ESO Compensation: The Roles of Default Risk and Over-Confidence

Charles Chang¹
Cornell University

Cheng-Der Fuh
Academia Sinica

Ya-Hui Hsu
University of Illinois Urbana-Champaign

January 17, 2006

Abstract

This paper derives a pricing model for employee stock options (ESO) that expands on Ingersoll (2002) and that includes default risk and considers employee over-confidence. Results show that the more risk-averse employees are and the more stock restrictions employees have, the smaller the value of ESOs. The model also incorporates potential subjective mis-pricing on the part of over-confident employees and finds that, under normal calibrations, employees who over-estimate firm returns by more than 5% will prefer ESOs over cash compensation. The results of this study may impact relevant accounting regulations regarding the valuation of ESOs as well as compensation decisions.

JEL: G11, G13, G32, G35

Keywords: Stock options, over-confidence, default model, jump diffusion

¹435 Statler Hall; Ithaca, NY 14853; charlesc@cornell.edu; 607-255-9882. Opinions expressed are the authors’ alone and all errors are our own.
1. Introduction

Employee stock options (henceforth ESOs) are an extensively employed method of compensation in today’s business world. Academic research and anecdotal evidence alike illuminate the potential for such a strategy to help retain talent and reduce agency costs. However, though a deep literature addresses pricing of general options, a robust ESO valuation technique remains elusive due to a number of additional modeling considerations: ESOs are generally not publicly traded, cannot be redeemed for a fixed period of time, and are increasingly popular for small, early stage firms that may have non-continuous value functions. Our paper seeks to address these particular qualities; we develop a jump-diffusion model where investors seek to optimize portfolios of partially un-tradable ESO endowments, a tradable market portfolio, and the risk free rate in the presence of default risk. We find that under-diversification induced by the non-tradeability of ESOs causes investors to subjectively value ESOs below the market price. This subjective value is decreasing in risk aversion and increasing in tradeability, and employees should in general prefer to receive the market value of the options in cash rather than the options themselves. However, we further find that this subjective discount can be offset by over-confidence on the part of employees. Under normal levels of risk aversion, when employees over-estimate the value firm’s returns by more than 5%, their subjective valuation begins to exceed the market value of the option. In such cases, employees will prefer options in lieu of cash. Results imply that a company’s decision to issue ESOs as a compensation component should depend on a number of factors, including employee risk-aversion, the degree of tradeability of the ESO and underlying stock, and the employees’ subjective over-confidence regarding the firm’s returns.

As will be pointed out later, though our solution methodology captures many qualities of ESOs, our findings can be further generalized in important ways. Our model is potentially able to capture and quantify liquidity premiums in imperfectly tradable assets such as stocks with large minority holders and newly listed stocks. We further know of no other paper that quantifies the trade-off between risk aversion and over-confidence in this manner. It is intuitively sensible that one’s distaste for risk might be offset by one’s mis-interpretation of firm-specific characteristics such as risk-adjusted returns; however, we believe our paper to be the first that can quantify this tradeoff.

Jensen and Meckling (1976) and Hall and Liebman (1998) among others demonstrate the popularity of ESO issuance as a form of agency-cost-reducing compensation. However, as Ingersoll (2002) points out, it is important to note that the objective value of the ESOs (the
cost to shareholders) is not generally equal to the subjective value of the ESO (the value to employees) since employees are constrained agents who cannot choose to divest holdings whenever they please\(^1\). For this reason, ESO valuations hinged on market value or unconstrained maximization considerations leave a part of the task unanswered.

In addition, Fama and French (2005) finds that, contrary to traditional pecking order considerations, ESO's are most popular with small cap, high growth companies. Given that default risk for small enterprises is higher than that of large ones and further considering recent concern over bankruptcy rates discussed in the Basel Capital Accord, it seems reasonable that a full discussion of ESO pricing should also take into account the possibility of default. Several options pricing models involving default risk have been proposed. Hull and White (1995) discusses vulnerable options (options whose writer may default); Litterman and Iben (1991) studies a discrete-time model where there is a zero payoff in the case of default; and Hsu, Saar-Requejo, and Santa-Clara (2002) proposes a structural model where bankruptcy is explicitly modeled.

Taking into account these two considerations, our model considers two components of uncertainty: a continuous part modeled by a standard Brownian motion and a discontinuous part modeled by a compound Poisson process\(^2\). It solves the optimal portfolio investment strategy for an employee facing a selection decision between a risk free asset, the market portfolio, and a firm-specific ESO. We derive closed-form solutions for the market and subjective values of ESOs (the difference henceforth to be called the ESO premium) and demonstrate that the latter is decreasing in risk aversion \(1 - \gamma\) and increasing in degree of tradeability \(1 - \alpha^J\), where \(\alpha^J\) can be thought of as the proportion of the employee’s endowment in company stock/ESO that is un-tradable/un-redeemable. The model is also able to address the issue of employee over-confidence by testing the sensitivity of subjective value to over-estimations of firm return. For normal calibrations of each parameter, we find equilibrium subjective under-valuations of 35%. Such valuations would require idiosyncratic over-confidence of 5% in order for employees to value ESOs as highly as the market.

The issue of ESO valuation is an important one both from a firm-level perspective as well

---

\(^1\)One might consider market value to be the value an unconstrained agent would pay for the ESO, objective value to be what an unconstrained agent would pay for the ESO if the exercise decision is made by employees and not by shareholders, and subjective value to be what a constrained agent would pay for the ESO if exercise decision is made by himself/herself. Both the inability to determine exercise and non-transferability cause erosion of value.

\(^2\)As in Merton (1976).
as from a regulatory one. At the firm level, ESOs are given as part of a total compensation package. Employees, in turn, value those options subjectively and decide whether the compensation package dominates their opportunity cost. Understanding how employees might value ESOs differently from the market is important to contract design. Additionally, it has been proposed by the Financial Accounting Standards Board (FASB) that ESOs be taxed. As such, several pricing models for ESOs have been proposed. The Accounting Principles Board (APB) in 1972 suggested adoption of the intrinsic method of accounting for employee stock options. FASB 123, published in 1995, encourages companies to adopt the fair-value method which applies an adapted Cox, Ross, and Rubinstein (1979) model where forfeiture can only occur when employees leave during the vesting period and which ignores restrictions on trading. Hull and White (2004) develops an enhanced FAS 123 model where employee stock options are viewed as barrier options. Kulatilaka and Marcus (1994), Huddart (1994), Rubinstein (1995), and Carpenter (1998) propose utility-maximizing models that model un-diversifiable risk as an early exercise parameter and calibrate empirical models to study early exit rates. Meulbroek (2001) presents a Sharpe ratio approach that ignores employee preferences and utilizes a lower bound for the cost of undiversified risk to price ESOs. Lambert et al. (1991) and Hall and Murphy (2002) propose a certainty equivalent approach to estimate subjective value of employee stock option. Like Ingersoll (2002), our model differs from these studies in that we propose a utility-maximizing model that derives subjective and market values for an illiquid (trade-restricted) holding in the ESO which incorporates a market portfolio into the employee’s wealth process. Our model, however, additionally admits default risk and parameterizes employee over-confidence. These additions provide the unique contributions of our paper.

The main intuition behind our findings is quite simple and is not unique to our model alone: ESOs represent non-tradable holdings in stock that cannot be diversified and hence force employees into sub-optimal portfolio holdings. In some sense, one might consider the ESO premium to be a form of an illiquidity premium. Indeed, Brenner et al (2001) find that illiquid currency options are valued as much as 20% less than liquid ones. Illiquidity in ESOs may arise not only from contractual obligations at the firm-level but also from legal restrictions such as SEC Rule 144 that limit the trading of corporate insiders or affiliates. Huddart and Lang (1996) and Carpenter (1998) provide empirical evidence that such rules induce lowered liquidity at the top of the corporate ladder. Particularly in the case of small firms, executives may have pressures, explicit or otherwise, that restrict their ability to short or sell their stock
holdings. For firms with unlisted stocks or in closed-end ventures, the problem would persist. Indeed, in general, any compensation scheme linked directly and un-tradable to the stock value’s governing diffusion would suffice. Because of the inclusion of a market portfolio in the investment optimization, the non-tradeability coefficient $\alpha^J$ is always a binding constraint. We find that subjective values are decreasing in $\alpha^J$ and that, in the absence of non-tradeability, subjective value equals market value. Intuitively, employees who have small existing holdings or who receive small ESO offerings find ESOs more valuable.

Employees should only prefer ESOs to receiving the market value in cash if they over-estimate the returns of the firm enough to be willing to over-weight that stock and absorb the illiquidity premium. Recent work by Bergman and Jenter (2005) highlight this possibility, and we find here that an over-confidence level of about 5% would be necessary to explain the ESO premium. Namely, employees would have to over-estimate firm risk-adjusted returns by 5% in order to value ESOs as highly as the market.

The remainder of this paper is divided as follows: Section 2 develops the model and solves the principle maximization problem. Section 3 presents the closed-form solution of market and subjective values, calculates those values for usual parameterizations, and derives a sensitivity analysis for each parameter. Section 4 parameterizes and discusses the over-confidence measure, and Section 5 provides a simple case example and discusses model extensions. Section 6 concludes.

2. Employee Stock Option Valuation

2.1. The Constrained Portfolio Problem

In our model, employees allocate wealth between three assets: the company stock $S$, the market portfolio $M$, and the risk-free bond $B$. For simplicity we assume that the continuous time CAPM model holds so that the efficient portfolio is the market. Note that this is for the convenience in our discussion, other equilibrium models can be used as well. Before retirement, the employee is constrained to allocate a fixed fraction $\alpha^J$ of her wealth to company stock (via some form of ESO). Define the diffusion process for our assets as follows:

$$\begin{align*}
\frac{dS}{S} &= (\mu_s - d_s + \lambda)dt + \nu_m d\omega_m + \nu_s d\omega_s - dq, \\
\frac{dM}{M} &= (\mu_m - d_m)dt + \sigma_m d\omega_m, \\
\frac{dB}{B} &= \mu_b dt.
\end{align*}$$

(1)

The dividend yields, $d_s$ and $d_m$, do not affect the consumption-portfolio choice problem but are
useful for constructing the pricing formula developed later. The Brownian motion process \( d\omega_m \) represents the Normal systematic risk of the market portfolio; the Brownian motion process \( d\omega_s \) represents the same for the stock. The jump process \( dq \) captures the default risk of company stock and follows a Poisson distribution with average frequency \( \lambda \) such that \( dq \sim P(\lambda dt) \). We assume here that if default occurs stock value goes to zero so the discontinuous part of the evolution of company stock, \((Y - 1)dq\), simplifies to \(-dq\) with zero jumping size \( Y \). \( \mu_b, \mu_s, \mu_m \) are instantaneous expected rates of return for the risk-free bond, stock, and market portfolio, respectively. \( \nu_m \) and \( \sigma_m \) are the Normal systematic portions of total volatility for the stock and the market portfolio, respectively; \( \nu_s \) is the Normal unsystematic volatility of the stock. The two Brownian motions and jump process are presumed independent. The expected returns and covariance matrix of the two risky assets are (See Appendix A):

\[
\begin{align*}
\mu &= \left( \mu_b + \beta(\mu_m - \mu_b) \right) \\
\Omega &= \begin{pmatrix}
\beta^2 \sigma^2_m + \nu^2_s + \lambda & \beta \sigma^2_m \\
\beta \sigma^2_m & \sigma^2_m 
\end{pmatrix},
\end{align*}
\] (2)

where \( \beta = \nu_m / \sigma_m \) is the standard beta. Let \( W \) denote the wealth process and \( C \) denote the consumption process, then the optimal portfolio selection choice problem becomes

\[
\begin{align*}
J[W(t), t] &= \max_{\{C, w_s, w_m, w_b\}} E_t \int_t^{T_2} e^{-\rho_s s} U(C(s)) ds + B[W(T_2), T_2], \\
\text{s.t. } J[W(T_2), T_2] &= B[W(T_2), T_2], \\
w_s + w_m + w_b &= 1,
\end{align*}
\] (3)

where \( J[W(t), t] \) is the employee’s total utility at time \( t \). The employee’s utility function \( U(\cdot) \) is assumed to satisfy constant relative risk aversion and can be represented \( U(C) = C^{\gamma} \). The coefficient of relative risk aversion is \( R_U(C) = -\frac{CU''(C)}{U'(C)} = 1 - \gamma > 0 \). \( B[W(T_2), T_2] \) is the bequest function at the date of termination \( T_2 \).

