n^{-1}(\theta)G_n(\theta) \left( \frac{\partial}{\partial \theta_3} F(\theta) \right) G_n(\theta)'T_n^{-1}(\theta)H(\theta) \right).
\end{align*}
\]
Therefore, for (A.37) to hold, it suffices to show that
\[
\begin{align*}
    \text{tr}\left( T_n^{-1}(\theta)G_n(\theta) \left( \frac{\partial^2}{\partial \theta_3^2} \Sigma(\theta) \right) G_n(\theta)'T_n^{-1}(\theta)H(\theta) \right) \\
    = O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1),
\end{align*}
\]
and
\[
\begin{align*}
    \text{tr}\left( T_n^{-1}(\theta)G_n(\theta) \left( \frac{\partial}{\partial \theta_3} F(\theta) \right) G_n(\theta)'T_n^{-1}(\theta)H(\theta) \right) \\
    = O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1),
\end{align*}
\]
uniformly in \( \Theta \), where the first equation follows from an argument similar to the proof of (A.34). We then remain to prove (A.96). By (2.4) and (2.22), for (A.96) to hold, it suffices to show that
\[
\begin{align*}
    n^{-2(1-\delta)} & \text{tr}\left( T_n^{-1}(\theta)\Sigma_\eta(\theta)T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1), \\
    n^{-4(1-\delta)} & \text{tr}\left( T_n^{-1}(\theta)G_n^{-1}(\theta)\Sigma_\eta(\theta)G_n^{-1}(\theta)'T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1), \\
    n^{-2(1-\delta)} & \text{tr}\left( T_n^{-1}(\theta)\Sigma_\eta(\theta)G_n(\theta)'T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1), \\
    n^{-4(1-\delta)} & \text{tr}\left( T_n^{-1}(\theta)(ve_1 + e_1v')T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1), \\
    n^{-2(1-\delta)} & \text{tr}\left( T_n^{-1}(\theta)G_n^{-1}(\theta)(ve_1 + e_1v')T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1), \\
    n^{-2(1-\delta)} & \text{tr}\left( T_n^{-1}(\theta)ve_1T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1), \\
    n^{-2(1-\delta)} & \text{tr}\left( T_n^{-1}(\theta)v'v'T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1),
\end{align*}
\]
and
\[
\begin{align*}
    n^{-2(1-\delta)} & \text{tr}\left( T_n^{-1}(\theta)F(\theta)T_n^{-1}(\theta)H(\theta) \right) \\
    &= O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1),
\end{align*}
\]
uniformly in $\Theta$, where the first six equations follow from (2.1), (A.7), (A.10)-(A.13) and an argument similar to the proof of (A.32). We remain to prove (A.97). By (2.4) and (A.20), for (A.97) to hold, it suffices to show that

$$\begin{align*}
n^{-4(1-\delta)} & \text{tr}\left(T_n^{-1}(\theta)G_n^{-1}(\theta)\Sigma_n(\theta)G_n^{-1}(\theta)'T_n^{-1}(\theta)H(\theta)\right) \\
& = O((\theta_2 - \theta_{0,2})n^\delta) + O((\theta_3 - \theta_{0,3})n^\delta) + O(1), \\
& = O((\theta_2 - \theta_0)2n^\delta) + O((\theta_3 - \theta_0)3n^\delta) + O(1), \\
& = O((\theta_2 - \theta_0)2n^\delta) + O((\theta_3 - \theta_0)3n^\delta) + O(1), \\
& = O((\theta_2 - \theta_0)2n^\delta) + O((\theta_3 - \theta_0)3n^\delta) + O(1),
\end{align*}$$

uniformly in $\Theta$, which follow from (2.1), (A.7), (A.8), (A.10)-(A.12) and an argument similar to the proof of (A.32). Thus (A.37) is established. This completes the proof. \qed

**Proof of Lemma A.11.** First, we prove (A.39), (A.42). By (1.6) and applying $\frac{\partial}{\partial \Sigma} \log \det(\Sigma) = \text{tr}(\Sigma^{-1})$, we have

$$\begin{align*}
g_1(\theta) &= \text{tr}(\Sigma^{-1}(\theta)) - (\eta + \epsilon)\Sigma^{-2}(\theta)(\eta + \epsilon) \\
& \quad + \mu^T \frac{\partial}{\partial \theta_1}(\Sigma^{-1}(\theta)(I - M(\theta)))\mu \\
& \quad + 2\mu^T \frac{\partial}{\partial \theta_1}(\Sigma^{-1}(\theta)(I - M(\theta)))(\eta + \epsilon) \\
& \quad - (\eta + \epsilon)^T \frac{\partial}{\partial \theta_1}(\Sigma^{-1}(\theta)M(\theta))(\eta + \epsilon).
\end{align*}$$

(A.98)

Let $\theta_a = (\theta_{0,1}, \theta_2, \theta_3)'$ and $\hat{\theta}_a = (\hat{\theta}_{0,1}, \hat{\theta}_2, \hat{\theta}_3)'$. It follows by Lemma B.1 of Chan and Ing (2011), the first moment based theorem of Findley and Wei (1993), and arguments similar to the proof of (A.28) and (A.31) that

$$\begin{align*}
\sup_{\theta_a \in \Theta} \mu^T \frac{\partial}{\partial \theta_1}(\Sigma^{-1}(\theta)(I - M(\theta)))(\eta + \epsilon) &= o_p(n^\delta), \\
\sup_{\theta_a \in \Theta} (\eta + \epsilon)^T \frac{\partial}{\partial \theta_1}(\Sigma^{-1}(\theta)M(\theta))(\eta + \epsilon) &= O_p(p_n).
\end{align*}$$

