A CLOSED-FORM OPTION VALUATION FORMULA IN MARKOV JUMP DIFFUSION MODELS

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To improve the empirical performance of the Black-Scholes model, many alternative models have been proposed to address the leptokurtic feature of the asset return distribution, volatility smile and the effects of volatility clustering phenomenon. However, analytical tractability remains a problem for most of the alternative models. In this paper, we propose a Markov jump diffusion model, that can not only incorporate both the leptokurtic feature and volatility smile, but also present the economic features of volatility clustering and long memory. To evaluate derivatives prices, we apply Lucas’s general equilibrium framework to provide closed form formulas for option and futures prices. When the jump size follows a specific distribution, for instance a lognormal distribution and a default probability, we write explicit analytic formulas for the equilibrium prices. Through these formulas, we illustrate the effect of jumps, via stochastic intensity, on implied volatility and volatility surface as well as sensitivity analysis in stock option prices.

KEY WORDS: contingent claims, equilibrium analysis, European call option, long memory, Markov jump diffusion model, Markov modulated Poisson process, rational expectations, volatility clustering.

1. INTRODUCTION

The characterization of the arbitrage-free dynamics of stocks and interest rates, in the presence of both jumps and diffusion, has been developed by many authors in the financial literature, for instance, option pricing with Poisson type jumps (cf. Merton, 1976; Naik and Lee, 1990, and Kou, 2002), the pricing of interest rate derivatives (cf. Duffie, Pan, and Singleton, 2000, and Jarrow and Madan, 1995, 1999), and the marked point process framework (cf. Björk, Kabanov, and Runggaldier, 1997, and Glasserman and Kou, 2003). Empirical evidences and estimation methods for jump diffusion models can be found in Chernov and Ghysels (2000), Pan (2002), and Eraker (2004) among others. The motivation of including jumps along with diffusion models also be explained in the above mentioned articles and references therein.

In the general framework of marked point process developed by Björk, Kabanov, and Runggaldier (1997), they investigated arbitrage and completeness theory, and proved the existence and uniqueness

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of a martingale measure. Further developments for interest rates were in Glasserman and Kou (2003), allowing randomness in jump sizes and dependence between jump sizes, jump times and interest rates. They also gave option pricing formulas via arbitrage theory in the setting of Poisson type jump diffusion models, to which explicit option price formulas for stock market have been established by Merton (1976), Naik and Lee (1990), and Kou (2002). Note that although the Poisson type jump diffusion model can reveal the empirical phenomena of both the leptokurtic feature and volatility smile, it can not explain volatility clustering and long memory phenomena which arising in empirical studies, due to the independent increment assumption for both the diffusion and jump. The primary goal of this paper is to fill in the gap. We present a Markov jump diffusion model, to be specified in (2.5), for stock log return distribution, to which it can not only capture the empirical features, including leptokurtic phenomena, volatility smile/surface, volatility clustering, and long memory effect, but it can also provide explicit option pricing formulas in the framework of Lucas’s general equilibrium setting. When the jump size follows a specific distribution, for instance a lognormal distribution and a default probability, we write explicit analytic formulas of the equilibrium prices for European call option and futures.

![Figure 1](image.png)

**Figure 1** The dynamic process of the underlying asset price under a jump diffusion model

There are two other aspects to study the Markov jump diffusion model beyond the motivation of capturing empirical phenomena and of having closed form option price formulas. First, the arrival rates of new information, good or bad news, are different from the ‘abnormal’ vibrations of the asset price dependent on the current situation. In the jump diffusion model, as described in Merton (1976) and Kou (2002), the ‘abnormal’ vibrations in price are only due to the arrival of important information about the stock that has more than a marginal effect on price. In the Markov jump diffusion model with two states, the so-called switch jump diffusion model, the ‘information’ also depends on the economic status, expansion and contraction, for instance. In Figures 1 and 2, we compare the dynamic processes of the asset price under a jump diffusion model and a switch jump diffusion model. In the jump diffusion model, the jump rate is averaged in the years as shown in Figure 1. While in the switch jump diffusion model, the jump rates are different in different states as shown in Figure 2. It is a large one in one state and a small one in the other state.

Secondly, the jump diffusion model as well as the Markov jump diffusion model can be used to describe defaultable risk in a financial market. This provides further motivation for including jumps.
In the jump diffusion model, there is a positive probability (default probability) of immediate ruin, i.e., if the Poisson event occurs, the stock price goes to zero (cf. Samuelson, 1973; Merton, 1976, and Duffie and Singleton, 1999). In the switch jump diffusion model, we describe the risk with two different probabilities. There is a high default probability and a low default probability of immediate ruin, and the jump rates are modeled by a two-states Markov modulated Poisson process.

The remainder of this paper is organized as follows. In Section 2 we introduce the structure of the model, and make necessary assumptions in a general equilibrium framework. In Section 3 we explore the empirical phenomena of the switch jump diffusion model, which includes leptokurtic feature, and volatility clustering. In the case of the transition probability depends on the time scale, we also prove the long memory phenomenon. In Section 4 we first present a general equilibrium framework of Lucas (1978) in the Markov jump diffusion model. Then we provide a formula for European call option price under the Markov jump diffusion model with a general jump size distribution. In particular, a closed form solution is given in the case of a default risk, and a lognormal jump size distribution. By making use of the closed form option price formula, in this section, we also study numerical analysis for implied volatility and volatility surface. Sensitivity analysis of the parameters to the option price is also investigated. Conclusions are given in Section 5. All proofs are given in the Appendices.

2. GENERAL FRAMEWORK OF THE MODEL

We consider a general equilibrium framework of Lucas (1978), where there is a representative consumer in the rational expectations economy who maximizes an objective function of the form,

$$
\max_c E \left[ \int_0^\infty U(c(t), t) dt \right],
$$

where $E$ is the unconditional expectations operator, and $U(c(t), t)$ is the utility function, which is continuously differentiable, strictly concave, and strictly increasing of the consumption process $c(t)$. Throughout this paper, for simplicity, we consider the power utility function. Our assumptions are listed as follows.
Assumption 1. The power utility function. Let the utility function be

\[ U(c, t) = \begin{cases} 
  e^{-\theta t c^a} & \text{if } 0 < a < 1, \\
  e^{-\theta t} \log c & \text{if } a = 0,
\end{cases} \]

(2.2)

where \( \theta \) is the positive discount rate and \( a \) denotes the risk aversion parameter.

There exists an exogenous endowment process, denoted by \( \delta(t) \), available to the investor. If \( \delta(t) \) is Markovian, it can be shown (cf. Stokey and Lucas, 1989) that the rational expectations equilibrium price of the security, \( p(t) \), must satisfy the Euler equation

\[ p(t) = \frac{E(U_c(\delta(T), T)p(T)|\mathcal{F}_t)}{U_c(\delta(t), t)p(t)}, \quad \text{for all } T \in [t, T_0], \]

(2.3)

where \( U_c \) is the partial derivative of \( U \) with respect to \( c \), and \( T_0 \) denotes a finite liquidation date of the security. Instead, in equilibrium the investor finds it optimal to just consume the exogenous endowment, \( \delta(t) \), i.e., \( c(t) = \delta(t) \) for all \( t \geq 0 \). Under equation (2.2), or for more general utility functions, the rational expectations equilibrium price in equation (2.3) becomes

\[ p(t) = \frac{E(e^{-\theta T(\delta(T))^{a-1}p(T)|\mathcal{F}_t})}{e^{-\theta t(\delta(t))^{a-1}}}. \]

(2.4)

Assumption 2. The stochastic differential equation of the endowment. Under the physical measure \( \mathbb{P} \), the endowment follows a Markov jump diffusion model,

\[ \frac{d\delta(t)}{\delta(t-)} = \mu_1(t)dt + \sigma_1 dW_1(t) + d \left( \Phi(t) \sum_{n=1}^{\Phi(t)} (\tilde{Y}_n - 1) \right), \]

(2.5)

where \( \delta(t-) \) denotes the endowment at time \( t- \), \( \delta(t) \) denotes the endowment at time \( t \), the drift \( \mu_1(t) \) is the instantaneous return of \( \delta(t) \) at time \( t \), the volatility \( \sigma_1 \) of the stock price is assumed to be constant, \( W_1(t) \) is assumed to be a one-dimensional standard Wiener process under physical measure \( \mathbb{P} \), \( \Phi(t) \) is a Markov modulated Poisson process with finite state \( \mathcal{X} \), and \( \tilde{Y}_n \) is a sequence of jump size when the jump event happens, and is assumed to be independent for the sequence, where the endowment is from \( \delta(t-) \) to \( \tilde{Y}_n \delta(t-) \).

The resulting sample path for the endowment process will be continuous except on finite points in time, where jumps occur with the new information and the jump rate depends on various economic status. The drift term is a deterministic function of time in (2.5), while the volatility is assumed to be a constant. This setting extends the previous work by Naik and Lee (1990), and Kou (2002), in which \( \Phi(t) = N(t) \) is a Poisson process, and \( \mu_1(t) = \mu_1 \) is a constant. We call a Markov jump diffusion model as a switch diffusion model when there are only two states in the underlying Markov chain. Under the general equilibrium framework, we consider the possibility that asset prices are influenced by exogenous endowment through a series of correlated noise, which are characterized by Brownian motion and jump size.
Assumption 3. The stochastic differential equation of the underlying asset price. The price of an underlying asset $S(t)$ is also following a Markov jump diffusion model, denoted

$$
\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma dW(t) + d\left(\sum_{n=1}^{\Phi(t)} (Y_n - 1)\right)
$$

$$
= \mu(t)dt + \sigma \left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right) + d\left(\sum_{n=1}^{\Phi(t)} (\tilde{Y}_n^b - 1)\right),
$$

(2.6)

(2.7)

where $W_2(t)$ is a Brownian motion independent of $W_1(t)$, $\rho$ is the constant correlation coefficient of the underlying asset and the endowment, the drift $\mu(t)$ is the instantaneous return of $S(t)$ at time $t$, $\sigma$ denotes the constant volatility of the asset, $\{Y_n\}$ is a sequence of jump size when the jump event happens, which is connected through a power function $b \in (-\infty, \infty)$, where $Y_n = \tilde{Y}_n^b$, and $\Phi(t)$ is the same Markov modulated Poisson process as the endowment process.