After retirement, the optimal consumption and portfolio weights can be derived from the first order condition of equation (3) and is simply (See Appendix B):

\[
\begin{align*}
C^* &= \left[ \frac{1}{1 - \gamma} W \right]^{1 - \gamma}, \\
w_s^* &= 0, \\
w_m^* &= \frac{\mu_m - \mu_b}{(1 - \gamma) \sigma^2_m}, \\
w_b^* &= 1 - \frac{\mu_m - \mu_b}{(1 - \gamma) \sigma^2_m}.
\end{align*}
\]

\(^3\)We use a reduced form approach to treat default as an event with zero payoff and which is governed by an exogenous jump process. Then we estimate three values of the ESO using the rational expectations approach such that the terminal value of the option is discounted by the marginal rate of substitution following Lucas (1978).
The total utility function is

\[ J = e^{-\rho t} \int_0^T e^{\lambda s} ds, \]

where

\[ b^J(t) = \left\{ \frac{1+(A'[b^J(T_2)]^{1/2})e^{A'[t-T_2]}}{A'} \right\}^{1-\gamma}, \]

\[ A^J = \frac{-\gamma}{1-\gamma} \left[ \frac{1}{\gamma} - \mu_b - \frac{1}{2} \frac{(\mu_m-\mu_b)^2}{(1-\gamma)^2\sigma_m^2} \right]. \] (4)

For computational simplicity, \( T_1 \) is both the vesting period of the employee’s European-style options and the retirement period for the employee.

During the period \( T_1 \leq t < T_2 \), employees do not face allocation/tradeability restrictions. The total utility function is

\[ J[W(t), t] = b^J(t)e^{-\rho t} W_{T_2} \gamma, \]

as the employee will allocate her wealth only in the market portfolio and risk-free asset, in accordance with usual two-fund separation results. No additional investment is made in stock.

Before vesting, with restricted holding of company stock \( \tilde{w}_s^* = \alpha^J \), the optimal consumption and portfolio can be similarly derived from the first order condition of (3) as follows:

\[ \tilde{C}^{\alpha} = \tilde{b}^J(t) W_{T_2} \gamma, \tilde{w}_s^* = \alpha^J, \tilde{w}_m^* = \frac{\mu_m - \mu_b}{(1-\gamma)^2\sigma_m^2} - \alpha^J \beta, \tilde{w}_b^* = 1 - \frac{\mu_m - \mu_b}{(1-\gamma)^2\sigma_m^2} - \alpha^J (1-\beta), \]

where

\[ \tilde{b}^J = \left\{ \frac{1+(A'[b^J(T_1)]^{1/2})e^{A'[t-T_1]}}{A'} \right\}^{1-\gamma}, \]

\[ \tilde{A}^J = \frac{-\gamma}{1-\gamma} \left[ \frac{1}{\gamma} - \mu_b - \frac{1}{2} \frac{(\mu_m-\mu_b)^2}{(1-\gamma)^2\sigma_m^2} + (1-\gamma)(\nu_s^2 + \lambda)(\alpha^J)^2 \right]. \] (5)

The total utility function is \( J[W(t), t] = \tilde{b}^J(t)e^{-\rho t} W_{T_2} \gamma, \ T < t < T_1 \). Illiquidity in the stock holding results in higher unsystematic risk; employees have incentive to reduce this risk by investing less in the market portfolio and more in the risk-free asset such that \( \tilde{w}_m^* < w_m^*, \tilde{w}_b^* > w_b^* \). This holds as long as \( \beta > 0 \), which is supported empirically particularly for small and medium-sized enterprises which may generally have high systematic risk components (\( \beta > 1 \)).

If additionally risk aversion is larger than 1 (\( \gamma < 0 \)), then \( \tilde{A}^J < A^J, \tilde{b}^J > b^J \) and optimal consumption and utility for the restricted employee is lower than that of the unrestricted case.

2.2. Evolution of the Employee’s Marginal Utility

By Ito’s lemma for jump processes and the evolution of wealth, employee’s marginal utility can be derived as (See Appendix C):

\[ \frac{dJ_W}{J_W} = \left\{ \begin{array}{ll} -\hat{\rho}dt - \frac{\mu_m-\mu_b}{\sigma_m} dW_m - (1-\gamma)\alpha^J v_s d\omega_s + [(1-\alpha^J)^{\gamma-1} - 1](dq - \lambda dt), & t < T_1, \\
-\mu_b dt - \frac{\mu_m-\mu_b}{\sigma_m} dW_m, & T_1 \leq t < T_2, \end{array} \right. \] (6)

where \( J_W = \frac{\partial J[W(t), t]}{\partial W(t)} \) is the marginal utility, and the adjusted interest rate, \( \hat{\rho} \), is \( \hat{\rho} = \mu_b - [(1-\alpha^J)^{\gamma-1} - 1]\lambda - (1-\gamma)[(\alpha^J)^2 v_s^2 + \frac{1}{2}(\alpha^J)^2 \gamma \lambda - \alpha^J \lambda]. \) The rational equilibrium value of the ESO
$F(S, t)$ then satisfies the Euler equation

$$F(S, t) = \frac{E(J_W[W(T), T]F(S, T)|F_t)}{J_W[W(t), t]},$$  \hspace{1cm} (7)

where $T$ is the maturity date of the contract. The marginal rate of substitution, $\frac{J_W[W(T), T]}{J_W[W(t), t]}$, can be viewed as a stochastic discount factor, pricing kernel, change of measure, or state-price density. The product $J_W[W(t), t]F(S, t)$ is a martingale under $F_t$ so, given $E\{d(J_W[W(t), t]F(S, t))\} = 0$, one can substitute the stochastic discount factor into the following result of Ito’s lemma

$$0 = E[d(J_W F(S, t))] = E[dJ_W F + J_W dF + dJ_W dF].$$  \hspace{1cm} (8)

Two partial differential equations can then be derived (See Appendix D). After Retirement, the following PDE holds

$$\frac{1}{2}F_{SS}S^2\sigma_N^2 + (\mu_b + \lambda - d_s)SF_S - (\mu_b + \lambda)F + F_t = 0.$$  \hspace{1cm} (9)

This partial differential equation will be recognized as the Merton’s (1976) result\(^4\). Market value of the employee stock option is

$$F(S, \tau) = C(Se^{-d_s\tau}, \tau; K, \sigma_N^2, \mu_b + \lambda) = Se^{-d_s\tau}N(d_1) - Ke^{-(\mu_b + \lambda)\tau}N(d_2),$$  \hspace{1cm} (10)

where $\mu_b + \lambda$ is the market (objective) rate of return, $\sigma_N = (\nu_m^2 + \nu_s^2)^{1/2}$ is the Normal volatility of the stock, $d_1 = \frac{\ln \frac{S}{K} + (\mu_b + \lambda - d_s + \frac{\sigma_N^2}{2})\tau}{\sigma_N\sqrt{\tau}}$, $d_2 = d_1 - \sigma_N\sqrt{\tau}$, $\tau = T - t$ is time to maturity, and $K$ is the exercise price. Upon inspection, one finds that companies with a positive probability of immediate ruin have more valuable employee stock options than those that do not. Consider that, in a risk-neutral economy, the risk-free rate can only maintained by changing the probability measure. If one maintains the same probability measure, default risk will generate additional volatility as $\sigma^2 = (\sigma_N^2 + \lambda) > \sigma_N^2$ and hence additional risk premium.

As expected, option value tends to increase with volatility.

Before Retirement, the following PDE holds

$$\frac{1}{2}F_{SS}S^2\sigma_N^2 + (\bar{r} - \bar{d}_s)SF_S - \bar{r}F + F_t = 0,$$  \hspace{1cm} (11)

where the subjective rate of return is $\bar{r} = \mu_b + \lambda - (1 - \gamma)[(\alpha^J)^2\nu_s^2 + \frac{1}{2}(\alpha^J)^2\gamma\lambda - \alpha^J\lambda]$ and the subjective dividend yield is $\bar{d}_s = d_s + (1 - \gamma)[\alpha^J(1 - \alpha^J)\nu_s^2 - \frac{1}{2}(\alpha^J)^2\gamma\lambda + \alpha^J\lambda]$. Both are

\(^4\)See Merton (1976, p.133). Merton’s PDE is $0 = \frac{1}{2}F_{SS}S^2\sigma^2 + (r - \lambda k)SF_S - F_r - rF + \lambda E(F(Y, \tau) - F(S, \tau))$. This result and equation (9) are the same after substituting $k = E[Y - 1] = -1$ into Merton’s PDE.
increasing in \( \lambda \). Subjective value of the employee stock option is

\[
\tilde{F}(S, \tau) = C(S \tilde{d}_s^\tau, \tau; K, \sigma_N^2, \tilde{\sigma}) = Se^{-\tilde{d}_s^\tau N(\tilde{d}_1)} - Ke^{-\tilde{\sigma}\tau N(\tilde{d}_2)},
\]

where \( \tilde{d}_1 = \left( \frac{\ln S}{\tilde{\sigma}} + (\mu_b + \lambda - d_s - (1 - \gamma)\alpha^J_2 + \sigma^2_{N})\tau \right) - \tilde{d}_2 = \tilde{d}_1 - \sigma_N\sqrt{\tau} \). Note that when \( \alpha^J = 0 \) (12) reduces to (10), and \( \tilde{\sigma} = \mu_b + \lambda \).

3. The Simulated ESO Premiums

3.1 Subjective Values vs Market Values

Table 1 shows subjective values; all are smaller than the market value; all ESO premiums are positive. The more risk averse the employee (smaller \( \gamma \)) and more stock restrictions/illiquidity in the stock holding (larger \( \alpha^J \)), the higher the ESO premium and the less employees are inclined to accept ESOs in lieu of cash compensation. In essence, the ESO premium becomes a nontransferable cost that employees must bear. Indeed, the discrepancy can be quite large for even reasonable calibrations. Consider \( \gamma = -1 \), and \( \alpha^J = 0.10 \): subjective value is only 60.7% of market value. As illiquidity increases, ESO premiums rapidly increase. We infer that employee stock option plans are not the most effective method to motivate employees, especially for those that are more risk averse or who face stricter stock restrictions and illiquidity.

