ASYMPTOTIC EXPANSIONS ON MOMENTS OF THE FIRST LADDER
HEIGHT IN MARKOV RANDOM WALKS WITH SMALL DRIFT

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Let \( \{ (X_n, S_n), n \geq 0 \} \) be a Markov random walk, in which \( X_n \) takes values in a general state space and \( S_n \) takes values in the real line \( \mathbb{R} \). In this paper, we present some results that are useful in the study of asymptotic approximations of boundary crossing problems for Markov random walks. The main result is asymptotic expansions on moments of the first ladder height in Markov random walks with small positive drift. In order to have the asymptotic expansions, we study a uniform Markov renewal theory which relates to the rate of convergence for the distribution of overshoot, and present an analysis of the covariance between the first passage time and the overshoot.

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1. Introduction. Let \( \{X_n, n \geq 0\} \) be a Markov chain on a general state space \( \mathcal{X} \) with \( \sigma \)-algebra \( \mathcal{A} \). Suppose an additive component \( S_n = \sum_{t=1}^{n} \xi_t \) with \( S_0 = \xi_0 = 0 \), taking values in the real line \( \mathbb{R} \), is adjoined to the chain such that \( \{(X_n, S_n), n \geq 0\} \) is a Markov chain on \( \mathcal{X} \times \mathbb{R} \) with

\[
\begin{align*}
  P\{ (X_n, S_n) \in A \times (B + s) | (X_{n-1}, S_{n-1}) = (x, s) \} &= P\{ (X_1, S_1) \in A \times B | (X_0, S_0) = (x, 0) \} = P(x, A \times B),
\end{align*}
\]

for all \( x \in \mathcal{X}, s \in \mathbb{R}, A \in \mathcal{A} \) and \( B \in \mathcal{B} \) (:= Borel \( \sigma \)-algebra on \( \mathbb{R} \)). The chain \( \{(X_n, S_n), n \geq 0\} \) is called a Markov random walk. For an initial distribution \( \nu \) on \( X_0 \), let \( P_\nu \) denote the probability measure under the initial distribution \( \nu \) on \( X_0 \) and let \( E_\nu \) denote the corresponding expectation. If \( \nu \) is degenerate at \( x \), we shall simply write \( P_x(E_x) \) instead of \( P_\nu(E_\nu) \). Suppose the Markov chain \( \{X_n, n \geq 0\} \) has an invariant probability \( \pi \). Let

\[
\tau = \tau_+ = \inf\{ n \geq 1 : S_n > 0 \}
\]
be the first ascending ladder epoch of $S_n$, and denote the first positive value taken by the Markov random walk, $S_{\tau_+}$, as the first ladder height. For given $a > 0$, we want to give asymptotic expansions, in terms of $\theta$ as $\theta \to 0$, of

\begin{equation}
\mu_\theta E_{\pi_+}^\theta (\tau_+ S_{\tau_+}^a) \text{ and } E_{\pi_+}^\theta S_{\tau_+}^a.
\end{equation}

Here $\mu_\theta$ denotes the mean value, and $E_{\pi_+}^\theta$ denotes the expectation under invariant probability $\pi_+$ of the ladder Markov chain in a family of distributions indexed by $\theta$, which will be defined precisely in Section 2.

The asymptotic expansions of (1.3) have important applications in deriving asymptotic approximations of boundary crossing problems for Markov random walks. In the case of simple random walks, Siegmund (1979, 1985) developed the so-called corrected diffusion approximations, by computing correction terms in the diffusion approximation, to approximate the first passage probabilities of $S_n$, and the expected values of first ladder height appeared in (1.3). That is, he considered the first ladder height in an exponential family of distributions $\{F_\theta : \theta \in \Theta\}$, which may be written in the form $F_\theta(dx) = \exp(\theta x - \Lambda(\theta))F_0(dx)$, where $\Lambda(\theta)$ is the cumulant generating function. Let $P_\theta$ and $E_\theta$ denote probability and expectation when the distribution of $X_n$ is $F_\theta$. Under some regularity conditions, Siegmund (1979) showed that for $a > 0$,

\begin{equation}
E_\theta S_{\tau_+}^a = E_0 S_{\tau_+}^a + \frac{a}{a+1} (E_0 S_{\tau_+}^{a+1}) \theta + o(\theta)
\end{equation}

as $\theta \downarrow 0$. Chang (1992) extended (1.4) to a high order asymptotic expansion, with the $o(\theta)$ replaced by $c_a \theta^2 + O(\theta^3)$, where $c_a$ is a constant depends on $a$, the distribution of overshoot, and the renewal function of the descending ladder random variables. Further refinements of (1.4) can be found in Lotov (1996), and Chang and Peres (1997) for Gaussian random walks, to which the coefficients are related to the celebrated Riemann zeta function. In the case of a finite state ergodic Markov chain, Asmussen (1989b) derived a first-order corrected diffusion approximation for one-barrier ruin problems in risk theory, while Fuh (1997) studied one-barrier and two-barrier boundary crossing probabilities, and derived a second-order corrected diffusion approximation in Markov random walks. To have the approximations, they also derived first order asymptotic expansions of (1.3). For general account on ruin probabilities, the reader is refereed to Asmussen (2000) and references therein.

An alternative approximation of the corrected diffusion approximation is the so-called large deviations approximation. In the case of Markov random walks. By using the idea of
large deviations, and construct a Markov chain extension of the classical Wald martingale family. Miller (1962a,b) derived the asymptotic behavior of $P\{\tau_b < \infty | X_0 = x\}$, based on a Markov Wiener-Hopf factorization, where $\tau_b = \inf\{n \geq 1 : S_n > b\}$ for $b > 0$. This technique was further developed by Arndt (1980), to study asymptotic properties of the distribution of the supremum of a random walk on a Markov chain. Local limit theorems for the joint distribution $P\{\tau_b = n, S_{\tau_b} - b \leq s, X_{\tau_b} \in dy | X_0 = x\}$ were derived by Lalley (1984). Hoglund (1991) combined these techniques with the idea of large deviations, to study the ruin problems for a finite state Markov random walk. Chan and Lai (2003) provided saddle point approximations and nonlinear boundary crossing probabilities for a general state Markov random walk.

Our motivation of providing asymptotic expansions for (1.3) comes from the approximation of boundary crossing probabilities in reflected Markov random walks. That is, let $W_n = S_n - \min_{0 \leq k \leq n} S_k$ be the reflected Markov random walk with reflecting barrier at 0. For $b > 0$ define the stopping time $t = t_b = \inf\{n : W_n > b\}$. In a variety of contexts, for given $m \leq \infty$, we need to approximate the first passage probabilities

$$P\{t \leq m\} \quad \text{and} \quad P\{t \leq m | S_m = \zeta\}, \quad \zeta < b.$$  

(1.5)

It is known that, with some proper identifications, the first part of (1.5) is the probability that at least one among the first ($m$) customers in a single server Markov-modulated queue have a waiting time exceeding $b$ (cf. Burman and Smith, 1986, and Asmussen, 1989a,b, 2000). And the approximation of (1.5) is an essential step to approximate the distribution of the run length of a CUSUM test in autoregressive models and state space models (cf. Basseville and Nikiforov, 1993, and Fuh, 2003).

The remainder of the paper is organized as follows. In Section 2, we formulate the problem and state our main result: asymptotic expansions on moments of the first ladder height in Markov random walks with small positive drift. Due to the Markovian structure, we can characterize the constants appeared in the asymptotic expansions via expected values of the ladder heights, and solutions of the Poisson equations. As an application of our main result, we also give a asymptotic approximation of the first term in (1.5). Motivated by the approximations of (1.3), we study a uniform Markov renewal theory which relates to the rate of convergence for the distribution of overshoot in Section 3, and the rate of convergence for the covariance between first passage time and overshoot in Section 4. The proofs of our main result in Section 2 are given in Section 5.
2. Asymptotic expansions on moments of the first ladder height. Let 
\{(X_n, S_n), n \geq 0\} be the Markov random walk on \(\mathcal{X} \times \mathbb{R}\) defined in (1.1), with transition probability kernel \(P(x, A \times B)\). The corresponding \(m\)-step transition kernel will be denoted by \(P^m\). For ease of notation, write \(P(x, A) = P(x, A \times \mathbb{R})\) as the transition probability kernel of \(\{X_n, n \geq 0\}\). For two transition probability kernels \(Q(x, A), K(x, A), x \in \mathcal{X}, A \in \mathcal{A}\) and for all measurable functions \(h(x), x \in \mathcal{X}\), define \(Qh\) and \(QK\) by \(Qh(x) = \int Q(x, dy)h(y)\) and \(QK(x, A) = \int K(x, dy)Q(y, A)\), respectively. Let \(\mathcal{N}\) be the Banach space of measurable functions \(h: \mathcal{X} \to \mathbb{C}(\mathbb{R})\) with norm \(\|h\| < \infty\). We also introduce the Banach space \(\mathcal{B}\) of transition probability kernels \(Q\) such that the operator norm 
\[\|Q\| = \sup\{\|Qg\|: \|g\| \leq 1\}\] 
is finite. Some prototypical norms considered in the literature are the supremum norm, \(L_p\) norm, the weighted variation norm and the bounded Lipschitz norm, among others. The reader is referred to Kartashov (1996), and Fuh (2004) for details. 

Define also the Cesaro averages \(P^{(n)} = \sum_{j=0}^{n} P_j/n\), where \(P_j\) is a \(j\)-fold power of \(P\), \(P^0 = P(0) = I\) and \(I\) is the identity operator on \(\mathcal{B}\). A Markov chain \(\{X_n, n \geq 0\}\) is said to be uniformly ergodic with respect to a given norm \(\|\cdot\|\), if there exists a stochastic kernel \(\Pi\) such that \(P^{(n)} \to \Pi\) as \(n \to \infty\) in the induced operator norm in \(\mathcal{B}\). The Markov chain \(\{X_n, n \geq 0\}\) is called \(w\)-uniformly ergodic in the case of weighted variation norm.

The following assumptions will be used throughout this paper.

C1. Assume \(\{X_n, n \geq 0\}\) is aperiodic, irreducible (with respect to a maximal irreducible measure \(\varphi\) on \((\mathcal{X}, \mathcal{A})\)), and \(\{(X_n, S_n), n \geq 0\}\) satisfies the minorization condition (cf. Ney and Nummelin, 1987): there exist \(k \geq 1\), a probability measure \(\Psi\) on \(\mathcal{X} \times \mathbb{R}\), and a measurable function \(h\) on \(\mathcal{X}\) such that \(\int h(x)d\nu(x) > 0\), \(\Psi(\mathcal{X} \times \mathbb{R}) = 1\), \(\int \Psi(dx \times \mathbb{R})h(x) > 0\), and \(P^k(x, A \times B) \geq h(x)\Psi(A \times B)\), for all \(x \in \mathcal{X}, A \in \mathcal{A}\) and \(B \in \mathcal{B}\).

C2. Assume \(\{X_n, n \geq 0\}\) is uniformly ergodic with respect to a given norm \(\|\cdot\|\), and it has an invariant probability \(\pi\).

C3. Assume \(\sup_{\|g\| \leq 1} \|E[\varphi(X_1)|X_0 = x]\| < \infty\), and for some \(r \geq 3\), \(\sup_{\|h\| \leq 1} \left|\int h(x)E_x[\varphi|d\nu(x)]\right| < \infty\), where \(\nu\) is an initial distribution of the Markov chain \(\{X_n, n \geq 0\}\). Furthermore, we assume that there exists \(\Theta \subset \mathbb{R}\) containing an interval of zero such that for all \(\theta \in \Theta\), \(\sup_{\|h\| \leq 1} \|E[\exp(\theta \xi_1)h(X_1)|X_0 = x]\| \leq C < \infty\), for some \(C > 0\).

C4. Assume \(\mu := E_\pi\xi_1 = 0\), \(\sup_x E_x[\xi_1]^{a+3} < \infty\), for some \(a > 0\), and there exists \(\varepsilon > 0\) such that \(\inf_x P_x\{\xi_1 \leq -\varepsilon|X_1 = x\} > 0\).

C5. There exists a \(\sigma\)-finite measure \(M\) on \((\mathcal{X}, \mathcal{A})\) such that for all \(x \in \mathcal{X}\), the probability measure \(P_x\) on \((\mathcal{X}, \mathcal{A})\) defined by \(P_x(A) = P(X_1 \in A|X_0 = x)\) is absolutely continuous with respect to \(M\), so that \(P_x(A) = \int_A p(x, y)\,dM(y)\) for all \(A \in \mathcal{A}\), where \(p(x, \cdot) = dP_x/dM\).
C6. Assume that for some $n_0 \geq 1$, $\int_{-\infty}^{\infty} \int_{x \in \mathcal{X}} |E_x \{\exp(i\theta \xi_1)\}|^{n_0} d\pi(x) d\theta < \infty$.