We notice that the same Markov modulated Poisson process $\Phi(t)$ affects both the endowment process and the asset price process, and the jump sizes are related through a power function, where the power $b \in (-\infty, \infty)$ is an arbitrary constant, $Y_n = \tilde{Y}_n^b$. Through the diffusion coefficients and the Brownian motion part of the endowment process and the asset price process should be arbitrary different, under the Markovian assumption of the diffusion and of $\Phi(t)$, the Markov jump diffusion model can be embedded in the rational expectations equilibrium requirement in Section 4.

The Markov modulated Poisson process $\Phi(t)$ forms a particular class of doubly stochastic Poisson processes where the underlying state is governed by a homogeneous Markov chain (cf. Last and Brandt, 1995). Specifically, we consider a series of nonnegative numbers $\{\lambda_1, \lambda_2, \cdots, \lambda_I\}$, where $\lambda_i$ denotes the intensity of the doubly stochastic Poisson process $\Phi$ if the underlying Markov chain $X(t)$ is at state $i$ at time $t$. Here $\{X(t), \{P_i : i \in \mathcal{X}\}\}$ is a Markov jump process on the state space $\mathcal{X} = \{1, \cdots, I\}$, with transition rate $\Psi(i, j)$ defined as

$$
\Psi(i, j) = \begin{cases} 
\alpha(i, j), & i \neq j, \\
-\sum_{j \neq i} \alpha(i, j), & \text{otherwise}, 
\end{cases}
$$

(2.8)

for $i, j \in \mathcal{X}$. Such $\Phi$ is called the Markov modulated Poisson process. Namely, the conditional distribution of a point process $\Phi$ is $P$-almost surely equal to the distribution of a Poisson distribution with the intensity function $t \rightarrow \lambda_X(t)$. That is

$$
P(\Phi(t) = n|X) = \frac{\left(\int_0^t \lambda_X(s)ds\right)^n}{n!} \exp[-\int_0^t \lambda_X(s)ds] \quad P-a.s..
$$

(2.9)

Due to the Markovian structure of $X(t)$, we can get the joint probability of $X(t)$ and $\Phi(t)$ via Laplace inverse transform, which is denoted as $P_{ij}(n, t) := P_i(X(t) = j, \Phi(t) = n) := P(X(0) = i, X(t) = j, \Phi(t) = n)$ at time $t$ with initial $X(0) = i$ for $n \in \mathbb{Z}^+$. Define $\Psi := (\Psi(i, j))$ and $P(n, t) := (P_{ij}(n, t))$, and denote $\Lambda$ as an $I \times I$ diagonal matrix with diagonal elements $\lambda_i$. For $0 \leq z \leq 1$, define

$$
P^*(z, t) = \sum_{n=0}^{\infty} P(n, t)z^n,
$$

(2.10)
with \( P(n, 0) = (1_{n=0} D_{ij}) \), where \( D_{ij} = 1 \), if \( i = j \); = 0, otherwise. Hence \( P^*(z, 0) = (D_{ij}) \). By using the Kolmogorov’s forward equation, the derivative of \( P(n, t) \) becomes
\[
\frac{d}{dt} P(n, t) = P(n, t)(\Psi - \Lambda) + 1_{\{n \geq 1\}} P(n-1, t) \Lambda.
\]
And its unique solution is
\[
P^*(z, t) = e^{[\Psi - (1-z) \Lambda] t},
\]
where
\[
e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n,
\]
for any \((I \times I)\)-matrix \( A \) and \( A^0 := (D_{ij}) \). By making use of the Laplace inverse transform (2.10) and the solution (2.11), we get the joint distribution of \( X \) and \( \Phi \) at the time \( t \) as
\[
P(n, t) = \frac{\partial^n}{n! \partial z^n} P^*(z, t)|_{z=0}.
\]

To compute (2.11), we use the numerical inversion method proposed by Abate and Whitt (1992), to which it presents a version of the Fourier-series method for numerically inverting probability generating function. And obtains a simple algorithm with a convenient error bound from the discrete Poisson summation formula.

**Assumption 4. Jump size distribution.** We define \( \zeta_1^{(a)} := E(\tilde{Y}^a - 1) \), and assume
\[
\zeta_1^{(a-1)} < \infty,
\]
\[
\zeta_1^{(a+b-1)} < \infty,
\]
\[
E(\prod_{n=0}^{\Phi(t)} \tilde{Y}^a - 1) = \sum_{n=0}^{\infty} \sum_{i=1}^{I} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n, t) < \infty, \text{ for all } t > 0,
\]
and
\[
E(\prod_{n=0}^{\Phi(t)} \tilde{Y}^a + b - 1) = \sum_{n=0}^{\infty} \sum_{i=1}^{I} (\zeta_1^{(a+b-1)} + 1)^n \pi_i P_{ij}(n, t) < \infty, \text{ for all } t > 0,
\]
where \( \pi_i \) denotes the stationary distribution at state \( i \), and \( a \in [0, 1) \) and \( b \in (-\infty, \infty) \) are defined in equations (2.2) and (2.7), respectively.

The assumption of (2.13) and (2.14) implies that the means of the jump sizes are finite for the endowment and the asset price under the first derivative of the power utility function. Equations (2.15) and (2.16) guarantee that the means of the jump sizes in the Markov modulated Poisson process are finite under the first derivative of the power utility function.

Two specific jump size distributions will be considered in the following sections. We first take the immediate ruin as the jump event occurs (cf. Samuelson, 1973, and Merton, 1976). That is, if the
Markov modulated Poisson event happens, the stock price goes down to zero as follows:

$$\tilde{Y}^b_n = \begin{cases} 0, & \text{if event happens,} \\ 1, & \text{if event does not happen.} \end{cases}$$ \hfill (2.17)

Next, the random variable $\tilde{Y}^b_n$ is assumed to have a log-normal distribution as the jump event occurs. Let $\sigma^2_y$ denote the variance of the logarithm of $\tilde{Y}_1$, and $\mu_y$ be the mean of the logarithm of $\tilde{Y}_1$. Note that Assumption 4 is satisfied in both cases.

**Assumption 5.** The discount rate $\theta$ of the utility function. The discount rate $\theta$ should be large enough, so that

$$\theta > -(1-a)\mu_1(t) + \frac{1}{2} \sigma^2_1(1-a)(2-a) + \tilde{\eta}(t), \quad \text{for } t \in [0, T],$$ \hfill (2.18)

where $\tilde{\eta}(t) := d\log\{E[\prod_{n=1}^{\Phi(t)} \tilde{Y}_n^{a-1}]\}/dt$.

This assumption guarantees that the term structure of deterministic interest rate is positive, which will be discussed in details in Section 4. Note that when $\lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda$ and as $\tilde{\eta}(t) = d\log\{E[\prod_{n=1}^{\Phi(t)} \tilde{Y}_n^{a-1}]\}/dt = \lambda \zeta_1^{(a-1)}$, Assumption 5 reduces to $\theta > -(1-a)\mu_1(t) + \frac{1}{2} \sigma^2_1(1-a)(2-a) + \lambda \zeta_1^{(a-1)}$, the parallel assumption appeared in Kou (2002).

**Assumption 6.** The deterministic interest rate. Let $B(t,T)$ be the price of a zero coupon bond with maturity date $T$. We assume that the interest rate

$$r(t) = \lim_{T \to t} \frac{-d\log(B(t,T))}{dT}$$ \hfill (2.19)

is a deterministic function of $t$. Therefore

$$B(t,T) = e^{-\int_t^T r(s)ds}.$$

(2.20)

Note that in Assumption 3, the asset return contains the risk premium from the Brownian motion and the Markov jump risk, which is the function of $t$. In equilibrium setting, we need to connect the relationship between the asset return and interest rate, therefore the interest rate is assumed to be a deterministic function of time in Assumption 6. The details is in Section 4.

### 3. EMPIRICAL PERFORMANCE

3.1 Leptokurtic and volatility clustering features

Recall that $S(t)$ is defined in Equation (2.6). Solving this stochastic differential equation of the asset price gives the dynamics of the asset price as follows:

$$S(t) = S(0) \exp \left\{ \int_0^t (\mu(s) - \frac{1}{2} \sigma^2)ds + \sigma W(t) \right\} \prod_{n=1}^{\Phi(t)} Y_n.$$ \hfill (3.1)
By using Equation (3.1), if the time interval $\Delta t$ is small, as in the case of daily observations, the return can be approximated, ignoring the terms with order higher than $\Delta t$ and using the expansion $e^x \approx 1 + x + x^2/2$, then the dynamic return of the asset for a small $\Delta t$ is

$$
\frac{\Delta S(t)}{S(t)} \approx (\mu(t) - \frac{1}{2}\sigma^2)\Delta t + \sigma(W(t + \Delta t) - W(t)) + \sum_{n=\Phi(t)}^{\Phi(t+\Delta t)} \log Y_n + \frac{1}{2}\sigma^2(W(t + \Delta t) - W(t))^2
$$

$$
\approx \mu(t)\Delta t + \sigma Z \sqrt{\Delta t} + \sum_{n=\Phi(t)}^{\Phi(t+\Delta t)} \log Y_n,
$$

where $Z$ is the standard normal random variable. Note that the probability of the Markov modulated Poisson process $\Phi(t)$ having one jump is

$$
\sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i P(X(t) = i, X(t + \Delta t) = j, \Phi(t + \Delta t) = 1) = \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i P_{ij}(1, \Delta t).
$$

And the probability of having more than one jump is $o(\Delta t)$.

