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A Note on Characterizations of Multistate Systems

By

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In this note some new characterization results for multistate systems are obtained. In particular, a new characterization of the Natvig class of multistate systems is presented, which provides us an easy-to-use method of classifying multistate systems. Examples are given to illustrate our result.

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Abstract: In this note some new characterization results for multistate systems are obtained. In particular, a new characterization of the Natvig class of multistate systems is presented, which provides us an easy-to-use method of classifying multistate systems. Examples are given to illustrate our result.

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1. INTRODUCTION

A binary coherent system comprising $n$ components is described by a structure function $\phi : \{0,1\}^n \mapsto \{0,1\}$. The structure function $\phi$ must satisfy the conditions (i) $\phi(x)$ is nondecreasing in each of its arguments and (ii) for each component $i$ there exists a vector $(\cdot_i, x)$ such that $\phi(1_i, x) = 1$ and $\phi(0_i, x) = 0$, where $(\cdot_i, x) = (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$. It follows from (i) and (ii) that (iii) $\phi(j) = j$ for $j = 0, 1$, where $j = (j, j, \ldots, j)$. The class of binary structure functions can be defined as the class of functions that can be represented as: $\phi(x) = \max_{1 \leq j \leq r} \min_{i \in P_j} x_i$ for all $x \in \{0,1\}^n$, where $P_1, \ldots, P_r$ are the $r$ minimal path sets of the system (see [1] for a full account of the theory of binary coherent systems).
Next we introduce some basic notations for multistate systems. Let the set \( S = \{0, 1, \ldots, M\} \) represent the \( M + 1 \) distinct levels of performance of the system and its \( n \) components, varying from the perfect functioning level \( M \) down to the complete failure level 0. A function \( \phi: S^n \mapsto S \) is said to be a multistate monotone structure function (MMS) [3] if (i) \( \phi \) is nondecreasing in each of its arguments and (ii) \( \phi(M) = M, \phi(0) = 0 \). A vector \( x \) is called an upper (lower) vector of \( \phi \) for level \( k \) (\( k > 0 \)) if \( \phi(x) \geq k \) (\( < k \)). An upper (lower) vector is called a critical upper (lower) vector of \( \phi \) for level \( k \) if \( \phi(x) \geq k \) and \( \phi(y) < k \) for all \( y < x \), where \( y > x \) means \( y_i \geq x_i \) for each \( i \) and strict inequality holds for some \( i \). We denote by \( U_k \) the collection of all upper vectors of \( \phi \) for level \( k \), i.e., \( U_k = \{ z \in S^n | \phi(z) \geq k \} \); and by \( CU_k \) the collection of all critical upper vectors of \( \phi \) for level \( k \), i.e., \( CU_k = \{ z \in S^n | \phi(z) \geq k; \phi(y) < k \ \forall y \prec z \} \). When \( M = 1 \) the MMS reduces to binary monotone systems, and the critical upper vectors reduce to minimal path vectors of binary systems (for more details about basic notations of multistate systems we refer to [2,3,4,8] and references there in).

The component relevance condition (ii) for binary systems has been generalized to multistate systems in many different ways by various authors. For example, the following are three successively stronger conditions:

(I) weak relevance [6]: for each \( i \), \( \exists (\cdot_i, x) \) such that \( \phi(M_i, x) > \phi(0_i, x) \).

(II) type 1 relevance [8]: for each \( i \) and level \( k > 0 \), \( \exists (\cdot_i, x) \) such that \( \phi(k_i, x) \geq k > \phi((k - 1)_i, x) \).

(III) strong relevance [4,5]: for each \( i \), \( \exists (\cdot_i, x) \) such that \( \phi(k_i, x) = k \) for all \( k \in S \).