REMARKS: 1. Under Conditions C1 and C2, and $\varphi$ is $\sigma$-finite, Theorem 1.1 of Kartashov (1996) shows that $P$ has a unique stationary projector $\Pi$ in the sense that $\Pi^2 = \Pi = P\Pi = \Pi P$, and $\Pi(x, A) = \pi(A)$ for all $x \in \mathcal{X}, A \in \mathcal{A}$. And Theorem 2.2 of Kartashov (1996) states that a Markov chain $\{X_n, n \geq 0\}$ with transition kernel $P$ is uniformly ergodic with respect to a given norm if and only if there exists $0 < \rho < 1$ such that $\|P^n - \Pi\| = O(\rho^n)$, as $n \to \infty$.

2. C3 is a moment condition in the sense of the corresponding norm, of $\xi_1$. C4 implies that $E_x S_{\tau_+}^{\theta +} < \infty$ for $a > 0$ (cf. Theorem 5 of Fuh and Lai, 1998). The existence of the transition probability density in C5 will be used to construct time-reversed Markov chains. It holds in most applications.

3. C6 implies that for all $n \geq n_0$, $S_n$ has a bounded probability density function for given $X_n$. Instead of assuming C6 hold, we may assume the following extension of the Cramér’s (strong nonlattice) condition: $g(\theta) := \inf_{|v| > 0} |1 - E_x \{\exp(ivS_1)\}| > 0$ for all $\theta > 0$. In addition, we also assume that the conditional Cramér’s (strong nonlattice) condition: there exists $m \geq 1$ such that $\limsup_{|\theta| \to \infty} |E\{\exp(i\theta S_m)|X_0, X_m\}| < 1$. Here, we assume C6 hold for simplicity.

Recall $\tau_+$ defined in (1.2), and denote $\tau^n_+ = \inf\{k \geq \tau^{n-1}_+ : S_k > S_{\tau^{n-1}_+}\}$ as the $n$th ascending ladder epoch of $S_n$. Let $\tau_+ = \inf\{n \geq 1 : S_n > 0\}$ be the first descending ladder epoch of $S_n$, and $\tau^n_- = \inf\{k \geq \tau^{n-1}_- : S_k > S_{\tau^{n-1}_-}\}$ be the $n$th descending ladder epoch of $S_n$, for $n = 2, 3, \cdots$. If $\mu > 0$, $\tau^n_+$ are finite almost surely under the probability $P\{X_{\tau^n_+} \in A|X_0 = x\}$ and therefore, the associated ladder heights $S_{\tau^n_+}$ are well-defined positive random variables. Furthermore, $\{(X_{\tau^n_+}, S_{\tau^n_+}^n), n \geq 0\}$ is a Markov chain, and it is the so-called ladder Markov random walk. When $\mu = 0$, we can still define the ladder Markov chain via the property of uniform integrability in Theorem 5 of Fuh and Lai (1998). It is know (cf. Alsmeyer, 2000) that ladder Markov random walk $\{(X_{\tau^n_+}, S_{\tau^n_+}), n \geq 0\}$ satisfies Condition C1. It is assumed throughout this paper that $P_x(\tau_+ < \infty) = 1$ for all $x \in \mathcal{X}$ and that the ladder random walk is uniformly ergodic with respect to a given norm and satisfies Condition C2. The uniformly strong nonlattice for the ladder random walk, and the exponential moment condition C3 for the ladder random walk can be found in Lemma 13 and Lemma 14 of Fuh (2004) respectively. Let $\pi_+$ denote the invariant measure of the kernel $P_+(x, A \times \mathbf{R}) := P\{X_{\tau^n_+} \in A, S_{\tau^n_+} \in \mathbf{R}|X_0 = x\}$, which is assumed to be irreducible and aperiodic. In Section 2 of Fuh and Lai (2001), and Section 3 of Fuh (2004), we showed how uniform ergodicity of the ladder Markov chain and finiteness of moments of $S_{\tau_+}$ can
be established in some interesting examples.

For \( b > 0 \), define the first passage time of the Markov random walk \( S_n \) as

\[
\tau_b = \tau(b) = \inf\{n : S_n > b\} \quad (\tau_+ = \tau(0)),
\]

and the residual at \( b \) is defined as

\[
R_b = R(b) = S_{\tau_b} - b.
\]

Under Conditions C1-C6, we can apply Corollary 2.2 of Alsmeyer (1994) to show that

\[
\text{as } b \to \infty, \quad (X_{\tau_b}, S_{\tau_b} - b) \text{ has the limiting distribution } (X_\infty, R_\infty),
\]

which is defined by

\[
P\{X_\infty \in A, R_\infty > s\} = \int_s^\infty P_{\pi_+}\{X_{\tau_+} \in A, S_{\tau_+} > u\} du / E_{\pi_+} S_{\tau_+}
\]

for every \( A \in \mathcal{A} \) and \( s > 0 \).

To state our main result, we need to consider time-reversed descending ladder Markov random walks, study solutions of Poisson equations, and define twist transformation of the transition probability operator. They are given in the following three paragraphs, respectively.

By Condition C5, the invariant probability measure \( \pi \) of the Markov chain \( \{X_n, n \geq 0\} \) has a positive density function with respect to \( M \). Without any confusion, we still denote it as \( \pi \) here and in the sequel. As that in Section 4 of Fuh and Lai (1998), we consider the time-reversed (dual) process \( \{\tilde{(X}_n, \tilde{S}_n), n \geq 0\} \) of \( \{(X_n, S_n), n \geq 0\} \) with transition kernel

\[
\tilde{P}(y, dx \times ds) = \frac{\pi(x)}{\pi(y)} P(x, dy \times ds).
\]

Note that \( \{(\tilde{X}_n, \tilde{S}_n), n \geq 0\} \) and \( \{(X_n, S_n), n \geq 0\} \) have the same invariant probability measure \( \pi \). Let \( \tilde{\tau}_+^0 = 0 \), and \( \tilde{\tau}_+^n = \inf\{n \geq 1 : \tilde{S}_n \leq 0\} \), and for \( n > 1 \) define the nth weakly descending ladder epoch as \( \tilde{\tau}_n = \inf\{k \geq \tilde{\tau}_n-1 : \tilde{S}_k \leq \tilde{S}_{\tilde{\tau}_n-1}\} \). The same assumptions will be made for \( \{(\tilde{X}_{\tilde{\tau}_n}, \tilde{S}_{\tilde{\tau}_n}), n \geq 0\} \) as those for \( \{(X_{\tau_+^n}, S_{\tau_+^n}), n \geq 0\} \). For \( x \in \mathcal{X} \) define the renewal measure

\[
\tilde{U}_{x,-}(A, B) := \sum_{n=0}^\infty \tilde{P}_x \{\tilde{\tau}_n < \infty, \tilde{X}_{\tilde{\tau}_n} \in A, \tilde{S}_{\tilde{\tau}_n} \in B\}
\]

for all \( A \in \mathcal{A} \) and Borel subsets \( B \subset [0, \infty) \), and let

\[
\tilde{U}_{x,-}(A, v) := \sum_{n=0}^\infty \tilde{P}_x \{\tilde{\tau}_n < \infty, \tilde{X}_{\tilde{\tau}_n} \in A, \tilde{S}_{\tilde{\tau}_n} \leq v\},
\]
be the renewal function corresponding to the renewal process \(\{-\tilde{S}_{n}, n = 0, 1, \cdots\}\). We simply denote \(\tilde{U}_{x,-}(A, v)\) as \(\tilde{U}_{x,-}(v)\) if \(A = \mathcal{X}\) in (2.6). Define

\[
\alpha_{\pi}^{a} = \int_{0,\infty} (E_{x} R_{b}^{a} - E_{\pi} R_{\infty}^{a}) \tilde{U}_{x,-}(db) \quad \text{and} \quad \alpha^{a} = \int_{\mathcal{X}} \alpha_{\pi}^{a} d\pi(x).
\]

Let \(g_{1} : \mathcal{X} \to \mathbb{R}\) be a solution of the Poisson equation

\[
g_{1}(x) - E_{x} g_{1}(X_{\tau_{+}}) = E_{x} S_{\tau_{+}} - E_{\pi} S_{\tau_{+}}
\]

for almost every (with respect to \(M\)) \(x \in \mathcal{X}\), with \(E_{\pi} g_{1}(X_{\tau_{+}}) = 0\), and let \(g_{2}\) be a solution of the Poisson equation

\[
g_{2}(x) - E_{x} g_{2}(X_{\tau_{+}})
\]

for almost every \(x\), with \(E_{\pi} g_{2}(X_{\tau_{+}}) = 0\). Note that under Conditions C1-C4, the solutions of (2.8) and (2.9) exist via (2.2) and (2.3) of Fuh and Zhang (2000), and Theorem 17.4.2 of Meyn and Tweedie (1993). Now define

\[
\alpha_{1}(X_{\tau_{+}}) = g_{1}(X_{\tau_{+}}) - g_{1}(X_{0}),
\]

\[
\alpha_{2}(X_{\tau_{+}}) = g_{2}(X_{\tau_{+}}) - g_{2}(X_{0}).
\]

For \(z \in \mathbb{C}\), define linear operators \(P_{z}, P, \nu_{s}\) and \(Q\) on \(N\) by

\[
(P_{z} h)(x) = E[h(X_{1}) e^{z\Lambda}] |X_{0} = x], \quad (P h)(x) = E[h(X_{1}) |X_{0} = x],
\]

\[
\nu_{s} h = E_{\pi} \{h(X_{0})\}, \quad Q h = \int h(y) \pi(dy).
\]

Proposition 1 of Fuh (2004) showed that there exists sufficiently small \(\delta > 0\) such that for \(|z| \leq \delta, N = N_{1}(z) \oplus N_{2}(z)\) and

\[
P_{z} Q_{z} h = \lambda(z) Q_{z} h, \quad \text{for all } h \in N,
\]

where \(N_{1}(z)\) is a one-dimensional subspace of \(N\), \(\lambda(z)\) is the eigenvalue of \(P_{z}\) corresponding eigenspace \(N_{1}(z)\) and \(Q_{z}\) is the parallel projection of \(N\) onto the subspace \(N_{1}(z)\) in the direction of \(N_{2}(z)\). Let \(h_{1} \in N\) be the constant function \(h_{1} \equiv 1\) and let \(r(x; z) = (Q_{z} h_{1})(x)\). From (2.13), it follows that \(r(\cdot; z)\) is an eigenfunction of \(P_{z}\) associated with the eigenvalue \(\lambda(z)\), i.e., \(r(\cdot; z)\) generates the one-dimensional eigenspace \(N_{1}(z)\). In particular, when \(z = \theta \in \mathbb{R}\) such that there exists \(\delta > 0\) and \(|\theta| \leq \delta\). Define the “twisting” transformation

\[
P^{\theta}(x, dy \times ds) = \frac{r(y; \theta)}{r(x; \theta)} e^{-\Lambda(\theta) + \theta s} P(x, dy \times ds), \quad \text{where} \ \Lambda = \log \lambda.
\]
Then $P^\theta$ is the transition probability of a Markov random walk $\{(X_n^\theta, S_n^\theta), n \geq 0\}$, with invariant probability $\pi^\theta$. Let $E^\theta_\nu$ be the expectation under $P^\nu$. The function $\Lambda(\theta)$ is normalized so that $\Lambda(0) = \Lambda(0) = 0$, where $\dot{\Lambda}$ denotes the first derivative of $\Lambda$ with respect to $\theta$. Then $P^0 = P$ is the transition probability of the Markov random walk $\{(X_n, S_n), n \geq 0\}$ with invariant probability $\pi$. Here and in the sequel, we denote $P^\theta$ as the probability measure of the Markov random walk $\{(X_n, S_n^\theta), n \geq 0\}$ with transition probability kernel (2.14), and has initial distribution $\nu^\theta$. For simplicity of notation, we denote $\nu^\theta := \nu$ and $\pi^\theta := \pi$, and delete $\theta$ in $\{(X_n, S_n^\theta), n \geq 0\}$ if it is under $P^\theta$ or $E^\theta$. Since $r(x; 0) = 1$, the continuity property of $r(x; \theta)$ implies that there exists $\delta > 0$, and for $|\theta| \leq \delta$ we have $r(x; \theta) > 0$ (or $1/r(x; \theta) < \infty$) uniformly for all $x \in \mathcal{X}$. For simplicity of notation, we let $\pi^0_+ = \pi_+$ be the invariant measure for the ascending ladder Markov random walk $\{(X_{\tau^+}^\theta, S_{\tau^+}^\theta), n \geq 0\}$. Note that $\pi_+$ has a probability density with respect to $M$, and abuse the notation a little bit, we denote it as $\pi_+$ again.