If the probability of one jump is smaller than $o(\Delta t)$, then we can ignore the multiple jumps and have

$$
\sum_{n=\Phi(t)}^{\Phi(t+\Delta t)} \log Y_n \approx \begin{cases} 
\log Y_n, & \text{with probability } \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i P_{ij}(1, \Delta t), \\
0, & \text{with probability } 1 - \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i P_{ij}(1, \Delta t). 
\end{cases}
$$

Taking summation, the return can be approximated for small $\Delta t$ in distribution by

$$
\frac{\Delta S(t)}{S(t)} \approx \mu(t)\Delta t + \sigma Z \sqrt{\Delta t} + HV,
$$

(3.2)

where $H$ is a Bernoulli random variable with $P(H = 1) = \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i P_{ij}(1, \Delta t)$, $P(H = 0) = 1 - \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i P_{ij}(1, \Delta t)$, and $V = \log Y$ is a normal random variable with mean $\mu_y = \mathbb{E}(\log Y)$ and variance $\sigma_y^2 = \text{var}(\log Y)$. Note that dropping the last term in Equation (2.6) reduces to the classical model of geometric Brownian motion, with the return, $\Delta S(t)/S(t)$, being characterized approximately by a normal density.

The probability density of $\frac{\Delta S(t)}{S(t)}$ is given by

$$
f(x) = (1 - \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i P_{ij}(1, \Delta t)) \frac{1}{\sigma \sqrt{\Delta t}} \phi\left(\frac{x - \mu(t)\Delta t}{\sigma \sqrt{\Delta t}}\right)
$$

$$
+ \left(\sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i P_{ij}(1, \Delta t)\right) \frac{1}{\sqrt{\sigma^2 \Delta t + \sigma_y^2}} \phi\left(\frac{x - \mu(t)\Delta t - \mu_y}{\sqrt{\sigma^2 \Delta t + \sigma_y^2}}\right),
$$

(3.3)
where $\phi(\cdot)$ is the standard normal density function. The mean and variance of $\frac{\Delta S(t)}{S(t)}$ are

$$E\left(\frac{\Delta S(t)}{S(t)}\right) = \mu(t) \Delta t + \mu_y \left(\sum_{i=1}^{I} \sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t)\right),$$  \hspace{1cm} (3.4)$$

and

$$\text{var}\left(\frac{\Delta S(t)}{S(t)}\right) = \sigma^2 \Delta t + \sigma_y^2 \left(\sum_{i=1}^{I} \sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t)\right) + \mu_y^2 \left(\sum_{i=1}^{I} \sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t)\right)(1 - \left(\sum_{i=1}^{I} \sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t)\right)).$$  \hspace{1cm} (3.5)$$

An important feature of this asset pricing density is that, comparing to the normal density with the identical mean and variance, it has a higher peak around the mean, and two heavier tails, or in short, the leptokurtic feature. Moreover, note that the density is not symmetric, if the mean jump size $\mu_y$ is not zero; in fact, it is skewed to the left if $\mu_y > 0$. These features have been favored by many empirical investigations.
density with mean (3.4) and variance (3.5). Figure 3.1 compares the overall shapes of the two densities, Figure 3.2 details the shapes around the peak areas, and Figures 3.3 and 3.4 show the left and right tails. The dot line is used for the normal density, and the solid line is used for the switch jump diffusion model with probability density function \( f(x) \). The parameters used here are the state \( I = 2 \), \( \Delta t = 1 \) day = 1/250 year, \( \sigma = 20\% \) per year, \( \mu(t)\Delta t = 0.06\% \) per year, \( \lambda_1 = 10 \) per year, \( \lambda_2 = 1 \) per year, the transition rate \( \alpha_1 = 0.9 \), \( \alpha_2 = 0.1 \), the jump size \( \mu_y = -2\% \); the jump volatility \( \sigma_y = 2\% \).

In words, if the Markov chain stays at state 1, there are about 10 jumps per year with average jump size of \(-2\%\) and jump volatility of \(2\%\); while if Markov chain stays at state 2, there is about 1 jump per year with average jump size of \(-2\%\) and jump volatility of \(2\%\). The transition rate is \( \alpha_1 = 0.9 \) of leaving state \( i \) for \( i = 1, 2 \), and \( \alpha_2 = 0.1 \). The jump parameters used here seem to be quite reasonable for a U.S. stock market. The leptokurtic feature is quite evident under the switch jump diffusion model. The peak of the density \( f(x) \) is about 30.9, whereas that of the normal density is about 29. The density \( f(x) \) also has heavier tails than the normal density, especially for the left tail, which could reach \(-10\%)\; while the normal density is basically confined within \(-6\%)\; from Figures 3.1 to 3.4. Additional numerical plots suggest that the feature of higher peak and heavier tails become more significant if either \( |\mu_y| \) (the jump size), \( \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i P_{ij}(1,\Delta t) \) (the transition probability), or \( \sigma_y \) (the jump volatility) increases.

Volatility clustering phenomenon, introduced by Mandelbrot (1963), states that large values of volatility are usually followed by large values, and small values are followed by small ones. Consider the asset dynamic equation (3.1) with parameters \( \lambda_1 = 30, \lambda_2 = 0.1, \alpha_1 = 0.8, \alpha_2 = 0.5, \mu_y = -0.1, \) and \( \sigma_y = 0.8 \), we observe, in Figure 4.2, that there is volatility clustering phenomenon for the process of the return under the switch jump diffusion model. Due to the property of independent increment, there is no volatility clustering phenomenon under the Black-Scholes model with the same parameters as shown in Figure 4.1. The same phenomenon can also be found in Poisson-type jump diffusion models.

### 3.2 Long memory phenomenon

In this subsection, we will show analytically that the long memory property and structural change in the jumps are intimately related in a switch jump diffusion model. Denote \( \sum_{t=1}^{T} R(t) \) as the \( T \) sums of returns \( R(t) \). A definition (cf. Diebold and Inoue, 2001) of long memory involves the rate of growth of variance of partial sums

\[
\text{var}\left(\sum_{t=1}^{T} R(t)\right) = O(T^{2d+1}),
\]

where \( d \) denotes the decay rate of the variance for the partial sum. A covariance stationary process is said to have long memory when \( 0 < d < 1 \).

There is a tight connection between this variance of partial sum definition of long memory, and the spectral and autocorrelation definitions of long memory. Because the spectral density at frequency zero is the limit of \( (1/T) \sum_{t=1}^{T} R(t) \), a covariance stationary process has long memory in the generalized spectral sense of Heyde and Yang (1997) if and only if it has long memory for some \( d > 0 \) in the variance.
of partial sum sense (see also Barndorff-Nielsen and Cox, 1989, p.13.) Our subsequent analysis is heavily influenced by the variance of partial sum definition of long memory.

Consider the case of a two-states Markov modulated Poisson process, and rewrite Equation (2.6) in discrete time,

$$ R(t) = \begin{cases} 
\mu(t) \Delta t + \sigma \Delta t Z(t) + \sum_{n=1}^{N_1(\Delta t)} \log Y_n, & \text{if } X(t) = 1, \\
\mu(t) \Delta t + \sigma \Delta t Z(t) + \sum_{n=1}^{N_2(\Delta t)} \log Y_n, & \text{if } X(t) = 2,
\end{cases} 
$$

where $Z(t) \sim N(0,1)$, $\log Y_n \sim N(\mu_y, \sigma_y^2)$, $N_1(\Delta t)$ is the Poisson process with jump rate $\lambda_1 \Delta t$ in the interval time $\Delta t$ when the Markov chain $X(t)$ stays at state 1, and $N_2(\Delta t)$ is the Poisson process with jump rate $\lambda_2 \Delta t$ in the interval time $\Delta t$ when the Markov chain $X(t)$ stays at state 2. To simplify notations, denote $V_n, \mu, \sigma, N_1, \text{ and } N_2$ as $\log Y_n, \mu(t) \Delta t, \sigma \Delta t, N_1(\Delta t)$ and $N_2(\Delta t)$, respectively, here and in the sequel. Let $\hat{\xi}(t) = (1_{\{X(t)=1\}}, 1_{\{X(t)=2\}})', \hat{V}(t) = \sum_{n=1}^{N_1} V_n, \sum_{n=1}^{N_2} V_n)'$, where $\mathbf{1}$ denotes the indicator function and $'$ denotes the transpose. In a matrix form, $R(t) = \mu + \sigma Z(t) + \hat{\xi}(t)\hat{V}(t)$.
Proposition 1 Suppose that $\lambda_1 \neq \lambda_2$, $\mu_y \neq 0$, and that $p_{11} = 1 - c_1 T^{-\kappa_1}$ and $p_{22} = 1 - c_2 T^{-\kappa_2}$, with $\kappa_1, \kappa_2 > 0$ and $0 < c_0, c_1 < 0$. Then Equation (3.6) holds for the variance of partial sums of $R(t)$ with $d = (1/2) \max(\min(\kappa_1, \kappa_2) - |\kappa_1 - \kappa_2|, 0))$.

The proof of Proposition 1 is in Appendix A.

Proposition 1 proves a long memory property from theoretical point of view. To show the empirical properties of long memory, let $\pi_1 = 0.5$, $\pi_2 = 0.5$, $\lambda_1 = 0.002$, $\lambda_2 = 50$, $\mu = 0.00004$, $\sigma = 0.0126$, $\mu_y = -0.04$, and $\sigma_y = 0.001$. The sample autocorrelation, presented in Figure 5 with $p_{11} = p_{22} = 0.9995$ and $T = 10,000$, shows that the sequence has long memory property. We also plot a realization in Figure 6. Note that the volatility has changed several times in this particular realization. We plot the corresponding average log periodogram against log frequency in Figure 7. With $\sqrt{10,000} = 100$ periodogram ordinates and 10,000 replicated realizations of the process, Figure 7 shows that the linear approximation seems to work well for the log periodogram at low frequencies.