Clearly, (III) \( \Rightarrow \) (II) \( \Rightarrow \) (I).
The Barlow and Wu (BW) class [2] of MMS consists of structure functions which can be represented as
\[ \phi(x) = \max_{1 \leq j \leq r} \min_{i \in P_j} x_i \] for all \( x \in S^n \), where \( P_1, \ldots, P_r \) are minimal path sets of a binary coherent system. Another class of MMS, more general than the BW class, is introduced by Natvig [8]. An MMS \( \phi \) belongs to the Natvig class if there exist \( M \) binary coherent systems \( \phi_1, \ldots, \phi_M \) such that \( \phi(x) \geq k \iff \phi_k(I_k(x)) = 1 \) (for \( k = 1, \ldots, M \)), where \( I_k(x) \) denotes the binary vector whose \( i \)th component is 1 if and only if \( x_i \geq k \), \( i = 1, \ldots, n \). It then follows that \( \phi(x) = \sum_{k=1}^{M} \phi_k(I_k(x)) \) for all \( x \in S^n \), and \( \phi_k \geq \phi_{k+1} \) for \( k = 1, \ldots, M - 1 \). The BW class is a subclass of the Natvig class corresponding to the case when \( \phi_1 \equiv \cdots \equiv \phi_M \). It is easy to see that \( \phi \) satisfies the strong relevance condition (III) if \( \phi \) belongs to BW class, and the Natvig class of MMS satisfies the relevance condition (II).

Various characterizations of the BW class of MMS have been obtained by researchers. For example, Block and Savits [3] showed that an MMS \( \phi \) is of BW type (allowing for some irrelevant components) if and only if \( CU_k = k \cdot CU_1 \) holds for \( k = 1, \ldots, M \). For the rest of this note, instead of BW (Natvig) class, we say the BW (Natvig) type means all MMS \( \phi \) which can be represented as \( \phi(x) = \max_{1 \leq j \leq r} \min_{i \in P_j} x_i \) (\( \phi(x) = \sum_{k=1}^{M} \phi_k(I_k(x)) \)), but some component \( x_i \) \((I_k(x_i))\) might be irrelevant to \( \phi \) \((\phi_k)\).

Two new conditions for MMS were introduced in [7]:

(condition A) \( \phi(x + a \cdot 1) = \phi(x) + a \) for all \( x \in S^n \), \( a \geq 0 \) such that \( x + a \cdot 1 \in S^n \);

(condition B) \( \phi(a \cdot x) = a \cdot \phi(x) \) for all \( x \in S^n \), \( a \geq 0 \) such that \( a \cdot x \in S^n \).

We restate the following theorems obtained in [7].

**Theorem 1.1.** Let \( \phi : S^n \rightarrow S \) be an MMS. Then, \( \phi \) is of BW type if and
only if \( \phi \) satisfies conditions \( A \) and \( B \).

**Theorem 1.2.** Let \( \phi : S^n \rightarrow S \) be an MMS. Then, \( \phi \in BW \) class if and only if \( \phi \) satisfies conditions \( A \) and \( B \) and the weak relevance condition (I).

In the above two theorems condition \( B \) can be replaced by the condition that \( \phi(\{0, M\}^n) = \{0, M\} \) to hold the same conclusion.

Like the BW class, the Natvig class of MMS is defined in terms of binary coherent systems, and enjoys many nice properties inherited. In section 2 we introduce two new conditions for MMS. We show that the two conditions are satisfied by the Natvig type of MMS, which reveals some insight into the Natvig type of MMS. Furthermore, the two conditions can be used in practice as a fast-checking method in verifying whether an MMS belongs to the Natvig class. Then we present a main theorem of this note, which characterizes the Natvig type of MMS, and provides a procedure (easy to operate) to classify multistate systems. Examples are given to illustrate our results.

## 2. CHARACTERIZATIONS OF MULTISTATE SYSTEMS

It is easy to see that one part of the implications in Theorem 1.1 is apparent. That is, the two invariance conditions \( A \) and \( B \) are satisfied if the structure function \( \phi \) is of BW type. However, for the Natvig class of MMS, the following Lemma 2.1 is not so obvious, and it reveals some insight into structural characteristics of the Natvig class of MMS.

**Lemma 2.1.** Let \( \phi : S^n \mapsto S \) be an MMS, and \( \phi \) is of Natvig type. Then \( \phi \) satisfies both condition \( A1 \) and condition \( B1 \):