By Proposition 1 of Fuh (2004), it is known that $\Lambda$ is a strictly convex and real analytic function for which $\dot{\Lambda}(0) = E^\theta_\pi \xi_1^\theta$. Therefore, $E^\theta_\pi \xi_1^\theta < -\infty$, or $0 > \theta < -\infty$, or $\theta > 0$. Denote $\dot{\Lambda}(\theta_0) = \mu_0$ and $\dot{\Lambda}(\theta_1) = \mu_1$. For any value $\theta \neq 0$ and $|\theta| < \delta$, there is at most one value $\theta'$ with $|\theta'| < \delta$, necessarily of opposite sign, for which $\dot{\Lambda}(\theta) = \dot{\Lambda}(\theta')$. Assume such $\theta'$ exists, we may let that $\theta_0 = \min(\theta, \theta')$ and $\theta_1 = \max(\theta, \theta')$ such that $\theta_0 < 0 < \theta_1$ and $\dot{\Lambda}(\theta_0) = \dot{\Lambda}(\theta_1)$. Denote $\Delta = \theta_1 - \theta_0$. We also assume, without loss of generality, that $\sigma^2 = \dot{\Lambda}(0) = 1$, where $\ddot{\Lambda}$ denotes the second derivative of $\Lambda$ with respect to $\theta$.

**Theorem 1** Let $\{(X_n, S_n), n \geq 0\}$ be a Markov random walk satisfying C1-C6, and denote $\{(X_n^\theta, S_n^\theta), n \geq 0\}$ as the Markov random walk induced by (2.14) with $E^\theta_2 S_1 > 0$. Then for any $a > 0$, as $\theta \downarrow 0$, we have

\[
\begin{align*}
\mu_\theta E^\theta_\pi (\tau_+ + S^a_{\tau_+}) &= \frac{1}{a + 1} E_{\pi_+} S^a_{\tau_+} + \left(\frac{1}{a + 2} E_{\pi_+} S^a_{\tau_+} + E_{\pi_+} (S^a_{\tau_+} \alpha_1(X_{\tau_+})) + a^2\right) \theta + O(\theta^2). \\
\end{align*}
\]

Hence as $\theta \downarrow 0$,

\[
\begin{align*}
E^\theta_\pi S^a_{\tau_+} &= E_{\pi_+} S^a_{\tau_+} + \left(\frac{a}{a + 1} E_{\pi_+} S^a_{\tau_+} + E_{\pi_+} (S^a_{\tau_+} \alpha_1(X_{\tau_+}))\right) \theta \\
&\quad + \frac{1}{2} \left(\frac{a}{a + 2} E_{\pi_+} S^a_{\tau_+} + E_{\pi_+} (S^a_{\tau_+} \alpha_1(X_{\tau_+}) + S^a_{\tau_+} \alpha_2(X_{\tau_+})) - a^2\right) \theta^2 + O(\theta^3).
\end{align*}
\]

**REMARK:** To compare the asymptotic expansion (2.16) to (4.3) of Chang (1992), we observe that there is one more term $E_{\pi_+} (S^a_{\tau_+} \alpha_1(X_{\tau_+}))$, the joint moment of the first ladder
height and the solution of the Poisson equation (2.8), appeared in the first order approximation; and one more term \( E_{\pi_+}(S_{\tau_+}^{a+1} \alpha_1(X_{\tau_+}) + S_{\tau_+}^a \alpha_2(X_{\tau_+})) \), the joint moment of the first ladder height and the solution of the Poisson equations (2.8) and (2.9), appeared in the second order approximation. An interpretation for these extra terms can be described as follows: note that

\[
(2.17) \quad \frac{a}{a+1}E_{\pi_+}S_{\tau_+}^{a+1} + E_{\pi_+}(S_{\tau_+}^a \alpha_1(X_{\tau_+})) = \frac{-1}{a+1}E_{\pi_+}S_{\tau_+}^{a+1} + E_{\pi_+} \left( S_{\tau_+}^a (\tau_{\pi_+} + \alpha_1(X_{\tau_+})) \right).
\]

Since \( \mu = 0 \), \( S_n + \alpha_1(X_n) \) forms a martingale (cf. Theorem 17.4.3 of Meyn and Tweedie, 1993). And \( \alpha_1(X_n) = 0 \) in the case of simple random walk. Therefore, we may consider the extra constant \( \alpha_1(X_{\tau_+}) \) is due to Markovian dependence, to reflect the martingale structure of Markov random walks. The same interpretation can also be applied to \( S_n^2 + \alpha_2(X_n) \) which forms a quadratic martingale (cf. Theorem 3 of Fuh and Zhang, 2000). Note that the constant \( \alpha^n \) depends on the distribution of overshoot, and the renewal function for the time-reversed descending ladder Markov chains.

By using the tilting formula (2.14) and Theorem 1, we give an asymptotic approximation of the first term in (1.5) as follows: Under the conditions as Theorem 1. Assume \( \theta_0 \uparrow 0, b \to \infty \) and \( m \to \infty \) in such a way that for some \( \delta > 0 \) and some \( k \) we have

\[
(2.18) \quad |\theta_0|^{1+\delta}b \to \infty, \quad |\theta_0|^k m \to \infty \quad \text{and} \quad m \geq \frac{b}{\mu_1}(1+\delta).
\]

Then

\[
(2.19) \quad P^{\theta_0}_\pi \{t_b \leq m\} \approx \exp[\Delta(b + \rho_+ - \rho_-)] \times \left( \frac{\Delta|\mu_0|(m - b + (\rho_+ + \zeta_+) - (\rho_- + \zeta_-)) + 3(1 + \Delta \bar{E}_{\pi_+} \alpha_1(X_{\tau_+})) - \frac{2}{3} \gamma \Delta}{\mu_1} \right),
\]

where \( \mu_0 = \hat{\lambda}(\theta_0), \mu_1 = \hat{\lambda}(\theta_1), \gamma = E_{\pi_1}^3 \), and

\[
\rho_+ = \frac{E_{\pi_+}S_{\tau_+}^2}{2E_{\pi_+}S_{\tau_+}}, \quad \rho_- = \frac{E_{\pi_-}S_{\tau_-}^2}{2E_{\pi_-}S_{\tau_-}}, \quad \zeta_+ = \frac{E_{\pi_+}(S_{\tau_+}^2 \alpha_1(X_{\tau_+}))}{E_{\pi_+}S_{\tau_+}}, \quad \zeta_- = \frac{E_{\pi_-}(S_{\tau_-}^2 \alpha_1(X_{\tau_-}))}{E_{\pi_-}S_{\tau_-}},
\]

with \( \tilde{\alpha}_1(\tilde{X}_{\tau_-}) \) is defined as \( \alpha_1(X_{\tau_+}) \) in (2.10) for the time-reversed ladder Markov random walk \( \{(\tilde{X}_{\tau_-}, \tilde{S}_{\tau_-}), n \geq 0\} \).

Note the approximation (2.19) generalizes Theorem 3 of Siegmund (1988) to the case of reflected Markov random walks. The formal proof of (2.19) along with a high order asymptotic approximation of (1.5), and its applications to CUSUM and other change point detection procedures will be published in a separate paper.
3. Rate of convergence for the distribution of overshoot. By using the same notations as those in Section 2, we define the renewal measure for Markov random walks as

\[(3.1) \quad U_\nu(A,B) = \sum_{n=0}^{\infty} P_\nu\{X_n \in A, S_n \in B\},\]

where \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\). When \(\nu\) is degenerated at \(x\), we simply denote it as \(U_x(A,\cdot)\). Then under Condition C5, for fixed \(B \in \mathcal{B}\), \(U_x(A,B)\) is absolutely continuous with respect to \(M\), with density function \(u(y;x,B)\) so that \(U_x(A,B) = \int_A u(y;x,B) dM(y)\). It is simple to note that the measure \(P_+\{x,\cdot \times \mathbb{R}\}\) on \((\mathcal{X},\mathcal{A})\) is also absolutely continuous with respect to \(M\) for every \(x \in \mathcal{X}\). Likewise, the renewal measure \(U_{x,+}\), defined as (3.1), associated with the ascending ladder random walk initialized at \((x,0)\) also has density function \(u_+(\cdot;x,B)\) with respect to \(M\), i.e., \(U_{x,+}(A,B) = \int_A u_+(y;x,B) dM(y)\). When \(B = (-\infty,t]\), we simply denote it as \(u_+(y,t,B)\). C5 also implies that for every \(x \in \mathcal{X}\), there exists \(p_-(x,\cdot)\) such that \(P(X_{\tau_x} \in A|X_0 = x) = \int_A p_-(x,y) dM(y)\) for all \(A \in \mathcal{A}\). For given \(B \in \mathcal{B}((-\infty,0])\) define \(G_-(x,y,B) = p_-(x,y)P_{\mathcal{S}_{\tau_x}} \in B|X_0 = x, X_{\tau_x} = y\). Consider the time-reversed (dual) process \(\{\tilde{X}_n, \tilde{S}_n\}, n \geq 0\) and define \(\hat{p}_-\) and \(\hat{G}_-\) for the dual process in the same way as \(p_-\) and \(G_-\) are defined for \(\{(X_n, S_n), n \geq 0\}\). Let \(\hat{G}_-(x,y,B) = \hat{G}_-(x,y,B)\pi(y)/\pi(x), \hat{U}_- = \sum_{n=0}^{\infty} \hat{G}^{(n)}_{\tau_x}\), where \(\ast\) denotes convolution of two transition kernels \(F_1(x,y,\cdot)\) and \(F_2(x,y,\cdot)\), with \(\hat{G}^{+1}_\ast = \hat{G}\) and \(\hat{G}^{\ast 0}\) being the kernel that puts all its mass at 0.

The next result states the rate of convergence for the distribution of overshoot.

**Theorem 2** Let \(\{(X_n, S_n), n \geq 0\}\) be a Markov random walk satisfying C1-C6, and denote \(\{(X_n^0, S_n^0), n \geq 0\}\) as the Markov random walk induced by (2.14) with \(E_x^0 S_1 > 0\). Then there exist \(r > 0, \theta^* > 0\) and \(C\) such that

\[(3.2) \quad |P_x^\theta\{X_{\tau_x} \in A, R_0 \leq s\} - P_\theta\{X_\infty \in A, R_\infty \leq s\}| \leq Ce^{-r(s+b)}\]

for all \(s,b \geq 0, A \in \mathcal{A}\), and \(\theta \in [0, \theta^*]\).

**PROOF.** For given \(\theta > 0\), define \(u_+^\theta(y;x,t)\) as that in the first paragraph of this section for the Markov random walk \(\{(X_n^0, S_n^0), n \geq 0\}\). Since by (3.15) of Fuh and Lai (2001),

\[(3.3) \quad P_x^\theta\{X_{\tau_x} \in A, S_{\tau_x} - b > s\} = \int_{\mathcal{X}} \int_{[0,b]} P_x^\theta\{X_{\tau_x} \in A, S_{\tau_x} > b + s - t\} u_+^\theta(y;x,dt) dM(y),\]

and since by (2.3),

\[(3.4) \quad P_x^\theta\{X_\infty \in A, R_\infty > s\} = \int_{s}^{\infty} F_x^\theta\{X_{\tau_x} \in A, S_{\tau_x} > t\} \frac{dt}{E_x^\theta S_{\tau_x}} = \int_{\mathcal{X}} \int_{b}^{\infty} F_x^\theta\{X_{\tau_x} \in A, S_{\tau_x} > b + s - t\} \frac{\pi_+^\theta(y) dt}{E_x^\theta S_{\tau_x}} dM(y),\]
we have
\[
P_x^0 \{ X_{\tau_b} \in A, S_{\tau_b} - b > s \} - P_x^0 \{ X_{\infty} \in A, R_{\infty} > s \}
\]
\[
= \int_{\mathcal{X}} \int_{(0,b)} P_y^0 \{ X_{\tau_+} \in A, S_{\tau_+} > b + s - t \} \left\{ u_+^\theta(y; x, dt) - \frac{\pi_+(y) dt}{E_{\pi_+} \pi(S_{\tau_+})} \right\} dM(y)
\]
\[
- \int_{\mathcal{X}} \int_{-\infty}^0 P_y^0 \{ X_{\tau_+} \in A, S_{\tau_+} > b + s - t \} \frac{\pi_+(y) dt}{E_{\pi_+} \pi(S_{\tau_+})} dM(y) := J_1 - J_2.
\]