Interestingly, the sensitivity of price to time-to-maturity can be either positive or negative as \( \tilde{d} = -\frac{\partial \tilde{F}(S, \tau)}{\partial \tau} = -\frac{\tilde{d}N_e^{-\tilde{d}_s^\tau N(\tilde{d}_1)}}{2\sqrt{\tau}} + \tilde{d}Se^{-\tilde{d}_s^\tau N(\tilde{d}_1)} - \tilde{\sigma}Ke^{-\tilde{\sigma}\tau N(\tilde{d}_2)}. \) However, the inclusion of the jump diffusion generates important innovations in the rates of return and dividend processes. Note that

\[
d(\text{market}) - d(\text{subjective}) = -(1 - \gamma)[\alpha^J(1 - \alpha^J)\nu_s^2 - \frac{1}{2}(\alpha^J)^2\gamma\lambda + \alpha^J\lambda] < 0.
\]

Therefore, employees will tend to overvalue dividend yield compared with that determined from the perspective of outside investors. Indeed, a novel finding in our paper is that

\[
[r(\text{market}) - d(\text{market})] - [r(\text{subjective}) - d(\text{subjective})] = (1 - \gamma)\alpha^J\nu_s^2 > 0.
\]

That is, the premium over the dividend yield is greater for the market than for the employee. Again the reward or premium that is perceived by employees to be earned above the dividend received is smaller. The following section discusses additional sensitivity measures.

3.2. Sensitivity Analysis

\(^5\)S: stock value, \( E \): exercise price, \( \tau \): riskless interest rate, \( \sigma_N \): normal systematic volatility, \( \tilde{\sigma} \): dividend yield, \( \tau \): time to maturity, \( \lambda \): frequency of default event.
3.2.1. Delta

The incentive effect of the employee stock option is the change in value as perceived by the employee relative to the change in stock value; therefore, subjective delta is the proper index to measure the incentive effect.

\[
\text{Market delta (}\delta\text{)} = e^{-d_s \tau} N(d_1), \\
\text{Subjective delta (}\tilde{\delta}\text{)} = e^{-d_s \tau} N(\tilde{d}_1).
\]

(13)

Since \(N(\tilde{d}_1) < N(d_1)\) and \(\tilde{d}_s > d_s\), the subjective delta is smaller than the market delta. That is, the market delta overestimates the true incentive provided by the employee stock option, and the incentive effect as perceived by employees is smaller than the market incentive effect: the more risk averse the employee (smaller \(\gamma\)) and more illiquid the position (larger \(\alpha^J\)), the less the incentive effect. Furthermore, the subjective incentive effect decreases as time-to-maturity increases. The difference between the market and subjective deltas can be viewed as the gap between the employer’s cost of incentivizing and the employee’s realized incentive. As the gap widens, employers are incurring increasingly large costs to provide decreasingly valued incentives.

3.2.2. Gamma

The variation in incentive effect, delta, per unit change in stock price is defined as gamma. The usual intuition holds: if gamma is small, it is not necessary for the company to adjust its compensation plan frequently to maintain the incentive effect.

\[
\text{Market gamma (}\Gamma\text{)} = \frac{e^{-d_s \tau} \sigma N'(d_1)}{S \sigma \sqrt{\tau}}, \\
\text{Subjective gamma (}\tilde{\Gamma}\text{)} = \frac{e^{-d_s \tau} \sigma N'(\tilde{d}_1)}{S \sigma \sqrt{\tau}}.
\]

(14)

Interestingly, the subjective gamma is smaller than the market gamma: the more risk averse the employee (smaller \(\gamma\)) and the more illiquid the position (larger \(\alpha^J\)), the smaller the variation in incentive effects. As such, firms need to update incentive plans less often they would otherwise expect. This, however, is in part a result of the fact that the incentive effect (delta) is smaller subjectively than it is in the market to begin with. That then results in a lower convexity as well. See Table 2 Panels A and B for details on delta and gamma, respectively.

3.2.3. Vega

It is a well-known and oft-studied intuition that, while options may provide incentives for employees to work harder, it can also induce suboptimal risk-taking behavior. The investigation of subjective vegas, or sensitivities with respect to risk levels, adds to this discussion. Here,
it is illuminating to discuss these effects with regard to total risk as well as to systematic and nonsystematic risk separately.

With respect to total risk, consider the total variance of the stock price:

$$\text{Var}(\frac{dS}{S}) \equiv \sigma_s^2 = \nu_s^2 + \lambda = \sigma_N^2 + \lambda.$$  

Holding unsystematic risk constant, the market total-risk vega and subjective total-risk vega are

$$\text{Market total-risk vega (}\Lambda_{\sigma_s|\nu_s}, \sqrt{\lambda}) = S \frac{\sigma_s}{\sigma_N} \sqrt{\tau} e^{-d_s \tau} N'(d_1) > 0,$$

$$\text{Subjective total-risk vega (}\tilde{\Lambda}_{\sigma_s|\nu_s}, \sqrt{\lambda}) = S \frac{\sigma_s}{\sigma_N} \sqrt{\tau} e^{-\tilde{d}_s \tau} N'(\tilde{d}_1) > 0.$$  \hspace{1cm} (15)

Both are positive as is consistent with the general options pricing result that value should increase with risk. However, the subjective total-risk vega is smaller than the market total-risk vega: the more risk averse the employee (smaller \(\gamma\)) and the more illiquid the position (larger \(\alpha\)), the smaller the increase in option value from risk-taking behavior. Hence, employees are less likely to engage in value-destroying risk-taking behavior given their subjective views than they would otherwise be taking. Investigation of the systematic component alone yields a similar finding as follows:

$$\text{Market systematic risk vega (}\Lambda_{\beta|\nu_s}, \sqrt{\lambda}) = \Lambda_{\sigma_s|\nu_s}, \sqrt{\lambda} \frac{\beta \sigma_s^2}{\sigma_N} = S \sqrt{\tau} e^{-d_s \tau} N'(d_1) \frac{\beta \sigma_s^2}{\sigma_N} > 0,$$

$$\text{Subjective systematic risk vega (}\tilde{\Lambda}_{\beta|\nu_s}, \sqrt{\lambda}) = \tilde{\Lambda}_{\sigma_s|\nu_s}, \sqrt{\lambda} \frac{\beta \sigma_s^2}{\sigma_N} = S \sqrt{\tau} e^{-\tilde{d}_s \tau} N'\tilde{d}_1) \frac{\beta \sigma_s^2}{\sigma_N} > 0.$$  \hspace{1cm} (16)

However, when looking at unsystematic risk, one must separate the Normal risk measure from the default risk measure. Looking first at the Normal component, holding default risk and total risk constant, Normal nonsystematic risk has no influence on the market value of the option but has a negative influence on subjective value.

$$\tilde{\Lambda}_{\nu_s|\sigma_s}, \sqrt{\lambda} = 0$$

$$\tilde{\Lambda}_{\nu_s|\sigma_s}, \sqrt{\lambda} = \frac{\partial F}{\partial \nu_s} \frac{\partial \nu_s}{\partial \nu_s} + \frac{\partial F}{\partial \tilde{d}_s} \frac{\partial \tilde{d}_s}{\partial \nu_s}$$

$$= -2\tau(1 - \gamma)\alpha^J \nu_s[\alpha^J Ke^{-\tilde{\tau} \tau} N(\tilde{d}_2) + (1 - \alpha^J)Se^{-d_s \tau} N(d_1)] < 0.$$  \hspace{1cm} (17)

The first of these conclusions is simple since nonsystematic risk can, by definition, be diversified away. However, the latter drives at the very heart of the conclusions made in this paper: because employees cannot diversify away some portion of the nonsystematic risk, option value will tend to be eroded when idiosyncratic risk increases.

On the contrary, holding Normal nonsystematic risk and total risk constant, increasing default risk will increase market values while its effect on subjective values is indefinite. On
one hand, increasing default risk erodes value for the same reason as increasing nonsystematic risk: the illiquidity of the position makes it impossible to diversify away the excess risk. On the other, increasing default risk increases the total risk of the stock and hence increases value for the aforementioned reason:

\[
\Lambda_{\sqrt{\lambda}(\sigma_s, \nu_s)} = 2\sqrt{\lambda} Ke^{-(r+\lambda)\tau} N(d_2) > 0,
\]

\[
\tilde{\Lambda}_{\sqrt{\lambda}(\sigma_s, \nu_s)} = \frac{\partial \tilde{F}}{\partial \tilde{r}} \frac{\partial \tilde{d}}{\partial \tilde{d}} + \frac{\partial \tilde{F}}{\partial \tilde{d}^2} \frac{\partial \tilde{d}}{\partial \tilde{d}} = 2\sqrt{\lambda}Ke^{-\tilde{r}\tau} N(\tilde{d}_2) - \tau(1-\gamma)\alpha J \sqrt{\lambda}(2-\alpha J \gamma) \tilde{F}(S, \tau).
\]

As before, the subjective vega is smaller than the market vega. Note also that the impact of default risk is stronger than that of Normal unsystematic risk when determining subjective value except for in cases of small risk aversion and high liquidity. See Table 3 Panels A, B, and C for details regarding total-risk vega, Normal-unsystematic-risk vega, and default-risk vega, respectively.

In summary, investigation of sensitivities finds that subjective incentive levels are lower those that implied by the market. Because of that, subjective values are less convex than market values and are less sensitive to changes in total risk. As idiosyncratic risk increases, subjective value decreases where as market value does not owing to the illiquidity of the employee’s holdings, and increases in default risk may either increase or decrease subjective value.

4. The Role of Employee Over-confidence

Our study has so far shown that subjective value should be substantially lower than market value for usual parameter calibrations. However, the fact that employees continue to accept and indeed sometimes prefer ESO compensation implies that some other forces may be at work. We contend here that employee over-confidence may be that force. Simply, employees, believing themselves to be insiders, over-estimate the return of the firm and believe the firm to be under-valued. As such, they overvalue their ESOs (undervalue differential strike price). Making a simple amendment to our pricing results, we seek here to discover what over-confidence level \( s \) would be necessary to equate subjective value to market value. How much higher do employees need to believe the expected rate of return to be such that their resulting subjective valuations would be equal to the valuation of the market? How much extra return is required to offset the illiquidity and risk aversion factors that would otherwise make subjective value lower? Define
this question as follows:
\[
\begin{align*}
\frac{dS}{S} &= ((\mu_s + s) - d_s + \lambda) dt + \nu_m d\omega_m + \nu_s d\omega_s - dq, \\
\frac{dM}{M} &= ((\mu_m - d_m) dt + \sigma_m d\omega_m, \\
\frac{dB}{B} &= \mu_d dt.
\end{align*}
\]

Here, parameter \(s\) corresponds to the return adjustment that must exist in order to equate the market value with the subjective value under the real return process. Drawing from the same solution methodology as before, the PDE for subjective value should be adjusted to:

\[
\frac{1}{2} F_{SS} S^2 \sigma_N^2 + (\tilde{r} - \tilde{d}^* S F_{S} - \tilde{r} F + F_t = 0.
\]

Therefore, the subjective value from the employee’s perspective is (See Appendix E):

\[
\tilde{F}^s(S, \tau) = C(SE^{-\tilde{d}^* _\tau}, \tau; K, \sigma_N^2, \tilde{r}^s) = SE^{-\tilde{d}^* _1} N(\tilde{d}^*_1) - KE^{-\tilde{r}^*_\tau} N(\tilde{d}^*_2),
\]

where \(\tilde{r}^s = \mu_b + \lambda - (1 - \gamma)[(\alpha J)^2 \nu_s^2 + \frac{1}{2}(\alpha J)^2 \gamma \lambda - \alpha J^1 s = \tilde{r} + \alpha J s\) is the subjective interest rate, and the subjective dividend rate is \(\tilde{d}^s = d_s + (1 - \gamma)[(\alpha J)(1 - \alpha J)^2 \nu_s^2 - \frac{1}{2}(\alpha J)^2 \gamma \lambda + \alpha J^1] - s(1 - \alpha J) = \tilde{d}_s - s(1 - \alpha J),\) where \(\tilde{d}^*_1 = \frac{lnS}{K} + (\mu_b + \lambda - (d_s - s) - (1 - \gamma)\alpha J^1 \nu_s^2 + \sigma_N^2 \tau}{\sigma_N \sqrt{\tau}} < d_1,\) and \(\tilde{d}^*_2 = \tilde{d}^*_1 - \sigma_N \sqrt{\tau}.\) It is important to note here that the sentiment level \(s\) is treated as a CAPM alpha. That is, employees perceive their information to be insider information and not accounted for by market risk in the traditional CAPM sense.