- (condition \( A1 \)) \( \phi(x+a \cdot 1) \leq \phi(x)+a \) for all \( x \in S^n \), \( a \geq 1 \) s.t. \( x+a \cdot 1 \in S^n \);
- (condition \( B1 \)) \( \phi(a \cdot x) \leq a \cdot \phi(x) \) for all \( x \in S^n \), \( a \geq 1 \) s.t. \( a \cdot x \in S^n \).
Proof. (condition A1) We assume that $\phi(x + a \cdot 1) > a$ to avoid triviality. Let $\phi(x + a \cdot 1) = k$, then $\phi_k(I_k(x + a \cdot 1)) = 1$. So, $\phi_{k-a}(I_k(x + a \cdot 1)) = 1$, since $\phi_{k-a} \geq \phi_k$. By noting that $I_{k-a}(x) = I_k(x+a \cdot 1)$, we obtain $\phi_{k-a}(I_{k-a}(x)) = 1$. It then follows that $\phi(x) \geq k - a$. (condition B1) Assume that $\phi(x) = k$ and $ak < M$. It suffices to show that $\phi(a \cdot x) < ak + 1$. Toward this end, we assume the contrary and let $\phi(a \cdot x) \geq ak + 1$, and hence $\phi_{ak+1}(I_{ak+1}(a \cdot x)) = 1$. Note that $ax_i \geq ak + 1$ implies $x_i \geq k + 1$, since $x_i$'s are all integers. Hence $I_{ak+1}(a \cdot x) \leq I_{k+1}(x)$, and $\phi_{ak+1}(I_{k+1}(x)) = 1$ follows. It in turn implies that $\phi_{k+1}(I_{k+1}(x)) = 1$ and then $\phi(x) \geq k + 1$, a contradiction.

The following lemmas shall be needed to derive more results. Furthermore, they can be used as a checking method in verifying whether a particular multistate monotone structure is of Natvig type.

**Lemma 2.2.** Let $\phi : S^n \mapsto S$ be an MMS. If condition A1 is satisfied, then

(i) $\phi(k) = k$ for all $k \in S$.

(ii) $\phi((x - a \cdot 1) \lor 0) \geq \phi(x) - a$, for all $x \in S^n$, $a \geq 1$ for which $(x - a \cdot 1) \lor 0 \in S^n$, and where $x \lor y$ is the componentwise maximum.

(iii) $\phi(k^A, x) - \phi((k-a)^A, x) \leq a$ for all $x \in S^n$, all subsets $A$ of $\{1, \ldots, n\}$, and $0 < a \leq k \leq M$, where $(k^A, x)$ means that the component level is $k$ when this component is in $A$. In particular, $\phi(k^A, 0) = k \Rightarrow \phi(j^A, 0) = j$ for all $j < k \leq M$, and $\phi(\{0, M\}^n) = \{0, M\} \Rightarrow \phi(\{0, k\}^n) = \{0, k\}$ for all $k = 1, \ldots, M - 1$.

Proof. (i) Note that $\phi(k) = \phi(0 + k1) \leq \phi(0) + k$, and hence $\phi(k) \leq k$. Also, $\phi(M) = \phi((k + (M-k)1) \leq \phi(k) + M - k$, and hence $\phi(k) \geq k$. The conclusion is obtained.
(ii) Consider the vector \((x - a \cdot 1) \lor 0\). By condition A1,
\[
\phi((x - a \cdot 1) \lor 0) + a \geq \phi((x - a \cdot 1) \lor 0 + a \cdot 1).
\]
Next note that the \(i\)th component of \((x - a \cdot 1) \lor 0 + a \cdot 1\) equals \(a\) if \(x_i \leq a\); and\(\)
equals \(x_i\) if \(x_i > a\). Hence, by monotonicity, \(\phi((x - a \cdot 1) \lor 0 + a \cdot 1) \geq \phi(x)\).

(iii) By (ii), \(\phi((k - a)^A, x) \geq \phi(((k^A, x) - a \cdot 1)) \lor 0) \geq \phi(k^A, x) - a\). The rest of the proof is easy and is omitted.

We state without proof the following useful lemma. This necessary condition can be used in verifying the Natvig type of MMS.

**Lemma 2.3.** Let \(\phi: S^n \to S\) be an MMS. If condition B1 is satisfied, then \(\phi(1^A, 0) = 0 \Rightarrow \phi(k^A, 0) = 0\) for all \(k = 1, \ldots, M\) and all \(A \subset \{1, \ldots, n\}\).

The following theorem is our main characterization result for the Natvig type of multistate structure functions.