This together with (A.30) and (A.98) follows that

$$g_1(\hat{\theta}_a) = -h_1(\hat{\theta}_a) + h_2(\hat{\theta}_a) + O_p(n^\delta) + O_p(1),$$

(A.99)

where

$$\begin{align*}
h_1(\theta) &= (\eta + \epsilon)\Sigma^{-2}(\theta)(\eta + \epsilon) - \text{tr}(\Sigma^{-2}(\theta)\Sigma(\theta_0)), \\
h_2(\theta) &= \text{tr}(\Sigma^{-1}(\theta)) - \text{tr}(\Sigma^{-2}(\theta)\Sigma(\theta_0)).
\end{align*}$$

Therefore, for (A.39) and (A.42) to hold, it suffices to show that

$$\begin{align*}
n^{-1/2}h_1(\theta_0) & \stackrel{d}{\rightarrow} N(0, 2\theta_{0,1}^2), \\
h_2(\theta_0) &= O(n^{1+\delta/2-r_2}) + O(n^\delta) + O(1), \\
\sup_{\theta_a \in \Theta_n} |h_1(\theta_a) - h_1(\theta_0)| &= o_p(n^{1+\delta/2-r_2}) + o_p(n^\delta) + o_p(n^{r_3-r_2}),
\end{align*}$$

(A.100)

(A.101)

(A.102)

where

$$\Theta_n \equiv \{ \theta \in \Theta : |\theta_1 - \theta_{0,1}| \leq \log(n)n^{-r_1}, |\theta_2 - \theta_{0,2}| \leq \log(n)n^{-r_2}, \\
|\theta_3 - \theta_{0,3}| \leq \log(n)n^{-r_3} \}.$$  

For (A.100), applying the Lindeberg’s theorem (e.g., Theorem 27.2 of Billingsley, 1995),

$$\frac{1}{(2\text{tr}(\Sigma^{-2}(\theta_0)))^{1/2}} h_1(\theta_0) \stackrel{d}{\rightarrow} N(0, 1).$$
Since from (2.2) and (A.17),
\[
\text{tr}(\Sigma^{-2}(\theta_0)) = \frac{1}{\theta_{0,1}} \left\{ \text{tr}(\Sigma^{-1}(\theta_0)) - \text{tr}(\Sigma^{-2}(\theta_0)\Sigma_\eta(\theta_0)) \right\}
\]
\[
= \frac{1}{\theta_{0,1}} \left\{ (n - \text{tr}(\Sigma^{-1}(\theta_0)\Sigma_\eta(\theta_0))) - \text{tr}(\Sigma^{-2}(\theta_0)\Sigma_\eta(\theta_0)) \right\} \tag{A.103}
\]
we obtain (A.100). For (A.101),
\[
h_2(\theta) = \text{tr}(\Sigma^{-2}(\theta)(\Sigma(\theta) - \Sigma(\theta_0)) = \text{tr}(\Sigma^{-2}(\theta)(\Sigma_\eta(\theta) - \Sigma_\eta(\theta_0))).
\]
This together with (A.32) gives (A.101). For (A.102), it follows by Lemma B.1 of Chan and Ing (2011), and the first moment based theorem of Findley and Wei (1993) that
\[
E \left\{ \sup_{\theta \in \Theta} |h_1(\theta_a) - h_1(\theta_0)|^2 \right\}
\leq C \sup_{\theta \in \Theta} \left\{ (\theta_2 - \theta_{0,2})^2 \text{var} \left( \frac{\partial}{\partial \theta_2} h_1(\theta_a) \right) + (\theta_3 - \theta_{0,3})^2 \text{var} \left( \frac{\partial}{\partial \theta_3} h_1(\theta_a) \right) \right. \tag{A.104}
\]
\[
+ (\theta_3 - \theta_{0,3})^2 \text{var} \left( \frac{\partial^2}{\partial \theta_2 \partial \theta_3} h_1(\theta_a) \right) \right\},
\]
for some constant $C > 0$. Since from (2.2) and (A.17),
\[
\text{var} \left( \left( \frac{\partial}{\partial \theta_2} h_1(\theta) \right)^2 \right)
\leq 2 \text{tr} \left( (\Sigma^{-2}(\theta)\Sigma_\eta(\theta)\Sigma^{-1}(\theta) + \Sigma^{-1}(\theta)\Sigma_\eta(\theta)\Sigma^{-2}(\theta)\Sigma(\theta_0))^2 \right)
\leq 2 \left\{ \text{tr} \left( (\Sigma^{-2}(\theta)\Sigma_\eta(\theta)\Sigma^{-1}(\theta)\Sigma(\theta_0))^2 \right)
+ 2 \text{tr} (\Sigma^{-2}(\theta)\Sigma_\eta(\theta)\Sigma^{-1}(\theta)\Sigma(\theta_0)) \text{tr} (\Sigma^{-1}(\theta)\Sigma_\eta(\theta)) \text{tr} (\Sigma^{-2}(\theta)\Sigma(\theta_0)) \right\}
\leq C \text{tr} (\Sigma^{-1}(\theta)\Sigma_\eta(\theta))
\leq O(n(1+\delta)/2),
\]
uniformly in $\Theta$, for some constant $C > 0$, and similarly from (2.2), (A.17) and (A.24),
\[
\sup_{\theta \in \Theta} \text{var} \left( \frac{\partial}{\partial \theta_3} h_1(\theta) \right) = O(n^\delta),
\]
\[
\sup_{\theta \in \Theta} \text{var} \left( \frac{\partial^2}{\partial \theta_2 \partial \theta_3} h_1(\theta) \right) = O(n^\delta).
\]
It follows from (A.104) that
\[
E \left( \sup_{\theta \in \Theta} |h_1(\theta_a) - h_1(\theta_0)|^2 \right) = O((\log(n))^2) n^{(1+\delta)/2 - 2r_2} + O(\log(n))^2 n^{\delta - 2r_3}.
\]
We obtain (A.102). Thus (A.39) is established.