Define
\[
\eta_\theta(y; x, b) = u_+^\theta(y; x, b) - \frac{\pi_+(y)b}{E_{\pi_+} \pi(S_{\tau_+})} - \frac{\pi_+(y)E_{\pi_+} \pi^2}{2(E_{\pi_+} \pi(S_{\tau_+}))^2}.
\]

Then
\[
J_1 = \int_{\mathcal{X}} \int_{(0,b)} P_y^0 \{ X_{\tau_+} \in A, S_{\tau_+} > b + s - t \} \eta_\theta(y; x, dt) dM(y),
\]

and integration by parts gives
\[
J_1 = \int_{\mathcal{X}} \left( \eta_\theta(y; x, b) P_y^0 \{ X_{\tau_+} \in A, S_{\tau_+} > s \}
\]
\[
+ \frac{\pi_+(y)E_{\pi_+} \pi^2}{2(E_{\pi_+} \pi(S_{\tau_+}))^2} P_y^0 \{ X_{\tau_+} \in A, S_{\tau_+} > b + s \} \right) dM(y)
\]
\[
- \int_{\mathcal{X}} \int_{(0,b)} P_y^0 \{ X_{\tau_+} \in A, S_{\tau_+} > b + s - dt \} \eta_\theta(y; x, t) dM(y).
\]

To find an upper bound of \(|J_1|\), note that Theorem 1 of Fuh (2004) states that there exist \(C_1, \alpha_1 > 0\) \(\text{and} \ \theta^*_1 > 0\)

\[
|\eta_\theta(y; x, b)| \leq C_1 e^{-\alpha_1 b}
\]

for all \(x, y \in \mathcal{X}, b \) and all \(\theta \in [0, \theta^*_1]\). Also, by making use of the requirement that \(\Theta\) contain an open interval around 0, Lemma 14 of Fuh (2004) that there exist a \(\alpha^* > 0\) such that for any \(\alpha \in [0, \alpha^*], E_{\pi}^\theta \exp\{\alpha \xi\} < \infty\) implies \(E_{\pi}^\theta \exp\{\alpha S_{\tau_+}\} < \infty\), and Wald’s likelihood ratio identity for Markov random walks (cf. Fuh, 2004), one can show that there exist \(C_2, \alpha_2 > 0\) \(\text{and} \ \theta^*_2 > 0\) such that \(E_{\pi_+}^\theta \exp\{\alpha_2 S_{\tau_+} I(X_{\tau_+} \in A)\} \leq C_2\), for all \(\theta \in [0, \theta^*_2]\), so that for all \(y \in \mathcal{X}\)

\[
P_y^\theta \{ X_{\tau_+} \in A, S_{\tau_+} > s \} \leq C_2 \exp\{-\alpha_2 s\}, \ \text{for all} \ \theta \in [0, \theta^*_2], \ \text{for all} \ s \geq 0.
\]

Therefore, by (3.7) and (3.8), letting \(\alpha_3 = \min\{\alpha_1, \alpha_2\}\) and \(\theta^* = \min\{\theta^*_1, \theta^*_2\}\), there exists a \(C_3\) such that

\[
|\int_{\mathcal{X}} \left( \eta_\theta(y; x, b) P_y^0 \{ X_{\tau_+} \in A, S_{\tau_+} > s \}
\]
\[
+ \frac{\pi_+(y)E_{\pi_+} \pi^2}{2(E_{\pi_+} \pi(S_{\tau_+}))^2} P_y^0 \{ X_{\tau_+} \in A, S_{\tau_+} > b + s \} \right) dM(y)|
\]
\[
(3.9) \leq C_3 \exp\{-\alpha_3(b + s)\},
\]
for all \( s, b \geq 0 \) and \( \theta \in [0, \theta^*] \). For the last term on the right hand side of (3.6), since by Theorem 1 of Fuh (2004)

\[
\left| \int_{\mathcal{X}} \int_{[0,b]/2} P^\theta_{y} \{ X_{\tau_+} \in A, S_{\tau_+} \in b + s - dt \} \eta_\theta(y; x, t) dM(y) \right| \\
\leq C_1 P^\theta_{\pi^*_+} \{ X_{\tau_+} \in A, S_{\tau_+} \geq s + \frac{b}{2} \} \leq C_1 C_2 \exp\{ -\alpha_2 (s + b/2) \},
\]

and

\[
\left| \int_{\mathcal{X}} \int_{[b/2,b)} P^\theta_{y} \{ X_{\tau_+} \in A, S_{\tau_+} \in s - dt \} \eta_\theta(y; x, t) dM(y) \right| \\
\leq C_1 \exp\{ -\alpha_1 b/2 \} P^\theta_{\pi^*_+} \{ X_{\tau_+} \in A, S_{\tau_+} \geq s \} \leq C_1 C_2 \exp\{ -\alpha_1 b/2 + \alpha_2 s \},
\]

there exists a \( \alpha_4 > 0 \) and a \( C_4 \) such that the integral over \([0, b]\) is bounded by \( C_4 \exp\{ -\alpha_4 (b + s) \} \) for all \( b, s \geq 0 \) and \( \theta \in [0, \theta^*] \). Thus, by (3.6) there exist \( \alpha_5 > 0 \) and \( C_5 \) such that

\[(3.10) \quad |J_1| \leq C_5 \exp\{ -\alpha_5 (b + s) \}
\]

for all \( b, s \geq 0 \) and \( \theta \in [0, \theta^*] \).

For \( J_2 \), since \( E^\theta_{\pi^*_+} S_{\tau_+} \) is bounded below for \( \theta \geq 0 \), we have by Theorem 1 of Fuh (2004)

\[(3.11) \quad |J_2| \leq \int_{\mathcal{X}} \int_{-\infty}^{0} C_2 \exp\{ -\alpha_2 (b + s - t) \} \frac{dt}{C_6} \pi_+(y) dM(y) = C_7 \exp\{ -\alpha_2 (b + s) \}.
\]

Combining (3.5), (3.10) and (3.11), we have the proof. \( \square \)

Recall \( \theta_1 \) and \( \theta_0 \) defined in the paragraph before Theorem 1, and denote \( \Delta = \theta_1 - \theta_0 \).

By using the fact of \( E^\theta_{x} R^\theta_{b} = a \int_{0}^{\infty} s^{a-1} P^\theta_{y} \{ X_{\tau_n} \in \mathcal{X}, R_0 > s \} ds \) and Theorem 2, we have

**Corollary 1** Assume the conditions of Theorem 2 hold. Then for any \( a > 0 \) there exist \( r > 0, \theta^* > 0 \) and \( C \) such that

\[(3.12) \quad |E^\theta_{x} R^\theta_{b} - E^\theta_{\pi^*_+} R^\theta_{\pi^*_+}| \leq Ce^{-rb}
\]

for all \( b \geq 0 \), and \( \theta \in [0, \theta^*] \). Also there exist \( r > 0, \theta^* > 0 \) and \( C \) such that

\[(3.13) \quad |E^\theta_{\pi^*_+} (e^{-\Delta R_0}) - E^\theta_{\pi^*_+} (e^{-\Delta R_\infty})| \leq C\Delta e^{-rb}
\]

for all \( b \geq 0 \), and \( \theta_1 \in [0, \theta^*] \).

4. **Covariance between first passage time and overshoot.** For any \( u > 0 \), \( x \in \mathcal{X} \) and \( A \in \mathcal{A} \), let

\[(4.1) \quad Q^\theta_{x}(A, u; b) = \sum_{n=0}^{\infty} P^\theta_{x} \{ \tau_n > n, X_n \in A, S_n > b - u \},
\]

\[(4.2) \quad Q^\theta_{\pi^*_+} (A, u; \infty) = \left( E^\theta_{\pi^*_+} S_{\tau_+} \right)^{-1} \int_{-u}^{0} \int_{\mathcal{X}} \left\{ \int_{A} {\tilde{P}^\theta_{-} (z, y; (s, 0))} dM(y) \right\} d\pi_+(z) ds.
\]
We first present the following two lemmas which are necessary to study the rate of convergence for the covariance between first passage time and overshoot. Lemma 1 generalizes Lemma 3.1 of Fuh and Lai (2001), to have the rate of convergence.

Lemma 1 Let \( \{(X_n, S_n), n \geq 0 \} \) be a Markov random walk satisfying C1-C6, and denote \( \{(X^n_x, S^n_x), n \geq 0 \} \) as the Markov random walk induced by (2.14) with \( E^n_x S_1 > 0 \). Then for any \( x \in \mathcal{X} \) and \( A \in \mathcal{A} \), there exist \( C, r_1 > 0 \) and \( \theta^* > 0 \) such that

\[
|Q^n_x(A, u; b) - Q^n_{x+}(A, u; \infty)| \leq C e^{-r_1(b-u)},
\]

for all \( b > 0, 0 \leq u \leq b \) and \( \theta \in (0, \theta^*] \).

**PROOF.** First note that for any \( x, z \in \mathcal{X} \),

\[
P^n_x \{ \tau_b > n, X_n \in A, S_n > b-u \} = \sum_{k=0}^{n} \int_{(b-u, b)} P^n_x \{ S_i < S_k \text{ for all } i < k, S_k \in dt, S_j \leq S_k \text{ for all } k < j \leq n, S_n - S_k > b-u-t, X_n \in A \}
\]

\[
= \sum_{k=0}^{n} \int_{(0, u)} \int_{\mathcal{X}} P^n_x \{ S_i < S_k \text{ for all } i < k, S_k \in b-ds, X_k \in dz \}
\times P^n_x \{ \tau_+ > n-k, S_{n-k} > s-u, X_{n-k} \in A \}.
\]

Recall that \( \tau_{n+m}^+ \) be the \( m \)th ascending ladder epoch. Then \( \sum_{k=0}^{\infty} P^n_x \{ S_i < S_k \text{ for all } i < k, S_k \in b-ds, X_k \in dz \} = \sum_{m=0}^{\infty} P^n_x \{ X_{n+m}^+ \in dz, S_{n+m}^+ \in b-ds \} \).

For given \( A \in \mathcal{X} \) and \( B \in \mathcal{B}((-\infty, 0)) \), as shown by Fuh and Lai (1998, page 576), \( \sum_{m=0}^{\infty} P^n_x \{ \tau_+ > n, X_n \in A, S_n \in B \} = \int_A \int_B \mathcal{U}^\theta(z, y; B) dM(y) \), where \( \mathcal{U}^\theta(z, y; B) \) is defined as \( \mathcal{U}^\theta(z, y; B) \) in Section 3 for the Markov random walk \( \{(X^n_x, S^n_x), n \geq 0 \} \). Then

\[
Q^n_x(A, u; b) = \sum_{n=0}^{\infty} P^n_x \{ \tau_b > n, X_n \in A, S_n > b-u \}
\]

\[
= \sum_{j=0}^{\infty} \int_{(0, u)} \int_{\mathcal{X}} \sum_{m=0}^{\infty} P^n_x \{ S_{n+m}^+ \in b-ds, X_{n+m}^+ \in dz \} P^n_x \{ \tau_+ > j, S_j > s-u, X_j \in A \}
\]

\[
= \int_{(0, u)} \int_{\mathcal{X}} \left\{ \int_A \mathcal{U}^\theta(z, y; (s-u, 0)) dM(y) \right\} U^\theta_{x, +}(z, b-ds),
\]

where \( U^\theta_{x, +}(z, u) := \sum_{m=0}^{\infty} P^n_x \{ X_{n+m}^+ \in dz, S_{n+m}^+ \leq u \} \). Therefore,

\[
Q^n_x(A, u; b) - Q^n_{x+}(A, u; \infty)
\]

\[
= \int_{\mathcal{X}} \int_{0}^{u} \left\{ \int_A \mathcal{U}^\theta(z, y; (-t, 0)) dM(y) \right\} \left( t^\theta_{x, +}(z, b-u+dt) - U^\theta_{x, +}(z, b-u) - \frac{dt}{E^\theta_{\pi^+} S^+} d\pi^+(z) \right)
\]
Combining these representation with the rate of convergence for Markov renewal theorem (cf. (2.7) in Theorem 1 of Fuh (2004)) to have \(|U_{X,+}^{\theta}(z, b) - U_{X,+}^{\theta}(z, b - u) - u/E_{x+S_{\tau+}}^{\theta} = o(e^{-r_1(b-u)})\), and \(|U_{x,+}^{\theta}(z, b - u + t) - U_{X,+}^{\theta}(z, b - u) - t/E_{x+S_{\tau+}}^{\theta} = o((e^{-r_1(b-u)})\). This yields (4.3).

\[\text{Lemma 2}\] Let \(\{(X_n, S_n), n \geq 0\}\) be a Markov random walk satisfying C1-C6, and denote \(\{(X_n^{0}, S_n^{0}), n \geq 0\}\) as the Markov random walk induced by (2.14) with \(E_{x}^{\theta}S_1 > 0\). Then, there exists \(\theta^* > 0\) such that for all \(a > 0\), and \(\theta \in (0, \theta^*)\),

\[
\sup_{b \geq 0} E_{x}^{\theta}R_b^a \leq \frac{a + 2 E_{x}^{\theta}(\xi_1^{+})^{a+1}}{a + 1} E_{\xi_1}^{\theta}.
\]

When the initial distribution of \(X_0\) is \(\nu\), (4.4) becomes that there exists a constant \(K > 0\) such that \(\sup_{b \geq 0} E_{x}^{\theta}R_b^a \leq (a + 2)E_{\xi_1}^{\theta}((a + 1)E_{\xi_1}^{\theta}) + K\).