**Theorem 2.4.** An MMS \(\phi\) is of Natvig type if and only if (i) condition A1 holds; and either (ii) or (iii) holds: (ii) \(\phi(\{0,M\}^n) = \{0,M\}\). (iii) if \(\phi(M^A, 0) = k\) for some \(A\) and \(0 < k < M\), then \(\phi(j^A, 0) = j\) for all \(j \leq k\); and \(\phi(M^A, x) < w\) for all \(w > k\) and all \(x < w^Ae\).

**Proof.** \((\Rightarrow)\) (i) follows from Lemma 2.1. (ii) holds if the MMS \(\phi\) is a special case of Natvig type, namely the BW type. Suppose that (ii) is not true and \(\phi(M^A, 0) = k\) \((0 < k < M)\) for some \(A\). Then \(\phi_k(I_k(M^A, 0)) = 1\), which in turn implies that \(\phi_j(I_j(M^A, 0)) = 1\) for all \(j \leq k\). Thus, \(\phi_j(I_j(j^A, 0)) = 1\), and \(\phi(j^A, 0) = j\) holds for all \(j \leq k\). Now it suffices to show that \(\phi_w(I_w(M^A, x)) = 0\) for all \(w > k\) and all \(x < w^Ae\) to complete the proof. This simply follows from the facts that \(\phi_w(I_w(M^A, x)) = \phi_w(I_w(M^A, 0))\) and that \(\phi(M^A, 0) < w\)
\[ \Rightarrow \phi_w(I_w(M^A, 0)) = 0. \]

(\iff) Suppose that (i) and (ii) hold. Then \( \phi \) must be of BW type. This can be proved by using essentially the same technique as that for Theorem 3.2 in [7, p.666], and we omit the proof. Now assume (i) and (iii) hold. Consider a vector \( x \in U_e \), that is, \( \phi(x) \geq e \ (0 < e \leq M) \). We shall construct a binary monotone structure \( \phi_e \) from all these upper vectors \( x \in U_e \), for each level \( 0 < e \leq M \). Then we show that the implications \( \phi(x) \geq e \iff \phi_e(I_e(x)) = 1 \) holds \((e = 1, \ldots, M)\), which completes the proof.

Let \( A_e(x) = \{ i | x_i \geq e \} \), and let \( r = \max_{i \in A_e(x)} x_i \). The set \( A_e(x) \) is nonempty since \( \phi(e) = e \) for all \( e \in S \), by (i) of Lemma 2.2. Then, by monotonicity and (ii) of Lemma 2.2,

\[
\phi(M^{A_e(x)}) \geq \phi((x - r \cdot 1) \lor 0) \geq \phi(x) - r > 0.
\]

**Case 1.** If \( \phi(M^{A_e(x)}, 0) = M \), then by (iii) of Lemma 2.2, \( \phi(1^{A_e(x)}, 0) = 1 \) and \( \phi(e^{A_e(x)}, 0) = e \) hold.

**Case 2.** Assume that \( \phi(M^{A_e(x)}, 0) = k < M \). Suppose that \( k < e \). Then by assumption (iii), \( x^{A_e(x)}e < e^{A_e(x)}e \Rightarrow \phi(M^{A_e(x)}, x^{A_e(x)}e) < e \). But, by monotonicity, \( \phi(M^{A_e(x)}, x^{A_e(x)}e) \geq \phi(x) \geq e \), which is a contradiction. Thus, \( e \leq k \) holds. It again implies that \( \phi(e^{A_e(x)}, 0) = e \) and \( \phi(1^{A_e(x)}, 0) = 1 \), by assumption (iii). We have shown that in either case 1 or case 2

\[
x \in U_e \Rightarrow \phi(e^{A_e(x)}, 0) = e \ and \ \phi(1^{A_e(x)}, 0) = 1.
\]

Now let the binary function \( \phi_e : \{0, 1\}^n \mapsto \{0, 1\} \) be defined as \( \phi_e(z) = 1 \) if \( z_i = 1 \) for all \( i \in A_e(x) \) and for some \( x \in U_e \), and let \( \phi_e(y) = 1 \) if and only if \( y > z \) for some \( z = (1^{A_e(x)}, 0) \); \( \phi(\cdot) = 0 \) otherwise. Note then, for all subsets \( B \subset \{1, \ldots, n\} \), \( \phi_e(1^B, 0) = 1 \) implies that \( \phi(e^B, 0) = e \), since \( B \supseteq A_e(x) \) for
some \( x \in U_e \). Now clearly, \( \phi(x) \geq e \Rightarrow \phi_e(1^{A_e}(x), 0) = 1 \Rightarrow \phi_e(I_e(x)) = 1 \).
Conversely, \( \phi_e(I_e(z)) = 1 \Rightarrow \phi(e \cdot I_e(z)) = e \Rightarrow \phi(z) \geq e \), since \( z \geq e \cdot I_e(z) \).
The proof is complete.