Next, we prove (A.40) and (A.43). By (1.6),
\[
g_2(\theta) = \frac{1}{\theta_3} \text{tr} \left( \Sigma^{-1}(\theta) \frac{\partial}{\partial \theta_3} \Sigma_\eta(\theta) \right)
\leq \frac{1}{\theta_3} (\eta + \epsilon)^3 \Sigma^{-1}(\theta) \frac{\partial}{\partial \theta_3} \Sigma_\eta(\theta) \Sigma^{-1}(\theta)(\eta + \epsilon)
+ \mu \frac{\partial}{\partial \theta_3} \Sigma^{-1}(\theta) (I - M(\theta)) \mu
+ 2\mu \frac{\partial}{\partial \theta_2} (\Sigma^{-1}(\theta) (I - M(\theta)))(\eta + \epsilon)
- (\eta + \epsilon)^3 \frac{\partial}{\partial \theta_2} (\Sigma^{-1}(\theta) M(\theta))(\eta + \epsilon),
\tag{A.105}
\]
Let \( \theta_h = (\theta_1, \theta_{0,2}, \hat{\theta}_3) \) and \( \hat{\theta}_h = (\hat{\theta}_1, \theta_{0,2}, \hat{\theta}_3) \). It follows by Lemma B.1 of Chan and Ing (2011), the first moment based theorem of Findley and Wei (1993), and arguments similar to the proof of (A.28) and (A.31) that

\[
\sup_{\theta_h} \mu' \frac{\partial}{\partial \theta_2} \left( \Sigma^{-1}(\theta)(I - M(\theta)) \right)(\eta + \epsilon) = o_p(n^\delta),
\]

\[
\sup_{\theta_h} (\eta + \epsilon)' \frac{\partial}{\partial \theta_2} \left( \Sigma^{-1}(\theta)M(\theta) \right)(\eta + \epsilon) = O_p(p_n).
\]

This together with (A.30) and (A.105) follows that

\[
g_2(\hat{\theta}_h) = - \frac{1}{\theta_{0,2}} (h_3(\hat{\theta}_h) + h_4(\hat{\theta}_h)) + O_p(n^\delta) + O_p(1), \tag{A.106}
\]

where

\[
h_3(\theta) = (\eta + \epsilon)' \Sigma^{-1}(\theta) \Sigma_\eta(\theta) \Sigma^{-1}(\theta)(\eta + \epsilon) - \text{tr}(\Sigma^{-1}(\theta) \Sigma_\eta(\theta) \Sigma^{-1}(\theta) \Sigma(\theta_0)),
\]

\[
h_4(\theta) = \text{tr}(\Sigma^{-1}(\theta) \Sigma_\eta(\theta) \Sigma^{-2}(\theta) \Sigma(\theta_0)) - \text{tr}(\Sigma^{-1}(\theta) \Sigma_\eta(\theta)).
\]

Therefore, for (A.40) and (A.43) to hold, it suffices to show that

\[
n^{-\left(1+\delta\right)/4} h_3(\theta_0) \overset{d}{\sim} N(0, 2^{-1/2} \theta_{0,1}^{-1/2} \theta_{0,2}^{-1/2}), \tag{A.107}
\]

\[
h_4(\theta_0) = O(n^{\left(1+\delta\right)/2-r_1}) + O(n^{\delta-r_3}) + O(1), \tag{A.108}
\]

\[
\sup_{\theta_h} |h_3(\theta) - h_3(\theta_0)| = o_p(n^{(1+\delta)/2-r_1}) + o_p(n^{\delta-r_3}). \tag{A.109}
\]

For (A.107), applying the Lindeberg’s theorem (e.g., Theorem 27.2 of Billingsley, 1995), we have

\[
\frac{1}{(2\text{tr}(\Sigma^{-1}(\theta_0) \Sigma_\eta(\theta_0))^2)^{1/2}} h_3(\theta_0) \overset{d}{\sim} N(0, 1).
\]

This together with (A.16) gives (A.107). For (A.108), we have

\[
h_4(\theta_0) = \text{tr}(\Sigma^{-1}(\theta) \Sigma_\eta(\theta) \Sigma^{-1}(\theta) \Sigma(\theta_0) - \Sigma(\theta))
\]

\[
= (\theta_{0,1} - \theta_1) \text{tr}(\Sigma^{-1}(\theta) \Sigma_\eta(\theta))
\]

\[
- \text{tr}(\Sigma^{-1}(\theta) \Sigma_\eta(\theta) \Sigma^{-1}(\theta) \Sigma(\theta_0) - \Sigma_\eta(\theta)).
\]

This together with (2.2), (A.17) and (A.33) gives (A.108). For (A.109), it follows by an argument similar to the proof of (A.102). Thus (A.40) is established.