REMARK: In the case of simple random walks, the upper bound (4.4) was given in Theorem 3 of Lorden (1970) by pathwise integration. In the case of Markov random walks, when \(a = 1\) the upper bound (4.4) was given in Fuh (2004). Since the proof of Lemma 2 is a simple consequence of Lemma 2 in Fuh (2004) via Theorem 3 in Lorden (1970), we omit it here.

\[\text{Theorem 3}\] Let \(\{(X_n, S_n), n \geq 0\}\) be a Markov random walk satisfying C1-C6, and denote \(\{(X_n^{0}, S_n^{0}), n \geq 0\}\) as the Markov random walk induced by (2.14) with \(E_{x}^{\theta}S_1 > 0\). Let \(\psi^{a}(\theta, v) = E_{x}^{\theta}R_b^a - E_{x}^{\theta}R_{b_{\infty}}^{a}\), and define

\[
C_{\alpha}(\theta) = \int_0^\infty \psi^{a}(\theta, v) \int_X \int \hat{U}_{x}^{\theta}(z, y; (-v, 0))dM(y)dv/E_{x=S_{\tau+}}^{\theta}.
\]

Then there exist \(A, r > 0\) and \(\theta^* > 0\) such that

\[
|\text{Cov}_{\alpha}^{\theta}(\tau_b, R_{b}^{a}) - C_{\alpha}(\theta)| \leq (E_{x=S_{\tau+}}^{\theta})^{-1}Ae^{-rb},
\]

for all \(b\) and for all \(\theta \in (0, \theta^*)\).

\[\text{PROOF.}\] First, we note that it is sufficient to show that there exist \(A_1, A_2, r > 0\) and \(\theta^* > 0\) such that

\[
|\text{Cov}_{\alpha}^{\theta}(\tau_b, R_{b}^{a}) - C_{\alpha}(\theta)| \leq (E_{x=S_{\tau+}}^{\theta})^{-1}(A_1 + A_2)e^{-rb},
\]

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for all \( b \) and for all \( \theta \in (0, \theta^*) \).

To prove \((4.7)\), we assume \( \theta > 0 \) is fixed, then
\[
(4.8) \quad \text{Cov}_x^\theta (\tau_b, R^\theta_b) = \int_{\zeta \in \mathcal{X}} \int_0^\infty \{ E^\theta_x R^\theta_v - E^\theta_x R^\theta_\infty \} Q^\theta_x (dz, dv; b).
\]

Next, (3.18) of Fuh and Lai (2001) showed (for the case of \( a = 1 \) but its generalizations to \( a > 0 \) is straightforward) that as \( b \to \infty \)
\[
(4.9) \quad \text{Cov}_x^\theta (\tau_b, R^\theta_b) \to \text{C}^\alpha (\theta) := \int_0^\infty \psi^\alpha (\theta, v) Q^\theta_x (dv; \infty),
\]
where \( Q^\theta_x (dv; \infty) = \int_{\zeta \in \mathcal{X}} Q^\theta_x (dz, dv; \infty) \).

Let \( Q^\theta_x (dv; b) = \int_{\mathcal{X}} Q^\theta_x (dz, dv; b) \). From \((4.8)\) and \((4.9)\), and noting that \( \int_{[0, \infty)} Q^\theta_x (dv; b) = E^\theta_x \tau_b \), we have
\[
(4.10) \quad \text{Cov}_x^\theta (\tau_b, R^\theta_b) - \text{C}^\alpha (\theta) = \int_0^\infty \left( E^\theta_x R^\theta_v - E^\theta_x R^\theta_\infty \right) \left[ Q^\theta_x (dv; b) - Q^\theta_x (dv; \infty) \right] + \left( E^\theta_x R^\theta_v - E^\theta_x R^\theta_\infty \right) E^\theta_x \tau_b \nabla
\]
\[
+ \int_{\zeta \in \mathcal{X}} \int_0^\infty \left( \left( E^\theta_x R^\theta_v - E^\theta_x R^\theta_\infty \right) - \left( E^\theta_x R^\theta_v - E^\theta_x R^\theta_\infty \right) \right) Q^\theta_x (dz, dv; b).
\]

To show the second term on the right hand side of \((4.10)\) satisfies \((4.7)\), let \( C > 0, u > 0 \) and \( \theta^* > 0 \) be chosen such that for all \( v \geq 0 \) and \( \theta \in [0, \theta^*] \). By the assumption of the continuity of \( E^\theta_x S^\tau_x \) in \( \theta \) we have \( E^\theta_x S^\tau_x \geq C^{-1} \), and by Theorem 1 of Fuh (2004)
\[
(4.11) \quad \left| U_{\nu^\theta_x}^\theta (s) - \frac{s + E^\theta_x R^\theta_\infty}{E^\theta_x S^\tau_x} \right| \leq Ce^{-us},
\]
where \( U_{\nu^\theta_x}^\theta (s) = \sum_{n=0}^\infty P^\theta_x \{ X_{\nu^\theta_x} = \mathcal{X}, S^\tau_x \in [-\infty, s] \} \). By making use of Wald’s equation for Markov random walks, Lemma 2, and Corollary 1, we get
\[
(4.12) \quad \left| \left( E^\theta_x R^\theta_v - E^\theta_x R^\theta_\infty \right) E^\theta_x \tau_b \right| \leq Ce^{-ub} \frac{b + C}{E^\theta_x S^\tau_x},
\]
for all \( b \) and for all \( \theta \in (0, \theta^*) \).

Next we show that the first term on the right hand side of \((4.10)\) satisfies \((4.7)\). For this purpose, we split the first integral in \((4.10)\) into two subintervals: \([0, b/2)\) and \([b/2, \infty)\). The first part is
\[
(4.13) \quad \int_{[0, b/2)} \{ E^\theta_x R^\theta_v - E^\theta_x R^\theta_\infty \} \left[ Q^\theta_x (dv; b) - Q^\theta_x (dv; \infty) \right] = \left( E^\theta_x R^\theta_v - E^\theta_x R^\theta_\infty \right) \left[ Q^\theta_x (b/2; b) - Q^\theta_x (b/2; \infty) \right]
\]
\[
- \int_{[0, b/2)} \left[ Q^\theta_x (dv; b) - Q^\theta_x (dv; \infty) \right] dE^\theta_x R^\theta_v.
\]
where we have used $Q^0_x(0^-; b) = 0 = Q^0_x(0^-; \infty)$. However by Corollary 1 and Lemma 1, we have

$$|\{E^0_{\pi,v} R^a_{\pi,v} - E^0_{\pi,v} R^a_{\pi,\infty}\}|Q^0_x(b/2; b) - Q^0_x(b/2; \infty)| \leq 4C^2(E^0_{\pi,v} S_{\pi,v})^{-1}e^{-sb}.$$

(4.14)

Note that $\int_0^v aE^0_{\pi,v} R^a_{\pi,v}^{-1}dt = (E^0_{\pi,v} S_{\pi,v} U^0_{\pi,v} - E^0_{\pi,v} R^a_{\pi,v})$, and this implies

$$\int_{(0,b/2)} [Q^0_x (dv; b) - Q^0_x (dv; \infty)]dE^0_{\pi,v} R^a_{\pi,v}^{-1}$$

(4.15)

$$\leq \left( \sup_{0 \leq t < b/2} [Q^0_x(v; b) - Q^0_x(v; \infty)] \right) \left( (E^0_{\pi,v} S_{\pi,v} U^0_{\pi,v} + (b/2) + \int_0^{b/2} aE^0_{\pi,v} R^a_{\pi,v}^{-1}dv \right)$$

$$\leq (4C^2(E^0_{\pi,v} S_{\pi,v})^{-1}e^{-sb})[2(E^0_{\pi,v} S_{\pi,v} U^0_{\pi,v} + (b/2)]$$

$$\leq 8C^2(E^0_{\pi,v} S_{\pi,v})^{-1}[b/2 + C)e^{-sb}].$$

Combining (4.14) and (4.15) shows that (4.13) satisfies (4.7).

Finally, for the integral over $[b/2, \infty)$, we use Corollary 1 to write

$$\int_{[b/2,\infty)} \{E^0_{\pi,v} R^a_{\pi,v} - E^0_{\pi,v} R^a_{\pi,\infty}\}|Q^0_x (dv; b) - Q^0_x (dv; \infty)|$$

(4.16)

$$\leq \int_{[b/2,\infty)} C e^{-sv} Q^0_x (ds; b) + \int_{[b/2,\infty)} C e^{-sv} Q^0_x (dv; \infty).$$

It is easy to see that the last two integrals satisfy (4.7), using

$$\int_{(0,\infty)} Q^0_x (dv; b) = E^0_{\pi,v} \tau_b \leq (E^0_{\pi,v} S_{\pi,v})^{-1}(b + C)$$

(4.17)

and Lemma 3.1 of Fuh and Lai (2001), we prove the first term of (4.10) satisfies (4.7).

It remains to show that the third term on the right hand side of (4.10) satisfies (4.7). By Corollary 1, we have that for any $a > 0$, there exist $C, r > 0$ and $\theta^* > 0$ such that

$$|E^0_{\pi,v} R^a_{\pi,v} - E^0_{\pi,v} R^a_{\pi,\infty}| \leq |E^0_{\pi,v} R^a_{\pi,v} - E^0_{\pi,v} R^a_{\pi,\infty}| + |E^0_{\pi,v} R^a_{\pi,\infty} - E^0_{\pi,v} R^a_{\pi,v}| \leq 2C e^{-rv}$$

(4.18)

for all $v \geq 0, \theta \in [0, \theta^*)$, and uniformly for $z \in \mathcal{X}$. Using the same argument as (4.16) and (4.17), we have

$$\int_{\mathcal{X} \times (0,\infty)} |(E^0_{\pi,v} R^a_{\pi,v} - E^0_{\pi,v} R^a_{\pi,\infty}) - (E^0_{\pi,v} R^a_{\pi,v} - E^0_{\pi,v} R^a_{\pi,\infty})|Q^0_x (dz, dv; b)$$

(4.19)

$$\leq 3C e^{-rb}.$$

And this completes the proof.

Let

$$\tilde{C}(\theta_1) = \int_0^{\infty} \left[ E^0_{\pi,v}(e^{-\Delta R_v}) - E^0_{\pi,v}(e^{-\Delta R_v}) \right] Q_x^0 (dv; \infty).$$

(4.20)
Theorem 4 Let \( \{X_n, S_n\}, n \geq 0 \) be a Markov random walk satisfying C1-C6, and denote \( \{X^n_0, S^n_0\}, n \geq 0 \) as the Markov random walk induced by (2.14) with \( E^n_\pi S_1 > 0 \). Then there exist \( A, r > 0 \) and \( \theta^* > 0 \) such that

\[
|\text{Cov}^{\theta^1}(\tau_b, e^{-\Delta R_b}) - \tilde{C}(\theta_1)| \leq Ae^{-rb},
\]

for all \( b \) and for all \( \theta_1 \in (0, \theta^*] \).