Next, we use the log-periodogram regression estimator proposed by Geweke and Porter-Hudak (1983), and refined by Robinson (1994, 1995). Let $I(\omega_j)$ denote the sample periodogram at the $j$th Fourier frequency for $\omega_j = 2j\pi/T$, and $j = 1, 2, \cdots, \lfloor T/2 \rfloor$. The estimate of the fractional integration
Figure 7  Average of the log-periodogram ordinates

parameter, $d$, is based on the least square regression

$$\log[I(\omega_j)] = \beta_0 + \beta_1 \log(\omega_j) + \varepsilon_j,$$

where $j = 1, 2, \cdots, \tilde{T}$, and $\hat{d} = -1/2\hat{\beta}_1$. The least square estimates of $\beta_1, \hat{d}$, are asymptotically normal with standard error, $\pi(24\tilde{T})^{-1/2}$, depending only on the number of periodogram ordinates used. The use of $\tilde{T} = \sqrt{T}$ has emerged as a popular rule of thumb, which was adopted by Diebold and Inoue (2001). Here, we employ the test statistics $\hat{d}/\pi(24\tilde{T})^{-1/2}$ for the test of $d = 0$ and $\hat{d} - 1/\pi(24\tilde{T})^{-1/2}$ for the test of $d = 1$. When $p_{11} = p_{22} = 0.95$, the rejection frequencies eventually decrease as the sample size increases. It makes sense because the process is $I(0)$. In contrast, when both $p_{11}$ and $p_{22}$ are large, such as $p_{11} = p_{22} = 0.999$, the rejection frequency is increasing in $T$.

Table 1  A switch jump diffusion model: empirical sizes of nominal 5% of $d = 0$

<table>
<thead>
<tr>
<th>$T$</th>
<th>$p_{11}$</th>
<th>0.95</th>
<th>0.95</th>
<th>0.95</th>
<th>0.99</th>
<th>0.99</th>
<th>0.99</th>
<th>0.99</th>
<th>0.99</th>
<th>0.99</th>
<th>0.99</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{22}$</td>
<td>0.95</td>
<td>0.95</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
</tr>
<tr>
<td>100</td>
<td>0.272</td>
<td>0.223</td>
<td>0.208</td>
<td>0.283</td>
<td>0.268</td>
<td>0.219</td>
<td>0.245</td>
<td>0.223</td>
<td>0.172</td>
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<td></td>
</tr>
<tr>
<td>200</td>
<td>0.305</td>
<td>0.252</td>
<td>0.194</td>
<td>0.375</td>
<td>0.418</td>
<td>0.284</td>
<td>0.254</td>
<td>0.326</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>0.316</td>
<td>0.271</td>
<td>0.174</td>
<td>0.410</td>
<td>0.553</td>
<td>0.337</td>
<td>0.270</td>
<td>0.403</td>
<td>0.241</td>
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<tr>
<td>400</td>
<td>0.320</td>
<td>0.288</td>
<td>0.162</td>
<td>0.448</td>
<td>0.657</td>
<td>0.370</td>
<td>0.262</td>
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<td>500</td>
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<td>0.294</td>
<td>0.159</td>
<td>0.477</td>
<td>0.743</td>
<td>0.410</td>
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<tr>
<td>2000</td>
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<td>0.278</td>
<td>0.132</td>
<td>0.465</td>
<td>0.969</td>
<td>0.632</td>
<td>0.248</td>
<td>0.782</td>
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<tr>
<td>3000</td>
<td>0.165</td>
<td>0.263</td>
<td>0.115</td>
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<td>0.983</td>
<td>0.695</td>
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<td>0.746</td>
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<tr>
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<td>0.210</td>
<td>0.920</td>
<td>0.987</td>
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</table>

We tabulate the empirical sizes of nominal 5% tests of $d = 0$ and $d = 1$ in Tables 1 and 2, respectively. Although the persistence of the Markov switching model is increasing in $p_{11}$ and $p_{22}$, it turns out that one rejects $d = 1$ in this particular experimental design. From Tables 1 and 2, one rejects $d = 1$ and $d = 0$ when $p_{11} = 0.999$ and $p_{22} = 0.999$, therefore the value of $d$ is between 0 and 1. Hence, we conclude that the Markov jump diffusion model has long memory phenomenon via the GPH-test.
Table 2  A switch jump diffusion model: empirical sizes of nominal 5% of $d = 1$

<table>
<thead>
<tr>
<th>$T$</th>
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<th>0.95</th>
<th>0.95</th>
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<td>$p_{22}$</td>
<td>0.95</td>
<td>0.99</td>
<td>0.99</td>
<td>0.95</td>
<td>0.99</td>
<td>0.99</td>
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<tr>
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<td>0.929</td>
<td>0.946</td>
<td>0.963</td>
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<td>0.993</td>
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<td>0.998</td>
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</tr>
<tr>
<td>2000</td>
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<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
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<td>1</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
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</tbody>
</table>

4. OPTION PRICING: THEORY AND NUMERICAL ANALYSIS

4.1 General equilibrium for a Markov jump diffusion model

Suppose Assumptions 1-7 hold. In this subsection, we study the relationship of the deterministic interest rate, the discount rate, and the return of endowment in Lucas’s equilibrium setting. And we get a risk neutral probability measure which depends on the utility function.

**Proposition 2** (1) The relationship of the deterministic interest rate and the endowment return in equilibrium is

$$r(t) = \theta + (1 - a)\mu_1(t) - \frac{1}{2} \sigma^2_1(1 - a)(2 - a) - \tilde{\eta}(t) > 0 \text{ for all } t \in [0, T]. \quad (4.1)$$

(2) Recall that $\delta(t)$ follows Equation (2.5) at time $t$. Let $Z(t) := e^{\int_{0}^{t}(r(s) - \theta)ds}U_c(\delta(t), t) = e^{\int_{0}^{t}(r(s) - \theta)ds}(\delta(t))^{a-1}$. Then $Z(t)$ is a martingale under $\mathbb{P}$ at time $t$, and

$$\frac{dZ(t)}{Z(t-)} = -\tilde{\eta}(t)dt + \sigma_1(a - 1)dW_1(t) + d \left( \sum_{n=1}^{\Phi(t)}(\tilde{Y}_n^{a-1} - 1) \right). \quad (4.2)$$

Using $Z(t)$, one can define a new probability measure $d\mathbb{P}^*/d\mathbb{P} := Z(t)/Z(0)$. Under $\mathbb{P}^*$, Euler equation (2.4) holds if and only if the asset price satisfies

$$S(t) = E^*(B(t, T)S(T)|\mathcal{F}_t), \text{ for all } T \in [t, T_0]. \quad (4.3)$$

Furthermore, the rational expectations equilibrium price of an European option, with payoff $\psi_s(T)$ at the maturity $T$, is given by

$$\psi_s(t) = E^*(B(t, T)\psi_s(T)|\mathcal{F}_t)), \text{ for all } t \in [0, T]. \quad (4.4)$$
The proof of Proposition 2 is in Appendix B.

Remarks: 1. If \( \lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda \), and \( \tilde{\eta}(t) = \lambda \zeta_1^{(a-1)} \) is a constant. Let \( Z(t) := e^{rt}U_c(\delta(t), t) \). Then Equation (4.1) reduces to

\[
    r = \theta + (1 - a)\mu_1 - \frac{1}{2}\sigma_1^2(1 - a)(2 - a) - \lambda \zeta_1^{(a-1)} > 0,
\]

and Equation (4.2) reduces to

\[
    \frac{dZ(t)}{Z(t-)} = -\lambda \zeta_1^{(a-1)} dt + \sigma_1(a - 1)dW_1(t) + d \left( \sum_{n=1}^{N(t)} (\bar{Y}_n^{a-1} - 1) \right),
\]


2. If \( \alpha_k \to 0 \) for some given \( k \in \mathcal{X} \), and \( \alpha_i \to \infty \), where \( \{i : i \neq k, i \in \mathcal{X}\} \). Then \( \tilde{\eta}(t) = \lambda_k \zeta_1^{(a-1)} \), and the results in Remark 1 still hold.

Theorem 1 The model (2.6) or (2.7) satisfies the equilibrium requirement (2.4) for the zero coupon bond and the asset price if and only if

\[
    \mu(t) = r(t) + \sigma_1 \sigma \rho(1 - a) - \eta^*(t) = \theta + (1 - a)(\mu_1(t) - \frac{1}{2}\sigma_1^2(2 - a) + \sigma_1 \sigma \rho) - \eta^*(t) - \tilde{\eta}(t),
\]

where \( \eta^*(t) = d \log \left( E^*(\prod_{n=1}^{\Phi^*(t)} \bar{Y}_n^{eb}) \right) / dt. \) If (4.5) is satisfied, then under \( \mathbb{P}^* \) the equilibrium model of the asset price is

\[
    \frac{dS(t)}{S(t-)} = r(t) dt - \eta^*(t) dt + \sigma dW^*(t) + d \left( \sum_{n=1}^{\Phi^*(t)} (\bar{Y}_n^{eb} - 1) \right).
\]

Here, under \( \mathbb{P}^* \), \( W^*(t) \) is a new Brownian motion, \( \Phi^*(t) \) is a new Markov modulated Poisson process with transition probability,

\[
    Q_{ij}(m, t) = \frac{(\zeta_1^{(a-1)} + 1) \pi_i P_{ij}(m, t)}{\sum_{n=0}^{\infty} \sum_{i=1}^{I} \sum_{j=1}^{I} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n, t)},
\]

and \( \{\bar{Y}_n^*, n \geq 0\} \) are independent and identically distributed random variables with the \( \mathbb{P}^* \)-probability density

\[
    f_{\bar{Y}^*}(y) = \frac{1}{(\zeta_1^{(a-1)} + 1)} y^{a-1} f_{\bar{Y}}(y).
\]

The proof of Theorem 1 is in Appendix B.