Table 4 highlights our findings: the over-confidence levels necessary are substantial. If \(\alpha\) is 10% and \(\gamma = -1,\) the sentiment parameter is around 5%. Presuming that ordinary stocks yield returns of 10% on average, this would imply a necessary return of 15% or a multiplier of 1.5. In other words, in order for over-confidence to account for the gap between market and subjective values, employees must significantly over-estimate the capital gains of the company. As we can see, the discount due to under-diversification experienced by a liquidity-constrained, risk-averse agent can be offset by over-confidence. Intuitively, a person’s unwillingness to take on risk can be reduced if that person is sufficiently optimistic about the stock. In a sense, subjective beliefs can result in virtually any valuation depending upon the balance between risk preference and optimism regarding firm characteristics.

5. Case Example and Model Extensions

5.1 UMC Example

\[\tilde{F}_S = \frac{\partial^2 F}{\partial S^2}, \quad F_{SS} = \frac{\partial^2 F}{\partial S^2}, \quad F_t = \frac{\partial F}{\partial t}.\]
Having noted important discrepancies in incentive capacity and valuation when taking into account subjective values, it is also important not to ignore the economic and regulatory importance of the ESO premium. How ESOs are accounted for in financial statements and by firms can substantially impact company earnings and tax shields. The following example is a case study of just a single representative firm from Taiwan. Though the impact of such a discussion will differ for different companies, the qualitative findings are informative.

United Microelectronics Corporation (UMC) granted 1 billion stock options to its employees at an exercise price of NT$20 in 2002, which equaled the stock’s closing price on the grant date. Details are as follows:

<table>
<thead>
<tr>
<th>ESOs of United Microelectronics Corporation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(TAIEX: 2303, NYSE: UMC)</td>
</tr>
<tr>
<td>Shares of options</td>
</tr>
<tr>
<td>Life of options</td>
</tr>
<tr>
<td>Vesting period</td>
</tr>
<tr>
<td>Stock price</td>
</tr>
<tr>
<td>Exercise price</td>
</tr>
<tr>
<td>Riskless rate</td>
</tr>
<tr>
<td>Cash dividend</td>
</tr>
<tr>
<td>Number of shares</td>
</tr>
<tr>
<td>Dividend yield</td>
</tr>
<tr>
<td>Expected exit rate</td>
</tr>
</tbody>
</table>

Suppose that $\lambda = 0.01$ (implying an expected firm life of 100 years), normal volatility $\sigma_N$ is 30%, normal unsystematic volatility $\nu$ is 20%, relative risk aversion $(1 - \gamma)$ is 2, and illiquidity parameter $\alpha_J$ is 0.25. Note here that 13 directors and managers at UMC held 1,493,883,000 shares for a per person value of US$76.6 million—$\alpha_J$ might certainly be even higher than .25. Based on our pricing model, the market and the subjective values are 6.57 and 5.03 respectively. That is, the subjective value is only 76.63% of the market value. Applying the FAS 123 adjustment factor, the market and the subjective values are $6.57 \times (1 - 8\%)^4 = 4.70$ and $5.03 \times (1 - 8\%)^4 = 3.60$ so the total compensation costs are NT$4,700,000,000 and NT$3,600,000,000, respectively. Straight-line depreciation would imply annual expenses of $\frac{NT$4,700,000,000}{6} = NT$783,333,333 and $\frac{NT$3,600,000,000}{6} = 14$
NT$600,000,000. The annual difference in earnings depending on which form of valuation is used would be NT$183,333,333. Given a corporate tax rate of 25%, the annual tax shield is NT$45,833,333 which is nearly a third of the entire dividend paid in the preceding year and would attribute an additional 2.8 cents per share of earnings over the next 6 years. These are not economically insignificant in light of the fact that this issuance represents only a 6% or so dilution of shares outstanding.

5.2 Model Extensions

5.2.1. Recovery rate exists when the default event occurs

We assumed here that, if the default event occurs, the recovery value is 0. However, a general model might admit various recovery proportions. The modified model would be generalized:

\[
\begin{align*}
\frac{dS}{S} &= (\mu_s - \lambda_s k_s)dt + \nu_m d\omega_m + \nu_s d\omega_s + (Y_s - 1)dq_s, \\
\frac{dM}{M} &= \mu_md\omega_m, \\
\frac{dB}{B} &= \mu_b dt,
\end{align*}
\]

where the jump size, \( Y_s \), is Normal and independent of jump frequency, i.e., \( \ln Y_s \sim N(\hat{\mu}_s, \hat{\sigma}_s^2) \). Parameter \( k_s \) is the expected percentage change in stock price if the default event occurs and equals \( E[Y_s - 1] \).

The expected returns and covariance matrix of the two risky assets are

\[
\mu = \begin{pmatrix} \mu_s \\ \mu_m \end{pmatrix}, \quad \Omega = \begin{pmatrix} \nu_m^2 + \nu_s^2 + \lambda_s E[(Y_s - 1)^2] & \nu_m \sigma_m \\ \nu_m \sigma_m & \sigma_m^2 \end{pmatrix}.
\]

The same methodology as presented in this paper could be repeated to derive market and subjective values under this new specification.

5.2.2. Economic Recession

When a financial market suffers a recession, multiple defaults may occur together. In this case, the market as a whole may experience a jump process. The modified model is generalized:

\[
\begin{align*}
\frac{dS}{S} &= (\mu_s - \lambda_m k'_m - \lambda_s k_s)dt + \nu_m d\omega_m + \nu_s d\omega_s + (Y'_m - 1)dq_m + (Y_s - 1)dq_s, \\
\frac{dM}{M} &= (\mu_m - \lambda_m k_m)dt + \sigma_m d\omega_m + (Y_m - 1)dq_m, \\
\frac{dB}{B} &= \mu_b dt,
\end{align*}
\]

where the jump process \( dq_m \) captures the default risk of the market portfolio and follows a Poisson distribution with the average frequency \( \lambda_m \) for \( dt \) : \( dq_m \sim P(\lambda_m dt) \), would directly affect the return of market portfolio and indirectly affect the return of company stock. \( Y_m \)
and $Y'_m$ are jump sizes for the market portfolio and company stock, respectively, and are log-
normally distributed, i.e., $\ln Y_m \sim N(\hat{\mu}_m, \hat{\sigma}_m^2)$, and $\ln Y'_m \sim N(\hat{\mu}'_m, \hat{\sigma}'_m^2)$. $d\omega_m$, $d\omega_s$, $dq_s$ and $dq_m$ are assumed to be independent. The jump size of the stock $Y_s$ also follows a log-normal
distribution $\ln Y_s \sim N(\hat{\mu}_s, \hat{\sigma}_s^2)$. Jump sizes $Y_s, Y_m, Y'_m$, and jump frequencies $\lambda_s, \lambda_m$ are also independent. Finally, $k_s = E[Y_s - 1]$, $k_m = E[Y_m - 1]$, $k'_m = E[Y'_m - 1]$.

The expected returns and covariance matrix of the two risky assets are

$$
\mu = \begin{pmatrix}
\mu_s \\
\mu_m \\
\end{pmatrix} = \begin{pmatrix}
\mu_b + \beta(\mu_m - r) \\
\mu_m \\
\end{pmatrix},
\Omega = \begin{pmatrix}
\nu^2_m + \nu^2_s + \lambda_s E[(Y_s - 1)^2] + \lambda_m E[(Y_m - 1)^2] & \nu_m \sigma_m + \lambda_m k_m k'_m \\
\nu_m \sigma_m + \lambda_m k_m k'_m & \sigma^2_m + \lambda_m E[(Y_m - 1)^2]
\end{pmatrix}.
$$

Again, the same methodology as presented in this paper could be repeated to derive market
and subjective values under this new specification.

6. Conclusion

In the knowledge-based economy, the most important factor in determining enterprise suc-
cess is talent. Enterprises and employees may seek to share a joint perspective on shared future
benefits through an employee stock option plan. Indeed, in 2000, the Singaporean government
announced an entrepreneurial employee stock option scheme for all start-up companies, calling
for the exclusion from taxable income of up to 50% of gains realized on stock options granted
on or after June 1, 2000. In Taiwan, small and medium-sized enterprises accounted for 97.83%
of all businesses in 2003. Small and medium-sized enterprises can often not attract or retain
talent based on salary compensation alone; however, clever application of ESOs provide a real-
izable future capital gain possibility to employees that they may find attractive. The question
is: at what cost to the firm? Our paper seeks to illuminate this issue.

Since the default risk faced by small and medium-sized enterprises might be considered
higher than that of larger ones, our model employs a jump-diffusion methodology. The model’s
findings resemble the standard Black-Scholes model but with an adjusted rate of return and
dividend yield result. We solve for both the market value and subjective value of the ESO,
further investigate the usual Greeks, and parameterize an over-confidence parameter that we
posit may explain the ESO premium. We find the following:

1. Subjective value is substantially lower than market value for reasonable parameter cali-
brations – the cost to the issuing firm is significantly larger than the value received/perceived
by employees.

2. A study of delta finds that the incentive effect experienced by employees is significantly smaller than that intended by employers. Interestingly, we further find that, once an incentive plan is in place, it need not be updated or reviewed as often as expected because employees will perceive less change in value than employers.

3. Whereas increase risk introduced by default serves to increase market value, its effect on subjective value is indeterminant—it may serve to decrease value as the costs of illiquidity outweigh the benefits of increase volatility.

4. A substantial level of over-confidence must exist in order to make up for the ESO premium. At normal calibrations, employees would need to expect profitability levels 1.5 times that which the market expects in order to be compensated for the ESO premium.

We conclude that for normal levels of risk aversion and especially in the case of small firms that face substantial default risk, ESOs are worth significantly less to employees than they cost owners to issue. They do not offer the incentive levels implied by the usual options pricing literature and hence may not be an optimal financing/compensation choice. Only companies where default risk is low, employees are not risk averse, employees are strongly over-confident, and where ESOs represent a small portion of total compensation should ESOs be employed. From the regulatory perspective, market value methods will tend to overvalue ESOs and hence provide better tax shields but weaker earnings. Enterprises and regulatory bodies alike should take this into account.

Though our model intuition follows that of the standard ESO contract, our methodology and results are expandable to a more generalized class of illiquid securities. For any securities for which owners are likely to be liquidity-constrained, this model would hold. Pre-IPO holders of stock who face a moratorium on selling rights would fall into this class. Principals at firms for whom de facto constraints might be binding for signaling reasons would likewise be appropriately subsumed here. As with ESOs, our model would derive quantifiable implications for option discounts in those contexts.