Finally, we prove (A.41) and (A.44). By (1.6),

\[
g_3(\theta) = \text{tr} \left( \Sigma^{-1}(\theta) \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right)
\]

\[
- (\eta + \epsilon)' \Sigma^{-1}(\theta) \left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta)(\eta + \epsilon)
\]

\[
+ \mu' \frac{\partial}{\partial \theta_3} \left( \Sigma^{-1}(\theta)(I - M(\theta)) \right) \mu
\]

\[
+ 2 \mu' \frac{\partial}{\partial \theta_3} \left( \Sigma^{-1}(\theta)(I - M(\theta)) \right) (\eta + \epsilon)
\]

\[
- (\eta + \epsilon)' \frac{\partial}{\partial \theta_3} \left( \Sigma^{-1}(\theta) M(\theta) \right)(\eta + \epsilon).
\]

Let \( \theta_e = (\theta_1, \theta_2, \theta_{0,3}) \) and \( \hat{\theta}_e = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_{0,3}) \). It follows by Lemma B.1 of Chan and Ing (2011), the first moment based theorem of Findley and Wei (1993), and arguments similar to the proof of (A.28) and (A.31) that

\[
\sup_{\theta_e} \mu' \frac{\partial}{\partial \theta_3} \left( \Sigma^{-1}(\theta)(I - M(\theta)) \right)(\eta + \epsilon) = o_p(n^\delta),
\]

\[
\sup_{\theta_e} (\eta + \epsilon)' \frac{\partial}{\partial \theta_3} \left( \Sigma^{-1}(\theta) M(\theta) \right)(\eta + \epsilon) = O_p(p_n).
\]

This together with (A.30) and (A.110) follows that

\[
g_3(\hat{\theta}_e) = - h_5(\hat{\theta}_e) + h_6(\hat{\theta}_e) + O_p(n^\delta) + O_p(1), \tag{A.111}
\]
where
\[ h_5(\theta) = (\eta + \epsilon)'\Sigma^{-1}(\theta)\left( \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right) \Sigma^{-1}(\theta)(\eta + \epsilon) \]
\[ - \operatorname{tr}\left( \Sigma^{-1}(\theta_0) \frac{\partial}{\partial \theta_3} \Sigma(\theta) \Sigma^{-1}(\theta_0) \right), \]
\[ h_6(\theta) = \operatorname{tr}\left( \Sigma^{-1}(\theta) \frac{\partial}{\partial \theta_3} \Sigma(\theta) \Sigma^{-1}(\theta) \Sigma(\theta_0) \right) - \operatorname{tr}\left( \Sigma^{-1}(\theta) \frac{\partial}{\partial \theta_3} \Sigma(\theta) \right). \]

Therefore, for \((A.41)\) and \((A.44)\) to hold, it suffices to show that
\[ h_5(\theta_0) = O_p(n^{\delta/2}), \]  
\[ h_6(\theta_0) = O(n^\delta-r_1) + O(n^{\delta-r_2}) + O(1), \]
\[ \sup_{\theta \in \Theta_n} |h_5(\theta) - h_5(\theta_0)| = o_p(n^{\delta-r_1}) + o_p(n^{\delta-r_2}). \]  

For \((A.112)\), it follows from \((A.24)\) and an argument similar to the proof of \((A.100)\) that for \(\delta \in (0, 1),\)
\[ n^{-\delta/2} h_5(\theta_0) \xrightarrow{d} N(0, 2\theta_{0.3}^{-1}). \]

For \((A.113)\), we have
\[ h_6(\theta) = (\theta_0, - \theta_1) \operatorname{tr}\left( \Sigma^{-1}(\theta) \frac{\partial}{\partial \theta_3} \Sigma(\theta) \Sigma^{-1}(\theta) \right) \]
\[ + \operatorname{tr}\left( \Sigma^{-1}(\theta) \frac{\partial}{\partial \theta_3} \Sigma(\theta) \Sigma^{-1}(\theta) \left( \Sigma_{\eta}(\theta_0) - \Sigma_{\eta}(\theta) \right) \right). \]

Thus \((A.113)\) follows from \((A.26)\) and \((A.34)\). For \((A.114)\), it follows by an argument similar to the proof of \((A.102)\). Thus \((A.41)\) is established. This completes the proof.

**Proof of Lemma A.12.** First, we prove \((A.46)\). By \((A.98)\), we have
\[ g_{11}(\theta) = - \operatorname{tr}(\Sigma^{-2}(\theta)) + 2(\eta + \epsilon)'\Sigma^{-3}(\theta)(\eta + \epsilon) \]
\[ + \mu' \frac{\partial^2}{\partial \theta_1^2} (\Sigma^{-1}(\theta)(I - M(\theta))) \mu \]
\[ + 2\mu' \frac{\partial^2}{\partial \theta_1^2} (\Sigma^{-1}(\theta)(I - M(\theta)))(\eta + \epsilon) \]
\[ - (\eta + \epsilon)' \frac{\partial^2}{\partial \theta_1^2} (\Sigma^{-1}(\theta)M(\theta))(\eta + \epsilon). \]  

Let \(\theta^* = (\theta_{01}, \theta_2, \theta_3)'\) and \(\tilde{\theta}^* = (\theta_{01}, \tilde{\theta}_2, \tilde{\theta}_3)'\). It follows by Lemma B.1 of Chan and Ing (2011), the first moment based theorem of Findley and Wei (1993), and arguments similar to the proof of \((A.28)\) and \((A.31)\) that
\[ \sup_{\theta^* \in \Theta} \mu' \frac{\partial^2}{\partial \theta_1^2} (\Sigma^{-1}(\theta)(I - M(\theta)))(\eta + \epsilon) = o_p(n^{\delta}), \]
\[ \sup_{\theta^* \in \Theta} (\eta + \epsilon)' \frac{\partial^2}{\partial \theta_1^2} (\Sigma^{-1}(\theta)M(\theta))(\eta + \epsilon) = O_p(p_n). \]

This together with \((A.30)\) and \((A.115)\) follows that
\[ g_{11}(\theta^*_0) = h_5(\theta^*_0) + h_6(\theta^*_0) + O_p(n^{\delta}) + O_p(1), \]  
where
\[ h_5(\theta) = (\eta + \epsilon)'\Sigma^{-3}(\theta)(\eta + \epsilon) - \operatorname{tr}(\Sigma^{-3}(\theta)\Sigma(\theta_0)), \]
\[ h_6(\theta) = 2\operatorname{tr}(\Sigma^{-3}(\theta)\Sigma(\theta_0)) - \operatorname{tr}(\Sigma^{-2}(\theta)). \]