Remarks: 3. Due to the Markovian structure of the diffusion and \( \Phi(t) \), which is also an essential part in Lucas’s equilibrium setting, \( \Phi^*(t) \) is still a Markov modulated Poisson process with transition
probability $Q_{ij}(m,t)$. Therefore, it is an equilibrium model. Note that $P_{ij}(m,t)$ keeps the same in the new transition probability $Q_{ij}(m,t)$ of (4.7), and the change is only affected by the moments of the jump size. This is also coherent to a generalization of the Girsanov theorem, as in Björk, Kabanov, and Runggaldier (1997), that changing measure corresponds to a change of drift for the underlying Brownian motion and a change of the stochastic intensity for the Markov modulated Poisson process.

4. If $\lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda$, then $\eta^*(t) = \lambda(\zeta_1^{(a+b-1)} - \zeta_1^{(a-1)})$ is a constant. Hence, under $\mathbb{P}^*$, equation (4.6) reduces to

$$\frac{dS(t)}{S(t-)} = r dt - \lambda (\zeta_1^{(a+b-1)} - \zeta_1^{(a-1)}) dt + \sigma dW^*(t) + d \left( \sum_{n=1}^{N^*(t)} (\bar{Y}_n - 1) \right),$$

(4.8)

where $N^*(t)$ is the new Poisson process. This is the general equilibrium setting for jump diffusion models. cf. Kou (2002).

5. If $\alpha_k \to 0$ for some $k \in \mathcal{X}$, and $\alpha_i \to \infty$, where $\{i : i \neq k, i \in \mathcal{X}\}$. Then the new Markov modulated Poisson process reduces to the new Poisson process with jump rate $\lambda_k(\zeta_1^{(a-1)} + 1)$. Hence, $\eta^*(t) = \lambda_k(\zeta_1^{(a+b-1)} - \zeta_1^{(a-1)})$, and the results in Remark 4 still hold.

**Corollary 1** Suppose the family $\mathcal{Y}$ of distributions of the jump size $\bar{Y}$ for the endowment process $\delta(t)$ satisfies that

$$\bar{Y}^b \in \mathcal{Y} \quad \text{and} \quad \frac{1}{\zeta_1^{(a-1)} + 1} \cdot y^{a-1} f_{\bar{Y}}(y) \in \mathcal{Y},$$

(4.9)

then the jump sizes for the asset price $S(t)$ under $\mathbb{P}$, and the jump sizes for $S(t)$ under the rational expectations risk neutral probability $\mathbb{P}^*$ all belong to the same family $\mathcal{Y}$.

4.2 Option pricing formulas

By making use the results in Section 4.1, in this subsection, we will derive European call option price formula as well as a formula for European call option on a futures contract. The European put option price formulas can be obtained via put-call parity.

Denote $C(\cdot,\cdot,\cdot,\cdot)$ as the Black-Scholes option price formula, in which it includes 5 parameters, the asset price $S(0)$ at time 0, the strike price $K$, the maturity $T$, the interest rate $\frac{1}{T} \int_0^T r(t) dt$, and the volatility $\sigma$. Denote $B(0,T)$ as the bond price in Assumption 6, $S(T)$ is the asset price at time $T$. Then

$$C(S(0),K,T,\frac{1}{T} \int_0^T r(t) dt, \sigma) = S(0)N(d(+)) - Ke^{-\frac{1}{T} \int_0^T r(t) dt} N(d(-)),$$

(4.10)

where $d(\pm) = \frac{\ln(S(0)/B(0,T)) + 1/2\sigma^2 T}{\sigma \sqrt{T}}$. Let $T^*$ be the delivery date, and denote the futures price $F(t,T^*)$ as

$$F(t,T^*) = e^{\int_t^{T^*} r(s) ds} S(t) = \frac{S(t)}{B(t,T^*)},$$

(4.11)
and define \( L(T) = e^{-\int_0^T \eta^*(t) dt} = \frac{1}{\sum_{n=0}^{\infty} \sum_{i=1}^{I} \sum_{j=1}^{I} (\zeta + 1)^n \pi_i Q_{ij}(n, T) } \) with \( \zeta = E^*(Y - 1) = E^*(\bar{Y}^b) \).

**Theorem 2**

1. From Equation (4.4), the European call option is given by
   \[
   MJ^c(0) = \sum_{m=0}^{\infty} \left( E^*(C(S(0)L(T)\bar{V}_m^b, K, T, \frac{1}{T} \int_0^T r(t) dt, \sigma) | \Phi^*(T) = m \right) \sum_{i=1}^{I} \sum_{j=1}^{I} \pi_i Q_{ij}(m, T) \right). (4.12)
   \]
   where \( \bar{V}_m^b = \prod_{n=1}^{m} \bar{Y}_n^{*b} \).

2. The European call option on a futures contract is given by
   \[
   MJ^f(0) = \sum_{m=0}^{\infty} \left( E^*(C(F(0, T^*)L(T)\bar{V}_m^b, K, T, \frac{1}{T} \int_0^T r(t) dt, \sigma) | \Phi^*(T) = m \right) \sum_{i=1}^{I} \sum_{j=1}^{I} \pi_i Q_{ij}(m, T) \right). (4.13)
   \]

The proof of Theorem 2 is in Appendix C.

We present three degenerate cases of Equations (4.12) and (4.13) as follows.

**Corollary 2**

1. If \( b \to 0 \), or \( \bar{Y}^b \to 1 \) with probability 1, then the pricing formulas (4.12) and (4.13) reduce to the corresponding Black-Scholes formulas
   \[
   MJ^c(0) \to C(S(0), K, T, \frac{1}{T} \int_0^T r(t) dt, \sigma), \quad (4.14)
   \]
   \[
   MJ^f(0) \to C(F(0, T^*), K, T, \frac{1}{T} \int_0^T r(t) dt, \sigma). \quad (4.15)
   \]

2. When \( \lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda \), then the pricing formulas (4.12) and (4.13) reduce to the Merton’s formulas (cf. Merton, 1976) with jump rate \( \lambda(\zeta_{1}^{(a-1)} + 1) \),
   \[
   J^c(0) = \sum_{m=0}^{\infty} \left( E^*(C(S(0)e^{-\lambda(\zeta_{1}^{(a+b-1)} - \zeta_{1}^{(a-1)})T} \bar{V}_m^b, K, T, \frac{1}{T} \int_0^T r(t) dt, \sigma) | N^*(T) = m \right) 
   \frac{e^{-\lambda(\zeta_{1}^{(a-1)} + 1)T}(\lambda(\zeta_{1}^{(a-1)} + 1)T)^m}{m!} \right), \quad (4.16)
   \]
   \[
   J^f(0) = \sum_{m=0}^{\infty} \left( E^*(C(F(0, T^*)e^{-\lambda(\zeta_{1}^{(a+b-1)} - \zeta_{1}^{(a-1)})T} \bar{V}_m^b, K, T, \frac{1}{T} \int_0^T r(t) dt, \sigma) | N^*(T) = m \right) 
   \frac{e^{-\lambda(\zeta_{1}^{(a-1)} + 1)T}(\lambda(\zeta_{1}^{(a-1)} + 1)T)^m}{m!} \right), \quad (4.17)
   \]
where $N^*(T)$ is the new Poisson process with jump rate $\lambda(\zeta_1^{(a-1)} + 1)$.

If $\lambda = 0$, then (4.16) and (4.17) reduce to the Black-Scholes formulas (4.14) and (4.15).

(3) If $\alpha_k \to 0$ for some $k \in \mathcal{X}$, and $\alpha_i \to \infty$ for $i \in \mathcal{X}$ and $i \neq k$, then the pricing formulas (4.12) and (4.13) reduce to (4.16) and (4.17) with intensity $\lambda_k$.

When the jump size has distribution of a default one or a lognormal, we provide explicit formulas for (4.12) and (4.13) in Corollary 3 and Corollary 4, respectively.

**Corollary 3** Suppose that the jump size follows a positive default probability as Equation (2.17).

(1) The price of the European call option is

$$MJ^c_1(0) = C(S(0), K, T, \frac{1}{T} \int_0^T (r(t) - T\eta_1(t))dt, \sigma).$$

(4.18)

(2) The price of the European call option on a futures contract is given by

$$MJ^c_{F,1}(0) = C(F(0, T^*), K, T, \frac{1}{T} \int_0^T (r(t) - T\eta_1(t))dt, \sigma).$$

(4.19)

Note that when $\lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda$, the pricing formulas (4.18) and (4.19) reduce to the Merton’s formulas with jump rate $\lambda(\zeta_1^{(a-1)} + 1)$,

$$J^c_1(0) = C(S(0), K, T, \frac{1}{T} \int_0^T r(t)dt + \lambda(\zeta_1^{(a-1)} + 1), \sigma),$$

(4.20)

$$J^c_{F,1}(0) = C(F(0, T^*), K, T, \frac{1}{T} \int_0^T r(t)dt + \lambda(\zeta_1^{(a-1)} + 1), \sigma).$$

(4.21)

Specially, if $\lambda = 0$, Equations (4.12) and (4.13) reduce to the Black-Scholes formulas (4.14) and (4.15), respectively. If $\alpha_k \to 0$ for $k \in \mathcal{X}$, and $\alpha_i \to \infty$ for $i \in \mathcal{X}$ and $i \neq k$. Then the pricing formulas (4.18) and (4.19) reduce to (4.20) and (4.21) with jump rate $\lambda_k(\zeta_1^{(a-1)} + 1)$.