**Theorem 2.5.** An MMS \( \phi \in \text{Natvig class if and only if (i) and (iii) (or (ii)) stated in Theorem 2.4 hold, and (iii) for each component } i \text{ and for each level } k > 0, \exists (\cdot^i, x) \text{ such that } \phi(k_i^i, x) \geq k > \phi((k - 1)_i^i, x).$$

**Proof.** By Theorem 2.4, all we need to show is that for all \( 1 \leq k \leq M \), each component \( I_k(x_i) \) is relevant to \( \phi_k \). It is easy to see that, from the relevance assumption, there exists a critical upper vector \((k_i, z) \in CU_k \), and \( \phi(k_i, z) \geq k > \phi((k - 1)_i, z) \). Thus, \( \phi_k(I_k(k_i, z)) > \phi_k(I_k((k - 1)_i, z)) \). Hence the conclusion.

In the following three illustrative example are given.

**Example 2.6.**

Let \( \phi : \{0, 1, 2\}^2 \mapsto \{0, 1, 2\} \) be defined as follows.

\[
\begin{align*}
\phi(0, 0) &= 0, \quad \phi(1, 0) = 1, \quad \phi(2, 0) = 1, \\
\phi(0, 1) &= 1, \quad \phi(1, 1) = 1, \quad \phi(2, 1) = 1, \\
\phi(0, 2) &= 2, \quad \phi(1, 2) = 2, \quad \phi(2, 2) = 2.
\end{align*}
\]

The structure \( \phi \) satisfies the weak relevance assumption (i), but not (ii). It is also easily verified that \( \phi \) satisfies condition A1 and (iii) of Theorem 2.4. Note that \( \phi(x) \geq 1 \iff x_1 \lor x_2 \geq 1 \), and \( \phi(x) \geq 2 \iff x_2 = 2 \). Thus, \( \phi(x) = \phi_1(I_1(x)) + \phi_2(I_2(x)) \), where \( \phi_1(I_1(x_1, x_2)) = I_1(x_1) \lor I_1(x_2) \) and \( \phi_2(I_2(x_1, x_2)) = I_2(x_2) \) for all \( x_1, x_2 \). The structure \( \phi \) is of Natvig type, but the first component \( x_1 \) is irrelevant to \( \phi \) for level 2, and \( I_2(x_1) \) is irrelevant to \( \phi_2 \). Thus, \( \phi_2 \) is not coherent, and \( \phi \not\in \text{Natvig class} \).

**Example 2.7.** (from [8, p.440]) Let \( \phi : \{0, 1, 2\}^2 \mapsto \{0, 1, 2\} \) be defined as
follows.

\begin{align*}
\phi(0, 0) &= 0, \ \phi(1, 0) = 1, \ \phi(2, 0) = 1. \\
\phi(0, 1) &= 1, \ \phi(1, 1) = 1, \ \phi(2, 1) = 1. \\
\phi(0, 2) &= 1, \ \phi(1, 2) = 1, \ \phi(2, 2) = 2.
\end{align*}

It is easily seen that \( \phi \) satisfies condition A1 and the relevance condition (II). Next, using the procedure in Theorem 2.4, we first find the boundary points \( \phi(0, 2) < 2, \ \phi(2, 0) < 2 \) (if \( \phi(0, 2) = \phi(2, 0) = 2 \), then \( \phi \) is of BW type). From (iii) of Theorem 2.4, we see \( \phi(0, 1) = 1 \) and \( \phi(1, 0) = 1 \) and conclude that \( \phi \in \text{Natvig class.} \)

**Example 2.8.** Let \( \phi : \{0, 1, 2, 3, \}^3 \mapsto \{0, 1, 2, 3\} \) be the structure function of a multistate system. Suppose that the structure is only partially observable, and we know that \( \phi(1, 2, 1) = 1 \) and \( \phi(2, 3, 2) = 3 \). Using condition A1, a quick conclusion can be made that the structure is not of the Natvig type.

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