Therefore, it suffices to show that
\[ h_5(\theta_0) = O_p(n^{1/2}), \]  
\[ h_6(\theta^*_0) = \frac{n}{\theta_{01}^2} + o_p(n), \]
\[ \sup_{\theta^* \in \Theta} |h_7(\theta^*_0) - h_7(\theta_0)| = o_p(n). \]
For (A.117), it follows by an argument similar to the proof of (A.100). For (A.118), we have
\[ h_7(\theta) = 2\text{tr}(\Sigma^{-3}(\theta)(\Sigma(\theta_0) - \Sigma(\theta))) + \text{tr}(\Sigma^{-2}(\theta)) \]
\[ = 2(\theta_{0,1} - \theta_1)\text{tr}(\Sigma^{-3}(\theta)) + 2\text{tr}\left(\Sigma^{-3}(\theta)(\Sigma(\theta_0) - \Sigma(\theta))\right) \]
\[ + \text{tr}(\Sigma^{-2}(\theta)). \]

Thus (A.118) follows from (2.2), (A.17), and (A.103). For (A.109), it follows by Lemma B.1 of Chan and Ing (2001), the first moment based theorem of Findley and Wei (1993), and arguments similar to the proof of (A.120) that
\[ E\left(\sup_{\theta \in \Theta} |h_7(\theta) - h_7(\theta_0)|^2\right) \]
\[ \leq C\sup_{\theta \in \Theta} \left\{ E|h_7(\theta^*_n) - h_7(\theta_0)|^2 + (\theta_2 - \theta_{0,2})^2\text{var}\left(\frac{\partial}{\partial \theta_2}^2 h_7(\theta^*_n)\right) \right\}, \tag{A.120} \]
for some constant $C > 0$. Since from (2.2), (A.17) and (A.24), we have
\[ \text{var}\left(\frac{\partial}{\partial \theta_2} h_7(\theta^*_n)\right) = O(n^{1+\delta}/2), \]
\[ \text{var}\left(\frac{\partial}{\partial \theta_3} h_7(\theta^*_n)\right) = O(n^\delta), \]
\[ \text{var}\left(\frac{\partial}{\partial \theta_2^2} h_7(\theta^*_n)\right) = O(n^\delta), \]
uniformly in $\Theta$, and follows from an argument similar to (2.15) that
\[ E|h_7(\theta^*_n) - h_7(\theta_0)|^2 = 2\text{tr}\left(\Sigma(\theta_0)\left(\Sigma^{-3}(\theta^*_n) - \Sigma^{-3}(\theta_0)\right)\right)^2 = O(n), \]
uniformly in $\Theta$. It follows from (A.120) that
\[ E\left(\sup_{\theta \in \Theta} |h_7(\theta^*_n) - h_7(\theta_0)|^2\right) = O(n). \]

We then obtain (A.119). Thus (A.46) is established.

Next, we prove (A.47). By (A.105),
\[ g_{22}(\theta) = -\frac{1}{\theta_3^2} \text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_2} \Sigma(\theta)\right)^2\right) \]
\[ + 2\frac{\partial}{\partial \theta_3}(\eta + \epsilon)\left(\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_2} \Sigma(\theta)\right)\Sigma^{-1}(\theta)(\eta + \epsilon) \]
\[ + \mu' \frac{\partial}{\partial \theta_2}(\Sigma^{-1}(\theta)(I - M(\theta)))\mu \]
\[ + 2\mu' \frac{\partial}{\partial \theta_2}(\Sigma^{-1}(\theta)(I - M(\theta)))(\eta + \epsilon) \]
\[ - (\eta + \epsilon)' \frac{\partial}{\partial \theta_2^2}(\Sigma^{-1}(\theta)M(\theta))(\eta + \epsilon). \tag{A.121} \]

Let $\theta^*_n = (\theta_1, \theta_{0,2}, \theta_3)'$ and $\hat{\theta}^*_n = (\hat{\theta}_1, \hat{\theta}_{0,2}, \hat{\theta}_3)'$. It follows by Lemma B.1 of Chan and Ing (2011), the first moment based theorem of Findley and Wei (1993), and arguments similar to the proof of (A.28) and (A.31) that
\[ \sup_{\theta^*_n \in \Theta} \mu' \frac{\partial}{\partial \theta_2}(\Sigma^{-1}(\theta)(I - M(\theta)))(\eta + \epsilon) = o_p(n^\delta), \]
\[ \sup_{\theta^*_n \in \Theta} (\eta + \epsilon)' \frac{\partial}{\partial \theta_2^2}(\Sigma^{-1}(\theta)M(\theta))(\eta + \epsilon) = O_p(p_n). \]

This together with (A.30) and (A.121) follows that
\[ g_{22}(\hat{\theta}^*_n) = \frac{1}{\theta_{0,2}^*} \left\{ 2h_9(\hat{\theta}^*_n) + h_{10}(\hat{\theta}^*_n) \right\} + O_p(n^\delta) + O_p(1), \tag{A.122} \]
where
\[ h_9(\theta) = (\eta + \epsilon)'(\Sigma^{-1}(\theta)|\Sigma_\eta(\theta)|^2 \Sigma^{-1}(\theta)(\eta + \epsilon) \]
\[ - \text{tr}\left((\Sigma^{-1}(\theta)|\Sigma_\eta(\theta)|^2 \Sigma^{-1}(\theta)|\Sigma(\theta_0)\right), \]
\[ h_{10}(\theta) = 2\text{tr}\left((\Sigma^{-1}(\theta)|\Sigma_\eta(\theta)|^2 \Sigma^{-1}(\theta)|\Sigma(\theta_0)\right) - \text{tr}\left((\Sigma^{-1}(\theta)|\Sigma_\eta(\theta)|^2 \Sigma^{-1}(\theta)ight). \]