When we consider two assets, one is that the underlying asset follows a diffusion model

$$\frac{dS_2(t)}{S_2(t)} = \mu_2(t)dt + \sigma dW(t),$$

(4.22)

and the other is that the underlying asset follows a Markov jump diffusion model with the default probability as

$$\frac{dS(t)}{S(t-)} = \mu(t)dt + \sigma dW(t) + d \left( \sum_{n=1}^{\Phi(t)} (Y_n - 1) \right),$$

(4.23)
where \( Y = \tilde{Y}^b \) satisfies Equation (2.17). In equilibrium, the deterministic return is \( \mu_2(t) = r(t) + \sigma_1 \rho (1 - a) \), and the European call option price formula is \( C(S(0), T, K, \frac{1}{T} \int_0^T r(t) dt, \sigma) \) in the asset (4.22). In the asset (4.23), the deterministic return is \( \mu(t) = r(t) + \sigma_1 \rho (1 - a) - \eta_1(t) \), where \( \eta_1(t) < 0 \). Therefore, we have \( \mu(t) > \mu_2(t) \), for all \( t > 0 \) in equilibrium, i.e., the second asset has the higher risk premium from the jump risk. Hence, in the Markov jump diffusion model, the European call option price formula Equation \( C(S(0), T, K, \frac{1}{T} \int_0^T (r(t) - T \eta_1(t)) dt, \sigma) \), with \( \eta_1(t) < 0 \), will be higher than that of the asset (4.22) with no jump. In particular if \( \lambda_1 = \lambda_2 = \cdots = \lambda \), the return of the asset (4.23) is \( \mu(t) = r(t) + \sigma_1 \rho (1 - a) + \lambda (\zeta^{a-1}_1 + 1) > \mu_2(t) \), and \( C(S(0), T, K, \frac{1}{T} \int_0^T r(t) dt + \lambda (\zeta^{a-1}_1 + 1), \sigma) > C(S(0), T, K, \frac{1}{T} \int_0^T r(t) dt, \sigma) \). As shown in Merton (1973, 1976), the option price is an increasing function of interest rate, therefore the option on a stock that has a positive default probability is more valuable than the one that has no default probability.

**Corollary 4** If the jump size follows a lognormal distribution with location parameter \( \mu_y \) and scale parameter \( \sigma^2_y \).

1. **The European call option price is given by**

\[
MJ^C_2(0) = \sum_{m=0}^{\infty} \left( C(S(0), K, T, 1 \int_0^T r(m, t, T) dt, \sigma(m)) \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i Q^*_ij(m, T) \right), \quad (4.24)
\]

where the deterministic interest rate \( r(m, t, T) = r(t) - T \eta_2(t) + m \gamma \) of the jump \( m \) times with the parameter \( \gamma = \mu_y + \frac{1}{2} \sigma^2_y \), and the variance of the asset price \( \sigma^2(m) = \sigma^2 + m \sigma^2/T \) with jump \( m \) times, \( Q^*_ij(m, T) \) is the new transition probability of the jump \( m \) times from the state \( i \) at time 0 to the state \( j \) at time \( T \), denoted as \( Q^*_ij(m, T) = (\zeta + 1)^m P_{ij}(m, T)/\sum_{n=0}^{\infty} \sum_{i=1}^{I} \sum_{j=1}^{J} (\zeta + 1)^n \pi_i P_{ij}(n, T) \), and the predictable process is \( \eta_2(t) = d \log \left( \sum_{n=0}^{\infty} \sum_{i=1}^{I} \sum_{j=1}^{J} (\zeta + 1)^n \pi_i P_{ij}(n, t) \right)/dt \).

2. **The European call option price on a futures contract is given by**

\[
MJ^C_{F2}(0) = \sum_{m=0}^{\infty} \left( C(F(0, T^*), K, T, 1 \int_0^T r(m, t, T) dt, \sigma(m)) \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i Q^*_ij(m, T) \right). \quad (4.25)
\]

Note that if \( \zeta \to 0 \), or \( \mu_y \to 0 \) and \( \sigma^2_y \to 0 \), the pricing formulas (4.24) and (4.25) reduce to the Black-Scholes model (4.14) and (4.15).

4.3 Volatility smile and surface

To illustrate that the Markov jump diffusion model can produce “volatility smile”, we consider a real data set in option market. The underlying asset is the IBM stock and its price is 62.66 at 2/1 in February, the maturity date at 7/19 in February, bond price 0.9851 using the Treasury Bill, the exercise
price and call value data. If we get the applicable parameter $\lambda_1 = 10$, $\lambda_2 = 5$, $\alpha_1 = 0.9$, $\alpha_2 = 0.1$, $\mu_y = -0.02$, and $\sigma_y = 0.02$ from the stock data, then we can show “volatility smile” in Figure 8.1 by using the option data and the pricing formula in Corollary 4, under the switch jump diffusion model with lognormal distribution for the jump size.

![Figure 8.1 Implied volatility under the switch jump diffusion model](image1)

Figure 8.1 Implied volatility under the switch jump diffusion model

![Figure 8.2 Volatility surface under the switch jump diffusion model](image2)

Figure 8.2 Volatility surface under the switch jump diffusion model

Figure 8 “Volatility smile” and “volatility surface” under the switch jump diffusion model

Similarly, we can show volatility surface against both maturity and strike in a three-dimensional plot. That is we consider $\sigma(S,t)$ as a function of $S$ and $t$. One is shown in Figure 8.2 that IBM call option and from 2002/3/15 with five expiry of nine different strikes. There are call option trades with an expiry of five months and strikes of 100, 105, 110, 115, 120, 125, 130, 135, 140. This implied surface represents the constant value of volatility that gives each traded option a theoretical value equal to the market value. We can see how the time dependence in implied volatility could be turned into a volatility of the underlying that was time dependent. In the case of $\sigma(S,t)$ can be deduced from volatility surface at a specific time $t_\ast$. This local volatility surface can be thought of as the market’s view of the future value of volatility when the asset price is $S$ at time $t$.

We should emphasize that the examples presented in Figures 8.1 and 8.2 are not an empirical test of the switch jump diffusion model, it is only an illustration to show that the model can produce a close fit to the empirical phenomenon.
4.4 Sensitivity analysis

Consider a sensitivity analysis for option prices under various parameters change. Table 3 reports the sensitivity of the parameters to the option prices, where the value is the stock price with small perturbation, of a 10% increase, for the indicated parameter; while all other parameters are fixed.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Perturbed Value</th>
<th>NIM $SJ^c$</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>1.1</td>
<td>12.5998</td>
<td>-0.0042</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>1.1</td>
<td>12.6072</td>
<td>0.0032</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>5.5</td>
<td>12.6220</td>
<td>0.0180</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>1.1</td>
<td>12.6044</td>
<td>0.0004</td>
</tr>
<tr>
<td>$-\ln B(t,T)$</td>
<td>0.022</td>
<td>12.6721</td>
<td>0.0681</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.22</td>
<td>13.0057</td>
<td>0.4017</td>
</tr>
<tr>
<td>$\mu_y$</td>
<td>-0.022</td>
<td>12.6230</td>
<td>0.0190</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>0.022</td>
<td>12.6168</td>
<td>0.0128</td>
</tr>
<tr>
<td>$T$</td>
<td>0.55</td>
<td>12.8726</td>
<td>0.2686</td>
</tr>
</tbody>
</table>

The parameters of base valuation are $S(0) = 100$, $K = 90$, $T = 0.5$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 5$, $\lambda_2 = 1$, $-\ln(B(0,0.5)) = 0.02$, $\sigma = 0.2$, $\mu_y = -0.02$, $\sigma_y = 0.02$, and truncated by 15, $SJ^c$ is the option price when the parameters increase 10%, and ‘Difference’ denotes the difference of the option price between base valuation and $SJ^c$ when the parameters increase 10%.

Note that in Table 3, the volatility $\sigma$ has most significant effect, although it is fixed and known in most time period. For the parameter $\alpha_1$, if it has a 10% increases, then the Markov chain will leave state 1 rapidly, so that the decreasing of the jump rate makes the option price decreasing, because the risk premium decreases in the jump risk. Similarly, if $\alpha_1$ increases 10%, then the Markov chain will leave state 2 rapidly, so that the increasing of the jump rate makes the option price increases. For the concern of the parameters $\lambda_1$ and $\lambda_2$, if one increases 10% while the other one is fixed, the increasing of the jump rate causes the option price increases. If the yield $-\ln B(t,T) = 0.02$ increases to $-\ln B(t,T) = 0.022$, then the European option price increases. For the whole time varying parameters such as $\lambda_1$, $\lambda_2$, $\alpha_1$, $\alpha_2$, $B(t,T)$, $\mu_y$, and $\sigma_y$, we observe that the parameter $\sigma$ has the most effect than any other parameters.

5. CONCLUSIONS

In this paper, we propose a Markov jump diffusion model, which can capture leptokurtic and asymmetric features, volatility clustering, and volatility smile. Under the general equilibrium setting of Lucas for the Markov jump diffusion model, closed form formulas for the European option prices and futures contract have been developed. When the jump size of a switch jump diffusion model follows a lognormal
distribution, we report detailed numerical analysis, and give a computation method via the numerical inversion method to compute the option prices.

An exact closed form formula provides useful insight into European option pricing in the Markov jump diffusion model. It not only explains the impact of regime switching in the jump rate on option pricing, but also sheds light on analytical approximation, where it accelerates the computation of European option pricing in the Markov jump diffusion model. It has a number of further applications: for instance, it can be used to compute hedge ratios and implied Markov jump diffusion model parameters, i.e., to calibrate the parameters by using the implied volatility surface. The approximation approach can facilitate empirical studies on index options, which are, in many cases, European in style. This feature is worthy of further exploration and may have many uses in other applications.

APPENDIX A: LONG MEMORY FOR THE SWITCH JUMP DIFFUSION MODEL

Proof of Proposition 1.