PROOF. By using the same method for (4.10), we have

\[
\text{Cov}^{\theta^1}(\tau_b, e^{-\Delta R_b}) - \tilde{C}(\theta_1) = \int_0^\infty [E^{\theta^1}_{x^*}(e^{-\Delta R_v}) - E^{\theta^1}_{x^*}(e^{-\Delta R_0})] [Q^{\theta^1}_x(dv; b) - Q^{\theta^1}_x(dv; \infty)]
\]

\[
+ [E^{\theta^1}_{x^*}(e^{-\Delta R_0}) - E^{\theta^1}_{x^*}(e^{-\Delta R_v})] \text{ Cov}_{x^*} \tau_b
\]

\[
+ \int_{x^*} \int_0^\infty [(E^{\theta^1}_{x^*}(e^{-\Delta R_v}) - E^{\theta^1}_{x^*}(e^{-\Delta R_0})) - (E^{\theta^1}_{x^*}(e^{-\Delta R_0}) - E^{\theta^1}_{x^*}(e^{-\Delta R_v}))] Q^{\theta^1}_x(dz, dv; b).
\]

The rest of the proof is the same as those of Theorem 3.

\[ \square \]

Theorem 5 Let \( \{X_n, S_n\}, n \geq 0 \) be a Markov random walk satisfying C1-C6, and denote \( \{X^n_0, S^n_0\}, n \geq 0 \) as the Markov random walk induced by (2.14) with \( E^n_\pi S_1 > 0 \). Then there exists \( \theta^* > 0 \) such that for \( \theta \in (0, \theta^*] \), we have

\[
\text{Cov}^{\theta^1}_{x^*}(\tau_b, R^n_b) = \frac{1}{E^n_\pi S_1} [E^n_\pi R^n_b + K \eta],
\]

where

\[
K \eta = E^n_\pi \left( \frac{\dot{r}(X_{\tau^*_b}; \theta)}{r(X_{\tau^*_b}; \theta)} - \frac{\dot{r}(x; \theta)}{r(x; \theta)} \right) - (E^n_\pi \dot{r}(X_{\tau^*_b}; \theta) - \dot{r}(x; \theta))(E^n_\pi R^n_b).
\]

PROOF. For given \( \theta > 0 \), by Wald’s likelihood ratio identity for Markov random walks, we have

\[
E^n_\pi R^n_b = E_x \left\{ R^n_b \exp[\theta S_{\tau^*_b} - \tau_b \Lambda(\theta)] \frac{r(X_{\tau^*_b}; \theta)}{r(x; \theta)} \right\}.
\]

Using the dominated convergence theorem to interchange differentiation with expectation to obtain

\[
\dot{E}^n_\pi R^n_b = E_x \left\{ R^n_b \exp[\theta S_{\tau^*_b} - \tau_b \Lambda(\theta)] \frac{r(X_{\tau^*_b}; \theta)}{r(x; \theta)} \right\}
\]

\[
\left( S_{\tau^*_b} - \tau_b \dot{\Lambda}(\theta) \right) + R^n_b \exp[\theta S_{\tau^*_b} - \tau_b \Lambda(\theta)] \frac{\dot{r}(X_{\tau^*_b}; \theta) r(x; \theta) - r(X_{\tau^*_b}; \theta) \dot{r}(x; \theta)}{r^2(x; \theta)}
\]

\[
= E_x \left\{ R^n_b \exp[\theta S_{\tau^*_b} - \tau_b \Lambda(\theta)] \frac{r(X_{\tau^*_b}; \theta)}{r(x; \theta)} \right\}
\]

\[
\left( S_{\tau^*_b} - \tau_b \dot{\Lambda}(\theta) \right) + \frac{\dot{r}(X_{\tau^*_b}; \theta) r(x; \theta) - r(X_{\tau^*_b}; \theta) \dot{r}(x; \theta)}{r(X_{\tau^*_b}; \theta) r(x; \theta)}
\]

\[
= E_x \left\{ R^n_b S_{\tau^*_b} - \tau_b E^n_\pi S_1 + G \right\},
\]
Lemma 10.27 of Siegmund (1985) to Markov random walks. In Lemma 4, we give the rate
first. Lemma 3 provides first order approximations of (2.15) and (2.16), which extends

5. Proof of Theorem 1. To prove Theorem 1, we need the following three lemmas
first. Lemma 3 provides first order approximations of (2.15) and (2.16), which extends
Lemma 10.27 of Siegmund (1985) to Markov random walks. In Lemma 4, we give the rate
of convergence for the renewal measures; while in Lemma 5, we give the rate of convergence for the distributions of overshoot on the descending ladder Markov random walk.

For given $\theta > 0$, and for $x \in \mathcal{X}$, define the renewal measure $\bar{U}_{x, -}^\theta$ by

$$\bar{U}_{x, -}^\theta(A, B) := \sum_{n=0}^{\infty} P_x^\theta\{\tau_n^+ < \infty, \tilde{X}_n < A, -\tilde{S}_n^+ < B\}$$

for all $A \in \mathcal{A}$ and Borel subsets $B \subset [0, \infty)$. We simply denote it as $\bar{U}_{x, -}^\theta(B)$ if $A = \mathcal{X}$, and denote it as $\bar{U}_{x, -}^\theta(v)$ if $B = [0, v)$.

**Lemma 3** Let $\{(X_n, S_n), n \geq 0\}$ be a Markov random walk satisfying C1-C6, and denote $\{(X_n^\theta, S_n^\theta), n \geq 0\}$ as the Markov random walk induced by (2.14) with $E_x^\theta S_1 > 0$. Then for any $a > 0$, we have

$$\lim_{\theta \downarrow 0} \mu_\theta E_{\pi_+}^\theta(\tau_+ S_{\tau_+}^a) = \frac{1}{a + 1} E_{\pi_+}^\theta S_{\tau_+}^{a+1}. \quad (5.2)$$

Hence as $\theta \downarrow 0$,

$$E_{\pi_+}^\theta S_{\tau_+}^a = E_{\pi_+}^\theta S_{\tau_+}^a + \left(\frac{a}{a + 1} E_{\pi_+}^\theta S_{\tau_+}^{a+1} + E_{\pi_+}^\theta S_{\tau_+}^a \alpha_1(X_\tau_+)\right) \theta + o(\theta). \quad (5.3)$$

**PROOF.** To prove (5.2) hold for $a > 0$, we first assume $\theta > 0$ is fixed. Recalling $Q_x^\theta(dv, dz; b)$ defined in (4.1) for $b = 0$

$$Q_x^\theta(dz, dv; 0) = \sum_{n=0}^{\infty} P_x^\theta\{\tau_+ > n, X_n \in dz, -S_n \in dv\}. \quad (5.4)$$

Then by (4.8) with $b = 0$ we have

$$Q_x^\theta(dz, dv; 0) = \bar{U}_{x, -}^\theta(dx, dv), \quad (5.5)$$

Applying duality to obtain $Q_x^\theta(dz, dv; 0) = \bar{U}_{x, -}^\theta(dx, dv)$, and this implies that

$$\text{Cov}_{\pi_+}^\theta(\tau_+, S_{\tau_+}^a) = \int_{x \in \mathcal{X}} \int_{z \in \mathcal{X}} \int_0^\infty \int_0^\infty (E_x^\theta R_v^a) \bar{U}_{x, -}^\theta(dx, dv) d\pi_+(z)$$

$$- \int_{x \in \mathcal{X}} \int_{z \in \mathcal{X}} \int_0^\infty \int_0^\infty (E_x^\theta S_{\tau_+}^a) Q_x^\theta(dz, dv; 0) d\pi_+(z)$$

$$= \int_{x \in \mathcal{X}} \int_0^\infty (E_x^\theta R_v^a) \bar{U}_{x, -}^\theta(dx, dv) d\pi_+(z) - (E_x^\theta S_{\tau_+}^a)(E_x^\theta E_{\pi_+}^\theta S_{\tau_+}^a). \quad (5.6)$$

where $\bar{U}_{x, -}^\theta(dx) := \int_x \bar{U}_{x, -}^\theta(dx, dv).$ By making use of (5.6) and the definition of $\text{Cov}_{\pi_+}^\theta(\tau_+, S_{\tau_+}^a)$, we have

$$\mu_\theta E_{\pi_+}^\theta(\tau_+ S_{\pi_+}^a) = \mu_\theta \int_{x \in \mathcal{X}} \int_0^\infty (E_x^\theta R_v^a) \bar{U}_{x, -}^\theta(dx, dv) d\pi_+(z)$$

$$= \mu_\theta (E_{\pi_+}^\theta R_v^a) E_{\pi_+}^\theta \tau_+ + \mu_\theta \int_{x \in \mathcal{X}} \int_0^\infty (E_x^\theta R_v^a - E_{\pi_+}^\theta R_v^a) \bar{U}_{x, -}^\theta(dx, dv) d\pi_+(z)$$

$$= \frac{1}{a + 1} E_{\pi_+}^\theta S_{\tau_+}^{a+1} + O(\theta). \quad (5.7)$$
Note that the last equality in (5.7) uses $E^\theta_x R^a_{\infty} = E^\theta_x S^{a+1}/(a+1)E^\theta_x S_\tau$, Corollary 1 and $\mu_\theta = \theta + o(\theta)$.

Next, we prove (5.3). From

$$E^\theta_x S^a_{\tau_+} = E_x \left\{ S^a_{\tau_+} \exp[\theta S_{\tau_+} - \tau_+ \Lambda(\theta)] \frac{r(X_{\tau_+};\theta)}{r(x;\theta)} \right\}. \tag{5.8}$$

By the dominated convergence theorem it follows that $\lim_{\theta \downarrow 0} E^\theta_x S^a_{\tau_+} = E_x S^a_{\tau_+}$. Also $f(\theta) = E_x(S^a_{\tau_+})$ is continuously differentiable for small positive $\theta$ and that

$$\dot{f}(\theta) = E_x \left\{ S^a_{\tau_+} \left[ (S_{\tau_+} - \tau_+ \mu_\theta) + \frac{\dot{r}(X_{\tau_+};\theta) r(x;\theta) - r(x_{\tau_+};\theta) \dot{r}(x;\theta)}{r(X_{\tau_+};\theta) r(x;\theta)} \right] \right\}. \tag{5.9}$$

Since $f(\theta_1) = f(\theta) + (\theta_1 - \theta) \dot{f}(\theta) + \int_0^\theta [\dot{f}(\alpha) - \dot{f}(\theta)] d\alpha$, we get the result by letting $\theta \to 0$. \hfill \Box

**Lemma 4** Let $\{(X_n, S_n), n \geq 0\}$ be a Markov random walk satisfying C1-C6, and denote $\{(X_n^\theta, S_n^\theta), n \geq 0\}$ as the Markov random walk induced by (2.14) with $E^\theta_x S_1 > 0$. Then for all $A \in \mathcal{A}$ and Borel subsets $B \subset [0, \infty)$,

$$\tilde{U}^\theta_{x,-}(A, B) \to \tilde{U}^\theta_{x,-}(A, B) \text{ as } \theta \downarrow 0, \tag{5.10}$$

where $\tilde{U}^\theta_{x,-}(A, B)$ is defined in (5.1) and $\tilde{U}^\theta_{x,-}(A, B)$ is defined in (2.5).

**PROOF.** For given $A \in \mathcal{A}$ and Borel subsets $B \subset [0, \infty)$. By Wald’s likelihood ratio identity for Markov random walks, for any $n \geq 0$, we have

$$P^\theta_x \{ \tilde{\tau}_-^n < \infty, \tilde{X}_{\tilde{\tau}_-^n} \in A, -\tilde{S}_{\tilde{\tau}_-^n} \in B \} \tag{5.11}$$

where $\tilde{\Lambda}_-(\theta)$ and $\tilde{r}_-(x;\theta)$ are defined as (2.14) for the transition probability of the Markov random walk $\{(\tilde{X}_{\tilde{\tau}_-^n}, \tilde{S}_{\tilde{\tau}_-^n}), n \geq 0\}$. However, By Proposition 1 of Fuh (2004), $\tilde{r}_-(x;\theta) \to 1$ as $\theta \downarrow 0$. Therefore, as $\theta \downarrow 0$,

$$\exp[\theta \tilde{S}_{\tilde{\tau}_-^n} - \tilde{\tau}_-^n \tilde{\Lambda}_-(\theta)] \frac{\tilde{r}_-(\tilde{X}_{\tilde{\tau}_-^n};\theta)}{\tilde{r}_-(x;\theta)} \to 1. \tag{5.12}$$

Hence, as $\theta \downarrow 0$

$$P^\theta_x \{ \tilde{\tau}_-^n < \infty, \tilde{X}_{\tilde{\tau}_-^n} \in A, -\tilde{S}_{\tilde{\tau}_-^n} \in B \} \to P_x \{ \tilde{X}_{\tilde{\tau}_-^n} \in A, -\tilde{S}_{\tilde{\tau}_-^n} \in B \}. \tag{5.13}$$
Lemma 5

Let \( n \) any nonnegative measures for each \( 0 < \theta < \delta \)

\[(5.15) \quad \int \left| f - \theta - \tilde{\Lambda}(\theta) \right| < 1 + c\theta \text{ for } 0 < \theta < \delta. \]

Along this with \( \exp[\rho S - \tilde{\Lambda}(\theta)] \) \( \uparrow 1 \) as \( \theta \downarrow 0 \), we get for \( 0 < \theta < \delta \),

\[(5.14) \quad P_x(\tilde{T}^n < \infty, \tilde{X} \in A, -\tilde{S} \in B) \leq (1 + c\theta)P_x(\tilde{X} \in A, -\tilde{S} \in B). \]

Note that \( \tilde{X} \in (A, B) < \infty \) for each \( B \subset [0, \infty) \) with finite measure. Using (5.11), summing over \( n \geq 0 \), and applying the dominated convergence theorem, we have (5.9). \( \square \)

**Lemma 5** Let \( \{X_n, S_n\}, n \geq 0 \) be a Markov random walk satisfying C1-C6, and denote \( \{(X_n, S_n), n \geq 0\} \) as the Markov random walk induced by (2.14) with \( E_S > 0 \). Then for any \( x \in \mathcal{X} \)

\[(5.13) \quad \int_{[0, \infty)} (E_x^{b} - E_{\pi^*} R^\infty, \tilde{U}_x^{db} = \alpha_x + O(\theta), \quad \text{as } \theta \downarrow 0, \]

where \( \alpha_x \) is defined in (2.7).