Therefore, it suffices to show that
\[ h_9(\theta_0) = O_p(n^{(1+\delta)/4}), \quad (A.123) \]
\[ h_{10}(\theta_0) = \left(\frac{\theta_0}{8\theta_{0,1}}\right)^{1/2} n^{(1+\delta)/2} + o_p(n^{(1+\delta)/2}), \quad (A.124) \]
\[ \sup_{\theta \in \Theta} |h_9(\theta_0^*) - h_9(\theta_0)| = o_p(n^{(1+\delta)/2}). \quad (A.125) \]

For (A.123), it follows by an argument similar to the proof of (A.117). For (A.125), it follows by an argument similar to the proof of (A.119). For (A.124), we have
\[ h_{10}(\theta) = 2\text{tr}\left((\Sigma^{-1}(\theta)|\Sigma_\eta(\theta)|^2 \Sigma^{-1}(\theta)|\Sigma(\theta_0) - \Sigma(\theta)\right) + \text{tr}\left((\Sigma^{-1}(\theta)|\Sigma_\eta(\theta)|^2 \Sigma^{-1}(\theta)ight) \]
\[ = 2(\theta_1 - \theta_{0,1})\text{tr}\left((\Sigma^{-1}(\theta)|\Sigma_\eta(\theta)|^2 \Sigma^{-1}(\theta)\right) \]
\[ + 2\text{tr}\left((\Sigma^{-1}(\theta)|\Sigma_\eta(\theta)|^2 \Sigma^{-1}(\theta)|\Sigma(\theta_0) - \Sigma(\theta)\right) \]
\[ \quad + \text{tr}\left((\Sigma^{-1}(\theta)|\Sigma(\theta)|^2 \Sigma^{-1}(\theta)\right). \]

This together with (2.2), (A.16), (A.17) and (A.35) gives (A.124). Thus (A.47) is established.

Finally, we prove (A.48). By (A.110),
\[ g_{33}(\theta) = - \text{tr}\left((\Sigma^{-1}(\theta)|\partial_{\theta_1} \Sigma(\theta)|^2 \right) + \text{tr}\left((\Sigma^{-1}(\theta)|\partial_{\theta_2} \Sigma(\theta)|^2 \right) \]
\[ + 2(\eta + \epsilon)'\left((\Sigma^{-1}(\theta)|\partial_{\theta_3} \Sigma(\theta)|^2 \right) \Sigma^{-1}(\theta)(\eta + \epsilon) \]
\[ - (\eta + \epsilon)'\Sigma^{-1}(\theta)(\partial_{\theta_3} \Sigma(\theta)) \Sigma^{-1}(\theta)(\eta + \epsilon) \]
\[ + \mu' \partial_{\theta_3} \left(\Sigma^{-1}(\theta)|I - M(\theta)|\right) \mu \]
\[ + 2\mu' \partial_{\theta_3} \left(\Sigma^{-1}(\theta)|I - M(\theta)|\right)(\eta + \epsilon) \]
\[ - (\eta + \epsilon)' \partial_{\theta_3} \left(\Sigma^{-1}(\theta)|M(\theta)|\right)(\eta + \epsilon). \quad (A.126) \]

Let \( \theta^*_1 = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_{0,1}^*)' \) and \( \theta^*_2 = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_{0,3}^*)' \). It follows by Lemma B.1 of Chan and Ing (2011), the first moment based theorem of Findley and Wei (1993), and arguments similar to the proof of (A.28) and (A.31) that
\[ \sup_{\theta \in \Theta} \mu' \partial_{\theta_3} \left(\Sigma^{-1}(\theta)|I - M(\theta)|\right)(\eta + \epsilon) = o_p(n^\delta), \]
\[ \sup_{\theta \in \Theta} (\eta + \epsilon) \partial_{\theta_3} \left(\Sigma^{-1}(\theta)|M(\theta)|\right)(\eta + \epsilon) = O_p(n^\delta). \]

This together with (A.30) and (A.126) follows that
\[ g_{33}(\theta^*_2) = 2h_{11}(\theta^*_2) - h_{12}(\theta^*_2) + h_{13}(\theta^*_2) + O_p(n^\delta) + O_p(1). \quad (A.127) \]
where

\[
\begin{align*}
\quad h_{11}(\theta) &= (\eta + \epsilon')\left(\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)^2 \Sigma^{-1}(\theta)(\eta + \epsilon) \\
& \quad - \text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)^2 \Sigma(\theta_0)\right), \\
\quad h_{12}(\theta) &= (\eta + \epsilon')\Sigma^{-1}(\theta)\left(\frac{\partial^2}{\partial \theta_3^2} \Sigma(\theta)\right)\Sigma^{-1}(\theta)(\eta + \epsilon) \\
& \quad - \text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial^2}{\partial \theta_3^2} \Sigma(\theta)\right)^2 \Sigma(\theta_0)\right), \\
\quad h_{13}(\theta) &= 2\text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)^2 \Sigma^{-1}(\theta)\Sigma(\theta_0)\right) \\
& \quad - 2\text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)^2\right) + \text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial^2}{\partial \theta_3^2} \Sigma(\theta)\right)^2 \Sigma(\theta_0)\right) \\
& \quad - \text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial^2}{\partial \theta_3^2} \Sigma(\theta)\right)\Sigma^{-1}(\theta)\Sigma(\theta_0)\right) + \text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)^2\right).
\end{align*}
\]