The variance of the partial sum of the return is

\[
\text{var}(\sum_{t=1}^{T} R(t)) = \text{var}(\sum_{t=1}^{T} \mu + \sigma Z(t) + \xi(t)\hat{V}(t)) = \text{var}(\sum_{t=1}^{T} \xi(t)\hat{V}(t)) + \sigma^2 T
\]

\[
= \sum_{t=1}^{T} \text{var}(\xi(t)\hat{V}(t)) + 2 \sum_{j=1}^{T} (T-j)\text{cov}(\xi(t)\hat{V}(t), \xi(t-j)\hat{V}(t-j)) + \sigma^2 T. \tag{A1}
\]

To evaluate (A1), we first compute \(\text{var}(\xi(t)\hat{V}(t))\). Note that

\[
\text{var}(\xi(t)\hat{V}(t)) = \text{var}(1_{\{X(t)=1\}} \sum_{n=1}^{N_1} V_n + 1_{\{X(t)=2\}} \sum_{n=1}^{N_2} V_n)
\]

\[
= \frac{(1-p_{11})}{(2-p_{11} - p_{22})} \lambda_1 \sigma_y^2 + \frac{(1-p_{22})}{(2-p_{11} - p_{22})} \lambda_2 \sigma_y^2 + \frac{(1-p_{11})(1-p_{22})}{(2-p_{11} - p_{22})} (\lambda_1 - \lambda_2)^2 \mu_y. \tag{A2}
\]

Next, we compute \(\text{cov}(\xi(t)\hat{V}(t), \xi(t-j)\hat{V}(t-j))\), which equals

\[
= E((1_{\{X(t-j)=1\}} \sum_{n=1}^{N_1} V_n + 1_{\{X(t-j)=2\}} \sum_{n=1}^{N_2} V_n)(1_{\{X(t)=1\}} \sum_{n=1}^{N_1} V_n + 1_{\{X(t)=2\}} \sum_{n=1}^{N_2} V_n)) \tag{A3}
\]

\[
-(E(1_{\{X(t-j)=1\}} \sum_{n=1}^{N_1} V_n + 1_{\{X(t-j)=2\}} \sum_{n=1}^{N_2} V_n)E((1_{\{X(t)=1\}} \sum_{n=1}^{N_1} V_n + 1_{\{X(t)=2\}} \sum_{n=1}^{N_2} V_n)).
\]

To compute (A3), denote \(M^k_{ij}\) as the \(k\)-period-ahead transition probabilities from \(i\) to \(j\), and let \(\nu_i, \ i = 1, 2\), be the eigenvalues of the transition matrix,

\[
P = \begin{bmatrix}
p_{11} & 1-p_{11} \\
1-p_{22} & p_{22}
\end{bmatrix}.
\]
APPENDIX B: GENERAL EQUILIBRIUM FOR MARKOV JUMP DIFFUSION MODELS

Proof of Proposition 2.

(1) Since \( B(T,T) = 1 \), Equation (2.4) yields

\[
B(t,T) = e^{-\theta(T-t)} E(\{ (\delta(T))^{a-1} | F_t \}) / (\delta(t))^{a-1}. \tag{B1}
\]

By making use of

\[
\left( \frac{\delta(T)}{\delta(t)} \right)^{a-1} = \exp\{ (a - 1) \int_t^T (\mu_1(s) - \frac{1}{2} \sigma_1^2) ds + \sigma_1(a - 1)(W_1(T-t)) \} \left( \prod_{n=1}^m \tilde{Y}_n^{a-1} \right),
\]

\[
E\left( \prod_{n=1}^m \tilde{Y}_n^{a-1} | F_t \right) = E\left( E\left( E\left( \prod_{n=1}^m \tilde{Y}_n^{a-1} | X(0) = i, X(t) = j, \Phi(T-t) = m \right) \right) \right)
= \sum_{i=1}^I \sum_{j=1}^I \sum_{n=0}^\infty (\zeta_i^{(a-1)} + 1)^n \pi_i P_{ij}(n, T-t),
\]

where \( \zeta_i^{(a-1)} = (a - 1) \int_0^T \mu_1(s) ds + \sigma_1(a - 1)(W_1(T-t)) \).
\[ B(t, T) = \exp\left\{ -(T-t)\theta + \int_t^T (a - 1)(\mu_1(s) - \frac{1}{2}\sigma_1^2)ds + \frac{1}{2}\sigma_1^2(a - 1)^2(T-t) \right\} \]
\[ \times \left( \sum_{i=1}^I \sum_{j=1}^J \sum_{n=0}^\infty (\zeta_i^{(a-1)}) + 1)^n \pi_j(n, T-t) \right) \]
\[ = \exp\{-\int_t^T r(s)ds\}. \]

Taking logarithm, differentiating \( T \) and limiting \( T \to t \), we get
\[ r(t) = \theta + (1 - a)\mu_1(t) - \frac{1}{2}\sigma_1^2(1 - a)(2 - a) \]
\[ d\log \left( \sum_{i=1}^I \sum_{j=1}^J \sum_{n=0}^\infty (\zeta_i^{(a-1)}) + 1)^n \pi_j(n, T-t) \right) \]
\[ - \lim_{T \to t} \frac{d\log\left( E\left( \prod_{n=1}^{\Phi(T-t)} \tilde{Y}_n^{a-1} \right) \right)}{dT} \]
\[ = \theta + (1 - a)\mu_1(t) - \frac{1}{2}\sigma_1^2(1 - a)(2 - a) - \lim_{T \to t} \frac{d\log\left( E\left( \prod_{n=1}^{\Phi(T-t)} \tilde{Y}_n^{a-1} \right) \right)}{dT} \]
\[ = \theta + (1 - a)\mu_1(t) - \frac{1}{2}\sigma_1^2(1 - a)(2 - a) - \tilde{\eta}(t) > 0, \]

where \( \tilde{\eta}(t) = d\log \left( E\left( \prod_{n=1}^{\Phi(t)} \tilde{Y}_n^{a-1} \right) \right)/dt = d\log \left( \sum_{i=1}^I \sum_{j=1}^J \sum_{n=0}^\infty (\zeta_i^{(a-1)}) + 1)^n \pi_j(n, t) \right)/dt, \) or
\[ e^{\int_t^T \eta(s)ds} = 1/\left( \sum_{i=1}^I \sum_{j=1}^J \sum_{n=0}^\infty (\zeta_i^{(a-1)}) + 1)^n \pi_j(n, t) \right). \]

(2) Note that Equation (B1) implies that
\[ e^{-\int_t^T r(s)ds} = E(U_c(\delta(T), T)/U_c(\delta(t), t)|\mathcal{F}_t), \quad \text{(B2)} \]
which shows that \( Z(t) \) is a martingale under \( \mathbb{P} \). Furthermore, Assumption 2 and Equation (4.1) lead that
\[ Z(t) = (\delta(0))^{a-1} \exp\left\{ \int_0^t (r(s) - \theta)ds + (a - 1) \int_0^t (\mu_1(s) - 1/2\sigma_1^2)ds + \sigma_1(a - 1)W_1(t) \right\} \left( \prod_{n=1}^{\Phi(t)} \tilde{Y}_n^{a-1} \right) \]
\[ = (\delta(0))^{a-1} \exp\{-\int_0^t \tilde{\eta}(s)ds - \frac{1}{2}\sigma_1^2(a - 1)^2t + \sigma_1(a - 1)W_1(t) \} \left( \prod_{n=1}^{\Phi(t)} \tilde{Y}_n^{a-1} \right), \]
from which Equation (4.2) follows. Now by Equations (2.4) and (B2),
\[
\varphi_s(t) = \frac{E(U_c(\delta(T), T))}{U_c(\delta(t), t)} = e^{-\int_t^T r(s)ds}E\left(\frac{Z(T)}{Z(t)}\varphi_s(T)|\mathcal{F}_t\right) = B(t, T)E^*(\varphi_s(T)|\mathcal{F}_t).
\]

**Proof of Theorem 1.**

By using the Girsanov theorem for the Markov jump diffusion model (see Björk, T., Kabanov, Y., and Runggaldier, W., 1997) we have, under \( \mathbb{P}^* \), \( W^*_1(t) := W_1(t) - \sigma_1(a - 1)t \) is a Brownian motion. And under \( \mathbb{P}^* \) the transition probability of \( \Phi^*(t) = m \) given \( X(0) = i \) and \( X(t) = j \) is
\[
Q_{ij}(m, t) = \frac{(\zeta_1^{(a-1)} + 1)^{m} P_{ij}(m, t)}{\sum_{i=1}^{l} \sum_{j=1}^{l} \sum_{n=0}^{\infty} (\zeta_1^{(a-1)} + 1)^{n} \pi_i P_{ij}(n, t)},
\]
and \( \tilde{Y}_n^* \) has probability density \( f_{\tilde{Y}}(y) = \frac{1}{(\zeta_1^{(a-1)} + 1)^{a-1} f_{\tilde{Y}}(y)} \).

We compute the pricing measure by exponential embedding given \( \Phi(t) = m, X(0) = i \) and \( X(t) = j \),
\[
d\mathbb{P}^*(W^*_1(t), \Phi^*(t) = m, X(0) = i, X(t) = j, \tilde{Y}_1^*, \ldots, \tilde{Y}_m^*)
\] 
\[= \exp\left\{ -\frac{1}{2} \sigma_1^2(a - 1)^2 t - \int_0^t \tilde{\eta}(s)ds + \sigma_1(a - 1)W_1(t) \right\} \left( \prod_{n=1}^{m} \tilde{Y}_n^{a-1} \right) d\mathbb{P}(W_1(t)) \]
\[\frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(W_1(t) - \sigma_1(a - 1)t)^2}{2t} \right\} \left( \prod_{n=1}^{m} \tilde{Y}_n^{a-1} \right) d\mathbb{P}(\Phi(t) = m, X(0) = i, X(t) = j, \tilde{Y}_1, \ldots, \tilde{Y}_m) \]

Then the new Brownian is \( W^*_1(t) := W_1(t) - \sigma_1(a - 1)t \), and we integrate out the Brownian motion to have
\[
d\mathbb{P}^*(\Phi^*(t) = m, X(0) = i, X(t) = j, \tilde{Y}_1^*, \ldots, \tilde{Y}_m^*)
\] 
\[= \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(W_1(t) - \sigma_1(a - 1)t)^2}{2t} \right\} \left( \prod_{n=1}^{m} \tilde{Y}_n^{a-1} \right) d\mathbb{P}(\Phi(t) = m, X(0) = i, X(t) = j, \tilde{Y}_1, \ldots, \tilde{Y}_m) \].
For the jump sizes, note that \( \{\tilde{Y}_1, \cdots, \tilde{Y}_m\} \) are independent and identically distributed random variables, therefore

\[
d\mathbb{P}^x(\Phi^*(t) = m, X(0) = i, X(t) = j, \tilde{Y}_1^*, \cdots, \tilde{Y}_m^*) = \frac{1}{(\zeta_1^{(a-1)} + 1)^m} \prod_{i=1}^I \frac{1}{(\zeta_1^{(a-1)} + 1)^{y_i}} \prod_{j=1}^J \frac{1}{(\zeta_1^{(a-1)} + 1)^{y_j}} \prod_{n=0}^{\infty} \frac{1}{(\zeta_1^{(a-1)} + 1)^{\pi_i P_{ij}(n, t)}}
\]