**PROOF.** Denote \( f_x^{\theta} := E_x^{\theta} R^b - E_{\pi^*} R^\infty, f_x := E_x^{b} - E_{\pi^*} R^\infty \), and write

\[
\int_{[0, \infty)} (E_x^{b} - E_{\pi^*} R^\infty, \tilde{U}_x^{db}) = \alpha_x \]

\[= \int_{[0, \infty)} f_x^{\theta} U_x^{db} - \int_{[0, \infty)} f_x U_x^{db} \]

\[= \int_{[0, \infty)} [f_x^{\theta} - f_x] U_x^{db} - \int_{[0, \infty)} f_x U_x^{db} - \tilde{U}_x^{db} : = J_1 - J_2. \]

Next, we will show that \( J_1 \) and \( J_2 \) are \( O(\theta) \) as \( \theta \downarrow 0 \) to complete the proof.

For \( J_1 \), we rewrite \( f_x^{\theta}(b) - f_x(b) = \int_{0}^{\theta} f_x^{\theta} d_\eta \), where the dot denotes differentiation with respect to \( \eta \). By Corollary 2 there exist \( A, r > 0 \) and \( \theta^* \) such that \( |f_x^{\theta} b| \leq A e^{-rb} \) for all \( x \in \mathcal{X}, b \geq 0 \) and all \( \eta \in (0, \theta^* \) for \( \theta \in [0, \theta^* \) \). Therefore, letting \( \theta \in [0, \theta^* \) \), we have for all \( b \geq 0 \)

\[(5.14) \quad |J_1| \leq \theta A \int_{[0, \infty)} e^{-rb} \tilde{U}_x^{db} \leq \theta A(1 + c\theta) \int_{[0, \infty)} e^{-rb} \tilde{U}_x^{db}, \]

where the second inequality follows from (5.12) and Lemma 4. Thus \( J_1 = O(\theta) \).

For \( J_2 \), use Lemma 4 to observe that \( \tilde{U}_x^{db} \) and \( (1 + c\theta) \tilde{U}_x^{db} - \tilde{U}_x^{db} \) are nonnegative measures for each \( 0 < \theta < \delta \). From this and the bound from Corollary 1, \( |f_x(b)| \leq A e^{-r_1 b} \) for \( r_1 > 0 \) and \( x \in \mathcal{X} \), say, we obtain

\[(5.15) \quad |J_2| \leq A \int_{[0, \infty)} e^{-rb} [(1 + c\theta) \tilde{U}_x^{db} - \tilde{U}_x^{db}], \]

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for all $0 < r < r_1$.

Now there exists $\theta^* > 0$ and $r_2 > 0$ such that for $0 < r < r_2$, we can define $\tilde{\lambda}_\theta(r), \tilde{\nu}_s, -\hat{Q}_r$, and $\hat{Q}_r$, uniformly for $\theta \in (0, \theta^*)$, as (2.14) for the transition probability of the Markov random walk $\{(\tilde{X}_{n}, \tilde{S}_{n}) : n \geq 0\}$ under the condition of $\{\tilde{\tau}_- < \infty\}$. Let $\nu$ be an initial distribution degenerated at $x$, and $h_\nu(r) := I_{\{x \in A\}}$, we have

$$\int_{(0, \infty)} e^{-rb}\tilde{U}_{\tau_-, x}(db) = \sum_{n=0}^{\infty} \int_{(0, \infty)} e^{-rb}P_x^\theta(\tilde{\tau}_- < \infty, \tilde{X}_{n} \in X, -\tilde{S}_{n} \in db)$$

$$= \sum_{n=0}^{\infty} E_x^\theta(\exp(r\tilde{S}_{\tilde{\tau}_-}); \tilde{\tau}_- < \infty, \tilde{X}_{\tilde{\tau}_-} \in X)$$

$$= (1 - \tilde{\lambda}_\theta(r))^{-1} \tilde{\nu}_s, -\hat{Q}_r, h_X(r) + \eta^\theta(r),$$

where $\eta^\theta(r) = \sum_{n=0}^{\infty} \tilde{\nu}_s, -\hat{Q}_r, (I - \hat{Q}_r)h_X(r)$. Note that the last equality follows from Proposition 1 of Fuh (2004).

Choosing $r^* = \min\{r_1, r_2\}$ and consider $r \in (0, r^*)$. (5.16) implies that

$$\int_{(0, \infty)} e^{-rb}[(1 + c\theta)\tilde{U}_{\tau_-, x}(db) - \tilde{U}_{\tau_-, x}(db)]$$

$$= \tilde{\lambda}_\theta(r) - \tilde{\lambda}_\theta(r) [1 - \lambda_\theta(r)][1 - \hat{\lambda}_\theta(r)] \tilde{\nu}_s, -\hat{Q}_r, h_X(r) + \frac{c\theta}{1 - \lambda_\theta(r)} \tilde{\nu}_s, -\hat{Q}_r, h_X(r)$$

$$+ \frac{1}{1 - \lambda_\theta(r)} (\tilde{\nu}_s, -\hat{Q}_r, h_X(r) - \tilde{\nu}_s, -\hat{Q}_r, h_X(r)) + (1 + c\theta)\eta(r) - \eta^\theta(r).$$

By Proposition 1 of Fuh (2004), $\tilde{\lambda}_\theta(r), \tilde{\lambda}_\theta(r), \tilde{\nu}_s, -\hat{Q}_r, h_X(r), \tilde{\nu}_s, -\hat{Q}_r, h_X(r), \eta(r)$ and $\eta^\theta(r)$ (uniformly for $\theta \in [0, \theta^*)$) have continuous derivatives for $|r|$ near 0, and

$$\tilde{\lambda}_\theta(r) = 1 + (\tilde{E}_\nu, -S_\tau_r)r + O(r^2), \quad \tilde{\lambda}_\theta(r) = 1 + (\tilde{E}_\nu, -S_\tau_r)r + O(r^2).$$

Moreover, we have as $\theta \to 0$,

$$\tilde{\nu}_s, -\hat{Q}_r, h_X(r) \to \tilde{\nu}_s, -\hat{Q}_r, h_X(r) \quad \text{and} \quad \eta^\theta(r) \to \eta(r).$$

Now for fixed $r \in (0, r^*)$, (5.18) and (5.19) imply that as $\theta \to 0$, (5.17) equals

$$\frac{(E_{\nu, -S_\tilde{\tau}_r} - E_{\nu, -S_\tilde{\tau}_r})r}{(E_{\nu, -S_\tilde{\tau}_r}E_{\nu, -S_\tilde{\tau}_r})r^2} \tilde{\nu}_s, -\hat{Q}_r, h_X(r) + \frac{c\theta}{-E_{\nu, -S_\tilde{\tau}_r} + O(r^2)} \tilde{\nu}_s, -\hat{Q}_r, h_X(r)$$

$$+ \frac{1}{E_{\nu, -S_\tilde{\tau}_r} + O(r^2)} o(\theta) + c\theta \eta(r) + o(\theta).$$

By making use the same argument as (5.3) in Lemma 3, we have $E_{\nu, -S_\tilde{\tau}_r} = E_{\nu, -S_\tilde{\tau}_r} + O(\theta)$. Therefore (5.20) and (5.17) imply that $J_2 = O(\theta)$. This completes the proof. \qed
PROOF OF THEOREM 1. First, we show that (2.15) implies that (2.16) hold. Let $a > 0$ and denote $h_x(\theta) = E_x^0S_{\tau^+_x}^a$. From

\begin{equation}
E_x^0S_{\tau^+_x}^a = E_x\left\{S_{\tau^+_x}^a \exp[\theta S_{\tau^+_x} - \tau^+_x \Lambda(\theta)] \frac{r(X_{\tau^+_x}; \theta)}{r(x; \theta)}\right\}.
\end{equation}

By the dominated convergence theorem it follows that $\lim_{\theta \to 0} E_x^0S_{\tau^+_x}^a = E_xS_{\tau^+_x}^a$. Note that $h_x(\theta)$ is continuously differentiable for small positive $\theta$, as well as $r(x; \theta) = 1 + O(\theta)$ and $\hat{r}(x; \theta) = \dot{r}(x; 0) + O(\theta)$, we have

\begin{equation}
\hat{h}_x(\theta) = E_x^0\left\{S_{\tau^+_x}^a \left(\left(S_{\tau^+_x} - \tau^+_x \mu_0\right) + \frac{\dot{r}(X_{\tau^+_x}; \theta) r(x; \theta) - r(X_{\tau^+_x}; \theta) \dot{r}(x; \theta)}{r(X_{\tau^+_x}; \theta) r(x; \theta)}\right) \right\} + O(\theta).
\end{equation}

Integrating $x \in X$ with respect to $\pi_+$ in (5.22) to have

\begin{equation}
\hat{h}_{\pi_+}(\theta) = E_{\pi_+}^0S_{\tau^+_x}^{ao+1} - \mu_0 E_{\pi_+}^0(S_{\tau^+_x}^a) + E_{\pi_+}^0\left(S_{\tau^+_x}^a (\dot{r}(X_{\tau^+_x}; 0) - \dot{r}(X_0; 0))\right) + O(\theta).
\end{equation}

Since $h_x(\theta) = h_x(\theta^*) + (\theta - \theta^*) \dot{h}_x(\theta^*) + \int_{\theta^*}^\theta \dot{h}_x(\alpha) d\alpha$, we get, via (5.3) in Lemma 3, by letting $\theta^* \to 0$ that

\begin{equation}
E_{\pi_+}^0S_{\tau^+_x}^a = E_xS_{\tau^+_x}^a + \left(\frac{a}{a+1} E_xS_{\tau^+_x}^{ao+1} + E_x[S_{\tau^+_x} (\dot{r}(X_{\tau^+_x}; 0) - \dot{r}(x; 0))]\right) + o(\theta).
\end{equation}

Integrating $x \in X$ with respect to $\pi_+$ in (5.24) to have

\begin{equation}
E_{\pi_+}^0S_{\tau^+_x}^a = E_{\pi_+}S_{\tau^+_x}^a + \left(\frac{a}{a+1} E_{\pi_+}S_{\tau^+_x}^{ao+1} + E_{\pi_+}[S_{\tau^+_x} (\dot{r}(X_{\tau^+_x}; 0) - \dot{r}(x; 0))]\right) + o(\theta).
\end{equation}

To have further expansion of $\dot{h}_{\pi_+}(\theta)$, we first expand $r(x; \theta) = 1 + \dot{r}(x; \theta) + O(\theta^2)$ and $\hat{r}(x; \theta) = \dot{r}(x; 0) + \ddot{r}(x; 0) + O(\theta^2)$, and using $1/(1 - \theta) = 1 + \theta + O(\theta^2)$ to have

\begin{equation}
\frac{\dot{r}(X_{\tau^+_x}; \theta) r(x; \theta) - r(X_{\tau^+_x}; \theta) \dot{r}(x; \theta)}{r(X_{\tau^+_x}; \theta) r(x; \theta)}
\end{equation}

\begin{equation}
= (\dot{r}(X_{\tau^+_x}; 0) - \dot{r}(x; 0)) + ((\ddot{r}(X_{\tau^+_x}; 0) - \dot{r}^2(X_{\tau^+_x}; 0)) - (\ddot{r}(x; 0) - \dot{r}^2(x; 0))) + O(\theta^2).
\end{equation}