Therefore, it suffices to show that

\[
\begin{align*}
\quad h_{11}(\theta_0) &= O_p(n^{5/2}), \\
\quad h_{12}(\theta_0) &= O_p(n^{5/2}), \\
\quad \sup_{\theta \in \Theta} |h_{11}(\theta_*) - h_{11}(\theta_0)| &= o_p(n^3) + O_p(1), \\ 
\quad \sup_{\theta \in \Theta} |h_{12}(\theta_*) - h_{12}(\theta_0)| &= o_p(n^3) + O_p(1), \\
\quad h_{13}(\theta_*) &= \frac{n^5}{\theta_0 h_{13}} + o_p(n^5).
\end{align*}
\]

For (A.128), it follows by an argument similar to the proof of (A.117) and (A.119). For (A.129),

\[
\begin{align*}
\quad h_{13}(\theta) &= 2(\theta_{0,1} - \theta_1)\text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)^2 \Sigma^{-1}(\theta)\right) \\
& \quad + 2\text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)^2 \Sigma^{-1}(\theta)(\Sigma_\eta(\theta_0) - \Sigma_\eta(\theta))\right) \\
& \quad + (\theta_1 - \theta_{0,1})\text{tr}\left(\Sigma^{-1}(\theta)\left(\frac{\partial^2}{\partial \theta_3^2} \Sigma(\theta)\right)\Sigma^{-1}(\theta)\right) \\
& \quad + \text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial^2}{\partial \theta_3^2} \Sigma(\theta)\right)^2 \Sigma^{-1}(\theta)(\Sigma_\eta(\theta) - \Sigma_\eta(\theta_0))\right) \\
& \quad + \text{tr}\left(\left(\Sigma^{-1}(\theta)\frac{\partial}{\partial \theta_3} \Sigma(\theta)\right)^2\right).
\end{align*}
\]

This together with (A.24)-(A.27), (A.36) and (A.37) gives (A.129). Thus (A.48) is established. This completes the proof.

**Proof of Lemma A.13.** First, we prove (A.49). By (A.1), we have

\[
1'\Sigma^{-1}(\theta)1 = 1'G_n(\theta)'T_n^{-1}(\theta)G_n(\theta)1 \\
= ((1 - \rho_n)1 + \rho_n e_1)'T_n^{-1}(\theta)((1 - \rho_n)1 + \rho_n e_1) \\
= (1 - \rho_n)^2 1'T_n^{-1}(\theta)1 + 2\rho_n(1 - \rho_n)1'T_n^{-1}(\theta)e_1 + \rho_n^2 e_1'T_n^{-1}(\theta)e_1.
\]

Therefore, for (A.49) to hold, it suffices to show that

\[
\begin{align*}
1'T_n^{-1}(\theta)1 &= \frac{n^{2-\delta}}{2\theta_2} + o(n^{2-\delta}), \\
1'T_n^{-1}(\theta)e_1 &= O(n^{1-\delta}), \\
e_1'T_n^{-1}(\theta)e_1 &= O(1),
\end{align*}
\]

uniformly in \(\Theta\), where (A.130) follows from (A.12), and (A.131) follows from (A.130) and (A.132). We remain to prove
By (A.63),
\[ 1' \mathbf{T}_n^{-1}(\theta) 1 = \sum_{k=1}^{n} \sum_{\ell=1}^{n} C_n(k, \ell) \]
\[ = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{\rho_n^{k-\ell} f_2^{-\ell}(\theta)}{(f_2^k(\theta) - 4\rho_n^2 \theta_2^2)^{1/2}} + O\left( n^{(1-\delta)/2} \sum_{k=1}^{n} \sum_{\ell=1}^{n} f_2^{-(k+\ell)}(\theta) \right) \]
\[ + O\left( n^{(1-\delta)/2} \sum_{k=1}^{n} \sum_{\ell=1}^{n} f_2^{k+\ell-2n}(\theta) \right) \]
\[ = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{\rho_n^{k-\ell} f_2^{-\ell}(\theta)}{(f_2^k(\theta) - 4\rho_n^2 \theta_2^2)^{1/2}} + O(n^{3(1-\delta)/2}), \]
uniformly in \( \Theta \), where the second last equality follows from (A.65), (A.61), \( \rho_n < 1 \) and \( f_2^{-1}(\theta) < 1 \), and the last equality follows from (A.67). It follows that
\[ 1' \mathbf{T}_n^{-1}(\theta) 1 \]
\[ = \sum_{k=1}^{n} \left\{ \sum_{\ell=1}^{k} \frac{\rho_n^{k-\ell} f_2^{-\ell}(\theta)}{(f_2^k(\theta) - 4\rho_n^2 \theta_2^2)^{1/2}} + \sum_{\ell=k+1}^{n} \frac{\rho_n^{\ell-k} f_2^{k-\ell}(\theta)}{(f_2^k(\theta) - 4\rho_n^2 \theta_2^2)^{1/2}} \right\} + O(n^{3(1-\delta)/2}) \]
\[ = \frac{1}{(f_2^1(\theta) - 4\rho_n^2 \theta_2^2)^{1/2}} \sum_{k=1}^{n} \left\{ f_2(\theta) - \rho_n^{k-1} f_2(\theta) - \rho_n f_2(\theta) - \rho_n^{n-k} f_2(\theta) - \rho_n \right\} \]
\[ + O(n^{3(1-\delta)/2}) \]
\[ = \frac{(f_2(\theta) + \rho_n)}{(f_2^1(\theta) - 4\rho_n^2 \theta_2^2)^{1/2}} + O\left( n^{1-\delta} \sum_{k=1}^{n} f_2^{-k}(\theta) \right) \]
\[ + O\left( n^{1-\delta} \sum_{k=1}^{n} f_2^{k-n}(\theta) \right) + O(n^{3(1-\delta)/2}) \]
uniformly in \( \Theta \), where the second last equality follows from (2.4), (A.54), (A.61), \( \rho_n < 1 \) and \( f_2^{-1}(\theta) < 1 \), and the last equality follows from (A.67). This together with (2.4), (A.54) and (A.61) gives (A.130). This completes the proof of (A.49).