Integrating out the behavior of \( \tilde{Y}_n \) to obtain

\[
d\mathbb{P}^x(\Phi^*(t) = m, X(0) = i, X(t) = j) = \frac{(\zeta_1^{(a-1)} + 1)^m P_{ij}(m, t)}{\sum_{i=1}^I \sum_{j=1}^J \sum_{n=0}^{\infty} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n, t)}
\]

Therefore, the new transition probability is

\[
Q_{ij}(m, t) = \frac{(\zeta_1^{(a-1)} + 1)^m P_{ij}(m, t)}{\sum_{i=1}^I \sum_{j=1}^J \sum_{n=0}^{\infty} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n, t)}
\]

And the dynamics of \( S(t) \) is given by

\[
\frac{dS(t)}{S(t-)} = \mu(t) dt + \sigma \{ \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \} + d \left( \sum_{n=1}^{\infty} (\tilde{Y}_n^b - 1) \right)
\]

Since

\[
E^* \left( \prod_{n=1}^I \tilde{Y}_n^{ab} \mid \mathcal{F}_t \right) = E^* \left( E^* \left( E^* \left( \prod_{n=1}^m \tilde{Y}_n^{ab} \right) \mid X(0) = i, X(t) = j, \Phi(t) = m \right) \right)
\]

\[
= \sum_{i=1}^I \sum_{j=1}^J \sum_{n=0}^{\infty} \frac{(\zeta_1^{(a+b-1)} + 1)^n \pi_i Q_{ij}(n, t)}{(\zeta_1^{(a-1)} + 1)^n}
\]

26
we have
\[
\eta^*(t) = \frac{d\log\{E^\ast\left(\prod_{n=1}^{I} \tilde{Y}_{n}^b\right)\}}{dt} = \frac{d\log\left(\sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{n=0}^{\infty} (\zeta_1^{(a+b-1)} + 1)^n \pi_i P_{ij}(n,t)\right)}{dt},
\]

(B3)

or
\[
e^{-\int_{0}^{t} \eta^*(s)ds} = \frac{\sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{n=0}^{\infty} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n,t)}{\sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{n=0}^{\infty} (\zeta_1^{(a+b-1)} + 1)^n \pi_i P_{ij}(n,t)}.
\]

Hence, under the new Markov jump diffusion model, the dynamic process of \( S(t) \) is
\[
\frac{dS(t)}{S(t-)} = \{\mu(t) + \sigma_1 \sigma \rho (a - 1) + \eta^*(t)\}dt - \eta^*(t)dt + \sigma \{\rho dW_1^\ast(t) + \sqrt{1 - \rho^2} dW_2(t)\}
\]
\[
+ d \left( \sum_{n=1}^{I} (\tilde{Y}_{n}^b - 1) \right).
\]

(B4)

If \( S(t) \) satisfy (B4) in the equilibrium setting (4.3), we must have \( \mu(t) + \sigma_1 \sigma \rho (a - 1) + \eta^*(t) = r(t) \) from which (4.5) follows. On the other hand, if (4.5) is satisfied, under the measure \( \mathbb{P}^\ast \), then the dynamics of \( S(t) \) is given by
\[
\frac{dS(t)}{S(t-)} = r(t)dt - \eta^*(t)dt + \sigma \{\rho dW_1^\ast(t) + \sqrt{1 - \rho^2} dW_2(t)\} + d \left( \sum_{n=1}^{I} (\tilde{Y}_{n}^b - 1) \right),
\]
from which Equation (4.6) follows.

**APPENDIX C: OPTION PRICING FORMULAS**

**Proof of Theorem 2.**

(1) Under \( \mathbb{P}^\ast \) measure, the dynamic process of the asset price \( S(t) \) in Equation (4.6) becomes
\[
S(T) = S(0) \exp\left\{ \int_{0}^{T} (r(t) - \eta^*(t) - 1/2\sigma^2)dt + \sigma W^\ast(T) \right\} \left( \prod_{n=1}^{I} \tilde{Y}_{n}^b \right),
\]

(C1)

where \( W^\ast(T) \) is a normal random variable with mean 0 and variance \( T \), and the jump size \( \tilde{Y}_{n}^* \) are i.i.d. random variables from distribution \( f_{\tilde{Y}_{n}}(y) \). Under the conditions of \( X(0) = i \) and \( X(T) = j \), the
conditional jump times $\Phi^*(T) = m$, $\bar{V}_{m}^{sb} = \prod_{n=1}^{m} \bar{V}_{n}^{sb}$, and $Z$ is standard normal distribution, we can rewrite (C1) as

$$S(T) = S(0) \exp \{ \int_{0}^{T} (r(t) - \eta^*(t) - 1/2\sigma^2) dt + \sigma \sqrt{T} Z \} \bar{V}_{m}^{sb}$$

with transition probability $Q_{ij}(m, T)$.

Hence, under the rational expectations equilibrium price (4.4) of the call option in the Markov jump diffusion model is

$$MJ^c(0) = E^*(B(0, T)(S(T) - K)^+ | F_0)$$

$$= B(0, T)E^*(\{S(T)1_{\{S(T)>K\}} | F_0\}) - B(0, T)KE^*1_{\{S(T)>K\}} | F_0)$$

$$= \sum_{m=0}^{\infty} \left( E^*(\{(S(0) \exp \{ \int_{0}^{T} (\eta^*(t) - 1/2\sigma^2) dt + \sigma \sqrt{T} Z \} \bar{V}_{m}^{sb} \} 1_{\{Z< 1^{\infty} \exp \left( -\int_{0}^{T} \eta^*(t) dt \right) \bar{V}_{m}^{sb} \}) | \Phi^*(T) = m) \right)$$

$$- B(0, T)KE^*1_{\{S(T)>K\}} | X(0) = i, X(T) = j, \Phi(T) = m) \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i Q_{ij}(m, T)$$

$$= \sum_{m=0}^{\infty} \left( E^*(S(0) \exp \{- \int_{0}^{T} \eta^*(t) dt \} \bar{V}_{m}^{sb} 1_{\{Z< 1^{\infty} \exp \left( -\int_{0}^{T} \eta^*(t) dt \right) \bar{V}_{m}^{sb} \}) | \Phi^*(T) = m) \right)$$

$$- B(0, T)KE^*1_{\{Z< 1^{\infty} \exp \left( -\int_{0}^{T} \eta^*(t) dt \right) \bar{V}_{m}^{sb} \}) | \Phi^*(T) = m) \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i Q_{ij}(m, T)$$

$$= \sum_{m=0}^{\infty} \left( \frac{E(C(S(0)e^{-\int_{0}^{T} \eta^*(t) dt} \bar{V}_{m}^{sb}, K, T, \frac{1}{T} \int_{0}^{T} r(t) dt, \sigma) | \Phi^*(T) = m)}{\sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i Q_{ij}(m, T)} \right) .$$

Here $C(S(0)e^{-\int_{0}^{T} \eta^*(t) dt} \bar{V}_{m}^{sb}, K, T, \frac{1}{T} \int_{0}^{T} r(t) dt, \sigma)$ is the option price in the Black-Scholes formula with the stock price $S(0)e^{-\int_{0}^{T} \eta^*(t) dt} \bar{V}_{m}^{sb}$, the strike price $K$, the maturity day $T$, the deterministic interest rate $r(t)$, and the volatility of the stock price $\sigma$. Define $E^*$ as the expectation operator under the
distribution $\tilde{V}_{mb}$, and
\[
d^*(\pm) = \frac{\log \left( S(0)e^{-\int_0^T \eta^*(t)dt} \tilde{V}_{mb}/K \right) + \int_0^T r(t)dt \pm 1/2\sigma^2T}{\sigma\sqrt{T}}.
\]

(2) Using Equations (4.11) and (4.4), we obtain
\[
E^*(B(0,T)(F(T,T^*))) = E^*(B(0,T)(S(T)B(T,T^*) - K)\})
= \frac{1}{B(T,T^*)} E^*(B(0,T)(S(T) - B(T,T^*)K)\})
= \frac{1}{B(T,T^*)} \sum_{m=0}^{\infty} \left( E^*(S(0)\exp\{-\int_0^T \eta^*(t)dt\} \tilde{V}_{mb}^{* b}N(d^*_F(+)) - B(0,T)B(T,T^*)KN(d^*_F(-)) \right)
|\Phi^*(T) = m) \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i Q_{ij}(m,T) \right) \right)
= \sum_{m=0}^{\infty} \left( E^*(C(F(0,T^*)e^{-\int_0^T \eta^*(t)dt} \tilde{V}_{mb}^{* b}, K, T, \frac{1}{T} \int_0^T r(t)dt, \sigma)|\Phi^*(T) = m) \sum_{i=1}^{I} \sum_{j=1}^{J} \pi_i Q_{ij}(m,T) \right),
\]
where
\[
d^*_F(\pm) = \frac{\log \left( S(0)e^{-\int_0^T \eta^*(t)dt} \tilde{V}_{mb}^{* b}/(KB(0,T^*)) \right) + \int_0^T r(t)dt \pm 1/2\sigma^2T}{\sigma\sqrt{T}}
= \frac{\log \left( F(0,T^*)e^{-\int_0^T \eta^*(t)dt} \tilde{V}_{mb}^{* b}/K \right) + \int_0^T r(t)dt \pm 1/2\sigma^2T}{\sigma\sqrt{T}}.
\]
This completes the proof.

REFERENCES