Furthermore, our model is the first we know of that specifically and quantifiably addresses the offsetting effects of risk aversion and over-confidence. Holding liquidity constant, we can find the discount present due to under-diversification given different levels of risk aversion. Simultaneously, our model can then derive what levels of over-confidence as parameterized by
risk-adjusted return levels would be necessary to offset this discount. Intuitively, it is clear that over-confidence, that is the feeling that a stock is better than it is likely to be in equilibrium, might offset risk aversion, that is, one's unwillingness to take on risk. Our model is able to quantify these effects and can attribute an empirically testable over-confidence level that could confirm accepted estimations of risk-aversion. This is left a topic for further research. This direct and testable link is the first the authors know of in this field.
Appendix A

Consider the following solution:

$$\mu = \begin{pmatrix} E(\frac{dS}{S})/dt \\ E(\frac{dM}{M})/dt \end{pmatrix} = \begin{pmatrix} \mu_s \\ \mu_m \end{pmatrix} = \begin{pmatrix} \mu_b + \beta(\mu_m - \mu_b) \\ \mu_m \end{pmatrix},$$

where $\mu_s - \mu_b = \beta(\mu_m - \mu_b)$ according to the assumption of CAPM. By using the fact that the quadratic variation $[t, q] = 0$, we have $dt \cdot dq = d[t, q] = 0$. Since $q$ is a Poisson process, the following holds:

$$dq \approx \begin{cases} 1, & \text{with probability } \lambda dt, \\ 0, & \text{with probability } 1 - \lambda dt. \end{cases}$$

This implies that $(dq)^2 \approx dq$.

To get the covariance matrix, we first derive

$$Var(\frac{dS}{S}) = E[(\frac{dS}{S})^2] - (\mu_s dt)^2 \approx E[(\frac{dS}{S})^2] \approx (\nu_m^2 + \nu_s^2 + \lambda)dt,$$

$$Var(\frac{dM}{M}) = E[(\frac{dM}{M})^2] - (\mu_m dt)^2 \approx E[(\frac{dM}{M})^2] \approx \sigma_m^2 dt,$$

$$Cov(\frac{dS}{S}, \frac{dM}{M}) = E[(\frac{dS}{S})(\frac{dM}{M})] - \mu_s \mu_m (dt)^2 \approx E[(\frac{dS}{S})(\frac{dM}{M})] \approx \nu_m \sigma_m dt.$$

By making use the assumption of CAPM, we have

$$\beta = \frac{Cov(\frac{dS}{S}, \frac{dM}{M})}{Var(\frac{dM}{M})} = \frac{\nu_m}{\sigma_m}.$$

Therefore,

$$\Omega = \begin{pmatrix} Var(\frac{dS}{S})/dt & Cov(\frac{dS}{S}, \frac{dM}{M})/dt \\ Cov(\frac{dS}{S}, \frac{dM}{M})/dt & Var(\frac{dM}{M})/dt \end{pmatrix} = \begin{pmatrix} \beta^2 \sigma_m^2 + \nu_s^2 + \lambda & \beta \sigma_m^2 \\ \beta \sigma_m^2 & \sigma_m^2 \end{pmatrix}.$$
Appendix B

Consider the following wealth process at terminal time $T$

\[
W(T) = \left( w_s \frac{S(T) - S(t)}{S(t)} + w_m \frac{M(T) - M(t)}{M(t)} + w_b \frac{B(T) - B(t)}{B(t)} \right) (W(t) - C(t)(T_2 - t))
\]

where $\Delta S(t) = S(T_2) - S(t)$, $\Delta M(t) = M(T_2) - M(t)$ and $\Delta B(t) = B(T_2) - B(t)$. Then

\[
W(T_2) - W(t) = \left( w_s \frac{\Delta S(t)}{S(t)} + w_m \frac{\Delta M(t)}{M(t)} + (1 - w_s - w_m) \frac{\Delta B(t)}{B(t)} \right) (W(t) - C(t)(T_2 - t)) - C(t)(T_2 - t),
\]

\[
\approx \left( w_s^2 \frac{\Delta S(t)}{S(t)} + w_m^2 \frac{\Delta M(t)}{M(t)} + 2w_s w_m \frac{\Delta S(t)}{S(t)} \frac{\Delta M(t)}{S(t)} \right) W^2(t), \quad \text{as } \Delta t = T_2 - t \to 0.
\]

Taking expectation on the both side of the above equation to get

\[
E_t[W(T_2) - W(t)] \approx [w_s \mu_s + w_m \mu_m + (1 - w_s - w_m) \mu_b] W(t) \Delta t - C(t) \Delta t
\]

\[
\approx \left[ \mu_b + w' \left( \mu - \mu_b I \right) \right] W(t) - C(t) \Delta t,
\]

\[
E_t[W(T_2) - W(t)]^2 \approx [w_s^2 (\nu_s^2 + \nu_m^2 + \lambda) + w_m^2 \sigma_m^2 + 2w_s w_m \nu_m \sigma_m] W^2(t) \Delta t
\]

\[
= w \Omega w' W^2(t) \Delta t,
\]

where $E_t$ denotes the expectation at time $t$, $'$ denotes transpose, and

\[
w = \begin{pmatrix} w_s \\ w_m \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_s \\ \mu_m \end{pmatrix}, \quad \text{and } \Omega = \begin{pmatrix} \beta \sigma_m^2 + \nu_s^2 & \beta \sigma_m^2 \\ \beta \sigma_m^2 & \sigma_m^2 \end{pmatrix}.
\]

By using an argument similar to that in Merton (1969), we have

\[
J[W(t), t] = \max_{\{C, w_s, w_m, w_b\}} \left( \int_t^{T_2} e^{-\rho s} U(C(s)) ds + B[W(T_2), T_2] \right)
\]

\[
= \max_{\{C, w_s, w_m, w_b\}} \left( \int_t^{T_2} e^{-\rho s} U(C(s)) ds + J[W(T_2), T_2] \right)
\]

\[
= \max_{\{C, w_s, w_m, w_b\}} \left( e^{-\rho t_0} U(C(t_0))(T_2 - t) + J[W(T_2), T_2] \right),
\]

where $t_0 \in [t, T_2]$ is by the Mean Value Theorem. Next, we apply Taylor’s Expansion, as
\[ \Delta t = T_2 - t \to 0, \text{ in } (20) \text{ to have} \]

\[
J[W(t), t] \\
\approx \max_{\{C, w_s, w_m, w_b\}} \mathbb{E}_t \left\{ e^{-\rho t} U(C(t_0))(T_2 - t) + J[W(t), t] + J_t[W(t), t](T_2 - t) + J_W[W(t), t] \right. \\
\quad \left. (W(T_2) - W(t)) + \frac{1}{2} J_W W(t, t)(W(T_2) - W(t))^2 + O((T_2 - t)^2) \right\}
\]

\[
\approx \max_{\{C, w_s, w_m, w_b\}} \left\{ e^{-\rho t} U(C(t_0)) \Delta t + J[W(t), t] + J_t[W(t), t] \Delta t + ([r + w'(\mu - r I)]W(t) \\
- C(t))J_W[W(t), t] \Delta t + \frac{1}{2} J_W W(t, t)(W(T_2) - W(t))^2 + O((\Delta t)^2) \right\}.
\]

The employee’s total utility can be further simplified by

\[
0 = \max_{\{c, w_s, w_m, w_b\}} \left\{ e^{-\rho t} U(C(t)) + \frac{1}{2} w' \Omega w W^2 J_W W + ([\mu_b + w'(\mu - \mu_b I)]W(t) - C(t))J_W + J_t \right\}
\]

\[
eq \max_{\{c, w_s, w_m, w_b\}} \phi, \quad (21)
\]

The trivial solution \( J[W(t), t] = b(t)e^{-\rho t W^2} \) satisfies equation (21), and the first order condition (3). Therefore, the optimal portfolio selection and consumption rules can be rewritten as

\[
\left\{ \begin{array}{l}
\max_{\{C, w_s, w_m, w_b\}} \phi = 0, \\
\text{s.t. } J(W(T), T) = b(T)e^{-\rho T W^2},
\end{array} \right.
\]

where

\[
\phi = \frac{1}{\gamma} e^{-\rho t} C^\gamma + \frac{1}{2} \left\{ (\beta^2 \sigma^2_m + \nu^2_s + \lambda) w^2_s + 2 \beta \sigma^2_m w_s w_m + \sigma^2_m w^2_m \right\} W^2 J_W W \\
+ \left\{ [\mu_b + w_s(\mu_a - \mu_b) + w_m(\mu_m - r)]W - C \right\} J_W + J_t.
\]

In the period \([T_1, T_2]\), we have

\[
\left\{ \begin{array}{l}
\max_{\{C, w_s, w_m, w_b\}} \phi = 0, \\
\text{s.t. } J(W(T_2), T_2) = b(T_2)e^{-\rho T_2 W^2},
\end{array} \right.
\]

From the first order condition, we get

\[
\phi_c = \frac{\partial \phi}{\partial c} = 0, \quad C^* = (e^{\rho t} J_W)^{1-\gamma}, \\
\phi_{w_m} = \frac{\partial \phi}{\partial w_m} = 0, \quad w^*_m = -\frac{\mu_m - \mu_b}{\sigma^2_m W} J_W W - \frac{\mu_m w^*_s}{\sigma^2_m W}, \\
\phi_{w_s} = \frac{\partial \phi}{\partial w_s} = 0, \quad w^*_s = -\frac{\beta (\mu_m - r)}{\beta \sigma^2_s W} J_W W - \frac{\beta^2 \sigma^2_s + \nu^2_s + \lambda}{\beta \sigma^2_s W} w^*_s,
\]

where the above equation is from the assumption CAPM. Then

\[
w^*_s = 0, \text{ and } w^*_m = -\frac{\mu_m - \mu_b}{\sigma^2_m W} \frac{J_W}{J_W W}.
\]
Plugging in $\phi(C^*, w^*_s, w^*_m) = 0$ in (22), we get

$$1 - \gamma (e^{\rho t}) \gamma \frac{1}{\gamma - 1} J^\gamma W^{\gamma - 1} - \frac{\mu_m - \mu_b}{2\sigma^2_m} J^\gamma W_{WW} + \mu_b W J_W + J_t = 0.$$  
(23)

Hence, the optimal conditions can be written as

$$\begin{align*}
1 - \gamma (e^{\rho t}) \gamma \frac{1}{\gamma - 1} J^\gamma W^{\gamma - 1} - \frac{\mu_m - \mu_b}{2\sigma^2_m} J^\gamma W_{WW} + r W J_W + J_t &= 0, \\
C^* &= (e^{\rho t} J_W) \gamma \frac{1}{\gamma - 1}, \\
w^*_s &= 0, \\
w^*_m &= -\frac{\mu_m - \mu_b}{J^\gamma W_{WW}} J^\gamma W_{WW}, \\
\text{subject to } J[W(T_2), T_2] &= \varepsilon^{1 - \gamma} e^{-\rho T_2} \frac{W(T_2)}{\gamma}, \quad \text{for } 0 < \varepsilon \ll 1.
\end{align*}$$

We take a trial solution, $J[W(t), t] = b^J(t) e^{-\rho t} \frac{W^\gamma}{\gamma}$. By substitution, we could find that $b^J(t)$ must satisfy the following ordinary differential equation:

$$b^J'(t) = b^J(t) [\rho - \gamma (\mu_b + \frac{(\mu_m - \mu_b)^2}{2(1 - \gamma) \sigma_m^2})] - (1 - \gamma) [b^J(t)]^{\frac{-\gamma}{1 - \gamma}},$$

to which we find the solution

$$b^J(t) = \left\{ 1 + (A^J[b^J(T_2)] \frac{1}{\gamma} - 1) e^{A^J(t-T_2)} \right\}^{1 - \gamma},$$

where $A^J = \frac{\gamma}{1 - \gamma} \rho - \mu_b - \frac{(\mu_m - \mu_b)^2}{2(1 - \gamma) \sigma_m^2}$.