Denote

\begin{equation}
\beta_1(X_{\tau^+_x}) = \dot{r}(X_{\tau^+_x}; 0) - \dot{r}(X_0; 0),
\end{equation}

\begin{equation}
\beta_2(X_{\tau^+_x}) = (\ddot{r}(X_{\tau^+_x}; 0) - \dot{r}^2(X_{\tau^+_x}; 0)) - (\ddot{r}(X_0; 0) - \dot{r}^2(X_0; 0)).
\end{equation}
By making use of (2.15), (5.26), and (5.25), with $a$ replaced by $a + 1$, we have

\begin{align*}
(5.29) \quad \hat{h}_{\pi_+}(\theta) &= E_{\pi_+}S_{\tau_+}^a + \left(\frac{a + 1}{a + 2}E_{\pi_+}S_{\tau_+}^{a+2} + E_{\pi_+}[S_{\tau_+}^{a+1}(\tilde{r}(X_{\tau_+}; 0) - \tilde{r}(X_0; 0))]\right)\theta + o(\theta) \\
& \quad - \frac{1}{a + 1}E_{\pi_+}S_{\tau_+}^{a+1} - \left(\frac{1}{a + 2}E_{\pi_+}S_{\tau_+}^{a+2} + \alpha^a\right) + O(\theta^2) \\
& \quad + E_{\pi_+}\left(S_{\tau_+}^a(\tilde{r}(X_{\tau_+}; 0) - \tilde{r}(X_0; 0))\right) \\
& \quad + E_{\pi_+}\left(S_{\tau_+}^a(\tilde{r}(X_{\tau_+}; 0) - \tilde{r}(X_0; 0)) - (\tilde{r}^2(X_{\tau_+}; 0) - \tilde{r}^2(X_0; 0))\right)\theta + O(\theta^2).
\end{align*}

Since for some $\varepsilon > 0$, $h_{\pi_+}$ is continuously differentiable in $(0, \varepsilon)$ and continuous on $[0, \varepsilon]$, therefore, for small $\theta$, we have

\begin{align*}
(5.30) \quad h_{\pi_+}(\theta) &= h_{\pi_+}(0) + \int_0^\theta \hat{h}_{\pi_+}(\theta')d\theta'.
\end{align*}

Now, replacing $\hat{h}_{\pi_+}(\theta')$ in (5.30) by that in (5.29) and simple calculation leads that

\begin{align*}
(5.31) \quad E_{\pi_+}^\theta S_{\tau_+}^a &= h_{\pi_+}(\theta) \\
& = E_{\pi_+}S_{\tau_+}^a + \left(\frac{a}{a + 1}(E_{\pi_+}S_{\tau_+}^{a+1} + E_{\pi_+}(S_{\tau_+}^a\beta_1(X_{\tau_+}))\right)\theta + \frac{1}{2}\left(\frac{a}{a + 2}E_{\pi_+}S_{\tau_+}^{a+2} - \alpha^a\right)\theta^2 \\
& \quad + \frac{1}{2}E_{\pi_+}\left(S_{\tau_+}^{a+1}\beta_1(X_{\tau_+}) + S_{\tau_+}^a\beta_2(X_{\tau_+})\right)\theta^2 + o(\theta^2) \\
& = E_{\pi_+}S_{\tau_+}^a + \left(\frac{a}{a + 1}(E_{\pi_+}S_{\tau_+}^{a+1} + E_{\pi_+}(S_{\tau_+}^a\beta_1(X_{\tau_+}))\right)\theta + O(\theta^2).
\end{align*}

Now change $a$ to $a + 1$ in (5.31), and go back to calculate (5.29) again, we have

\begin{align*}
(5.32) \quad \hat{h}_{\pi_+}(\theta) &= \frac{a}{a + 1}E_{\pi_+}S_{\tau_+}^{a+1} + E_{\pi_+}\left(S_{\tau_+}^a\beta_1(X_{\tau_+})\right) + \left(\frac{a}{a + 2}E_{\pi_+}S_{\tau_+}^{a+2} - \alpha^a\right)\theta \\
& \quad + E_{\pi_+}\left(S_{\tau_+}^{a+1}\beta_1(X_{\tau_+}) + S_{\tau_+}^a\beta_2(X_{\tau_+})\right)\theta + O(\theta^2).
\end{align*}

Plugging (5.32) into (5.30) to get (2.16).

Next, we will prove (2.15) hold. First we assume $\theta > 0$ is fixed. Using the same argument as that from (5.4) to (5.7), we get

\begin{align*}
(5.33) \quad \mu_\theta E_{\pi_+}^\theta (\tau_+ S_{\tau_+}^a) &= \mu_\theta \int_{\tau_+}^{\infty} \int_0^\infty (E_{\omega}^\theta R_{\nu}^a)\tilde{U}_{\omega, -}^\theta (dv)d\pi_+(z) \\
& = \mu_\theta (E_{\pi_+}^\theta R_{\nu}^a)E_{\pi_+}^\theta \tau_+ + \mu_\theta \int_{\tau_+}^{\infty} \int_0^\infty (E_{\omega}^\theta R_{\nu}^a - E_{\pi_+}^\theta R_{\nu}^a)\tilde{U}_{\omega, -}^\theta (dv)d\pi_+(z) \\
& = \frac{1}{a + 1}E_{\pi_+}^\theta S_{\tau_+}^{a+1} + \alpha^a\theta + O(\theta^2).
\end{align*}
Note that the last equality in (5.33) uses $E_{\pi_+}^\theta R_{a_{\infty}}^\theta = E_{\pi_+}^\theta S_{\tau_+}^{a_{\infty}+1}/(a+1)E_{\pi_+}^\theta S_{\tau_+}$, Lemma 5 and $\mu = \theta + O(\theta^2)$. Now, we need to establish an expansion for $E_{\pi_+}^\theta S_{\tau_+}^{a_{\infty}+1}$ up to $O(\theta^2)$.

\[(5.34) \quad E_{\pi_+}^\theta (S_{\tau_+}^{a_{\infty}+1}) = E_{\pi_+}^\theta S_{\tau_+}^{a_2} - \mu_\theta E_{\pi_+}^\theta (\tau_+ S_{\tau_+}^{a_2}) + E_{\pi_+}^\theta \left(S_{\tau_+}^{a_2}(\hat{r}(X_{\tau_+}; 0) - \hat{r}(x; 0))\right) = E_{\pi_+}^\theta S_{\tau_+}^{a_2} - \frac{1}{a+2} E_{\pi_+}^\theta S_{\tau_+}^{a_2} + E_{\pi_+}^\theta \left(S_{\tau_+}^{a_2}(\hat{r}(X_{\tau_+}; 0) - \hat{r}(x; 0))\right) + O(\theta) = \frac{a+1}{a+2} E_{\pi_+}^\theta S_{\tau_+}^{a_2} + E_{\pi_+}^\theta \left(S_{\tau_+}^{a_2}(\hat{r}(X_{\tau_+} + 1) - \hat{r}(x; 0))\right) + O(\theta),\]

where the first equality is from (5.23), and the second and third equalities are from (5.7) and Lemma 3, respectively. Integrating the last display of (5.34) gives

\[(5.35) E_{\pi_+}^\theta (S_{\tau_+}^{a_{\infty}+1}) = E_{\pi_+}^\theta (S_{\tau_+}^{a_{\infty}+1}) + \left(\frac{a+1}{a+2} E_{\pi_+}^\theta S_{\tau_+}^{a_2} + E_{\pi_+}^\theta \left(S_{\tau_+}^{a_2}(\hat{r}(X_{\tau_+} + 1) - \hat{r}(x; 0))\right)\right) \theta + O(\theta^2).\]

Plugging (5.35) into (5.7) gives (2.15).

To complete the proof, it remains to show that (2.15) and (2.16) still hold when $\beta_1(X_{\tau_+})$ and $\beta_2(X_{\tau_+})$ are replaced by $\alpha_1(X_{\tau_+})$ and $\alpha_2(X_{\tau_+})$ defined in (2.10) and (2.11), respectively. This will be shown in the next lemma.

Recalling $g_1$ and $g_2$ defined in (2.8) and (2.9), respectively.

**Lemma 6** Assume the conditions of Theorem 1 hold. Then, $\hat{r}(x; 0)$ and $\bar{r}(x; 0)$ are bounded on $\mathcal{X}$, and for all $x \in \mathcal{X}$, there exist constants $c_1$ and $c_2$ such that

\[(5.36) \quad \hat{r}(x; 0) = g_1(x) + c_1,\]
\[(5.37) \quad \bar{r}(x; 0) - \hat{r}^2(x; 0) = g_2(x) + c_2.\]

**PROOF.** To establish (5.36) and (5.37), we simply assume that the random variable $\xi_1$ taking positive values, as the extension to general case is straightforward via ladder random variables. Since $r(\cdot; \theta)$ is an eigenfunction of $\lambda(\theta)$ with respect to the operator $P_\theta$, we have $P_\theta r(x; \theta) = \lambda(\theta)r(x; \theta)$, which implies that $E_x\{e^{\theta \xi_1}r(x; \theta)\} = \lambda(\theta)r(x; \theta)$. By Proposition 1 of Fuh (2004), there exists a $\delta > 0$ such that both $\lambda(\theta)$ and $r(\cdot; \theta)$ are analytic functions for $|\theta| < \delta$. Note that $\mu = \hat{\lambda}(0) > 0$. A one-term Taylor expansion for $\lambda(\theta)$ and $r(x; \theta)$ with respect to $\theta$ around $0$ entails $\lambda(\theta) = 1 + \mu \theta + O(\theta^2)$ and $r(x; \theta) = 1 + \hat{r}(x; 0)\theta + O(\theta^2)$. Therefore, for $x \in \mathcal{X}$

\[(5.38) \quad E_x\{1 + \xi_1 \theta + O(\theta^2)\}(1 + \hat{r}(X_1; 0)\theta + O(\theta^2)) = (1 + \mu \theta + O(\theta^2))(1 + \hat{r}(x; 0)\theta + O(\theta^2)).\]
Matching the coefficient of $\theta$ in (5.38) to get

\begin{equation}
\dot{r}(x;0) - E_x\dot{r}(X_1;0) = E_x\xi_1 - \mu. \tag{5.39}
\end{equation}

By conditions C1-C4, and $E_\pi|E_x\xi_1 - \mu| < \infty$, the existence and boundedness of the solution $\dot{r}(x;0)$ for the Poisson equation (5.38) follows from (17.38) and Theorem 17.4.2 of Meyn and Tweedie (1993). Furthermore, $|\dot{r}(x;0)| \leq |E_x\xi_1 - \mu|$. Hence, (5.36) follows from (5.39).

To prove (5.37). Note that $\mu = \dot{\Lambda}(0) > 0$ and $\sigma^2 = \ddot{\lambda}(0) - \mu^2$. A two-term Taylor expansion for $\lambda(\theta)$ and $r(x;\theta)$ with respect to $\theta$ around 0 entails $\lambda(\theta) = 1 + \mu\theta + \ddot{\lambda}(0)\theta^2/2 + O(\theta^3)$ and $r(x;\theta) = 1 + \dot{r}(x;0)\theta + \ddot{r}(x;0)\theta^2/2 + O(\theta^3)$. Therefore,

\begin{equation}
E_x(1 + \xi_1\dot{r} + \xi_1^2\theta^2/2 + O(\theta^3))(1 + \dot{r}(X_1;0)\theta + \ddot{r}(X_1;0)\theta^2/2 + O(\theta^3)) \\
= (1 + \mu\theta + \ddot{\lambda}(0)\theta^2/2 + O(\theta^3))(1 + \dot{r}(x;0)\theta + \ddot{r}(x;0)\theta^2/2 + O(\theta^3)). \tag{5.40}
\end{equation}

Matching the coefficient of $\theta^2$ in (5.40) to have

\begin{equation}
2E_x(\xi_1\dot{r}(X_1;0)) + E_x\ddot{r}(X_1;0) + E_x\xi_1^2 = \ddot{r}(x;0) + \mu\dot{r}(x;0) + \sigma^2 + \mu^2. \tag{5.41}
\end{equation}

A simple but tedious calculation via (5.41) leads that

\begin{equation}
(\ddot{r}(x;0) - \dot{r}^2(x;0)) - E_x(\ddot{r}(X_1;0) - \dot{r}^2(X_1;0)) \\
= E_x(\xi_1 - \mu + \dot{r}(X_1;0) - \dot{r}(x;0))^2 - E_x(\xi_1 - \mu + \dot{r}(X_1;0) - \dot{r}(X_0;0))^2. \tag{5.42}
\end{equation}

By conditions C1-C4, and because there exists a positive constant $c$ such that $E_\pi(E_x\xi_1 - \mu + E_x\dot{r}(X_1;0) - \dot{r}(X_0;0))^2 \leq c\sup_x E_x\xi_1^2 < \infty$, the existence and boundedness of the solution $\ddot{r}(x;0) - \dot{r}^2(x;0)$ of (5.42) follows from (17.38) and Theorem 17.4.2 of Meyn and Tweedie (1993). Furthermore, we have $|\dot{r}(x;0)| \leq (E_x\xi_1 - \mu + E_x\dot{r}(X_1;0) - \dot{r}(x;0))^2$. \(\square\)

References