Next, we prove (A.50). By (A.1), we have
\[ x' \Sigma^{-1}(\theta) 1 = x' G_n(\theta)' T_n^{-1}(\theta) G_n(\theta) 1 \]
\[ = (n^{-1} 1 + (1 - \rho_n) x' T_n^{-1}(\theta) ((1 - \rho_n) 1 + \rho_n 1) + \rho_n e_1) \]
\[ = n^{-1} (1 - \rho_n) 1' T_n^{-1}(\theta) 1 + (1 - \rho_n)^2 x' T_n^{-1}(\theta) 1 \]
\[ + n^{-1} \rho_n 1' T_n^{-1}(\theta) e_1 + \rho_n (1 - \rho_n) x' T_n^{-1}(\theta) e_1. \]
Therefore, by (A.130)-(A.132), for (A.50) to hold, it suffices to show that
\[ x' T_n^{-1}(\theta) 1 = \frac{n^{2-\delta}}{4\theta_2} + o(n^{2-\delta}), \quad \text{(A.133)} \]
\[ x' T_n^{-1}(\theta) e_1 = O(n^{1-\delta}), \quad \text{(A.134)} \]
uniformly in \( \Theta \). First, we prove (A.133). By (A.63),
\[ x' T_n^{-1}(\theta) 1 = \sum_{k=1}^{n} \sum_{\ell=1}^{n} C_n(k, \ell) n^{-1} \]
\[ = n^{-1} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{\ell \rho_n^{k-\ell} f_2^{-\ell}(\theta)}{(f_2^k(\theta) - 4\rho_n^2 \theta_2^2)^{1/2}} \]
\[ + O\left( n^{(1+\delta)/2} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \ell f_2^{-(k+\ell)}(\theta) \right) \]
\[ + O\left( n^{(1+\delta)/2} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \ell f_2^{\ell+k-2n}(\theta) \right) \]
\[ = n^{-1} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{\ell \rho_n^{k-\ell} f_2^{-\ell}(\theta)}{(f_2^k(\theta) - 4\rho_n^2 \theta_2^2)^{1/2}} + O(n^{3(1-\delta)/2}), \]
Therefore, by (A.65), (A.61), \( \rho_n < 1 \) and \( f_2^{-1}(\theta) < 1 \), and the last equality follows from (2.4), (A.54), (6.7) and

\[
\sum_{\ell=1}^{n} \ell f_2^{-\ell}(\theta) = O(n^{1-\delta}),
\]

\[
\sum_{\ell=1}^{n} \ell f_2^{\ell-n}(\theta) = O(n^{(3-\delta)/2}),
\]

uniformly in \( \Theta \). It follows from an argument similar to the proof of (A.67) that

\[
x' T_n^{-1}(\theta) 1 = \frac{1}{n} \left\{ \sum_{k=1}^{n} \left( n f_2^{k-n}(\theta) + \rho_n \right) \right\} + O\left( n^{(3-\delta)/2} \right)
\]

uniformly in \( \Theta \), where the second last equality follows from (2.4), (A.54), (A.61), \( \rho_n < 1 \) and \( f_2^{-1}(\theta) < 1 \), and the last equality follows from (6.7). It follows that

\[
x' T_n^{-1}(\theta) 1 = \frac{(n+1) f_2(\theta) + \rho_n}{2 (f_2(\theta) - 4 \rho_n^2 \theta_1^2)^{1/2} (f_2(\theta) - \rho_n)} + o(n^{2-\delta}),
\]

uniformly in \( \Theta \). This together with (2.4), (A.54) and (A.61) gives (A.133). It follows from an argument similar to the proof of (A.133) gives (A.134). Thus (A.50) is established.

Finally, we prove (A.51). By (A.1), we have

\[
x' \Sigma^{-1}(\theta) x = x' G_n(\theta)' T_n^{-1}(\theta) G_n(\theta) x = (n^{-1}1 + (1 - \rho_n)x)' T_n^{-1}(\theta)(n^{-1}1 + (1 - \rho_n)x)
\]

\[
= n^{-2}1' T_n^{-1}(\theta) 1 + 2n^{-1}(1 - \rho_n)1' T_n^{-1}(\theta) x + (1 - \rho_n)^2 x' T_n^{-1}(\theta) x.
\]

Therefore, by (A.130) and (A.133), for (A.51) to hold, it suffices to show that

\[
x' T_n^{-1}(\theta) x = \frac{n^{2-\delta}}{6 \theta_2^2} + o(n^{2-\delta}),
\]

(A.135)

uniformly in \( \Theta \). It follows from an argument similar to the proof of (A.133) that

\[
x' T_n^{-1}(\theta) x = \sum_{k=1}^{n} \frac{k^2 f_2(\theta) + \rho_n}{n^2 (f_2(\theta) - 4 \rho_n^2 \theta_1^2)^{1/2} (f_2(\theta) - \rho_n)} + o(n^{2-\delta})
\]

\[
= \frac{n(n+1)(2n+1)f_2(\theta) + \rho_n}{6n^2 (f_2(\theta) - 4 \rho_n^2 \theta_1^2)^{1/2} (f_2(\theta) - \rho_n)} + o(n^{2-\delta}),
\]

uniformly in \( \Theta \). This together with (2.4), (A.54) and (A.61) gives (A.135). Thus (A.51) is established. This completes the proof.