In the period $[0, T_1]$, we have

$$\max \{ C, w_s, w_m, w_b \} \phi = 0,$$

$$\text{s.t. } J(W(T_1), T_1) = b^J(T_1) e^{-\rho T_1} \frac{W^\gamma}{\gamma}.$$  

Under the restriction $w^*_s = \alpha^J$, the optimal consumption and investment portfolio can be derived

$$\tilde{b}^J(t) = \tilde{b}^J(t) [\rho - \gamma (\mu_b + \frac{(\mu_m - \mu_b)^2}{2(1 - \gamma) \sigma_m^2}) - \frac{1}{2}(1 - \gamma)(\nu^2 + \lambda) (\alpha^J)^2] - (1 - \gamma) [\tilde{b}^J(t)]^{\frac{-\gamma}{1 - \gamma}},$$


to which we find the solution

$$\tilde{b}^J = \left\{ 1 + (\tilde{A}^J[b^J(T_1)] \frac{1}{\gamma} - 1) e^{\tilde{A}^J(t-T_1)} \right\}^{1 - \gamma},$$

where $\tilde{A}^J = \frac{\gamma}{1 - \gamma} \rho - \mu_b - \frac{(\mu_m - \mu_b)^2}{2(1 - \gamma) \sigma_m^2} + \frac{1}{2}(1 - \gamma)(\nu^2 + \lambda) (\alpha^J)^2]$. 

22
Appendix C

The employee’s wealth process is defined as
\[ dW = \left( w_s \frac{dS}{S} + w_m \frac{dM}{M} + w_b \frac{dB}{B} \right) \left( W(t-) - C(t-) dt \right) - C(t-) dt, \]
where \( C \) is the consumption process. Then the evolution of the wealth process is
\[ \frac{dW}{W} = [\mu_b + w_s (\mu_s + \lambda - \mu_b) + w_m (\mu_m - \mu_b) - [b(t)]^{-1}] dt \]
\[ + (w_s v_m + w_m \sigma_m) d\omega_m + w_s \nu d\omega_s - w_s dq, \]
where
\[ b(t) = \begin{cases} b'(t), & T_1 \leq t < T_2, \\
\tilde{b}'(t), & t \leq T_1, \end{cases} \]
with \( b'(t) \) and \( \tilde{b}'(t) \) are defined in (4) and (5), respectively.

By using the same idea as pages 99 and 112 of Gukhal (2001), Under the risk-neutral probability measure \( Q \), the evolution of stock price can be written as
\[ dS_t = (\mu_b - \delta - \lambda k) S_t dt + \sigma S_t \tilde{\omega}_t + (Y - 1) S_t dq_t, \]
where \( \tilde{\omega} \) is the standard Brownian motion under \( Q \). Let \( Z_t = Z(S_t, t) \), then by Ito’s formula, we have
\[ Z_T = Z_0 + \int_0^T \frac{\partial Z(S_t, t)}{\partial S} [(\mu_b - \lambda k) S_t dt + \sigma S_t dq_t] \]
\[ + \int_0^T \left[ \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 Z(S_t, t)}{\partial S^2} + \frac{\partial Z(S_t, t)}{\partial t} \right] dt + \int_0^T [Z(Y_t S_{t-}, t) - Z(S_{t-}, t)] dq_t. \]
Write (24) in differential form to get
\[ dZ(S_t, t) \]
\[ = \frac{\partial Z(S_t, t)}{\partial S} [(\mu_b - \delta - \lambda k) S_t dt + \sigma S_t \tilde{\omega}_t] + \left[ \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 Z(S_t, t)}{\partial S^2} + \frac{\partial Z(S_t, t)}{\partial t} \right] dt \]
\[ + [Z(Y_t S_{t-}, t) - Z(S_{t-}, t)] dq_t \]
\[ = \frac{\partial Z(S_t, t)}{\partial S} (dS)^c + \frac{\partial Z(S_t, t)}{\partial t} dt + \frac{1}{2} \frac{\partial^2 Z(S_t, t)}{\partial S^2} [(dS)^c]^2 + [Z(Y_t S_{t-}, t) - Z(S_{t-}, t)] dq_t, \]
where \((dS)^c\) is the continuous part of \( dS \), and \((dS)^c = (\mu_b - \delta - \lambda k) S_t dt + \sigma S_t \tilde{\omega}_t \).
Therefore

\[ dJ_W[W(t), t] = J_WW(dW)^c + J_W^t dt + \frac{1}{2} J_WWW[(dW)^c]^2 + \{ J_W[(1 - w_s)W(t-), t] - J_W[W(t-), t-] \} dq \]

\[ = b(t)e^{-\rho t}W^{\gamma - 1}\left\{ (\gamma - 1)(\mu_b + w_m\mu_s + \lambda - \mu_b) + w_m(\mu_m - \mu_b) - [b(t)]^{-\frac{1}{\gamma - 1}} \right. \\
\left. + \frac{1}{2}(\gamma - 2)[(w_m\nu_m + w_m\sigma_m)^2 + w_s^2\nu_s^2] + \frac{b'(t)}{b(t)} - \rho \right\} dt \\
\left. + (\gamma - 1)(w_s\nu_m + w_m\sigma_m)dw_m + (\gamma - 1)w_s\nu_sd\omega_s + [(1 - w_s)^{\gamma - 1} - 1] dq \right\}, \]

where

\[ J_W = \frac{\partial J[W(t), t]}{\partial W(t)} , \quad J_WW = \frac{\partial^2 J[W(t), t]}{\partial W(t)^2} , \quad J_W^t = \frac{\partial^2 J[W(t), t]}{\partial W(t)\partial t} , \quad J_WWW = \frac{\partial^3 J[W(t), t]}{\partial W(t)^3} . \]

The continuous part of \( dW \), \( (dW)^c \), is

\[ (dW)^c = \left( r + w_s(\mu_s + \lambda - \mu_b) + w_m(\mu_m - \mu_b) - [b(t)]^{-\frac{1}{\gamma - 1}} \right) W dt \\
+ (w_s\nu_m + w_m\sigma_m) W d\omega_m + w_s\nu_s W d\omega_s . \]

Then

\[ \frac{b'(t)}{b(t)} - \rho = \begin{cases} \\
(1 - \gamma)\{\mu_b + \frac{(\mu_m - \mu_b)^2}{2(1 - \gamma)\sigma_m^2} + \frac{1}{2}\gamma(\nu_s^2 + \lambda)(\alpha^J)^2 - [b^J(t)]^{-\frac{1}{\gamma - 1}} \} - \mu_b - \frac{(\mu_m - \mu_b)^2}{2(1 - \gamma)\sigma_m^2} & t \leq T_1 , \\
(1 - \gamma)\{\mu_b + \frac{(\mu_m - \mu_b)^2}{2(1 - \gamma)\sigma_m^2} - [b^J(t)]^{-\frac{1}{\gamma - 1}} \} - \mu_b - \frac{(\mu_m - \mu_b)^2}{2(1 - \gamma)\sigma_m^2} & T_1 \leq t < T_2 , \end{cases} \]

and

\[ \frac{dJ_W}{J_W} = \begin{cases} \\
-\dot{\hat{r}} dt - \frac{\mu_m - \mu_b}{\sigma_m^2} d\omega_m - (1 - \gamma)\alpha^J\nu_s d\omega_s + [(1 - \alpha^J)^{\gamma - 1} - 1](dq + \lambda dt) & t \leq T_1 , \\
-\mu_b dt - \frac{\mu_m - \mu_b}{\sigma_m^2} d\omega_m & T_1 \leq t < T_2 , \end{cases} \]

where \( \hat{r} = \mu_b - [(1 - \alpha^J)^{\gamma - 1} - 1]\lambda - (1 - \gamma)[(\alpha^J)^2\nu_s^2 + \frac{1}{2}(\alpha^J)^2\gamma \lambda - \alpha^J \lambda] . \)
Appendix D

Consider the following stochastic differential equations

\[
\begin{align*}
\text{d}S &= (\mu_s - d_s + \lambda)S \text{d}t + \nu_mS \text{d}\omega_m + \nu_sS \text{d}\omega_s - S \text{d}q \\
\text{d}F(S, t) &= F_S(dS)^c + F_t \text{d}t + \frac{1}{2}F_{SS}[(dS)^c]^2 + (0 - F(S, t)) \text{d}q
\end{align*}
\]

When \( t \leq T_1 \), then

\[
\begin{align*}
\text{d}F \cdot \text{d}J_W &= -\left[\frac{\mu_m - \mu_b}{\sigma_m}\nu_m + (1 - \gamma)\alpha^J\nu_s^2\right]SF_SJ_W \text{d}t - [(1 - \alpha^J)^{\gamma - 1} - 1]FJ_W \text{d}q, \\
E[\text{d}J_W \cdot F] &= -\hat{r}FJ_W \text{d}t, \\
E[\text{d}F \cdot \text{d}J_W] &= [F_S(\mu_s - d_s + \lambda)S + F_t + \frac{1}{2}F_{SS}(\nu_m^2 + \nu_s^2)S^2 - \lambda F]J_W \text{d}t, \\
E[\text{d}F \cdot \text{d}J_W] &= -\left\{[\beta(\mu_m - \mu_b) + (1 - \gamma)\alpha^J\nu_s^2]SF_S + [(1 - \alpha^J)^{\gamma - 1} - 1]\lambda F\right\}J_W \text{d}t.
\end{align*}
\]

Plugging in the above identities into equation (8), we have

\[
-\hat{r}FJ_W \text{d}t + [F_S(\mu_s - d_s + \lambda)S + F_t + \frac{1}{2}F_{SS}(\nu_m^2 + \nu_s^2)S^2 - \lambda F]J_W \text{d}t - \left\{[\beta(\mu_m - r) + (1 - \gamma)\alpha^J\nu_s^2]SF_S + [(1 - \alpha^J)^{\gamma - 1} - 1]\lambda F\right\}J_W \text{d}t = 0. \tag{25}
\]

After a simple calculation, (25) becomes

\[
\frac{1}{2}F_{SS}S^2(\nu_m^2 + \nu_s^2) + [\mu_b - d_s + \lambda - (1 - \gamma)\alpha^J\nu_s^2]SF_S - [\hat{r} + (1 - \alpha^J)^{\gamma - 1}\lambda]F + F_t = 0. \tag{26}
\]

Denote

\[
\begin{align*}
\sigma_N^2 &= \nu_m^2 + \nu_s^2, \\
\hat{r} &= \hat{r} + (1 - \alpha^J)^{\gamma - 1}\lambda = \mu_b + \lambda - (1 - \gamma)[(\alpha^J)^2\nu_s^2 + \frac{1}{2}(\alpha^J)^2\gamma \lambda - \alpha^J \lambda], \\
\hat{d}_s &= d_s + (1 - \gamma)[(\alpha^J)^2(1 - \alpha^J)\nu_s^2 - \frac{1}{2}(\alpha^J)^2\gamma \lambda + \alpha^J \lambda] > d_s.
\end{align*}
\]

Then (26) implies that

\[
\frac{1}{2}F_{SS}S^2\sigma_N^2 + (\hat{r} - \hat{d})SF_S - \hat{r}F + F_t = 0,
\]

which is equation (11).

When \( T_1 \leq t < T_2 \), equation (9) can be derived by using a similar procedure.
Appendix E

Consider the following system of stochastic differential equations
\[
\begin{align*}
\frac{dS}{S} &= ((\mu_s + s) - d_s + \lambda)dt + v_m dw_m + v_s dw_s - dq, \\
\frac{dM}{M} &= (\mu_m - d_m)dt + \sigma_m dw_m, \\
\frac{dB}{B} &= \mu_b dt.
\end{align*}
\]

The expected returns and covariance matrix of the two risky assets are
\[
\mu^s = \begin{pmatrix} \mu_b + \beta(\mu_m - \mu_b) + s \\ \mu_m \end{pmatrix}, \quad \text{and} \quad \Omega = \begin{pmatrix} \beta^2 \sigma_m^2 + \nu_s^2 + \lambda & \beta \sigma_m^2 \\ \beta \sigma_m^2 & \sigma_m^2 \end{pmatrix}.
\]

Let \( \phi = \frac{1}{T}e^{-\rho t} C^\gamma + \frac{1}{2}((\beta^2 \sigma_m^2 + \nu_s^2 + \lambda)w_s^2 + 2\beta \sigma_m^2 w_s w_m + \sigma_m^2 w_m^2)w^2 J_{WW} + \{(\mu_b + w_s(\mu_s + s - \mu_b) + w_m(\mu_m - \mu_b))W - C\} J_{tW} + J_t \). Then the optimal portfolio selection and consumption rules in \([0, T]\) are
\[
\begin{align*}
\max_{C, w_s, w_m} \phi &= 0 \\
\text{s.t.} \quad J(W(T_1), T_1) = b(T_1)e^{-\rho T_1} \frac{W^\gamma}{W}.
\end{align*}
\]

From the first order condition, we find
\[
C^* = \left[\tilde{b}^J(t) \right]^{-1} \tilde{W}, \quad w^*_s = \alpha^J, \quad \text{and} \quad w^*_m = \frac{\mu_m - \mu_b}{(1 - \gamma) \sigma_m} - \alpha^J \beta.
\]

Since \( \phi(C^*, w^*_s, w^*_m) = 0 \), we have
\[
\frac{1}{2} F_{SS} S^2 \sigma_N^2 + (\tilde{r}^s - \tilde{d}^s) S F_S - \tilde{r}^s F + F_t = 0,
\]
and
\[
\tilde{b}''(t) = \tilde{b}'(t) \left[ \rho - \gamma(\mu_b + \alpha^J s + \frac{(\mu_m - \mu_b)^2}{2(1 - \gamma) \sigma_m^2} - \frac{1}{2}(1 - \gamma)(\nu_s^2 + \lambda)(\alpha^J)^2] \right] - (1 - \gamma) \tilde{b}'(t) \frac{1}{1 - \gamma}.
\]

The evolution of wealth process is
\[
\frac{dW}{W} = \left[ \mu_b + w_s(\mu_s + s + \lambda - \mu_b) + w_m(\mu_m - \mu_b) - \tilde{b}'(t) \frac{1}{1 - \gamma} \right] dt \\
+ (w_s v_m + w_m \sigma_m) dw_m + w_s \nu_s dw_s - w_s dq;
\]

while the evolution of employee’s marginal utility with sentimental effect is
\[
\frac{dJ_W}{J_W} = -\tilde{r}^s dt - \frac{\mu_m - \mu_b}{\sigma_m} dw_m - (1 - \gamma) \alpha^J \nu_s dw_s + [(1 - \alpha^J) \gamma^{-1} - 1](dq - \lambda dt),
\]
where \( \tilde{r}^s = \mu_b + \alpha^J s - [(1 - \alpha^J) \gamma^{-1} - 1] \lambda - (1 - \gamma)(\alpha^J)^2 \nu_s^2 + \frac{1}{2} (\alpha^J)^2 2 \gamma \lambda - \alpha^J \lambda \).
Considering the following equations

\[
\begin{align*}
\text{d}S &= (\mu_s + s - d_s + \lambda)S\text{d}t + \nu_m S\text{d}\omega_m + \nu_s S\text{d}\omega_s - S\text{d}q, \\
\text{d}F(S, t) &= F_S(dS) + F_t dt + \frac{1}{2} F_{SS} [(dS)^2 + (0 - F(S, t))\text{d}q] \\
&= [F_S(\mu_s + s - d_s + \lambda)S + F_t + \frac{1}{2} F_{SS}(\nu_m^2 + \nu_s^2)S^2]dt \\
&\quad + F_S\nu_m S\text{d}\omega_m + F_S\nu_s S\text{d}\omega_s - F\text{d}q \\
E[dJ \cdot F] &= -\tilde{r}^s F J_W dt, \\
E[dF \cdot J_W] &= [F_S(\mu_s + s - d_s + \lambda)S + F_t + \frac{1}{2} F_{SS}(\nu_m^2 + \nu_s^2)S^2 - \lambda F] J_W dt, \\
E[dF \cdot dJ_W] &= -\{(\beta(\mu_m - r) + (1 - \gamma)\alpha^J)\nu_s S F_S + [(1 - \alpha^J)\gamma - 1 - 1] F\} J_W dt.
\end{align*}
\]

Plug in the above identities in equation (8) to get

\[
\frac{1}{2} F_{SS} S^2 \sigma_N^2 + (\tilde{r}^s - \tilde{d}_s)SF_S - \tilde{r}^s F + F_t = 0,
\]

where

\[
\begin{align*}
\sigma_N^2 &= \nu_m^2 + \nu_s^2 = \beta^2 \sigma_m^2 + \nu_s^2, \\
\tilde{r}^s &= \mu_b + \lambda - (1 - \gamma)[(\alpha^J)\nu_s^2 + \frac{1}{2}\beta^J\nu_s^2 \gamma \lambda - \alpha^J \lambda] + \alpha^J s = \tilde{r} + \alpha^J s > \tilde{r}, \\
\tilde{d}_s &= d_s + (1 - \gamma)[(\alpha^J)(1 - \alpha^J)\nu_s^2 - \frac{1}{2}\beta^J\nu_s^2 \gamma \lambda + \alpha^J \lambda] - s(1 - \alpha^J) = \tilde{d}_s - s(1 - \alpha^J) < \tilde{d}_s.
\end{align*}
\]

Solving (27), the subjective value from the employee’s perspective is

\[
\hat{F}^s(S, \tau) = C(Se^{-d^*_s \tau}, \tau; E, \sigma_N^2, \tilde{r}^s) = Se^{-d^*_s \tau} N(d^*_1) - Ee^{-\tilde{r}^s \tau} N(d^*_2),
\]

where \(d^*_1 = \frac{\ln \left( \frac{S}{E} \right) + (\tilde{r}^s - \tilde{d}_s + \frac{\sigma_N^2}{2})\tau}{\sigma_N \sqrt{\tau}}, \ d^*_2 = \tilde{d}_s - \sigma_N \sqrt{\tau}.\)
References


Table 1: Subjective Value of ESOs

This table presents employee stock option values using equation (12):

\[
\tilde{F}(S, \tau) = C(Se^{-r \tau}, \tau; K, \sigma^2_N, \rho) = Se^{-r \tau}N(d_1) - K e^{-r \tau}N(d_2),
\]

\[
d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},
\]

\[
d_2 = d_1 - \sigma\sqrt{\tau}.
\]

with the following parameterizations:

- \( K = 100, \mu_b = 5\%, \sigma_N = 30\%, \nu_s = 20\%, d_s = 0\%, \lambda = 0.2. \)

Cell entries are subjective values; in parentheses is the value to cost ratio (Subjective Value/Market Value) which may also be seen as an efficiency measure and (1-ESO Premium).

| Panel A: S=100, \( \tau=10 \), Market Value=91.82 |
|---|---|---|---|---|
| \( \gamma \) | 0 | -0.5 | -1 | -1.5 | -2 |
| \( \alpha \) | 10% | 72.26(78.7%) | 63.62(69.3%) | 55.73(60.7%) | 48.57(52.9%) | 42.12(45.9%) |
| 25% | 51.19(55.8%) | 36.46(39.7%) | 25.16(27.4%) | 16.82(18.3%) | 10.90(11.9%) |
| 50% | 29.97(32.6%) | 14.17(15.4%) | 5.91(6.4%) | 2.17(2.4%) | 0.70(0.8%) |
| 75% | 18.43(20.1%) | 5.39(5.9%) | 1.19(1.3%) | 0.20(0.2%) | 0.02(0.0%) |

| Panel B: S=100, \( \tau=9 \), Market Value=89.51 |
|---|---|---|---|---|
| \( \gamma \) | 0 | -0.5 | -1 | -1.5 | -2 |
| \( \alpha \) | 10% | 72.08(80.5%) | 64.24(71.8%) | 56.99(63.7%) | 50.34(56.2%) | 44.26(49.4%) |
| 25% | 52.77(59.0%) | 38.83(43.3%) | 27.77(31.0%) | 19.31(21.6%) | 13.04(14.6%) |
| 50% | 32.51(36.3%) | 16.52(18.5%) | 7.49(8.4%) | 3.03(3.4%) | 1.09(1.2%) |
| 75% | 20.92(23.4%) | 6.89(7.7%) | 1.76(2.0%) | 0.35(0.4%) | 0.05(0.0%) |

| Panel C: S=115, \( \tau=9 \), Market Value=104.50 |
|---|---|---|---|---|
| \( \gamma \) | 0 | -0.5 | -1 | -1.5 | -2 |
| \( \alpha \) | 10% | 84.19(80.6%) | 75.06(71.8%) | 66.61(63.7%) | 58.85(56.3%) | 51.76(49.5%) |
| 25% | 61.70(59.0%) | 45.43(43.5%) | 35.52(31.1%) | 22.63(21.7%) | 15.30(14.6%) |
| 50% | 38.07(36.4%) | 19.38(18.5%) | 8.81(8.4%) | 3.54(3.4%) | 1.29(1.2%) |
| 75% | 24.55(23.5%) | 8.11(7.8%) | 2.07(2.0%) | 0.41(0.4%) | 0.06(0.1%) |

| Panel D: S=85, \( \tau=9 \), Market Value=74.55 |
|---|---|---|---|---|
| \( \gamma \) | 0 | -0.5 | -1 | -1.5 | -2 |
| \( \alpha \) | 10% | 59.98(80.5%) | 53.43(71.7%) | 47.39(63.6%) | 41.83(56.1%) | 36.76(49.3%) |
| 25% | 43.86(58.8%) | 32.24(43.2%) | 23.03(30.9%) | 15.99(21.4%) | 10.79(14.5%) |
| 50% | 26.96(36.2%) | 13.67(18.3%) | 6.18(8.3%) | 2.49(3.3%) | 0.91(1.2%) |
| 75% | 17.31(23.2%) | 5.68(7.6%) | 1.44(1.9%) | 0.28(0.4%) | 0.04(0.1%) |

31
Table 2: Incentive Effect Sensitivity Analysis

This table presents sensitivity measures for the following parameterization:

\[ S = 100, \ K = 100, \ \mu_s = 5\%, \ \sigma_N = 30\%, \ \nu_s = 20\%, \ d_s = 0\%, \ \lambda = 0.2. \]

Delta is calculated:

\[
\text{Market delta} (\delta) = e^{-\delta \cdot \tau} N(\delta), \\
\text{Subjective delta} (\tilde{\delta}) = e^{-\tilde{\delta} \cdot \tau} N(\tilde{\delta}).
\]

Gamma is calculated:

\[
\text{Market gamma} (\Gamma) = \frac{e^{-\delta \cdot \tau} N'(\delta)}{S\sigma_N \sqrt{\tau}}, \\
\text{Subjective gamma} (\tilde{\Gamma}) = \frac{e^{-\tilde{\delta} \cdot \tau} N'(\tilde{\delta})}{S\sigma_N \sqrt{\tau}}.
\]

Cell entries are subjective values; in parentheses is the employee to company sensitivity ratio.

<table>
<thead>
<tr>
<th>Panel A: Subjective Delta (Market Delta=0.999)</th>
<th>0</th>
<th>-0.5</th>
<th>-1</th>
<th>-1.5</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha^J )</td>
<td>10%</td>
<td>0.789(79.0%)</td>
<td>0.696(69.6%)</td>
<td>0.611(61.1%)</td>
<td>0.533(53.4%)</td>
</tr>
<tr>
<td></td>
<td>25%</td>
<td>0.562(56.3%)</td>
<td>0.402(40.3%)</td>
<td>0.279(27.9%)</td>
<td>0.187(18.8%)</td>
</tr>
<tr>
<td></td>
<td>50%</td>
<td>0.332(33.3%)</td>
<td>0.159(15.9%)</td>
<td>0.067(6.7%)</td>
<td>0.025(2.5%)</td>
</tr>
<tr>
<td></td>
<td>75%</td>
<td>0.207(20.7%)</td>
<td>0.062(6.2%)</td>
<td>0.014(1.4%)</td>
<td>0.002(0.2%)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Subjective Gamma (( \times 10^{-3} )) (Market Gamma=3.3)</th>
<th>0</th>
<th>-0.5</th>
<th>-1</th>
<th>-1.5</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha^J )</td>
<td>10%</td>
<td>3.0(90.0%)</td>
<td>2.8(84.6%)</td>
<td>2.6(79.2%)</td>
<td>2.5(73.7%)</td>
</tr>
<tr>
<td></td>
<td>25%</td>
<td>2.6(77.7%)</td>
<td>2.2(65.0%)</td>
<td>1.8(52.6%)</td>
<td>1.4(41.2%)</td>
</tr>
<tr>
<td></td>
<td>50%</td>
<td>2.1(62.7%)</td>
<td>1.4(40.5%)</td>
<td>0.76(22.8%)</td>
<td>0.38(11.2%)</td>
</tr>
<tr>
<td></td>
<td>75%</td>
<td>1.8(52.6%)</td>
<td>0.81(24.1%)</td>
<td>0.27(8.1%)</td>
<td>0.068(2.0%)</td>
</tr>
</tbody>
</table>
Table 3: Volatility Sensitivity Analysis

This table presents sensitivity measures for the following parameterization:

\( K = 100, \ \mu_v = 5\%, \ \sigma_v = 30\%, \ \nu_s = 20\%, \ \delta_s = 0\%, \ \lambda = 0.2. \)

Total-risk vega (TV) is calculated:

\[
\text{Market total-risk vega } (\Lambda_{v,\nu_s,\sqrt{\lambda}}) = S \frac{\sigma_s}{\sigma_N} \sqrt{\tau} e^{-\frac{\delta_s}{\tau}} N'(d_1) > 0,
\]

\[
\text{Subjective total-risk vega } (\hat{\Lambda}_{v,\nu_s,\sqrt{\lambda}}) = S \frac{\sigma_s}{\sigma_N} \sqrt{\tau} e^{-\frac{\delta_s}{\tau}} N'(d_1) > 0.
\]

Normal-unsystematic-risk vega (NUV) is calculated

\[
\Lambda_{v,\nu_s,\sqrt{\lambda}} = 0,
\]

\[
\hat{\Lambda}_{v,\nu_s,\sqrt{\lambda}} = -2\tau(1-\gamma)\alpha'\nu_s[\alpha'Ke^{-\frac{\nu_s}{\tau}}N(\hat{d}_2)+(1-\alpha')Se^{-\frac{\delta_s}{\tau}}N(\hat{d}_3)] < 0.
\]

Default-risk vega (DV) is calculated:

\[
\Lambda_{J,\nu_s,\sqrt{\lambda}} = 2\sqrt{\lambda} \tau Ke^{-\frac{\nu_s}{\tau}}N(d_1) > 0,
\]

\[
\hat{\Lambda}_{J,\nu_s,\sqrt{\lambda}} = 2\sqrt{\lambda} \tau Ke^{-\frac{\nu_s}{\tau}}N(d_1) - \tau(1-\gamma)\alpha'\sqrt{\lambda}(2-\alpha')\tilde{F}(S, \tau).
\]

Cell entries are subjective values; in parentheses is the employee to company sensitivity ratio.

| Panel A: Subjective TV (S=100, \( \tau=10 \), Market TV=1.800) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \alpha' \) | 0 | -0.5 | -1 | -1.5 | -2 |
| 10% | 1.619(90.0%) | 1.523(84.6%) | 1.425(79.2%) | 1.327(73.7%) | 1.228(68.2%) |
| 25% | 1.398(77.7%) | 1.171(65.0%) | 0.948(52.6%) | 0.741(41.2%) | 0.561(31.1%) |
| 50% | 1.129(62.7%) | 0.729(40.5%) | 0.411(22.8%) | 0.202(11.2%) | 0.087(4.8%) |
| 75% | 0.948(52.6%) | 0.434(24.1%) | 0.147(8.1%) | 0.036(2.0%) | 0.007(0.4%) |

| Panel B: Subjective NUV (S=100, \( \tau=10 \), Market NUV=0) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \alpha' \) | 0 | -0.5 | -1 | -1.5 | -2 |
| 10% | -28.67 | -37.93 | -44.39 | -48.46 | -50.54 |
| 25% | -43.49 | -46.65 | -43.20 | -36.35 | -28.47 |
| 50% | -36.49 | -26.38 | -14.97 | -7.03 | -2.80 |
| 75% | -20.49 | -9.48 | -2.95 | -0.65 | -0.10 |
Table 3: Volatility Sensitivity Analysis

Panel C: Subjective DV (S=100, τ=10, Market DV=72.29)

<table>
<thead>
<tr>
<th>α^J</th>
<th>0</th>
<th>-0.5</th>
<th>γ</th>
<th>-1</th>
<th>-1.5</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>-5.31</td>
<td>-34.16</td>
<td>-56.97</td>
<td>-74.30</td>
<td>-86.73</td>
<td></td>
</tr>
<tr>
<td>25%</td>
<td>-69.72</td>
<td>-96.35</td>
<td>-102.17</td>
<td>-94.49</td>
<td>-79.67</td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>-104.96</td>
<td>-91.70</td>
<td>-58.98</td>
<td>-30.48</td>
<td>-13.10</td>
<td></td>
</tr>
<tr>
<td>75%</td>
<td>-103.77</td>
<td>-57.64</td>
<td>-20.15</td>
<td>-4.81</td>
<td>-0.81</td>
<td></td>
</tr>
</tbody>
</table>

Panel D: Subjective DV (S=100, τ=9, Market DV=83.13)

<table>
<thead>
<tr>
<th>α^J</th>
<th>0</th>
<th>-0.5</th>
<th>γ</th>
<th>-1</th>
<th>-1.5</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>11.52</td>
<td>-16.33</td>
<td>-39.22</td>
<td>-57.47</td>
<td>-71.48</td>
<td></td>
</tr>
<tr>
<td>25%</td>
<td>-52.29</td>
<td>-82.94</td>
<td>-94.54</td>
<td>-92.59</td>
<td>-82.32</td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>-94.32</td>
<td>-91.79</td>
<td>-65.21</td>
<td>-37.42</td>
<td>-18.13</td>
<td></td>
</tr>
<tr>
<td>75%</td>
<td>-100.47</td>
<td>-64.29</td>
<td>-26.27</td>
<td>-7.51</td>
<td>-1.56</td>
<td></td>
</tr>
</tbody>
</table>
This table presents over-confidence levels necessary to resolve the ESO premium as quantified by the additional stock return required to justify subjective values calculated from:

\[ \tilde{r}^* = \mu_s + \lambda - (1 - \gamma) [(\alpha' - \gamma \lambda + \alpha' \lambda] + \alpha' s = \tilde{r} + \alpha' s > \tilde{r}, \]

\[ \tilde{d}_i' = d_i + (1 - \gamma) [(\alpha') (1 - \alpha') \nu^2 - \frac{1}{2} (\alpha')^2 \gamma \lambda + \alpha' \lambda] - s(1 - \alpha') = \tilde{d}_i - s(1 - \alpha') < \tilde{d}_i, \]

with the following parameterization:

- \( \gamma = 100 \), \( \mu_s = 5\% \), \( \sigma_s = 30\% \), \( \nu_s = 20\% \), \( \alpha_s = 0\% \), \( \lambda = 0.2 \).

Cell entries are extra percentage stock return.

<table>
<thead>
<tr>
<th>Panel A: S=100, ( \tau=10 ), Market Value=91.82</th>
<th>( \gamma )</th>
<th>0</th>
<th>-0.5</th>
<th>-1</th>
<th>-1.5</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha' )</td>
<td>10%</td>
<td>2.44%</td>
<td>3.75%</td>
<td>5.13%</td>
<td>6.56%</td>
<td>8.05%</td>
</tr>
<tr>
<td></td>
<td>50%</td>
<td>20.55%</td>
<td>35.11%</td>
<td>52.31%</td>
<td>72.05%</td>
<td>94.29%</td>
</tr>
<tr>
<td></td>
<td>75%</td>
<td>59.60%</td>
<td>107.96%</td>
<td>167.59%</td>
<td>238.46%</td>
<td>320.59%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: S=100, ( \tau=9 ), Market Value=89.51</th>
<th>( \gamma )</th>
<th>0</th>
<th>-0.5</th>
<th>-1</th>
<th>-1.5</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha' )</td>
<td>10%</td>
<td>2.39%</td>
<td>3.67%</td>
<td>5.02%</td>
<td>6.43%</td>
<td>7.90%</td>
</tr>
<tr>
<td></td>
<td>25%</td>
<td>6.97%</td>
<td>11.15%</td>
<td>15.82%</td>
<td>20.95%</td>
<td>26.55%</td>
</tr>
<tr>
<td></td>
<td>50%</td>
<td>20.01%</td>
<td>34.43%</td>
<td>51.57%</td>
<td>71.29%</td>
<td>93.54%</td>
</tr>
<tr>
<td></td>
<td>75%</td>
<td>58.11%</td>
<td>106.45%</td>
<td>166.08%</td>
<td>236.95%</td>
<td>319.08%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: S=115, ( \tau=9 ), Market Value=104.50</th>
<th>( \gamma )</th>
<th>0</th>
<th>-0.5</th>
<th>-1</th>
<th>-1.5</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha' )</td>
<td>10%</td>
<td>2.42%</td>
<td>3.72%</td>
<td>5.08%</td>
<td>6.50%</td>
<td>7.98%</td>
</tr>
<tr>
<td></td>
<td>25%</td>
<td>7.06%</td>
<td>11.28%</td>
<td>15.97%</td>
<td>21.13%</td>
<td>26.74%</td>
</tr>
<tr>
<td></td>
<td>50%</td>
<td>20.27%</td>
<td>34.74%</td>
<td>51.90%</td>
<td>71.63%</td>
<td>93.87%</td>
</tr>
<tr>
<td></td>
<td>75%</td>
<td>58.77%</td>
<td>107.12%</td>
<td>166.75%</td>
<td>237.62%</td>
<td>319.75%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel D: S=85, ( \tau=9 ), Market Value=74.55</th>
<th>( \gamma )</th>
<th>0</th>
<th>-0.5</th>
<th>-1</th>
<th>-1.5</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha' )</td>
<td>10%</td>
<td>2.35%</td>
<td>3.62%</td>
<td>4.95%</td>
<td>6.34%</td>
<td>7.79%</td>
</tr>
<tr>
<td></td>
<td>25%</td>
<td>6.85%</td>
<td>10.98%</td>
<td>15.61%</td>
<td>20.71%</td>
<td>26.28%</td>
</tr>
<tr>
<td></td>
<td>50%</td>
<td>19.65%</td>
<td>34.00%</td>
<td>51.12%</td>
<td>70.84%</td>
<td>93.09%</td>
</tr>
<tr>
<td></td>
<td>75%</td>
<td>57.21%</td>
<td>105.55%</td>
<td>165.17%</td>
<td>236.05%</td>
<td>318.17%</td>
</tr>
</tbody>
</table>